33-777

Today

I Energy Generation in Stars

II Nuclear Reaction Rates

In this lecture we go into more detail about energy generation inside stars. We then spend some time going over nuclear reaction rates.

The remaing bit of physics required to complete our relatively detailed picture of stellar interriors is a discussion of stars' energy source(s).

We have already mentioned one source of energy, gravitational energy: From the virial theorem,

$$E_{thermal}^{tot} = -\frac{E_{grav}}{2}$$

This tells us as a star collapses and the gravitational potential energy becomes more negative, the star must heat up. In addition to implying stars have a negative heat capacity, conservation of energy implies an equal amount of energy must be radiated away.

Although the transformation of gravitational energy into thermal energy via gravitational collapse is important in order to understand stellar evolution, clearly it can not be the only source of energy. This can be seen with an order of magnitude argument comparing the total amount of liberated gravitational energy from the Sun's formation to its current luminosity.

A rough estimate of the Sun's gravitational potential energy can be estimated assuming constant density.

$$\rho \simeq \bar{\rho}_\odot = \frac{M_\odot}{\frac{4}{3}\pi R_\odot^3}$$

And the Sun was assembled inside out by bringing shells together from an initial position at $r = \infty$.

In this case the gravitational potential energy is given by,

$$U_{grav} = -4\pi G \int_0^{R_\odot} M(r)\,\rho(r)\,r\,dr$$

$$= -\frac{3}{5}\frac{G M_\odot^2}{R_\odot}$$

Half of this energy must be accounted for by thermal energy with the Sun. The other half must be radiated away. The net change in energy is then

$$\Delta E = -\frac{1}{2} U_{grav}$$

By comparing this to the Sun's luminosity, we arrive at the Kelvin-Helmholtz timescale

$$t = \frac{\Delta E}{L_\odot} \simeq 10^7 \text{ yr.}$$

Given the age of the solar system ($> 10^9$ yr) there must be another source of energy available to stars that serves as a source of heat for pressure support against against runaway gravitational collapse.

We now know that this energy source is nuclear fusion. Again a rough order of magnitude shows fusion to be a copious energy source in stars.

Consider the formation of helium via the fusion of four hydrogen nuclei.

$$4\,{}^{1}_{1}H \rightarrow {}^{4}_{2}He + 2e^{+} + 2\nu_e + 2\gamma$$

The energy released by this reaction in the form of photons and neutrinos.

Note that in the above reaction, the two positrons will quickly annihilate with free electrons in the sun to produce photons.

$$e^- + e^+ \rightarrow \gamma + \gamma$$

It turns out the neutrinos produced will leave the Sun without interacting, serving as a slight energy loss mechanism. More about neutrinos later.

In general, the energy released by a nuclear reaction will be the difference in binding energy between the nuclear constituents on the left and right side.

$$E_{binding} = \left[Z m_p + (A-Z) m_n - M_{nucleous} \right] c^2$$

Here Z is the atomic number, A the atomic mass number, m_p the proton mass, m_n the neutron mass, and $M_{nucleous}$ the mass of the nucleous. For the above reaction, the total amount of energy released is

$$E = \Delta m c^2 = 26.731 \text{ MeV}$$

This is 0.7% of the rest mass of the four H nuclei on the left side of the reaction.

We can use this to make an analgous calculation to the Kelvin-Helmholtz timescale, the <u>nuclear timescale</u>,

$$t = \frac{0.007 M_\odot}{L_\odot} \simeq 10^{10} \text{ yr}$$

time to convert 1 $M_\odot$ of ${}^1_1H$ to ${}^4_2He$ at a rate of 1 $L_\odot$.

Recall one of our equations of stellar structure,

$$\frac{dL}{dr} = 4\pi r^2 \rho \varepsilon$$

energy generation rate.

We now know ε is primarily the energy generation rate from nuclear fusion. In general this will be a function of temperature, density, and chemical composition,

$$\varepsilon = \varepsilon(\rho, T, X_i)$$

which in general are all a function of radius inside a star. In order to understand how one may calculate ε in a stellar model, we must review the basics of nuclear fusion.

