Measuring the Acceleration of Gravity with a Kater Pendulum

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Contents

Abstract

The acceleration of gravity in Houston is measured using a Kater pendulum, which allows for a more accurate measurement of g than a simple pendulum. The pendulum has two axes, and when adjusted so that the period is the same about each axis, the period is equivalent to that of a simple pendulum with a length exactly that of the distance between the axes. The accuracy should be to about 1 parts in 1,000 given the limits of the equipment available (primarily the measurement of the distance between the two axes). Sources of deviation from the ideal system are quantified and incorporated into the calculations.

1. Introduction

1.1 Introduction: Background

A simple pendulum of length h can be used to measure g, by measuring its period $T = (I / mgh)^{1/2}$, but the accuracy is limited because it is hard to determine the moment of inertia and center of mass with any accuracy. In 1817 Henry Kater built a reversible pendulum after Bessel's suggestion, which allows a more accurate determination of g. A Kater pendulum has two axes, and when it is adjusted so that the period is the same for both axes (by positioning a small adjustable weight), the period is equivalent to that of a simple pendulum with a length exactly that of the distance between the axes. This allows a very accurate measure of g to be determined, because this length can be precisely measured.

Values for g are typically reported in **gals** (cm/sec^2), after Galileo; typical gravity values are in the range of 980 ± 2 gal. This experiment will provide a precision to about 0.1 gal.

Accurate measurements of g are useful in geology, for instance in locating underground oil deposits, ore deposits, potential fault lines, and other geological formations. They are typically used in conjunction with seismic and magnetic data. For instance, oil deposits are often trapped under salt domes, which have a lower density than surrounding rock. If the gravitational acceleration of an area is lower than surrounding areas, the odds are increased that oil would be found by drilling there. To investigate the shape of the earth or its internal structures a resolution of at least 1 mgal is required.

The earth is basically ellipsoidal in shape, and the gravity at the surface can be estimated from this shape with the additional affects of centripetal acceleration due to the earth's rotation. This is the basis of the International Gravity Formula. Comparing the actual gravity at some location with this theoretical gravity gives the so called "gravity anomaly", which can be used in determining subsurface structures. For instance, if you were over a salt dome, there would be a negative gravity anomaly because the density of salt is lower than normal crustal material, and the measured gravity would be lower than expected. In the Houston area, the gravity anomaly is about -15 mgals. Towards either pole the gravity increases due to reduced centripetal acceleration and being closer to the earth's center of mass; for instance, at Pt. Barrow, Alaska the gravity is about 982 gals, while in Panama near the equator the gravity is about 979 gals.

Determining the absolute value of g at some point with a resolution of 1 mgal or greater was difficult until relatively recently, but more precise values for the difference in gravity could be determined using the relation for a pendulum at two locations $g_1/g_2 = (T_2/T_1)^2$. In the middle part of the last century a world wide gravity network was set up, with pendulums being carried to various locations in a round trip, determining the change in gravity between locations (typically with a resolution of around 0.1 mgal), and then accounting for the pendulum's drift by comparing the value obtained back at the starting location. This was a painstaking procedure, but has been made obsolete by advances in gravimeters.

Modern gravimeters in use today use different techniques to measure gravity. **Mass-spring** systems measure the displacement of a mass on a spring to provide differences in local gravity values to a resolution of about 1 µgal. **Free-fall** systems, which are becoming more common, measure the position of a body in free-fall very accurately with a laser-interferometer to determine *absolute* gravity values to an accuracy of about 1 µgal. **Superconducting gravimeters** are also available, which use a tiny sphere suspended in a magnetic field to provide sensitivity to the 1 nanogal level. At this level of resolution, oscillations of the earth and tidal forces become discernible.

1.2 Introduction: Pendulum Mechanics

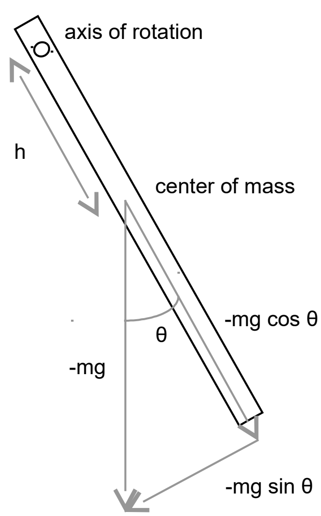


Figure 1. Pendulum under influence of gravity

For a pendulum, the torque τ about an axis is equal to the moment of inertia about the axis times the radial acceleration, which is just the rotational analog of Newton's Second Law, F = ma, i.e.

$$\tau = I\theta''$$

Torque is defined as force times the moment arm of the force with respect to the axis ($\tau \equiv$ Force · Moment Arm). For the pendulum, the force of gravity mg acts at the center of mass, which is h meters from the axis, but only the component $\sin\theta$ acts perpendicularly to the pendulum (the other component acts to stretch the pendulum), so

$$\tau = -mg\sin\theta \cdot h$$

Setting these equal we get

$$I\theta'' = -mg\sin\theta\, h$$

so

$$I\theta'' + mgh\sin\theta = 0$$

Solving this nonlinear differential equation in closed form is possible but requires the use of elliptical integrals. We can get a linear equation though if we use the small-angle approximation, i.e. $\sin\theta \approx \theta$ for small θ (by the Taylor expansion $\sin\theta = \theta - \theta^3/6 + \theta^5/120 + O(\theta^7)$). In this case we get

$$I\theta'' + mgh\,\theta = 0$$

or

$$\theta'' + (mgh/I)\,\theta = 0$$

This is a differential equation of the form $x'' + \omega^2 x = 0$. The solution is of the form $x = A\cos(\omega t + \phi)$ where A is the amplitude, ω the angular frequency, and ϕ the phase angle. So we have an angular frequency given by $\omega = (mgh / I)1/2$. The pendulum completes one period when $\omega T = 2\pi$, so the period $T = 2\pi / \omega$, so

$$T = 2\pi\,(I/mgh)^{1/2} \qquad (1)$$

This formula could be used to determine the gravity from the period, but it is hard to measure h and I with any accuracy. If the pendulum is reversible though, and the period is the same when measured from either axis, a more useful result is obtained.

