33-224

Today

- Main Sequence Stellar Evolution
- Isothermal Stellar Cores

In this lecture we will begin our discussion of stellar evolution.

For this section of the course I have relied on:

1. "An Introduction to Modern Astrophysics" by Carroll and Ostlie
   chapter 13, "main sequence and post-main sequence stellar evolution"
2. Gary Glatzmaier's online Astronomy 112 notes (UCSC)
3. Adam Burrow's online Astrophysics 514 notes (princeton)

Stars begin their lives on the main sequence. At this point, they are burning hydrogen in their cores either via the p-p chain or the CNO cycle. However, as stars convert hydrogen to helium in their cores their structure must change slightly to compinsate for the change in chemical composition. The mass sequence where stars begin on the main sequence is called the zero age main sequence (ZAMS). For stars of the same initial composition, spin, etc, the ZAMS would be a thin line on the HR diagram. However, because stars vary in their initial conditions, their is some associated thickness to even the ZAMS.

Let's begin our discussion of stellar evolution with the evolution of stars off the main sequence. In general stars with different initial masses follow different evolutionary paths. We will split stars into groups given their initial mass on the ZAMS when discussing evolution.

First, we will talk about low mass stars, $M_* \lesssim 1.2\, M_\odot$. These stars generally have convective envelopes and radiative cores. Very low mass stars, $M_* \lesssim 0.3\, M_\odot$, have convective envelopes that reach all the way down into the core and as a result have slightly different evolution.

As fusion proceeds in the core, the mean particle mass increases. Given the ideal gas law,

$$P = \frac{\mathcal{R}}{\mu} \rho T$$

As μ increases, the pressure must decrease. As a result, the core contracts thereby increasing the density and temperature (think about the virial theorem). Although the mass fraction of hydrogen decreases, the increase in density and temperature more than compensates, and the H-burning rate increases.

$$\varepsilon_{pp} \propto \rho X^2 \left(\frac{T}{10^6 K}\right)^4$$

As a result, the luminosity of the star must increase. For low mass stars, this increase in luminosity is accompanied by slight increases in both the effective temperature <u>and</u> radius.

Eventually, the core is depleted of hydrogen. At this point the core (now ~ pure helium) becomes inert and isothermal. Hydrogen burning moves to a shell around the core. Detailed calculations show that total energy production continues to increase during this phase (so we expect the luminosity to increase further) however, some of this energy goes into heating the envelope, causing ito to expand and T_{eff} to decrease.

Because the surface temperature drops, the star moves to the right on the HR diagram. As H shell burning continues, mass is added to the inert He core. Eventually, the isothermal core can no longer support the star. This point defines the end of the main sequence. At this point, the star will enter a new rapid phase of evolution. More on this in a moment...

More massive stars evolve off the ZAMS slightly differently. Because these stars have convective cores, the nuclear fuel and ashes (H + He) are constantly mixed.

Still, during core H-burning the luminosity will increase. However, the size increases more rapidly relative to the luminosity of lower mass stars during a similar phase. As a result, these stars move to the right on the H-R diagram. Eventually, the core is depleted of H and the entire star begins to contract. The effective temperature and luminosity increase. This defines the end of the main sequence phase for massive stars.

Eventually for both low and more massive stars the inert helium core can no longer support the star. Lets take a look at this issue in more detail.

As stars reach the end of their main sequence lifetime, they have built up a significant core of helium. However, at this point, the core temperature is not hot enough to fuse He via the triple-α process. Let's consider what happens in this case.

Consider two of our equations of stellar structure which deal with energy flux

$$① \quad \frac{dL(r)}{dr} = 4\pi r^2 \rho(r) \varepsilon$$

$$② \quad \frac{dT(r)}{dr} = \frac{-3}{4ac} \frac{\bar{\kappa} \rho(r)}{T(r)^3} \frac{L(r)}{4\pi r^2}$$

Let's apply these to our inhert He core. It turns out it will be safe to assume heat transport is radiative.

In the core, the temperature is too low to fuse He

$$\Rightarrow \quad \varepsilon = 0$$

$$\Rightarrow \quad \frac{dL}{dr} = 0$$

inside core

This implies the heat flow inside the core is constant, including at the surface, which implies the luminosity of the inert core is zero

$$L(r = R_{core}) = 0$$

$$\Rightarrow \quad \frac{dT}{dr} = 0 \qquad \text{(isothermal)}$$

This is great! We have shown that the core is isothermal and since $\frac{dT}{dr} = 0$ it is certainly safe to assume radiative and not convective heat transfer.

It turns out that a star can not keep adding mass to this isothermal core forever. At some point it starts to lack the ability to provide pressure support against the material pushing down from above.

