33-777

Today

- Review of Stellar Evolution
- Degenerate Matter in Stars

In this lecture we briefly review our picture of Stellar Evolution before a more detailed discussion of electron degeneracy pressure in stars

– Constraints on Stellar Evolution

Although fundamentally stars evolve according to the full set of stellar structure equations, we have covered at least two physical constraints on stellar evolution.

1. limit on pressure support from an isothermal core.

The maximum pressure at the surface of such a core is.

$$P_s^{max} = \frac{2187}{1024} \frac{\mathcal{R}^4}{\pi \alpha^3 G^3} \frac{T_{core}^4}{\mu_{core}^4 M_{core}^2}$$

This results in a maximum core mass,

$$\frac{M_{core}}{M_*} \lesssim 0.37 \left(\frac{\mu_{env}}{\mu_{core}} \right)^2$$

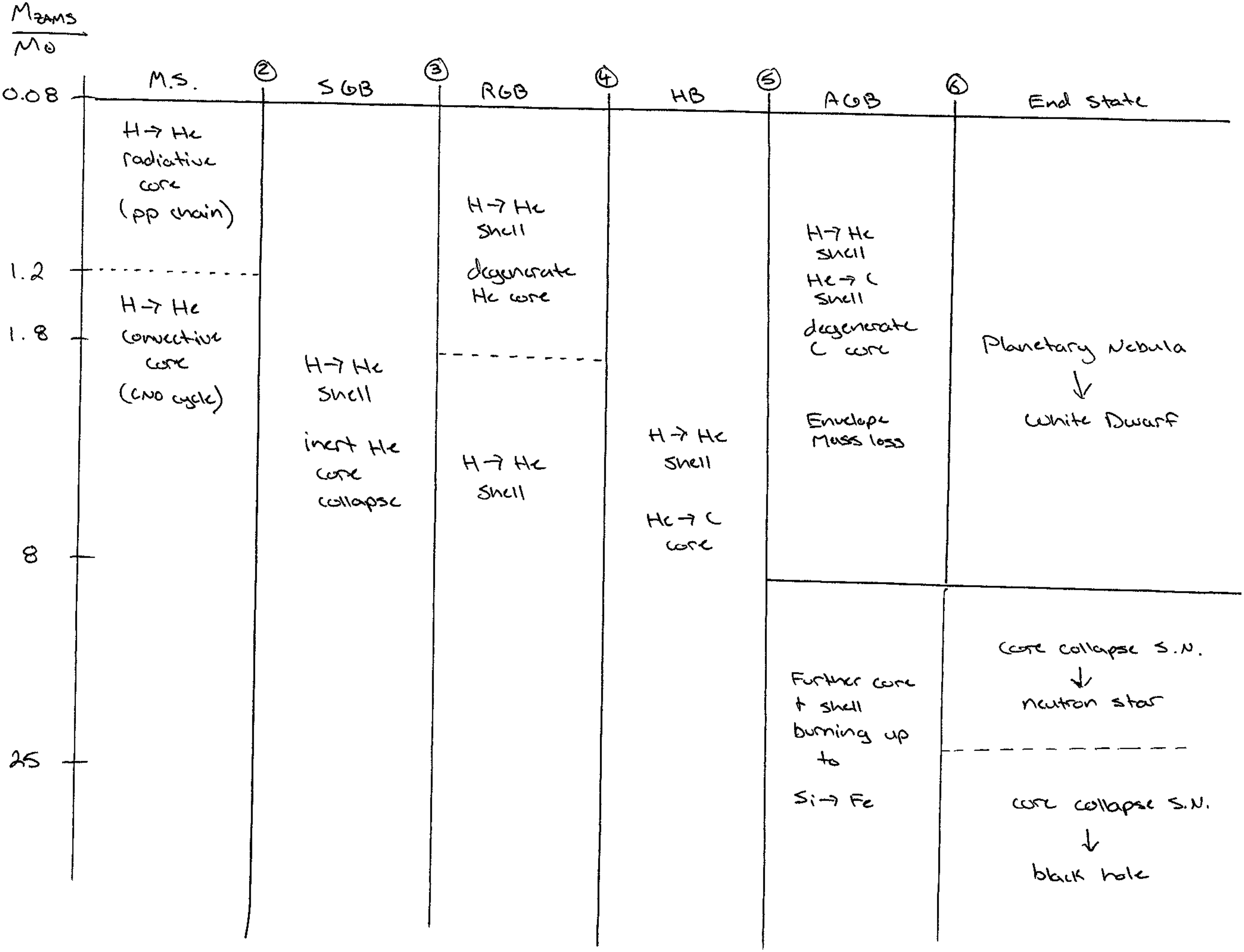
When an isothermal core exceeds this mass it will start to collapse on the Kelvin-Helmholtz timescale