By the parallel axis theorem, $I_1 = I_{cm} + mh_1^2$, where I_1 is the moment of inertia about axis 1, I_{cm} is the moment of inertia about the center of mass, m is the mass of the pendulum, and h_1 is the distance from the center of mass to axis 1. Plugging this into the equation for the period, we get

$$T_1 = 2\pi\sqrt{\frac{I_{cm} + mh_1^2}{mgh_1}} \qquad (2)$$

and similarly for T_2 about the second axis. When the periods are equal about either axis, $T_1 = T_2$, so we can rearrange to get

$$\frac{I_{cm} + mh_1^2}{mgh_1} = \frac{I_{cm} + mh_2^2}{mgh_2}$$

with some algebra this simplifies to

$$I_{cm} = mh_1h_2$$

This can be plugged into (2) to get

$$T_1 = 2\pi\sqrt{\frac{mh_1h_2 + mh_1^2}{mgh_1}} = 2\pi\sqrt{\frac{h_2 + h_1}{g}}$$

and so we get the final result,

$$T = 2\pi\sqrt{\frac{h}{g}}$$

where h = h1 + h2, i.e. the distance between the two axes. Note that this is the period for a simple pendulum of length h (i.e. a point mass at the end of a massless string).

We can now obtain g from the period by:

$$g = \frac{4\pi^2 h}{T^2}$$

2. Measurements

2.1 Measurements: Equipment

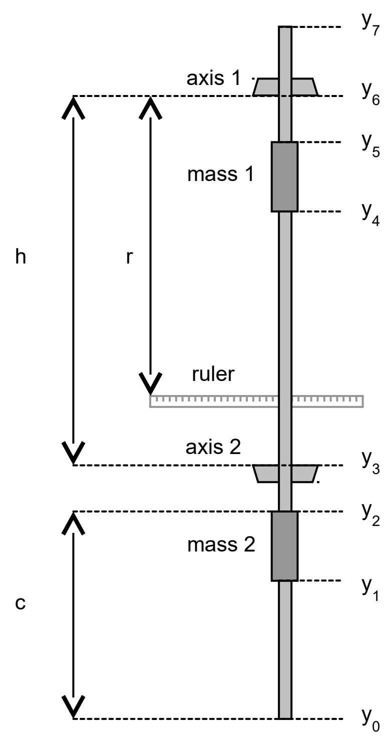


Figure 2. Pendulum Dimensions

The pendulum consists of a long steel bar with two knife-edge axes and two adjustable masses. The support is a metal rod with three adjustable feet, and the pendulum's knife-edge is placed in a holder at the top of the support so that the pendulum can swing freely.

The pendulum's dimensions were measured using a long wooden ruler, accurate to about 0.02cm, and a digital caliper, accurate to 0.01 cm (with a range of only about 20 cm though). The masses were removed from the pendulum to get an accurate reading of the dimensions, as parallax errors of up to 1mm resulted otherwise. The masses were determined with an electronic scale.

Table 1. Pendulum Dimensions

Dimension	Description	As measured
y_0		0.00 cm
y_1		will vary
y_2		will vary
y_3		33.78 ± 0.02 cm
y_4		106.55 ± 0.02 cm
y_5		116.70 ± 0.02 cm
y_6		133.20 ± 0.02 cm
y_7		166.90 ± 0.02 cm
h	Distance between axis 1 and 2 ($h=y_6-y_3$)	99.42 ± 0.02 cm
c	Distance from end of pendulum to adjustable mass	will vary
h_1	Distance from axis 1 to center of mass	used in derivations
h_2	Distance from axis 2 to center of mass	used in derivations
h_0	Length of pendulum ($h_0 = y_7$)	166.90 ± 0.02 cm
h_t	Distance from top of pendulum to axis 1 ($h_t=y_7-y_6$)	33.70 ± 0.02 cm
h_b	Distance from bottom of pendulum to axis 2 ($h_b=y_3$)	33.78 ± 0.02 cm
h_w	Width of pendulum bar	1.62 ± 0.01 cm
h_d	Depth of pendulum bar	0.60 ± 0.01 cm
r	Distance from axis to ruler used in amplitude measurements	66 cm
h_{diam}	Diameter of masses	10.12 ± 0.02 cm
h_{height}	Height of masses (roughly)	4.5 ± 0.5 cm
V_{bar}	Volume of pendulum bar ($V_{bar} = h_0 \cdot h_w \cdot h_d$)	162 cm^3
V_{mass}	Volume of mass ($V_{mass} = \pi/4 \cdot h_{diam}^2 \cdot h_{height}$)	362 cm^3
$V_{pendulum}$	Total volume of pendulum and masses ($V_{pendulum} = V_{bar} + 2V_{mass}$)	886 cm^3
m_{bar}	Mass of bar	1261.0 g
m_1	Mass of mass 1 ("1000g")	924.8 g
m_2	Mass of mass 2 ("1400g")	1402.0 g
$m_{pendulum}$	Total mass ($m_{pendulum} = m_{bar} + m_1 + m_2$)	3587.8 g

A meter stick is positioned horizontally so that readings of the amplitude can be made. A photogate is positioned on the floor so that the pendulum will block the beam of light as it reaches its highest amplitude of about 1.0 to 1.5 cm. The photogate is connected to a photogate timer with a resolution of 1 μsec and an accuracy of 50 ppm.

The pendulum is started swinging with an amplitude of about 1.5 to 2.0 cm, and allowed to go for one to two minutes so that transients can decay. Half the measurements are done with the pendulum facing one way, and half the other way, to reduce any systematic errors due to placement on the knife-edge.

The photogate timer is used in **Pulse Mode**, which measures the time between the falling edges of consecutive pulses. A **Period Mode** is available which measures the time between the falling edges of every other pulse, but this is more suited for a pendulum with a hole in it - using Pulse Mode allows us to use smaller amplitudes of oscillation. Typically, measurements are taken after one or two minutes, once the amplitude reaches 1.0 to 1.5 cm. Ten samples are taken of consecutive periods, and the mean and standard deviation computed.

2.2 Measurements: Determination of Period

Measurements were taken of the pendulum's period as the position of the moveable mass was varied over a range of 25 to 27 cm, as measured from the end of the pendulum to the top of the mass. At some point the periods will be equivalent about either axis, and this point is determined from the intersection on the graph.

