PageRank

Ben Burns, Dan Magazu, Lucas Chagas,
Thomas Webster, Trung Do

Fall 2021

Table of Contents

Motivation

- Problem: the internet has a *lot* of web pages
- A lot of the information out there either isn't relevant to us, or is inaccurate
- Motivation: we want a program that, when provided a phrase, returns webpages with information relevant to the input
- Intuitive solution: return back websites that either contain that phrase, or contain similar phrases
- This helps us find more *relevant* pages, but we can't know if we're getting the best information (much less *accurate* information) without manually going through each result
- We desire a stronger solution

PageRank

- Invented by Sergey Brin and Larry Page (1998)[1]
  - Publication marks them becoming co-founders of Google
- Idea: we want some way to numerically score each webpage based on how "important" it is
- Algorithm numerically scores each page p based on
  - How many other pages link to p (or "cite" it)
  - The "importance" of each of p's citations
- We then numerically order pages to rank them
- PageRank: the procedure for scoring each website
- Google: the database that indexes the PageRank of each website for search

[1] Many use the year of the original manuscript, 1996

Underlying Assumptions

- Running our basic search engine gives us a collection of pages with information relevant to our query
- Assumption: More "important" and useful websites will be the ones with proportionally more inbound links
- Pages with very reliable, primary information are likely to be cited by lots of website authors, and therefore will have lots of "flow" into them
- Assumption: websites with no outgoing links are treated as linking to every other website
- Else, these pages would just take in flow without ever redistrubting, and eventually all users get stuck on these pages

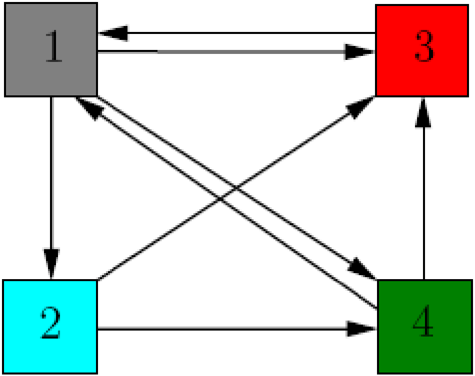
Finding Less Important Information

- A worry you may have is that we'll only find pages with lots of inbound links
- You can still find niche information by making your query more specific so that it won't match more general pages
- Searching "PageRank" will likely get you Wikipedia, but "Anatomy of PageRank architecture" gives you the original research literature.
- Your returned urls are still proportionally important results, your query just filtered out the numerically more "important", yet less relevant pages.

Formalizing the PageRank problem

- We're going to construct a directed graph $G = (V, E)$
- For each website we consider, we construct a node $v_i \in V$
- For two distinct nodes $v_i,\ v_j \in V$, the *directed* edge $v_i v_j \in E$ iff there is a link on website i that goes to website j.
- If v_i and v_j are not distinct (a website is linking to itself), we ignore the link and do not construct a loop edge.
  - G is not a psuedograph
- Multiple hyperlinks on page i to page j are all represented by the single, directed edge
  - G is not a multigraph.

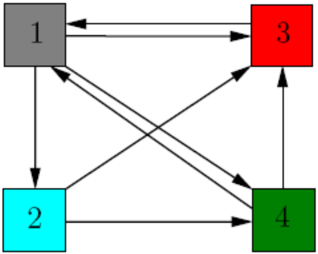
Visual Representation



- We have a set of 4 websites
- Each edge represents a hyperlink from the origin node to the destination node

Applying PageRank Values

- At first, we assign each vertex $v \in V$ with a weight of $\frac{1}{|V|}$.
- All vertices have equal weight, and our weights sum up to 1. The weight of a particular vertex v_i is denoted $PR(v_i)$.
- Consider the set O_i of vertices that v_i has an edge to:
  - $O_i = \{v_j | v_i v_j \in E\}$ (O for outbound)
- A user on page 1 can choose to click a link to traverse to either 2, 3, or 4. In other words, $O_1 = \{2, 3, 4\}$.



Traversing from Page to Page

- Assumption: A user on page i user is equally likely to choose to visit each vertex in O_i (our set of vertices that v_i cites)
- In our example from before, the probabilty that our user on page 1 visits page 2 is $P(v_2) = \frac{1}{|O_1|} = \frac{1}{3}$
- So a third of page 1's visitors will "flow" to page 2, a third to page 3, and a third to page 4
- From v_i, $P(v_j) = \frac{1}{|O_i|}$ if $v_j \in O_i$, and 0 else.

Iteration Model

- All at once, all users will click one of the links on their current page
- At "click 0", the PageRank value of all vertices $PR(v, 0) = \frac{1}{|V|}$
- On click 1, each page will equally split its PageRank value among the pages it cites
- Page 1 starts with $PR(v_1, 0) = \frac{1}{4}$, and contributes $\frac{1}{12}$ to each of 2, 3, and 4 on click 1
- Conversely, page 1 will receive nothing from page 2, all of $PR(v_3, 0)$, and half of $PR(v_4, 0)$. So after click 1,

$$\begin{aligned} PR(v_1, 1) &= 0 \cdot PR(v_2, 0) + PR(v_3, 0) + \frac{1}{2} PR(v_4, 0) \\ &= \frac{1}{4} + \frac{1}{8} = \frac{3}{8} \end{aligned}$$

What else do we need?