$$E = \frac{(h/\lambda)^2}{2\mu}$$

$$= \frac{\left(\frac{K_e 2\mu Z_1 Z_2 e^2}{h}\right)^2}{2\mu}$$

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note

$e \simeq 0.3\sqrt{\hbar c}$

$K_e = \frac{1}{4\pi}$

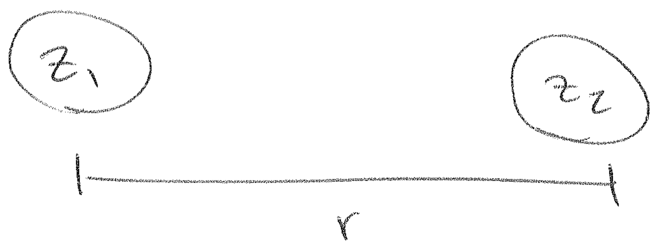
For the proton-proton case $Z_1 = Z_2 = 1$, $\mu = \frac{1}{2} m_p$

$$\Rightarrow \quad E = \left(\frac{0.3^2}{8\pi^2}\right)^2 m_p c^2 = 1.2 \times 10^3 \text{ eV}$$

$$\simeq 1 \text{ KeV}$$

This corresponds to a temperature of $\sim 10^7$ K, precisely the estimate we got for the solar core temperature.

Consider two nuclei with atomic numbers Z_1 and Z_2 seperated by a distance r



The electrostatic potential energy due to Coulomb repulsion is given by

$$\frac{1}{4\pi\varepsilon_0}\frac{Z_1 Z_2 e^2}{r}$$

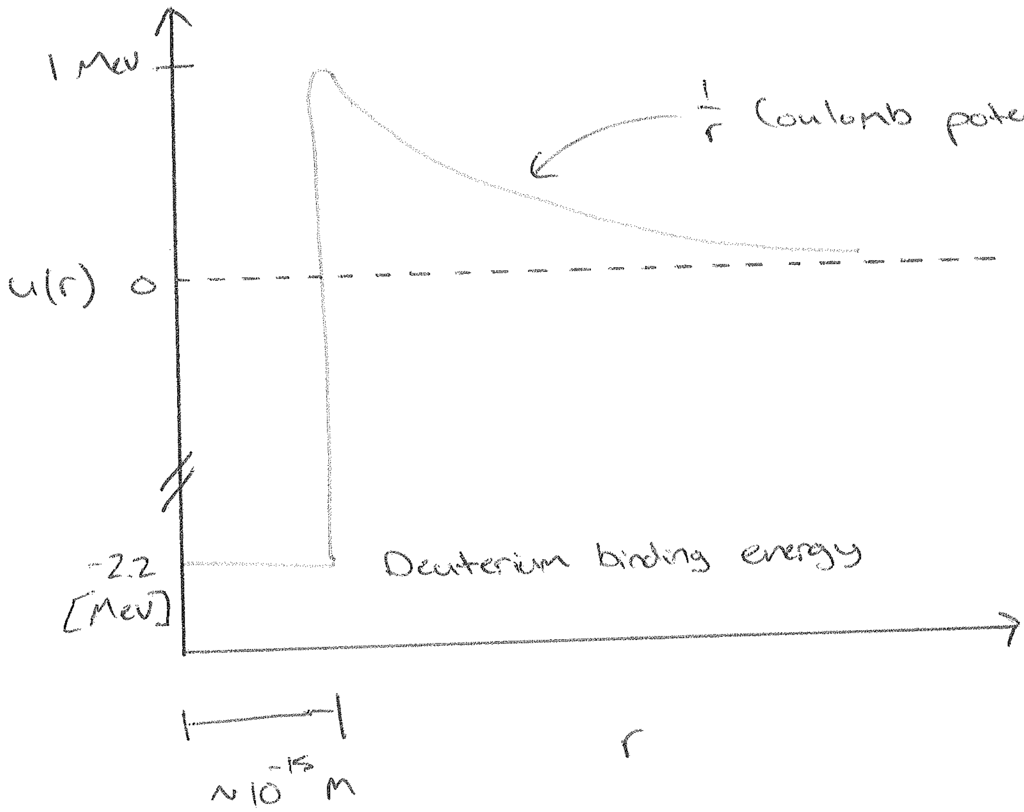
This suggests that the potential keeps rising as two nuclei are brought closer together. However, eventually, the strong force takes over and the potential must drop to

$$U < 0 \Rightarrow |U| = |E_b|$$

binding energy of new nucleous

the binding energy of the new nucleous with $Z = Z_1 + Z_2$.

For two ^1_1H nuclei, the potential energy curve will look like this,



At first blush, it appears that particles need about ~1 MeV in kinetic energy to overcome the Coulomb barrier. However, the typical energy of a proton in the Sun's core where $T \sim 10^7$ K is on the order of 1 KeV. Even accounting for the high energy tail

Maxwell-Boltzmann distribution, not a single particle in the Sun has the required energy.

Fortunatly, one of the famous predictions from quantum mechanics is that a particle may "tunnel" through a potential barrier. Without rederiving this result in detail, we can get a flavor for it with the following argument. Recall that the de Broglie wavelength of a massive particle is given by

$$\lambda = \frac{h}{p}$$

Now the idea is that instead of getting close enough to overcome the Coulomb potential, two particles need only be within $r \sim \lambda$ in order for a significant chance that a particle tunnels through the Coulomb potential!

