33-777

Today

- Post-main Sequence Stellar Evolution
- Hayashi Limit

In this lecture we continue our discussion of stellar evolution, including,

- subgiant branch
- red giant branch
- horizontal branch

- Post-Main Sequence Stellar Evolution

Up to this point, we have described stellar evolution within the main sequence. We did this for two catagories of stars

1. low mass stars
   $M < 1.2\ M_\odot$
2. intermediate | high mass stars
   $M > 1.2\ M_\odot$

The reason these two mass regimes have differing evolution on the main sequence is that low mass stars have radiative cores burning $H \rightarrow He$ via the pp chain. More massive stars have convective cores that burn $H \rightarrow He$ via the more temperature sensitive CNO cycle.

The main sequence is defined to be over for

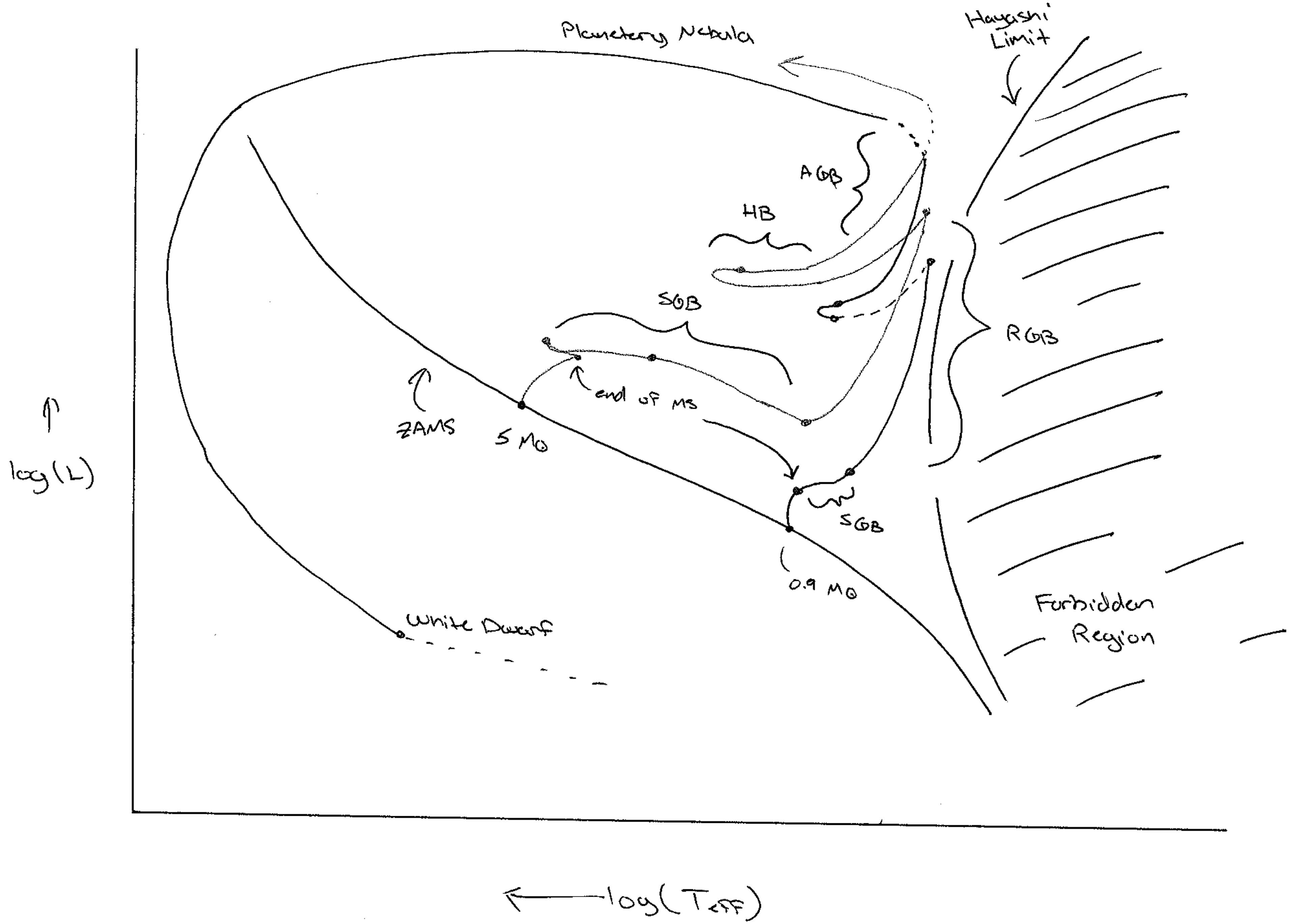
① low mass stars when the inert isothermal helium core can no longer provide enough pressure to support itself + the envelope above it

② intermediate|massive stars when the core is largely depleted of H and the entire star contracts.