- We want a function $f(v_i, v_j)$ defined as:

$$f(v_i, v_j) := \frac{\text{\# of links from page } i \text{ to page } j}{\text{total \# of links out of page } j}$$

- In our simplified case, this will be $\dfrac{1}{|O_j|}$ if j links to i, and 0 else
- Additionally, at some point, our user who is randomly clicking links will eventually stop clicking links
- At each iteration we consider a constant probability $d :=$ the probability that our user stops traversing the graph
  - Various studies have settled on $d \approx 0.85$

General PageRank equation

- For an arbitrary graph, we obtain the PageRank value of vertex v_i at iteration $t+1$ from the following:

$$PR(v_i, t+1) = \frac{1-d}{|V|} + d \sum_{v_j \in I_i} f(v_i, v_j) PR(v_j, t)$$

O_i is set of vertices with edges outbound from v_i

I_i is the set of vertices with edges inbound to v_i

$$f(v_i, v_j) = \begin{cases} \frac{1}{|O_j|} & i \in O_j \\ 0 & \text{else} \end{cases}$$

Matrix Construction

- The idea behind PageRank is construct and solve a matrix
- Each row i represents the PageRank equation for $PR(v_i, t)$
- Each column j represents the PageRank value contributed by j to each vertex v_i
- The solved for $t+1$ iteration PageRank values will therefore be a vector:

$$\begin{bmatrix} PR(v_1, t+1) \\ PR(v_2, t+1) \\ \vdots \\ PR(v_n, t+1) \end{bmatrix}$$

Boiling it Down to an Eigenvalue

- For our iteration process, we want to be able to input a vector of PageRank values at iteration t vector $\mathbf{R}(t)$, and get back a vector of PageRank values at iteration $t+1$ ($\mathbf{R}(t+1)$)
- In other words, we can determine PageRank values as
$$\begin{bmatrix} PR(v_1, t+1) \\ PR(v_2, t+1) \\ \vdots \\ PR(v_n, t+1) \end{bmatrix} = A \cdot \begin{bmatrix} PR(v_1, t) \\ PR(v_2, t) \\ \vdots \\ PR(v_n, t) \end{bmatrix}$$
for an $n \times n$ matrix A
- For our values to converge, our input vector will approximately equal our output vector
- In other words, we're searching for an eigenvector

Defining Our Matrix A

- Observe our general PageRank equation

$$PR(v_i, t+1) = \frac{1-d}{|V|} + d \sum_{v_j \in I_i} f(v_i, v_j) PR(v_j, t)$$

- We define the matrix A to the $n \times n$ matrix where $A_{ij} := f(v_i, v_j)$
- We then multiply this matrix by our t iteration PageRank vector
- Finally, we add the result by a n length vector of $\dfrac{1-d}{|V|}$ terms

$$\mathbf{R}(t+1) = \begin{bmatrix} (1-d)/|V| \\ (1-d)/|V| \\ \vdots \\ (1-d)/|V| \end{bmatrix} + \begin{bmatrix} f(v_1, v_1) & \dots & f(v_1, v_n) \\ f(v_2, v_1) & \dots & f(v_2, v_n) \\ \vdots & \ddots & \vdots \\ f(v_n, v_1) & \dots & f(v_n, v_n) \end{bmatrix} \cdot \mathbf{R}(t)$$

Determining Existance of *PR* Eigenvector

- Each column j of our matrix is the series of terms $f(v_1, v_j)$ through $f(v_n, v_j)$
- Recall that f is defined as

$$f(v_i, v_j) = \begin{cases} \frac{1}{|O_j|} & i \in O_j \\ 0 & \text{else} \end{cases}$$

- All entries are non-negative
  - $|O_j|$ entries will hold a value of $\frac{1}{|O_j|}$
  - All other values will be zero
- Further, the sum of the entries in any column j is

$$\sum_{v_i \in V} f(v_i, v_j) = |O_j| \frac{1}{|O_j|} = 1$$

and A is column stochasic matrix

Perron-Frobenius Theorem

Perron-Frobenius Theorem:
If A is a positive, column stochastic matrix, then
1. 1 is an eigenvalue of multiplicity one.
2. 1 is the largest eigenvalue: all the other eigenvalues have absolute value smaller than 1.
3. the eigenvectors corresponding to the eigenvalue 1 have either only positive entries or only negative entries. In particular, for the eigenvalue 1 there exists a unique eigenvector with the sum of its entries equal to 1.

Using this theorem, $Av = v$ will always have the (largest) eigenvalue equal 1.

Applications

- PageRank is perhaps the most famous search ranking algorithm. Googles high-quality search engine results are directly correlated to PageRank
- There are a remarkable wide variety of applications of the PageRank algorithm that apply to non-search engine contexts.
- Its simplicity and elegance allow PageRank to be a more general and powerful tool.
- Let's look at the applications of PageRank and its connection to Twitter.

Twitter

- In 2010, Twitter was lagging behind and was lacking a user recommendation service.
- This was perceived both externally and internally as a critical gap in Twitter's product offerings, so quickly launching a high-quality product was a top priority.
- Twitter is unique because of the asymmetric nature of the following relationship—a user can receive messages from another without reciprocation.
- This differs substantially from other social networks such as Facebook or LinkedIn, where social ties can only be established with the consent of both participating members.

Introducing PageRank

- This works well with PageRank because we can determine outbound and inbound links.
- A user u is likely to follow those who are followed by users that are similar to u.
- These users are in turn similar to u if they follow the same (or similar) users.
- Therefore using Page Rank, Twitter is able to offer unique recommendations for users.
- By analyzing who the user follows and who those users follow, the PageRank algorithm will allow Twitter to make specific recommendations for each user.
- Essentially, the more outbound links that an account receives correlate to a higher PageRank score.