2 The Hayashi Limit

There is a minimum temperature a star in hydrostatic equilibrium can maintain. This limit is set by the structure of a fully convective star. If the surface temperature were any lower (redword on the HR diagram) convection (being an effecient heat transport mechanism) would quickly increase the surface temperature to the Hayashi limit.

$$T_{eff} \gtrsim (2{,}000\ K)\left(\frac{M}{M_\odot}\right)^{0.14}\left(\frac{L}{L_\odot}\right)^{0.01}\left(\frac{z}{0.02}\right)^{-0.08}$$

Notice the very weak dependence on luminosity. This is effectively a verticle line on the HR diagram.



$M_{ZAMS}/M_\odot$
0.08
1.2
1.8
8
25
M.S.
2
SGB
3
RGB
4
HB
5
AGB
6
End State
H → He radiative core (pp chain)
H → He convective core (CNO cycle)
H → He Shell
inert He core collapse
H → He Shell
degenerate He core
H → He Shell
H → He Shell
He → C core
H → He Shell
He → C Shell
degenerate C core
Envelope Mass loss
Further core + shell burning up to
Si → Fe
Planetary Nebula
↓
White Dwarf
Core collapse S.N.
↓
neutron star
core collapse S.N.
↓
black hole

(1) Zero Age Main Sequence (ZAMS)

This begins for stars once their core starts to burn H → He either via the pp chain or the CNO cycle

Stars evolve away from the ZAMS as μ increases in the core as a result of H → He core burning

(2) End of Main sequence

- low mass stars ($M < 1.2 M_\odot$)
  isothermal core collapses
- higher mass stars ($M > 1.2 M_\odot$)
  entire star collapses after H depletion in core

  H → He shell burning eventually begins adding mass to the core. Eventually on the SGB the isothermal core collapses

(3) Red Giant Branch (RGB) begins

All stars run up against the Hayashi limit. At this point their luminosity and size increase

Inert He core continues to collapse as $H \rightarrow He$ shell burning continues

(4) End of RGB

- low mass stars ($M < 1.8 M_\odot$)

  He core is degenerate

  $He \rightarrow C$ burning begins with He flash

- higher mass stars ($M > 1.8 M_\odot$)

  He core is not (at least entirely) degenerate.

  $He \rightarrow C$ burning begins more gradually

(5) End of Horizontal Branch (H.B.)

The core is depleted of He

Isothermal core begins to collapse

$H \rightarrow He$ shell burning continues

$He \rightarrow C$ shell burning is episodic

(6) End of AGB

- low mass stars ($M < 8 M_\odot$)

Stars experience significant mass loss

Envelope is expelled during planetary nebula phase

- higher mass stars ($M > 8 M_\odot$)

Retain enough mass to continue multiple (rapid) phases of core and shell burning

Degenerate Matter in Stars

At various points in a star's life, the core may be supported by electron degeneracy pressure. Furthermore, the end states of some stars are composed of nearly entirely degenerate matter. In order to understand this aspect of stellar evolution, we need to look at degenerate matter in more detail.

Let's start by considering the equation of state. For a classical gas, we can derive the equation of state using the pressure integral.

$$P = \frac{1}{3}\int_0^\infty n(p)\, p\, v\, dp$$

pressure

$n(p)dp \equiv$ "distribution function"

By substituting in the Maxwell-Boltzmann distribution we can derive the ideal gas law.