Let's apply this to the proton-proton case. In the rest frame of the target proton, the kinetic energy of the other proton is given by,

$$\frac{1}{2}\mu v^2 = \frac{p^2}{2\mu} \qquad \mu = \frac{1}{2}M_p \quad \text{(reduced mass)}$$

$$= \frac{(h/\lambda)^2}{2\mu}$$

A proton with this energy will approach to a distance λ when

note $K_e = \frac{1}{4\pi\varepsilon_0}$

$$K_e \frac{Z_1 Z_2 e^2}{\lambda} = \frac{(h/\lambda)^2}{2\mu}$$

$$\Rightarrow \lambda = \frac{h^2}{K_e 2\mu Z_1 Z_2 e^2}$$

This corresponds to an energy,

Now that we have an idea about the conditions under which fusion may take place, let's take a closer look energy generation <u>rates</u>.

For an ideal gas, particles follow a Maxwell-Boltzmann distribution. The pairwise relative velocity distribution is also Maxwellian but with the particle mass replaced with the reduced mass μ.

$$f(v)\,dv = \left(\frac{\mu}{2\pi KT}\right)^{3/2} e^{\frac{-\mu v^2}{2KT}} \, 4\pi v^2 \, dv$$

where $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$

The kinetic energy $E = \frac{1}{2}\mu v^2$ distribution is then

$$f(E)\,dE = \frac{2}{\sqrt{\pi}} \frac{E^{1/2}}{(KT)^{3/2}} e^{-E/KT} \, dE$$

Let's let $\sigma(E)$ be the energy dependent nuclear reaction cross section. The reaction rate is then

$$r = n_1 n_2 \langle \sigma v \rangle \qquad \left[\frac{\#\ \text{reactions}}{\text{volume} \cdot \text{time}}\right]$$

number density of each species

where the average product of the cross-section and velocity is given by,

$$\langle \sigma v \rangle = \int_0^\infty \sigma(E)\, v\, f(E)\, dE$$

Here we can see that if we know the temperature and the density of each species, then all we need to calculate the reaction rate is the cross section, $\sigma(E)$. For this type of fusion reaction σ consists of two parts

$$\sigma = \underset{\substack{\uparrow \\ \text{classical} \\ \text{geometric} \\ \text{cross-section}}}{P_{collision}} \; \underset{\substack{\nwarrow \\ \text{QM tunneling} \\ \text{probability}}}{P_{tunnel}}$$

The tunneling probability, first calculated by Gamow is a famous result in quantum mechanics. We just quote the result here,

$$P_{tunnel} \propto \exp\left[-\frac{1}{2\varepsilon_0 \hbar} \left(\frac{\mu}{2}\right)^{1/2} \frac{Z_1 Z_2 e^2}{\sqrt{E}} \right]$$

The geometric cross-section should go as

$$P_{\text{collision}} \propto \underbrace{\lambda^2}_{\text{de Broglie wavelength}} \propto \frac{1}{E}$$

With these two parts, the nuclear cross-section is given by

$$\sigma(E) = \underbrace{S(E)}_{\text{proportionality constant}} \frac{1}{E} e^{-b/\sqrt{E}}$$