After the main sequence, both mass regimes of stars enter the sub-giant branch. This phase is characterized by hydrogen burning in a shell around an inert helium core.

After contraction in the intermediate and massive stars, the base of the envelope heats enough to begin H shell burning. Quickly, the isothermal He core reaches its limit.

At this point on the subgiant branch, the helium core contracts on the Kelvin-Helmholtz timescale. This releases gravitational energy. Most of this energy goes into expanding the envelope. As a result, the star gets significantly redder while maintaining approximatly fixed luminosity. This phase is very quick since this all happens on the Kelvin-Helmholtz timescale. The subgiant phase ends when the star approaches the Hayashi limit. At this point the star has become as red as it can become. the star now enters the red giant phase.



Planetary Nebula
Hayashi Limit
AGB
HB
SGB
RGB
end of MS
ZAMS
5 M☉
SGB
0.9 M☉
White Dwarf
Forbidden Region
log(L)
log(Teff)

As we have seen, the equations of stellar structure contain seven unknowns and equations,

- density, ρ
- temperature, T
- pressure, P
- Mass, M
- energy flux, L
- opacity, κ
- energy generation rate, ε

Typically, not much progress can be made solving these equations analytically. However, under some restrictive assumptions, some progress can be made. One such assumption is that pressure and density inside a star are related as

$$P = K_p \rho^{\gamma_p}$$

$$\gamma_p = 1 + \frac{1}{n}$$

"polytropic index"

This sort of equation of state is called a polytrope. If we put on hold for a moment the question of why this might be useful, we can see where this takes us.

From our first two equations of stellar structure

$$① \quad \frac{dP}{dr} = -\frac{GM(r)}{r^2}$$

$$② \quad \frac{dM}{dr} = 4\pi r^2 \rho(r)$$

we can obtain a single equation,

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

If we substitute in our polytropic equation of state we get,

$$\frac{(n+1)K_\rho}{4\pi G n}\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho^{(n-1)/n}}\frac{d\rho}{dr}\right) = -\rho$$

Let's define a new variable to absorb some of these constants,

$$\alpha^2 = \frac{K_p(n+1)}{4\pi G} \rho_c^{(1-n)/n}$$

central density, $\rho(r=0)$

And by choosing a set of variable substitutions,

$$\theta = \left(\frac{\rho}{\rho_c}\right)^{1/n} \qquad \text{"Polytropic temperature"}$$

$$\xi = \frac{r}{\alpha} \qquad \Big\} \text{ radius like variable}$$

$$\Rightarrow \quad \rho = \rho_c \theta^n, \quad P = P_c \theta^{n+1}, \quad r = \alpha \xi$$

With these substitutions, our hydrostatic condition equation simplifies to,

$$\frac{1}{\xi^2} \frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\theta^n$$

This equation is known as the Lane-Emden equation. Stellar models where this holds true are called polytropes.

There are two important central boundary conditions,

$$\left.\begin{aligned} \theta &= 1 \\ \frac{d\theta}{d\xi} &= 0 \end{aligned}\right\} \xi = 0 \qquad \text{boundary conditions}$$

There turns out to be analytic solutions for $n = 0, 1, 5$. Generally, for $n < 5$, θ monotonically decreases until it reaches zero at some finite value of $\xi = \xi_1$. At this point density and pressure go to zero, naturally defining the size of the star,

$$R = \alpha \xi_1$$

We can also write the total stellar mass as,

$$M = \int_0^R 4\pi r^2 \rho \, dr = 4\pi \alpha^3 \rho_c \int_0^{\xi_1} \xi^2 \theta^n \, d\xi$$

$$= 4\pi \alpha^3 \rho_c \int_0^{\xi_1} \left[-\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi} \right) \right] d\xi$$

$$= -4\pi \alpha^3 \rho_c \, \xi_1^2 \left(\frac{d\theta}{d\xi} \right)_{\xi_1}$$

The equations for stellar radius and mass can be combined to show,

$$R^{(3-n)/n} \, M^{(n-1)/n} = \frac{K_p}{G N_n}$$

where N_n depends only on the polytropic index

$$N_n \equiv \frac{(4\pi)^{1/n}}{n+1} \left(\left[-\xi^2 \frac{d\theta}{d\xi} \right]_{\xi_1} \right)^{(1-n)/n} \xi_1^{(n-3)/n}$$

From here it is easy to calculate the average density of the star as,

$$\bar{\rho} \equiv \frac{3M}{4\pi R^3} = \rho_c \frac{3}{\xi_1^3}\left[-\xi^2 \frac{d\theta}{d\xi}\right]_{\xi_1}$$

and the central pressure as,

$$P_c = K_p \rho_c^{(n+1)/n} = W_n \frac{GM^2}{R^4}$$

where W_n is related to N_n as,

$$W_n = \left(\frac{3}{4\pi}\frac{\rho_c}{\bar{\rho}}\right)^{(n+1)/n} N_n$$

Below, I provide solutions for the Lane-Emden equation:

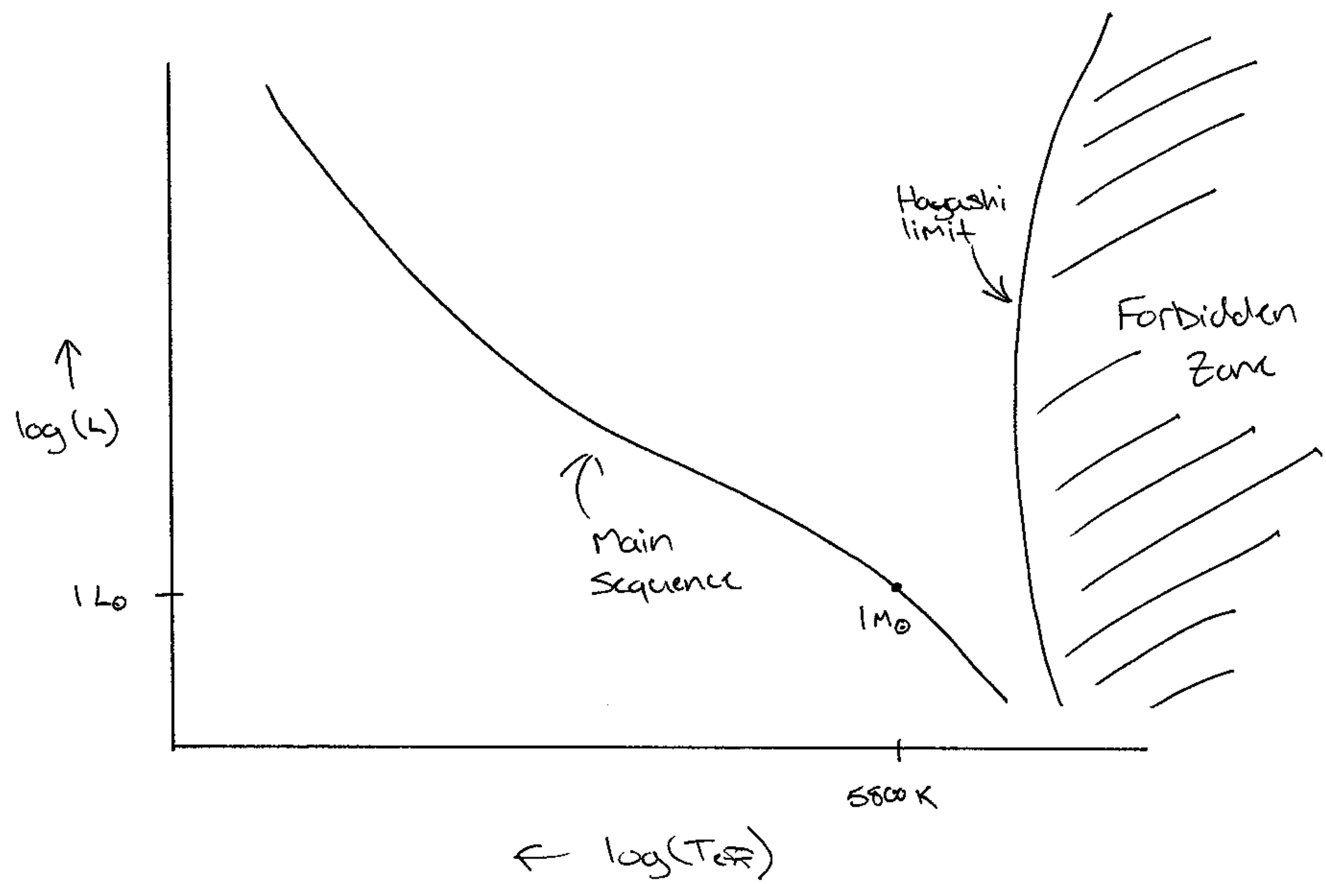
n	ξ_1	$-\xi_1^2 (d\theta/d\xi)_{\xi_1}$
1.0	3.14	3.14
1.5	3.65	2.71
2.0	4.35	2.41
2.5	5.36	2.19
3.0	6.90	2.02

Stars to the far right of the H-R diagram are fully convective. This defines the <u>Hayashi limit</u>.

Stars in hydrostatic equilibrium can not exist further to the right because their temperature gradients would exceed

$$\frac{d\ln(T)}{d\ln(P)} > 0.4 \qquad (\text{superadiabatic})$$

As we have seen whenever the temperature gradient exceeds this value, convection quickly brings it back down.



We can estimate the location of the Hayashi limit by considering a cool star (low T_{eff}) that is entirely <u>convective</u>.