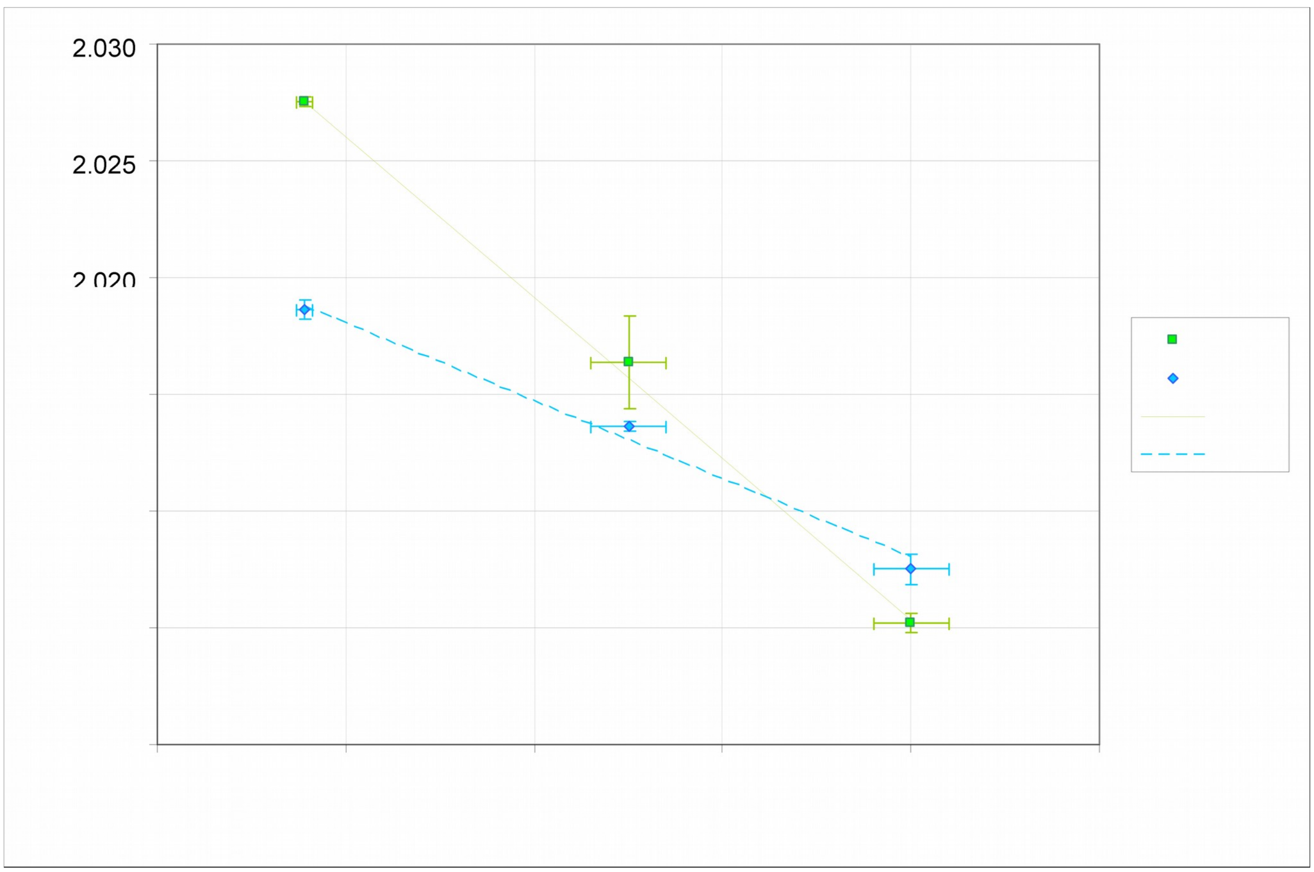
The period is measured about either axis for the different mass positions, the least-squares fit for both axes is found, and the intersection of these lines gives the period.

Table 2. Pendulum Period Data

c, position of mass (cm)	σ_c (cm)	T1, period about axis 1 (sec)	σ_{T1} (sec)	T2, period about axis 2 (sec)	σ_{T2} (sec)
25.39	0.02	2.02754270	0.00022954	2.01862918	0.00039810
26.25	0.10	2.01639800	0.00198401	2.01362700	0.00021783
27.00	0.10	2.00518880	0.00040244	2.00750545	0.00062598

Note: Position was originally measured with relatively large parallax errors (σ = 0.1cm), first position was remeasured with ruler flat against pendulum, after marking position and removing masses (σ = 0.02cm).

Graph 1. Period of Pendulum vs Position of Mass



The uncertanties in the position of the mass are much larger than those of the period (roughly 1 in 1,000 for position vs. 1 in 10,000 for period), so in performing the least-squares fit it is best to treat the position as the dependent variable, and the period as the independent variable (i.e. switch the graph so that y = position and x = period).

After switching the axes there still remain significant errors in the independent variable (i.e. the x-coordinate, or the period). We can include this in the standard least-squares calculations by estimating the effect of the errors in x on y. First we get an initial estimate for the least-squares line, then use the slope calculated to determine the linear relation between errors in x and errors in y. Following Bevington (p102), with dy/dx = m = slope of the least-squares line,

$$\sigma y_{indirect} = \sigma x * m$$

Then we combine this by quadrature with the direct contribution σy_{direct}, which is the measurement uncertainty in y, to get

$$\sigma y_{net}^2 = \sigma y_{indirect}^2 + \sigma y_{direct}^2$$

In effect this lets us treat the uncertainties in the x-coordinate as if they were in the y-coordinate only. By iterating the solutions you could presumably get more accurate results, but in this case we just do one iteration (see Appendix E for details of the calculations).

So we get the following lines as the best fit to the data (plotted in graph above):

Axis 1: Period = -0.013810686 * Position + 2.378200109
Axis 2: Period = -0.00668865 * Position + 2.188579799

The intersection of these two lines is at the point

(Position, Period) = (26.6255 cm, 2.01049 sec)

So our value for the observed period is

$$T_{observed} = 2.01049 \text{ sec}$$

The error in this value can be taken to be, roughly, the average standard deviation in the measured periods, which is $\sigma_T = 0.0006$ sec. Bevington (p109 and 122) has more detailed procedures, but for our purposes this rough estimate should suffice.