$$P = \frac{k}{\mu m_H}\rho T$$

Ideal Gas Law

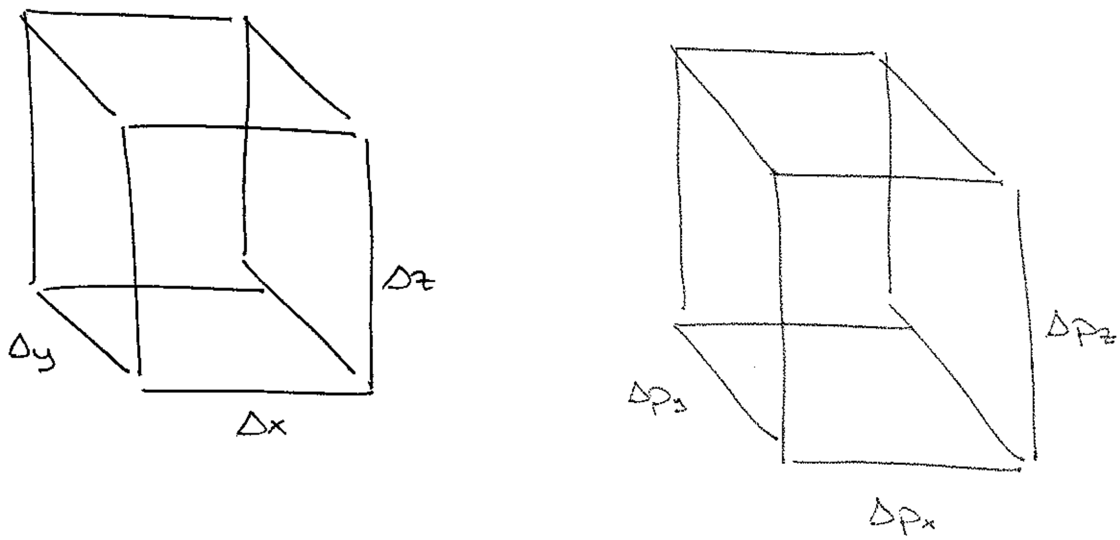
In addition to some simplifications assumed for an ideal gas (non-interacting particles) we know this equation of state must break down in two regimes:

1. $T \to \infty$
2. $T \to 0$

In the first case, as temperature increases particles become relativistic and no longer follow the Maxwell-Boltzmann distribution. In addition as particle energies become comparable to their rest mass, particle-anti particle creation becomes important.

It is the second case which is of interest to us today. As $T \to 0$, the ideal gas implies pressure goes to zero. However, at high densities quantum effects cause fermions to remain in non-zero momentum states. Inside stars, at certain densities, this effect causes electrons to provide more pressure support than the ideal gas law would predict.

To get a feel for a degenerate electron gas consider a small portion of position and momentum phase space



The volume of this phase space element is given by,

$$\underbrace{\Delta x \Delta y \Delta z}_{dV} \Delta p_x \Delta p_y \Delta p_z$$

From the Heisenberg uncertainty principle $\Delta x \Delta p_x > h$. Furthermore, let the total momentum,

$$p^2 = p_x^2 + p_y^2 + p_z^2$$

Thus, this element of phase space has a volume,

$$d^3p\, dV > h^3$$

This implies that it doesn't make much sense to split this phase space into chunks with less 6-D volume than $\sim h^3$.

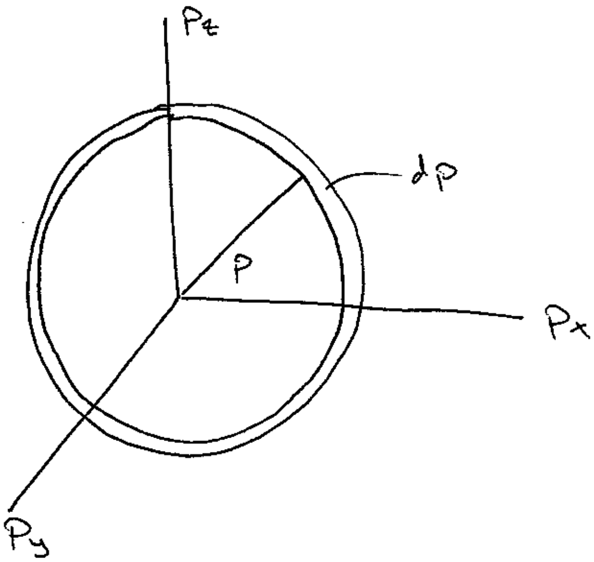
Consider an electron gas which fills some spatial volume V with electrons with momentum in the range $p, p+dp$. The number of quantum states available to these electrons is

$$N = \frac{2\, V\, d^3p}{h^3}$$

factor of two accounts for spin $\pm\frac{1}{2}$ electrons

No more electrons may be found in this part of phase space given the Pauli exclusion principle, i.e. no two fermions may have the exact same quantum state.

How big is d^3p? For a fixed magnitude p, d^3p is a shell in momentum space



$$d^3p = (4\pi p^2 dp)$$

The total number of states is then,

$$N_p = \frac{8\pi p^2 dp}{h^3} V$$

The number density is then,

$$n_e = \frac{N_p}{V} = \frac{8\pi p^2 dp}{h^3}$$

If we let the temperature of this electron gas go to zero, $T \to 0$, then the lowest energy (momentum) states will be occupied.