To see this we will begin with the virial theorem. We saw that for a star as a whole, hydrostatic equilibrium implied

$$\bar{P} = -\frac{1}{3}\frac{E_{grav}}{V}$$

A more general statement can be made

$$P_s V - \int_0^{m(r)} \frac{P}{\rho}\, dm = \frac{1}{3} U_{grav}$$

surface pressure

Here I am using Lagrangian coordinates to replace a dependence on position, r, with mass, m. These are related by

$$m(r) = \int_0^r 4\pi (r')^2 \rho(r')\, dr'$$

and

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

We can convince ourselves this is at-least okay by showing this gives the familiar result for the star as a whole by integrating from $m=0$ to $m=M$.

$$P_s V - \int_0^M \frac{P}{\rho}\, dm = \frac{1}{3} u_{grav}$$

surface pressure $P(r=R_*)=0$

$\frac{dm}{\rho} = dV$

recall that $dm = \underbrace{4\pi r^2 dr}_{dV}\, \rho$

$$\Rightarrow \quad 0 - \int P\, dV = \frac{1}{3} u_{grav}$$

$$\Rightarrow \quad \bar{P} = -\frac{1}{3} u_{grav} \checkmark$$

This is enlightening! We can only use this form because the surface pressure is zero (at least estimated to be) for stars. For the core $P_s \neq 0$ (not even close!).

As per usual in this class, we will make some assumptions in order to proceed with this virial argument.

(1) the gravitational binding energy can be estimated as

$$\Omega_{grav} = -\alpha \frac{G M_{core}^2}{R_{core}}$$

parameter to account for structure.
for uniform density $\alpha = \frac{3}{5}$

(2) the core is uniform ideal gas of pure ionized helium.

$$\frac{P}{\rho} = \left(\frac{R}{\mu_{core}}\right) T_{core}$$

gas constant

$\mu_{core} = 1.34$ (mean atomic mass)

pure He

note that $\mu_{\odot} \simeq 0.6$

The first estimate gives an expression for the right-hand side of the virial equation. The second replaces $\frac{P}{\rho}$ inside the integral.

$$P_s V_{core} - \int_0^M \frac{P}{\rho}\, dM = \frac{1}{3} U_{grav}$$

$$P_s V_{core} - \left(\frac{\mathcal{R}}{\mu_{core}}\right) T_{core} \int_0^M dM = \frac{1}{3} U_{grav}$$

$$\Rightarrow P_s V_{core} - \frac{\mathcal{R}}{\mu_{core}} T_{core} M_{core} = -\frac{\alpha}{3} \frac{G M_{core}^2}{R_{core}}$$

Solving for the surface pressure of the core gives,

$$P_s = \frac{3}{4\pi} \frac{\mathcal{R} T_{core}}{\mu_{core}} \frac{M_{core}}{R_{core}^3} - \frac{\alpha}{4\pi} \frac{M_{core}^2}{R_{core}^4}$$

note I have used $V_{core} = \frac{4\pi}{3} R_{core}^3$

This result is pretty interesting! Note that if we fix T_{core} + M_{core},

$$P_s \propto \frac{1}{R_{core}^3} - \frac{1}{R_{core}^4}$$

which for $R_{core} > 0$ has a local maximum.

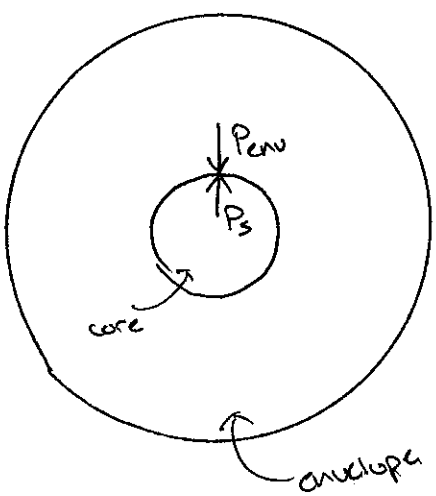
By taking the derivative wrt to R_{core} we can find the size of the core which can supply maximum pressure at its surface.

$$\frac{dP_s}{dR_{core}} = 0 \Rightarrow R_{core}^{max} = \frac{4\alpha \mu_{core} G}{9\mathcal{R}} \left(M_{core} / T_{core} \right)$$

This allows us to solve for the maximum pressure itself.

$$P_s^{max} = \frac{2187}{1024} \frac{\mathcal{R}^4}{\pi \alpha^3 G^3} \frac{T_{core}^4}{\mu_{core}^4 M_{core}^2}$$

Let's consider the consequences of this result.



Hydrostatic equilibrium tell us the surface pressure of the core balances that of the envelope

$$P_s = P_{env}$$

If the envelope is not applying much pressure, the core will be large. By examing the equation for P_s this implies the thermal contribution $\left(\propto \frac{1}{R^3}\right)$ is large compared to the self gravity of the core $\left(\propto \frac{1}{R^4}\right)$.

However, if P_s is large enough, the core's self gravity is significant. To account for this, the thermal pressure must increase by the core shrinking.