So

$$\mathbf{T_{observed} = 2.01049 \pm 0.0006 \text{ sec}}$$

2.2 Measurements: Determination of Damping Factor

The amplitude of the pendulum is measured over a period of about 25 minutes as it decays from a starting amplitude of about 8 cm. This amplitude is then converted to the angular value θ by θ = arctan(a / r), where a is the amplitude and r is the distance from the ruler to the axis. The error in amplitude a is then propagated into the angle θ by the following linear approximation, with x = a / r

$$\begin{aligned} \delta\theta &= \partial\theta/\partial x \cdot \delta x \\ &= 1 / (1 + x^2) \cdot \delta x \end{aligned}$$

Assuming that damping is proportional to velocity (which is usually true for low velocities), we modify the original equation of motion to include a damping term $2\beta\theta'$:

$$\theta'' + 2\beta\theta' + \omega^2\theta = 0$$

the solution is of the form $\theta = \theta_{max}\, e^{\lambda t}$, with

$$\lambda = -\beta \pm i\,(\omega^2 - \beta^2)^{1/2}$$

so

$$\theta = \theta_{max}\, e^{-\beta t} \cos(\omega_1 t)$$

where

$$\omega_1 = (\omega^2 - \beta^2)^{1/2}$$

Note that the damped frequency ω_1 is slightly smaller than the undamped frequency ω.

The dimensionless amplitude y is thus

$$y = \theta / \theta_{max} = e^{-\beta t}$$

or

$$\ln y = -\beta t$$

so plotting ln y vs t should give a line with a slope of $-\beta$, allowing us to determine β, since

$$\beta = -\frac{\ln(\theta / \theta_{\max})}{t}$$

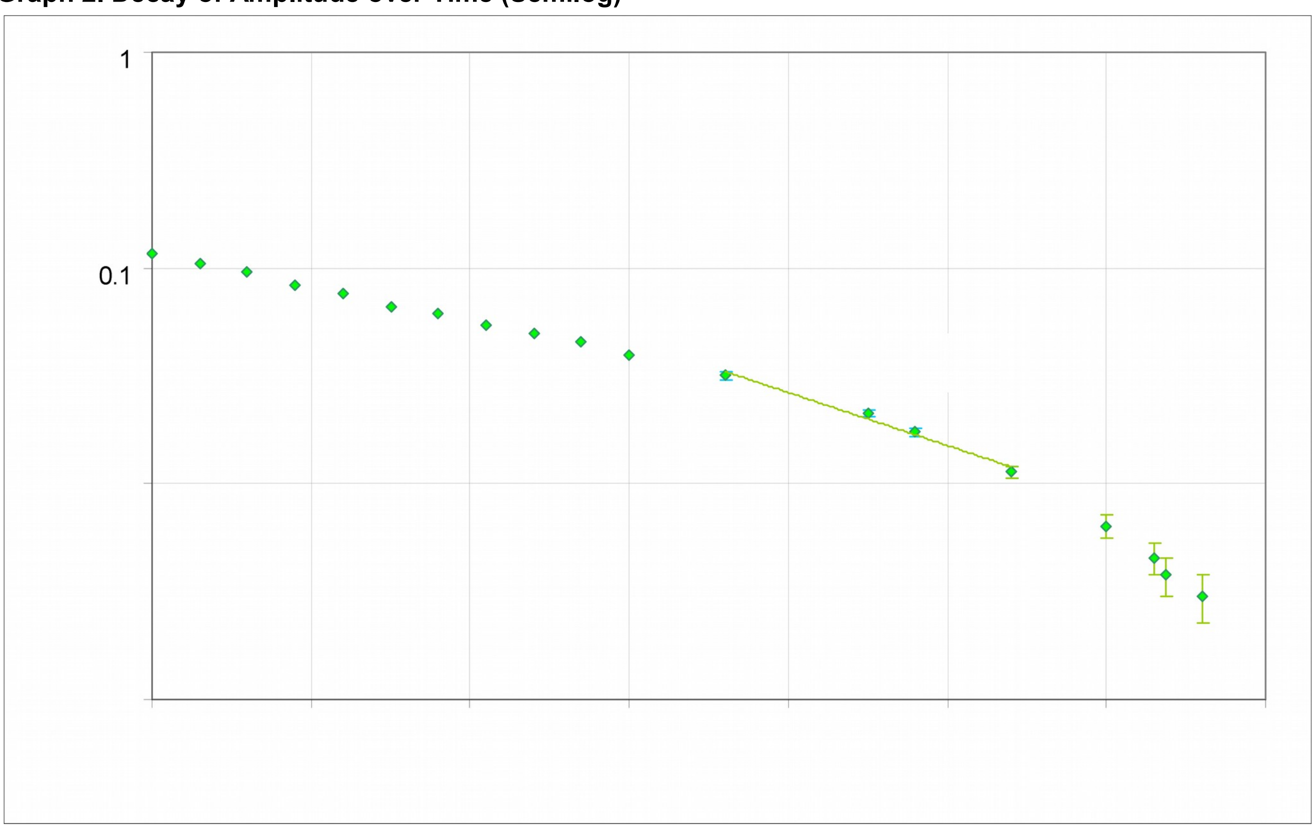
This will give us a rough determination of the damping factor.

Note that an "instantaneous" value for β can be obtained by

$$\beta_{inst} = -\frac{\ln(\theta_2 / \theta_1)}{t_2 - t_1}$$

where θ_1 and θ_2 are the amplitude at some times t_1 and t_2. We can use this to obtain a value for β at different regions on the following graph:

Graph 2. Decay of Amplitude over Time (Semilog)



Since the slope and hence β is not constant, our equation of motion must be incorrect or lacking sufficient detail. If we limit ourselves to the region where measurements are normally taken though, we can obtain a usable value for β. Measurements of the period of the pendulum were usually taken in the horizontal amplitude range of 1.0 to 1.5 cm, corresponding to a radial amplitude of about 0.015 to 0.020 radians (0.8 to 1.2 degrees). On the graph above this corresponds to the time range of about 800 to 1100 seconds, which is where we'll take our value for β from, obtaining (from the best exponential fit to the data)

β ≈ 0.0028

Appendix A has more on the damping constant as a function of velocity.

3. Calculations

3.1 Calculations: Corrections to Period

The observed period is that of a nonideal physical system. In order to determine the gravitational acceleration from the period, it is necessary to make adjustments to the observed period in order to approach the ideal period that would be observed with an ideal system. There are various corrections that can be added to the observed period, as detailed in the table below:

In the following, T_{ideal} is the period for an ideal pendulum in the gravity field. This is the value we will use to calculate g. We quantify various corrections to the ideal period to try to match the observed period, and add them together into a correction term ΔT_{net}:

$$T_{observed} = T_{ideal} + \Delta T_{net}$$

(Note: This is the opposite of the approach in Nelson and Olsson 1986 - this way allows us to say "damping increases the period", and have a positive ΔT.)