The total electron number density is then,

$$n_e = \int_0^{p_F} \frac{8\pi}{h^3} p^2 \, dp = \frac{8\pi}{3h^3} p_F^3$$

where p_F is the Fermi momentum, the highest momentum state filled by electrons in the $T=0$ gas.

$$p_F = \left(\frac{3h^3}{8\pi}\right)^{1/3} n_e^{1/3}$$

$$= \hbar \left(3\pi^2 n_e\right)^{1/3}$$

The corresponding Fermi energy is

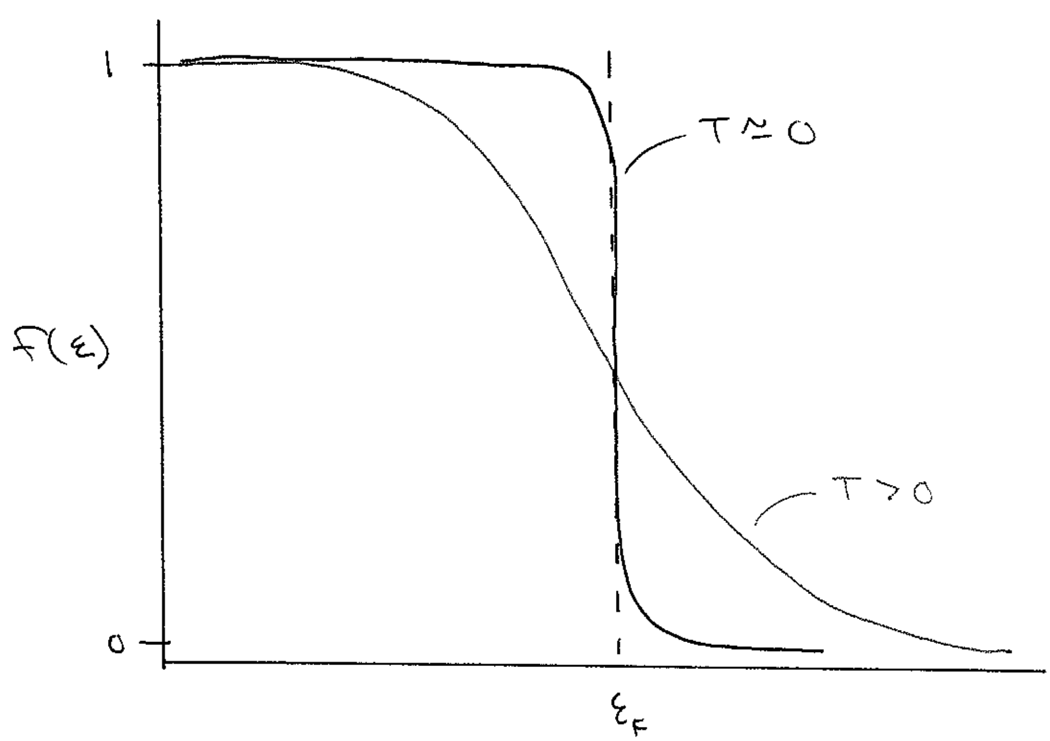
$$\varepsilon_F = \frac{p_F^2}{2m_e} = \frac{\hbar^2}{2m_e}\left(3\pi^2 n_e\right)^{2/3}$$

Although not derived here, for an electron gas, the occupation fraction of states with energy ε is given by the Fermi-Dirac distribution,

$$f(E) = \frac{1}{\exp\left[(\varepsilon - \mu(T))/KT\right] + 1}$$

Here $\mu(T)$ is the chemical potential of the gas. At low temperatures this approaches the Fermi energy,

$$\mu(T=0) \equiv \varepsilon_F$$



As a result, the Fermi energy serves as a good energy scale for degenerate matter.

Consider that the mean kinetic energy of a particle in an ideal gas is given by,

$$\langle KE \rangle = \frac{3}{2} KT \quad \Big\} \text{ For ideal gas}$$

It should be clear that degeneracy pressure becomes an important source of pressure when

$$\frac{3}{2} KT \sim \varepsilon_F$$

$$\Rightarrow \quad \frac{T}{n_e^{2/3}} \sim \underbrace{\frac{\hbar^2}{3 K m_e} \left(3\pi^2\right)^{2/3}}_{\text{constants}}$$

Thus degeneracy pressure is important at

1. low temperatures
2. high densities

We can rewrite this condition for degeneracy in terms of mass density.

$$n_e = \frac{Z}{A}\frac{\rho}{m_H}$$

$$\Rightarrow \quad \frac{T}{\rho^{2/3}} \sim \frac{\hbar^2}{3 k m_e}\left(3\pi^2 \frac{Z}{A}\frac{1}{m_H}\right)^{2/3}$$

This gives us a good idea about when electron degeneracy pressure is important inside of stars. Now we would like to know what kind of pressure this can provide.