Eventually, the core can not support itself against gravity because P_s can not increase beyond P_s^{max}.

Let's now estimate when this happens.

Starting with hydrostatic equilibrium,

$$\frac{dP}{dr} = - \frac{GM(r)\rho(r)}{r^2}$$

Dividing by mass conservation equation gives,

$$\frac{dr}{dM(r)} \frac{dP(r)}{dr} = - \frac{GM(r)\rho(r)}{4\pi r^2 \rho(r) r^2}$$

$$\Rightarrow \left.\frac{dP(r)}{dM(r)} = - \frac{GM(r)}{4\pi r^4}\right\} \text{ Hydrostatic Equilibrium in Lagrangian coordinates}$$

Integrating from the surface to the isothermal core

$$P_{env} = \int_0^{P_{env}} dP = - \int_{M_*}^{M_{core}} \frac{GM(r)}{4\pi r^4} dM(r)$$

$$= - \frac{G}{8\pi} \frac{1}{\langle r^4 \rangle} \left[M_{core}^2 - M_*^2 \right]$$

some mass averaged value of the stellar radius.

Here come the approximations! Let's assume

$$\langle r^4 \rangle \sim R_*^4$$

$$M_{core} \ll M_*$$

$$\Rightarrow \quad P_{env} \simeq \frac{G M_*^2}{8\pi R_*^4}$$

Setting $P_{env} = P_s^{max}$ gives the approximate condition a star must satisfy to remain in hydrostatic equilibrium

$$\frac{G M_*^2}{8\pi R_*^4} = \frac{2187}{1024} \frac{\mathcal{R}^4}{\pi \alpha^3 G^3} \frac{T_{core}^4}{\mu_{core}^4 M_{core}^2}$$

Finally, we can require the temperature and pressure over the core/envelope boundary to be finite

$$T_{env}(r = R_{core}) = T_{core}$$

$$P_{env}(r = R_{core}) = P_s$$

For and ideal gas envelope,

$$T_{env} = T_{core} = \frac{P_s \, \mu_{env}}{\mathcal{R} \, \rho_{env}(r = R_{core})}$$

Substituting in $P_s = P_{max}$,

$$\Rightarrow \quad T_{core} = \frac{\mu_{env}}{\mathcal{R} \, \rho_{env}(r = R_*)} \left(\frac{2187}{1024}\right) \frac{\mathcal{R}^4}{\pi \alpha^3 G^3} \frac{T_{core}^4}{\mu_{core}^4 M_{core}^4}$$

$$= \left(\frac{2187}{1024}\right)^{-1} \frac{\pi \alpha^3 G^3}{\mathcal{R}^3} \left(\mu_{core}^4 M_{core}^2\right) \left(\frac{\rho_{env}(r = R_{core})}{\mu_{env}}\right)$$

Again, let's make a rough estimate that

$$\rho_{env}(r = R_{core}) \simeq \frac{3 M_*}{4 \pi R_*^3}$$

$$\cancel{\Rightarrow \quad T_{core}^3 = \left(\frac{2187}{1024}\right)^{-1} \frac{\pi \alpha^3 G^3}{\mathcal{R}^3} \left(\mu_{core}^4 M_{core}^2\right)}$$

$$\cancel{T_{core}^3 \simeq \left(\frac{2^8}{3^6}\right)}$$

$$T_{core}^3 = \left(\frac{256}{729}\right) \frac{\alpha^3 G^3}{\mathcal{R}^3} \left(\mu_{core}^4 M_{core}^2\right) \frac{1}{\mu_{env}} \left(\frac{M_*}{R_*^3}\right)$$

Plugging this into our condition for stability ($P_s^{max} > P_{env}(r = R_{core})$) gives,

$$\frac{G M_*^2}{8\pi R^4} \gtrsim \left(\frac{2187}{1024}\right) \frac{\mathcal{R}^4}{\pi \alpha^3 G^3} \frac{1}{\mu_{core}^4 M_{core}^2} \left[\left(\frac{256}{729}\right) \frac{\alpha^3 G^3}{\mathcal{R}^3} \frac{\mu_{core}^4 M_{core}^2 M_*}{\mu_{env} R_*^3}\right]^{4/3}$$

$$\Rightarrow \frac{M_{core}}{M_*} \lesssim \underbrace{\left(\frac{27}{2048\,\alpha^3}\right)^{1/2}}_{\substack{\text{for } \alpha = \frac{3}{5} \\ \simeq 0.25}} \left(\frac{\mu_{env}}{\mu_{core}}\right)^2$$

This is pretty good! More detailed calculations arrive at,

$$\frac{M_{core}}{M_*} \leq 0.37 \left(\frac{\mu_{env}}{\mu_{core}}\right)^2$$

For a pure helium core $\mu = 1.3$ and for a solar composition envelope $\mu = 0.6$. This gives,

$$\frac{M_{core}}{M_*} \lesssim 0.1$$

A more massive core will collapse!