Table 3. Corrections to Period (including affect on g)

Correction	Description	ΔT/T	ΔT	Δg
Buoyancy of air	By Archimedes' principle, the weight of the pendulum is reduced by the weight of the displaced air. m_{air} = mass of air = $\rho \cdot V_{pendulum}$ ρ = density of air = 0.00129 g/cm^3 $V_{pendulum}$ = volume of pendulum = 162cm^3 Effectively lowers gravity, so period will be lengthed.	$\frac{1}{2}\frac{m_{air}}{m_{pendulum}}$	58.6 μs	- 56.6 mgal
Small-angle approximation	θ ≈ sin(θ). Need to keep amplitude of pendulum small.	$-\frac{\theta_{\max}^2}{16}$	- 50.3 μs	48.5 mgal
Above ground-level	Gravity decreases with altitude - will increase period. y = height above ground ≈ 4m r_{earth} = radius of earth	$\frac{y}{r_{earth}}$	1.26 μs	- 1.22 mgal
Damping	Combined effect from knife-edge friction, air resistance. Damping lengthens period.	$\frac{\beta^2 T^2}{8\pi^2}$	0.9 μs	- 0.9 mgal
Non-uniform gravity field	Gravity decreases with height, and pendulum swings through different g values. h_0 = length of pendulum	$\frac{2}{3}\frac{h_0}{r_{earth}}$	0.35 μs	-0.34 mgal
Gravity of building	$a = Gm_{building}/d^2$ $m_{building}$ ≈ 10,000 kg a = Gm/d^2 = 6.68e-7 cm/s^2 Will effectively decrease gravity, so	$\frac{1}{2}\frac{a}{g}$	0.001 μs	(negligible)

	increase period.			
Total			**11 μs**	

Other sources of systematic errors which can be quantified and corrected for here include non-rigid support, temperature variance, pendulum bending and stretching, magnetic fields, non-inertial reference frame, sun and moon's influence, and nonhomogeneous pendulum density. See Nelson and Olsson 1986 for discussion of some of these corrections.

We can now calculate the ideal period by

$$T_{ideal} = T_{observed} - \Delta T_{net}$$

so

$$T_{ideal} = 2.010490 \pm 0.000600 - 0.000011 \text{ sec}$$

or

$$\mathbf{T_{ideal} = 2.01048 \pm 0.0006 \text{ sec}}$$

Note that the corrections were effectively swamped by the error in the period, but for more accurate gravity measurements by pendulum, these corrections become important.

3.2 Calculations: Reference Value for Gravity

The NOAA has a website that allows you to enter a latitude and longitude and it will provide an estimate of g at that location, interpolating from an extensive database of absolute and relative gravity measurements (NGDC 1999). Woollard 1963 lists various gravity determinations in the World Gravity Network, including some made at Rice University and Hobby Airport. These values are shown on the map below.

Figure 3. Local values of gravity

So the reference value for the UH campus is

$$\mathbf{g_{reference} = 979.287 \pm 0.002 \text{ gals}}$$

Appendix B has more information on reference gravity values, including the International Gravity Formula and gravity anomalies.

3.3. Calculations: Determination of Gravity

The acceleration due to gravity at the ground level of the UH campus is calculated using

$$g_{observed} = 4\pi^2 h / T_{ideal}^2$$
$$T_{ideal} = 2.01048 \pm 0.0006 \text{ sec}$$
$$h = 99.42 \pm 0.02 \text{ cm}$$

so

$$\mathbf{g_{observed} = 971.031 \text{ gals}}$$

We can estimate the error in g due to errors in the measure of h and T by the following:

Error in g due to variation in h:

$$\sigma g_h = \sigma_h \cdot \partial g / \partial h$$
$$= \sigma_h \cdot 4\pi^2/T^2$$
$$= \sigma_h \cdot (9.767)$$
$$= 0.195$$

i.e. the variation in g is about 10x that of h for this pendulum

Error in g due to variation in T:

$$\sigma g_T = \sigma_T \cdot \partial g / \partial T$$
$$= \sigma_T \cdot - 8\pi^2 h/T^3$$
$$= \sigma_T \cdot (-965.9697)$$
$$= -0.579$$

i.e. the variation in g is about 1000x that of T for this pendulum

and the total error is found by quadrature,

$$\sigma_g = (\sigma g_h^2 + \sigma g_T^2)^{1/2} = (0.038 + 0.147)^{1/2}$$
$$= 0.612$$

Thus our final value for the observed gravity is

$$\mathbf{g_{observed} = 971.031 \pm 0.612 \text{ gals}}$$

4 Discussion

4.1 Discussion: Results

A comparison of the observed gravity with the reference value shows that our measurement is off by about 8 gals.

$$\mathbf{g_{observed} = 971.031 \pm 0.612 \text{ gals}}$$

$$\mathbf{g_{reference} = 979.287 \pm 0.002 \text{ gals}}$$

The percent error is 0.8% (i.e. the accuracy is 8 parts per 1,000).

A previous attempt using a smaller brass pendulum actually got closer (about 977 gals, 0.2% error). The larger steel pendulum was then used in an attempt to get more accurate results, but the opposite occurred. For the brass pendulum, we did more runs (about 6 different placements of the adjustable mass versus 3 for the steel pendulum), which might have helped get more accurate least-squares lines.

With the larger steel pendulum it was difficult to measure the position of the mass without parallax error, unless the position was marked somehow and the mass removed. Also the brass pendulum was small enough that digital calipers could be used, which may have allowed for more accurate measurements.

One possible cause for the brass pendulum's results being off is that the adjustable mass was a plastic ring that could be tilted slightly - making the actual position of the mass uncertain - this wasn't noticed until partway through the measurements. Also, the period of the pendulum was very sensitive to the position of the pendulum on the knife edge - if it was not exactly in the center of the axis hole then the period would vary. This was not noticed until the data was analyzed and large discrepencies between periods were noticed for the pendulum facing different directions (we were not careful enough in positioning the pendulum on the knife edge).

To improve the accuracy of this experiment then, multiple measurements of all distances should be taken, and the mean used for the value. This is one simple way of improving the accuracy of the measures h and c. Also, more runs with the adjustable mass at different positions should be done, and then do in more detail near the intersection of the two lines (e.g. move the mass by 1mm at a time). It might also help to do multiple runs at the same distance, to improve the accuracy of the period at each point.