Returning to the pressure integral,

$$P = \frac{1}{3}\int_0^\infty n(p)\, p v \, dp$$

We have already estimated (for $T \to 0$)

$$n_e(p)\,dp = \begin{cases} \dfrac{8\pi p^2 dp}{h^3}, & p \le p_F \\ 0, & p > p_F \end{cases}$$

Substituting this into the pressure integral

$$P = \frac{8\pi}{3h^3} \int_0^{p_F} v p^3 \, dp$$

Let's allow particles to be relativistic

$$v = \frac{p}{m\gamma} = \frac{pc^2}{E} = \frac{pc^2}{(p^2c^2 + m^2c^4)^{1/2}}$$

Lorentz factor

$$\Rightarrow \quad P = \frac{8\pi}{3h^3} \int_0^{p_F} \frac{p^4 c^2}{(p^2c^2 + m_e^2 c^4)^{1/2}} \, dp$$

Now, let's consider two regimes,

① non-relativistic

$$(p^2c^2 + m_e^2c^4)^{1/2} \simeq m_e c^2$$

② ultra-relativistic

$$(p^2c^2 + m_e^2c^4)^{1/2} \simeq pc$$

In the non-relativistic case:

$$P = \frac{8\pi}{15h^3 m_e} p_F^5$$

$$\boxed{P = K_1 \rho^{5/3}}$$

where $K_1 = \left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{20 m_e m_H^{5/3}} \left(\frac{Z}{A}\right)^{5/3}$

In the ultra-relativistic case:

$$P = \frac{2\pi c}{3h^3} p_F^4$$

$$\boxed{P = K_2 \rho^{4/3}}$$

where $K_2 = \left(\frac{3}{8\pi}\right)^{1/3} \frac{hc}{4 m_H^{4/3}} \left(\frac{Z}{A}\right)^{4/3}$

These are both pretty interesting! Notice that neither equation of state depends on temperature. While each of these are limits, it can be shown numerically there is a smooth transition from the non-relativistic case towards the ultra-relativistic case.

Let's continue to derive a couple important results for "stars" supported entirely via electron degeneracy pressure.

Both equations of state take the form

$$P = K \rho^{\gamma}$$

where for $\gamma = 1 + \frac{1}{n}$

$$n = \frac{3}{2} \quad \text{(non-relativistic)}$$

$$n = 3 \quad \text{(ultra-relativistic)}$$

These are of course polytropic equations of state where the structure of the star can be solved via the Lane-Emden equation.

Recall that for a polytrope star mass and radius are related via,

$$R^{(3-n)/n} M^{(n-1)/n} = \underbrace{\frac{K_p}{G N_n}}_{\text{constants}}$$

For the non-relativistic case ($n = 3/2$)

$$R \propto M^{-1/3}$$

For fully degenerate stellar cores or whole stars (white dwarfs) the size of the core/star decreases with mass.

For the ultra-relativistic case ($n = 3$) the same relation implies,

$$M \propto \text{const.}$$

i.e., mass is independant of size.

Filling in the constants and solving for the mass gives,

$$M_{ch} = 0.21 \left(\frac{Z}{A}\right)^2 \underbrace{\left(\frac{hc}{GM_H^2}\right)^{3/2}}_{} M_H$$

This quantity is a dimensionless constant $\frac{1}{\alpha_G}$

$$\alpha_G = \frac{GM_p^2}{\lambda_p} \frac{1}{M_p c^2} = \frac{GM_p^2}{(h/M_p c) M_p c^2} \simeq 10^{-39}$$

$$\lambda_p = \frac{h}{p} = \frac{h}{M_p c}$$

This is analgous to the fine structure constant for electromagnetism!

$$\boxed{M_{ch} = (1.4\, M_\odot) \left(\frac{Z}{A}\right)_{0.5}^2}$$

$$\left(\frac{Z}{A}\right)_{0.5} \equiv \left(\frac{Z/A}{0.5}\right)$$