where

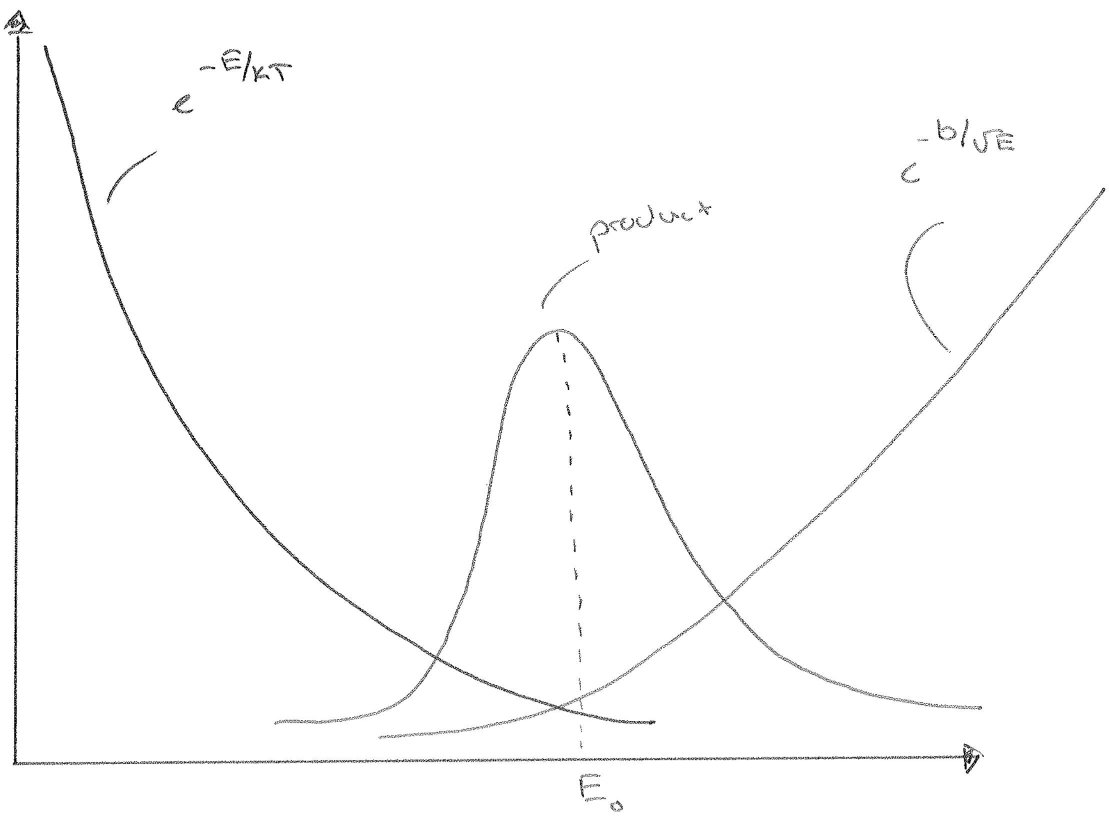
$$b = \frac{1}{2\varepsilon_0 \hbar} \left(\frac{\mu}{2}\right)^{1/2} Z_1 Z_2 e^2$$

We can substitute this into our equation for $\langle \sigma v \rangle$

$$\langle \sigma v \rangle = \frac{2^{3/2}}{\sqrt{\pi \mu}} \frac{1}{(kT)^{3/2}} \int_0^\infty S(E)\, e^{-E/kT} e^{-b/\sqrt{E}}\, dE$$

Let's notice a few things about this expression

* $S(E)$ is often assumed to be a slowly varying function of E, although this is not true around a nuclear resonance
* $e^{-E/kT}$ is the high energy tail of the Maxwell-Boltzmann distribution
* $e^{-b/\sqrt{E}}$ is called the Gamow factor and increases rapidly with E



E

Since the product of the two exponentials is only appreciably non-zero around a value E_0, we can pull $S(E)$ outside the integral and set it to its value at E_0

$$\langle \sigma v \rangle = \frac{2^{3/2}}{\sqrt{\pi \mu}} \frac{1}{(KT)^{3/2}} S(E_0) \, J$$

where

$$J = \int_0^\infty \underbrace{\exp\left(-\frac{E}{KT} - \frac{b}{\sqrt{E}}\right)}_{g(E)} dE$$

The value of E_0 is found by solving

$$\frac{dg}{dE} = 0 \Rightarrow E_0 = \left(\frac{1}{2} b KT\right)^{2/3}$$

we can then find the value of J by approximating $g(E)$ as a gaussian centered on E_0

$$J \simeq \frac{2}{3} KT \sqrt{\pi \tau} \, e^{-\tau}$$

where $\tau = -g(E_0) = \dfrac{3 E_0}{KT}$

If we put this all together we find,

$$\langle \sigma v \rangle \propto \frac{S(E_0)}{T^{2/3}} \exp\left[-3\left(\frac{e^4}{32\varepsilon_0^2 k \hbar^2} \frac{\mu Z_1^2 Z_2^2}{T}\right)^{1/3}\right]$$

Now recall that the relavent equation of stellar structure is

$$\frac{dL}{dr} = 4\pi r^2 \rho \varepsilon$$

The energy generation rate is

energy released per reaction

$$\rho \varepsilon = r \Delta E = n_1 n_2 \langle \sigma v \rangle \Delta E$$

reaction rate

Writing this in the astronomer-like way using Mass fractions

$$\varepsilon = \frac{X_1 X_2}{m_H^2 A_1 A_2} \langle \sigma v \rangle \Delta E$$

Putting this together we get the energy generation rate

$$\varepsilon = C \rho X_1 X_2 \frac{1}{T^{\frac{2}{3}}} \exp\left[-3 \left(\frac{e^4}{32 \varepsilon_0^2 k \hbar^2} \underbrace{\frac{\mu Z_1^2 Z_2^2}{T}}_{\text{tells us lighter elements burn first.}} \right)^{1/3} \right]$$