It would also be good to verify the accuracy of the old wooden ruler used to measure the pendulum lengths, and verify the accuracy of the photogate timer. The manual says that no calibration is required, but it's possible that it is off nonetheless.

Other sources of random errors and hard to quantify systematic errors include affect of **transients** on the period (which we tried to eliminate by waiting 1-2 minutes before taking period, but may need to wait longer, or devise a mechanism that will release pendulum evenly, and also wait the same length of time for each measurement, and make sure you start at the same amplitude), **nonplanar motion** (i.e. the steel pendulum was capable of elliptical motion rather than strictly planar motion), **wobble** (the brass pendulum was prone to wobble on the second axis - need to make sure the knife edge is perfectly level), **air currents** (the steel pendulum should be relatively immune to these, but the brass pendulum was lighter and especially on the second axis might be vulnerable to these effects), and **temperature** variance (affects length of pendulum - from handling, air temperature, being close to person, etc.).

Perhaps the most important effect that was not included was that of the **non-rigid support** - when the pendulum swings it causes the support to sway also, since it is not perfectly rigid. This could conceivably be corrected for in the period corrections, with a bit of work. The pendulum may also be **bending and stretching**, it may be affected by **magnetic fields**, the **non-inertial reference frame** that is the earth, and also the material of the pendulum may be **nonhomogeneous**, which would throw the derivation of the formula for g off.

4.2 Discussion: Future

Using a cathetometer to measure the interaxis distance and position of the adjustable mass could improve the accuracy of g by 10 fold (assuming a cathetometer accurate to 0.01mm over a range of 100 cm). An **optical interferometer** (accurate to about 1nm) would improve it even more (the period measurement would then become the limiting measurement).

Calibrating the phototimer by letting it run for a long period of time and comparing it with a standard time reference might be worthwhile. Crystal-controlled clocks may be affected by temperature (unless they are compensated with appropriate circuitry), and vibrations. Could also increase the accuracy of the period measurement by counting the number of swings over a longer time period. May need to reduce friction in the steel pendulum to do this though, since it tended to decay faster than the brass pendulum did.

Could try attaching a mirror to the pendulum and record the variations in a light beam to measure the amplitude and decay more accurately.

If time permitted a full 3D analysis would be interesting - i.e. derive the equations of motion by Newton, Lagrangian, Hamiltonian, inertia tensor, etc. Could try to account for wobble in pendulum. Also transients - how long should you wait before taking measurements? Use numerical methods to determine value of g, compare with determination by analytical means. Accuracy as function of pendulum length? i.e. when do you start bumping into errors from air resistance, etc.? Try to determine optimal pendulum length for most accuracy. Determine the onset of non-isochronism - when non-linearity starts, etc.

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Microsoft, 2000, Microsoft Expedia Streets & Trips 2000 [cdrom] [used to obtain latitude and longitude for UH SR1 building]

Nelson and Olsson, 1986, The Pendulum - Rich Physics from a Simple System, American Journal of Physics 54, 112 (1986) [applies corrections to observed period to obtain ideal period, lots of interesting math]

NGDC 1999, Land and Marine Gravity CD-ROMs, 1999. Compiled by David Dater, Dan Metzger, and Allen Hittelman. U.S. Department of Commerce, National Oceanic and Atmospheric Administration, National Geophysical Data Center, Boulder, CO. Gravity 1999 Edition on 2 CD-ROMs. Available at <http://www.ngdc.noaa.gov/seg/potfld/gravity> and <http://www.ngs.noaa.gov/TOOLS/Gravity/gravcon.html> viewed October 2002. ["Approximately 70 percent of the data are observed values - regional station data collections and absolute gravity measurements. Grids and other derived summary data sets represent another 30 percent of the data. Surface gravity is predicted by interpolation based on observed values. Observed gravity data in the National Geodetic Survey's Integrated Database are referenced to the International Gravity Standardization Net 1971, which is an absolute gravity datum."]

TEL-Atomic, 2000, "Student-friendly Precision Kater Pendulum" [manual]

Woollard and Rose, 1963, International Gravity Measurements, Society of Exploration Geophysicists. [good reference on the world gravimetric control net, describes location of local pendulum stations throughout the world, values for Houston Hobby Airport, gravity anomaly maps of Texas and US p 379-80, etc.]

Appendix A. Damping Factor

We can calculate an "instantaneous" damping factor β by

$$\beta_{inst} = - \ln (\theta_2 / \theta_1) / (t_2 - t_1)$$

where θ_1 and θ_2 are the amplitude at some times t_1 and t_2. Plotting this instantaneous β versus time shows that β increases sharply at long time (low amplitude), indicating that we might be able to model β as a function of amplitude.

At some time you have an amplitude θ_A - what is the maximal velocity at this point? Roughly, neglecting the damping term,