First, let's consider the atmosphere of such a star. From hydrostatic equilibrium,

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

and the optical depth,

$$\frac{d\tau}{dr} = -\kappa\rho$$

We get the pressure gradient in terms of optical depth,

$$\frac{dP}{d\tau} = \frac{1}{\kappa}\frac{GM(r)}{r^2}$$

Recall that the photosphere is defined to be at an optical depth of $\tau = 2/3$.

If we assume that the protosphere is thin, and it contributes a negligable mass to the star, and the opacity $\kappa = \bar{\kappa}$ is constant with radius we can integrate this equation to get the pressure at the protosphere.

$$P(\tau = \tfrac{2}{3}) = \frac{2}{3} \frac{GM}{R^2} \frac{1}{\bar{\kappa}}$$

It is safe to assume an ideal gas equation of state,

$$P = \frac{K}{\mu m_H} \rho T$$

In cool stellar atmospheres (including the Sun's) the dominant source of opacity is bound-free scattering with the H^- ion. The opacity of H^- increases rapidly with temperature until H^- can no longer form.

$$\bar{K}_{H^-} \simeq \underbrace{(2.5 \times 10^{-31}) \left(\frac{Z}{0.02}\right)}_{\bar{K}_0} \rho^{1/2} T^9 \quad \frac{cm^2}{g}$$

Substituting in for these two quantities,

$$\frac{K}{\mu m_H} \rho T = \frac{2}{3} \frac{1}{\bar{K}_0 \rho^{1/2} T^9} \frac{GM}{R^2}$$

And rearranging we find that,

$$\rho^{1.5} T^{10} = \frac{2}{3} \frac{\mu m_H}{K} \frac{G}{\bar{K}_0} \frac{M}{R^2}$$

Now, assuming the star is fully convective from the photosphere down to the core, we can model the stellar interrior as a polytrope with polytropic index, $n = 1.5$. This is appropiate for an adiabatic ideal gas where $\gamma = \frac{5}{3}$.

Now, we can relate the surface conditions to the interrior using the polytropic solutions we derived earlier.

For an adiabatic ideal gas

$$\frac{\rho}{T^{1.5}} = \text{const} \quad \Rightarrow \quad \frac{\rho}{T^{1.5}} = \frac{\rho_c}{T_c^{1.5}}$$

and for a polytrope with $n=1.5$

$$\rho_c = 5.99 \frac{3M}{4\pi R^3} \quad , \quad T_c = 0.539 \frac{\mu M_H}{K} \frac{GM}{R}$$

Applying these relations to the result for the atmosphere

$$T^{12.25} \simeq \frac{0.1}{K_0} \left(\frac{\mu M_H G}{K} \right)^{3.25} M^{1.75} R^{0.25}$$

At the photosphere $T = T_{eff}$ and we can replace R using the stefan-boltzmann law

$$L = 4\pi R^2 \sigma T_{eff}^4$$

$$T_{eff}^{12.75} \simeq \frac{0.07}{\bar{K}_0 \sigma^{1/8}} \left(\frac{\mu m_H G}{k} \right)^{3.25} M^{1.75} L^{1/8}$$

Collecting the constants and scaling the variables,

$$T_{eff} \simeq (1300 \text{ K}) \left(\frac{M}{M_\odot} \right)^{0.14} \left(\frac{L}{L_\odot} \right)^{0.01} \left(\frac{Z}{0.02} \right)^{-0.08}$$

This is not too far off of much more detailed calculations!

The key is that this places a nearly vertical (weak dependance on L) limit on how cool a star may be.

At the end of the subgiant branch, the stellar surface temperature has effectively become as cool as it can. At this point, H^- has become the main source of opacity in the stellar atmosphere. As the He core continues to collapse, the H burning shell heats up increasing the luminosity. the envelope expands near the Hayashi limit ($\sim$ fixed T_{eff}).

Next, for

1. massive stars ($> 1.8\ M_\odot$)
   The He core reaches the temperature needed to start the triple-α process.

2. less massive stars ($< 1.8\ M_\odot$)
   The He core becomes degenerate before this temperature is reached.

For a degenerate electron gas, the equation of state is

$$P_e = K\rho^{5/3} \quad \text{(non-relativistic)}$$

constant

More on this later, but notice P_e is not a function of temperature. When He burning starts in a degenerate core, there is no effective thermostat to regulate the reaction rate. He burning increases the temperature, but this does not increase the pressure. As a result, the He burning rate rapidly increases. This causes the <u>helium flash</u> in low mass stars at the tip of the RGB. The core luminosity may reach $10^{11}\ L_\odot$ for ~ 2 seconds. Some of this goes into lifting the degeneracy of the core, restoring the thermostat.

With He → C core burning, stars enter the horizontal branch. This is analogous to a He burning main sequence.