$$\theta = \theta_A \cos \omega_1 t$$

Taking the derivative,

$$\theta' = - \theta_A \omega_1 \cdot \sin \omega_1 t$$

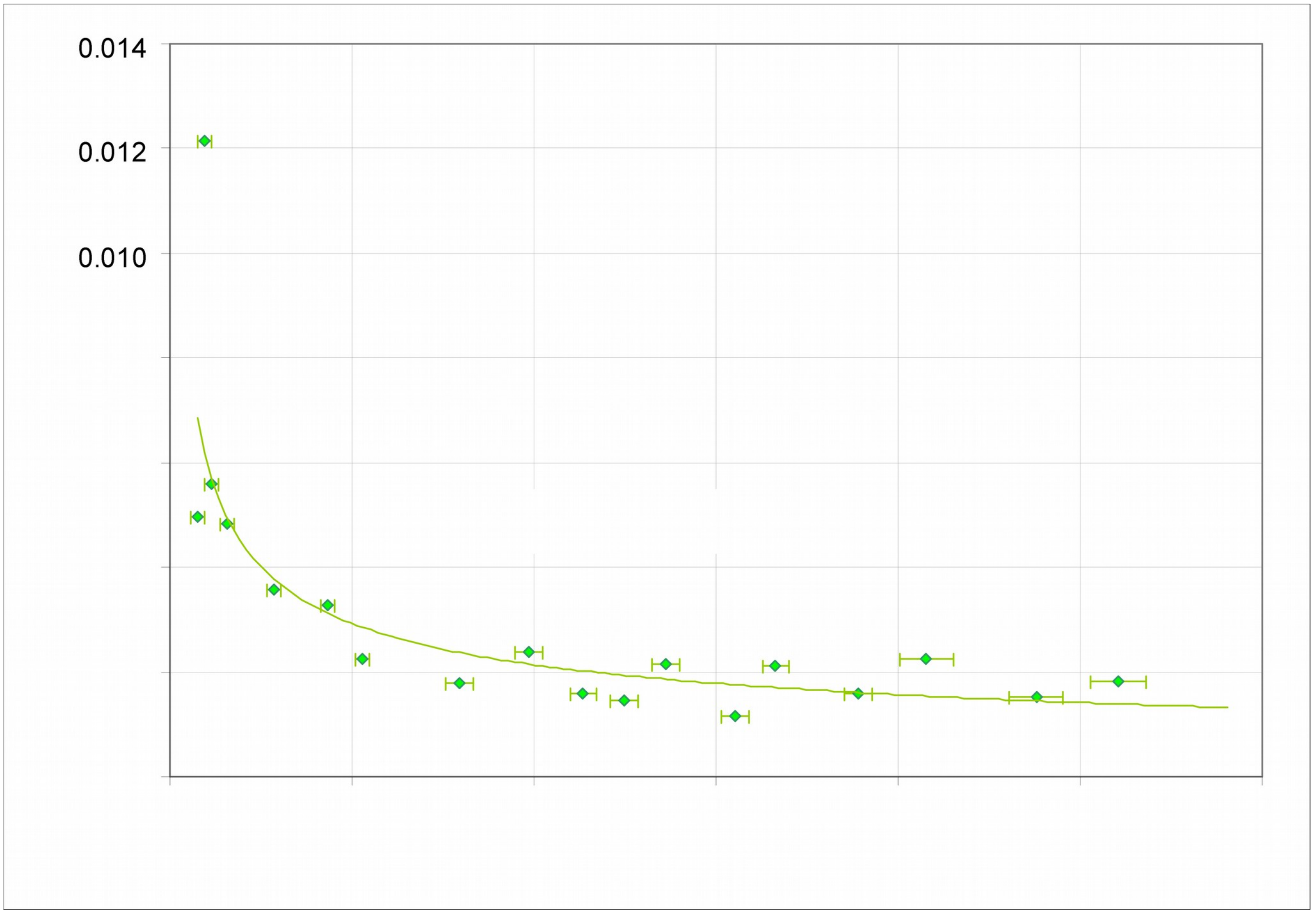
The velocity is maximum when sin is 1, so

$$\theta'_{max} = -\theta_A \omega_1$$

in other words the maximal velocity is proportional to the amplitude.

So to plot β vs maximal velocity, we can just plot β vs the amplitude:

Graph 3. Damping Factor as a Function of Velocity



.
An exponential curve fits the data fairly well, so we can say

$$\beta \approx 0.0005\ \theta^{-0.451}$$

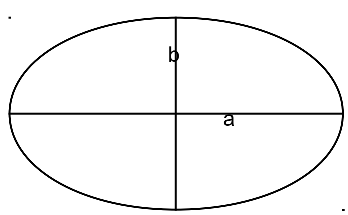
where θ is the amplitude in radians.

This presumably reflects the increasing friction as the velocity becomes lower, and the more time available for interatomic bonds to form, akin to static versus kinetic friction constants.

This is based on a rather sparse data set - could do with more runs to verify this relation. Also β might differ about the other axis.

Appendix B. International Theoretical Gravity Formula

The International Theoretical Gravity Formula is based on an approximation of the earth as a rotating ellipsoid.



a = semimajor axis (equatorial radius)
b = semiminor axis (polar radius)
f = flattening = (a-b)/a

Figure 4. Ellipsoidal Earth

A second order approximation to the gravity at a given latitude is given by the formula (derived in Heiskanen and Moritz 1967)

$$g_{normal} = g_{equator}\,(1 + \alpha \sin^2(\lambda) - \beta \sin^2(2\lambda))$$

where

g_{normal} = gravity at surface of ellipsoid = "normal" gravity or "theoretical" gravity
$g_{equator}$ = gravity as measured at equator
λ is latitude, α and β are constants derived from the reference ellipsoid

To be useful, this "normal" gravity must be adjusted to account for the elevation above the theoretical ellipsoid. This includes a "free-air" correction, which accounts for the additional elevation above the ellipsoid, and the Bouguer correction, which includes the effects of a slab of rock above the ellipsoid.

For instance, for the UH SR1 building, the latitude and longitude are approximately (estimated from Microsoft Streets and Trips 98 and a UH map)

Latitude λ = 29.7228˚ ± 0.0001˚ N
Longitude = 95.3457˚ ± 0.0001˚ W

(Note: an error in latitude of 0.0001˚ corresponds to distance of $\Delta\lambda \cdot 2\pi\, r_{earth} / 360^\circ = \Delta\lambda \cdot 111319.5$ m/˚ $\approx$ 11 m.)

The "theoretical" or "normal" gravity is determined from the International Gravity Formula, with

$g_{equator}$ = 978.0327
α = 0.0053024
β = 0.0000058

so

g_{normal} = 979.303 gals

This normal value is then adjusted for the height above the reference ellipsoid by applying what is called the **"free-air" correction**. Usually the elevation above sea level is used, though this is not necessarily the height above the ellipsoid, strictly speaking. The elevation for the UH campus is estimated from the NGDC 1999 dataset:

$h_{elevation}$ = 13.4 m
$\delta_{freeair}$ = - 0.3086 mgal/m $\cdot$ $h_{elevation}$
= - 4.135 mgal

Another correction is then added for the additional mass of rock between the reference ellipsoid and the location. This is called the **Bouguer correction**, and is estimated by assuming that the rock is an infinite slab. The density of rock is usually taken to be a standard value, the average for crustal rocks.

ρ = Density of rock = 2.67g/cm^3

δ_{bouger} = $2\pi\, G\, \rho\, h_{elevation}$
= + 0.1118 mgal/m $\cdot$ $h_{elevation}$
= + 1.498 mgal

Applying these corrections to the normal value for gravity gives you the calculated value (sometimes called the Bouguer value).

$g_{calculated} = g_{normal} + \delta_{freeair} + \delta_{bouguer}$
= 979.303 - 0.004135 + 0.001498 gals
= **979.300 gals**

This value can then be compared with actual measurements. The reference value for the UH campus is

$g_{reference}$ = **979.287 ± 0.002 gals** (NDGC 1999)

The difference between the calculated and the actual value is called the **gravity anomaly**, or Bouguer anomaly.

$$
\begin{aligned}
g_{anomaly} &= g_{reference} - g_{calculated} \\
&= 979.287 - 979.300 \\
&= -13 \text{ mgal}
\end{aligned}
$$

So the gravity anomaly for the UH campus is about - 13 mgals. In other words, the actual value of gravity is lower by 13 mgals than that predicted by the theoretical gravity formula. This might be caused by less dense material in crust - clay, etc.

If the terrain is not smooth, an additional correction may be needed (called the terrain correction), but this is not necessary for Houston.

Table 4. International Gravity Formulas

Formula	**Reference Ellipsoid**	$g_{equator}$	α	β
IGF 1930	International 1924	978.0490	0.0052884	0.0000059
IGF 1967	GRS 67	978.031846	0.005278895	0.000023462
IGF 1980	GRS 80	978.0327	0.0053024	0.0000058

Note: The constants α and β are calculated directly from the reference ellipsoid. Various reference ellipsoids have been determined through history.

Table 5. Reference Ellipsoids

Year	**Name**	**Semi-major axis (equatorial radius), a** (meters)	**Inverse flattening, 1/f, f = (a-b)/a**	**Origin**	**Geocentric Constant of Gravitation, GM** (x 10^{14} m^{3}/s^{2})	**Mean angular velocity of earth, ω** (x 10^{-5} rad/s)	**Notes**
1738	Bouguer, Maupertuis	6,397,300	216.8				
1800	Delambre	6,375,653	334.0				
1819	Walbeck	6,376,896	302.8				
1830	**Airy**	6,377,563.396	299.324 964 6				
1830	Everest	6,377,276	300.802				
1841	Bessel	6,377,397	299.153				
1866	Clarke	6,378,206	294.978				
1906	**Helmert**	6,378,200	298.3				
1909	Hayford	6,378,388	297				
1924	**International**	6,378,388	297				Based on Hayford 1909 ellipsoid. IUGG/IAG
1927	NAD 27	6,378,206.4	294.978	Meades Ranch, KS			Based on Clarke 1866 ellipsoid.
1948	Krassovsky	6,378,245	298.300				
1960	Fischer	6,378,155	298.3				
1960	WGS 60						First global reference frame.
1966	WGS 66	6,378,145	298.25				
1967	**GRS 67**	6,378,160	298.247 167 427				
1972	WGS 72	6,378,135	298.26				
1980	**GRS 80**	6,378,137	298.257 222 101				
1984	**WGS 84**	6,378,137	298.257 223 563	Geocentric			Designed for GPS. Based on Doppler observations of

							TRANSIT satellites.
1987	ITRS			Geocentric			
1993	Geoid 93			Geocentric			
1999	IAG 99	6,378,136.6		Geocentric	3.986004418(8) 0.002 ppm	7.2921150(1) 0.014 ppm	

Appendix C. Terms and Acronyms

The following terms and acronyms were encountered in researching this lab.

Term	Definition
accuracy	How close a measurement is to the true value of the quantity measured. Compare with precision.
Bouguer anomaly	See gravity anomaly
cathetometer	A telescope or microscope fitted with crosswires in the eyepiece and mounted so that it can slide along a graduated scale. Cathetometers are used for accurate measurement of lengths without mechanical contact. The microscope type is often called a travelling microscope. Typical accuracy of 0.01mm over range of 100cm.
datum	A reference frame - provides a point of origin and orientation for a coordinate system.
ellipsoid	A solid body formed when an ellipse is rotated about an axis. Used as an approximation for the geoid. A reference ellipsoid is chosen to be as close a fit to the geoid as possible.
geodesy	The science concerned with determining the size and shape of the Earth, the location of points upon its surface, and its gravity field. Uses gravity to define the geoid.
geodetic	Of or determined by geodesy.
geoid	The surface of the Earth's gravity field which best fits sea level, i.e. the gravitational equipotential surface corresponding to mean sea level. Devised by Gauss in 1828. Defined by elevation above or below a reference ellipsoid, and usually approximated by coefficients for the spherical harmonic expansion of the geopotential (eg Defense Mapping Agency model with terms to 180th power and 32,000 coefficients!).
geophysics	Uses gravity to learn about the density variations of the Earth's interior.
gradiometer	Any instrument that measures the gradient of a potential field rather than its absolute value (i.e. gravimeters that determine the change in g between locations rather than the absolute value of g).
gravimeter	An instrument for measuring the gravitational acceleration. Most field instruments are relative instruments, i.e. they determine the difference in gravitational acceleration between two or more points.
gravity anomaly	A difference between the measured value of g at some point and the expected value (as determined by the International Gravity Formula).
isochronous	Greek "equal time" - refers to the property of a pendulum that its period is roughly constant for small amplitudes. Non-isochronous refers to motion outside of this range, i.e. large amplitudes.
log-decrement	Equivalent to the instantaneous damping factor β_{inst} for a Δt of 1 second, ie δ = log decrement = $-\ln(x_2/x_1)/\Delta t$
optical interferometer	Measures lengths to fraction of a wavelength (about 1 nm), by counting interference patterns between start and end points.
precision	The number of significant figures quoted in a measurement. Compare with accuracy.
radius of gyration	The radius of gyration of a mass about a given axis is defined as $I = mk^2$, where I is the moment of inertia about the axis, and m is the mass. If no axis is specified the centroidal axis is assumed.
spheroid	Can refer to an ellipsoid or any shape that is close to a sphere.
theodolite	An instrument combining a telescope with a way to measure angles accurately, once used in doing surveys.

Acronym	Definition
EGM	Earth Gravity Model, e.g. EGM 96 global gravity model
GRS	Geodetic Reference System
IAG	International Association of Geodesy (part of IUGG)
IERS	International Earth Rotation Service, established 1987 by IUGG and IAU
ITRS	International Terrestrial Reference System, by IERS
IUGG	International Union of Geodesy and Geophysics, established 1919. Composed of 7 organizations.
NAD	North American Datum. eg NAD83 - based on GRS80 ellipsoid.
NGS	National Geodetic Survey. Part of NOAA. Defines and manages the National Spatial Reference System (NSRS) - the framework for latitude, longitude, height, scale, gravity, orientation and shoreline throughout the United States.
NOAA	National Oceanic and Atmospheric Administration
NSRS	National Spatial Reference System
WGS	World Geodetic System, from US Department of Defense