# Astro 519 HW Solutions #2

1. We can start by finding the momentum transferred to a charged particle through Thompson scattering.

   Consider a charged particle with mass $m$ and Thompson cross-section $\sigma_T$, scattering with an unpolarized incident photon with energy $E \ll mc^2$ (Thompson limit). Since the recoil of the mass from such an interaction will be small, we can approximate that the outgoing photon has the same *magnitude* of momentum $p_0$ as it had when it came in. If the photon is deflected by an angle $\phi$, then the momentum transferred to the charged particle *along the direction of the incident wavevector* $\hat{k}$ is

   $$\Delta p_k = p_0(1 - \cos\phi).$$

   Since the differential Thompson scattering cross-section is proportional to $1 + \cos^2\phi$, and is thus symmetric about $\phi = \pi/2$, we get

   $$\langle \Delta p_k \rangle = p_0$$

   averaged over all scattered photons. To find the resulting *force* from a light source with uniform $\hat{k}$, we would sum this over all of the interactions per second. Since the scattering cross-section doesn't depend on the momentum of the incoming photon, this is just the flux, divided by $c$ to convert to momentum, times $\sigma_T$.

   However, if we *don't* have a source with uniform $\hat{k}$, then only *some component* of the resulting force will be in the direction of interest. However, the resulting cosine factor is already included in the definition of momentum flux. So, to get the force from Thompson scattering in a given direction, we just need to compute

   $$F_T = p\sigma_T$$

   where $p$ is the momentum flux through a surface with normal vector in the desired direction.

   For the given case, the force will be in the $\hat{z}$ direction by rotational symmetry, so we just need to calculate the momentum flux through a surface parallel to (and along the axis of) the disk at the indicated distance $z$. In terms of surface brightness $B$, this is

   $$\begin{aligned} p &= \frac{B}{c}\int \cos^2\theta \, \mathrm{d}\Omega \\ &= \frac{2\pi B}{c}\int_0^{\theta_{\max}} \cos^2\theta \sin\theta \, \mathrm{d}\theta \\ &= \frac{2\pi B}{c}\int_{-1}^{-\cos\theta_{\max}} u^2 \, \mathrm{d}u \end{aligned}$$

   for $u \equiv -\cos\theta$. The integral then gives

   $$\begin{aligned} p &= \frac{2\pi B}{3c}\left(1 - \cos^3\theta_{\max}\right) \\ &= \frac{2\pi B}{3c}\left(1 - \frac{z^3}{(z^2 + R^2)^{3/2}}\right) \\ &= \frac{2\pi B}{3c}\left[1 - \left(1 + \frac{R^2}{z^2}\right)^{-3/2}\right]. \end{aligned}$$

For an object with uniform surface brightness, the total luminosity is $L = \pi B A$ where $A$ is the total surface area. Assuming the disk is physically thin, $A = 2\pi R^2$ (top and bottom), and so we have

$$B = \frac{L}{2\pi^2 R^2}.$$

Plugging this in,

$$p = \frac{L}{3\pi R^2 c}\left[1 - \left(1 + \frac{R^2}{z^2}\right)^{-3/2}\right].$$

The force is then $p\sigma_T$, as before.

Note: because the force falls off *slower than quadratically*, if this force is *initially* stronger than the gravitational attraction, then it will remain so as the particle moves away from the disk. The Eddington luminosity limit is thus given by

$$F_T = F_g$$

$$\frac{L\sigma_T}{3\pi R^2 c}\left[1 - \left(1 + \frac{R^2}{z^2}\right)^{-3/2}\right] = \frac{GMm}{z^2}$$

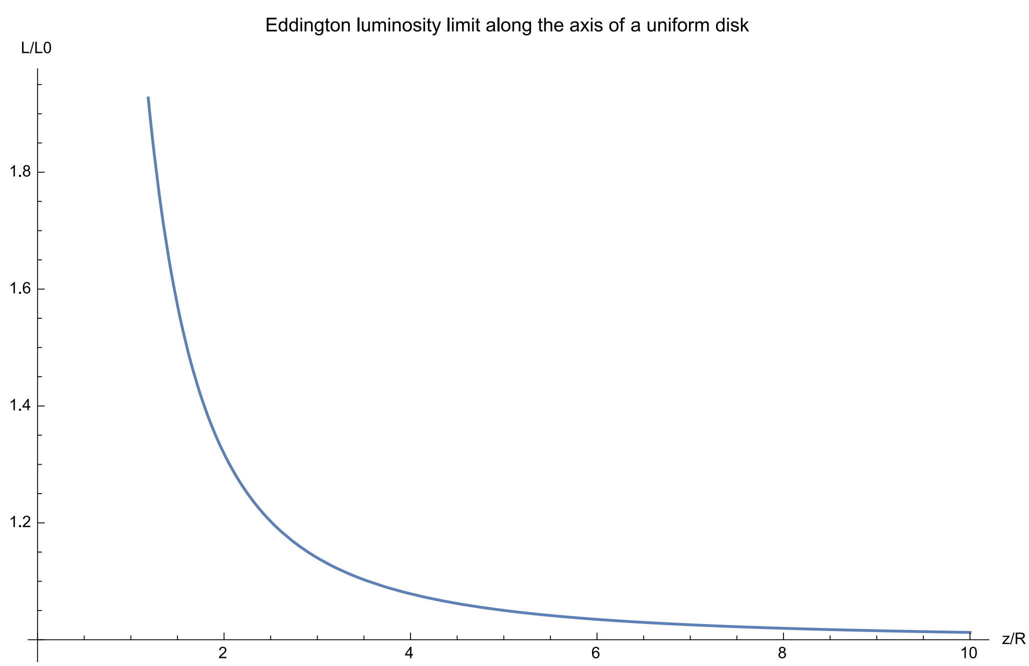
$$L_{\text{edd}}(z) = \frac{3\pi R^2 GMmc/\sigma_T}{z^2\left[1 - (1 + R^2/z^2)^{-3/2}\right]}.$$

In the limit $z \gg R$, this simplifies down to the asymptotic form

$$L_0 = 2\pi GMmc/\sigma_T$$

which is *half* of the limit for a point-source. This makes sense, because a disk viewed from far enough away along its axis takes up twice as much solid angle as a sphere with the same total surface area.

To see how the Eddington luminosity of the disk approaches this limit:




The Eddington limit equals that for a point source for $z/R \approx 1.14$.

2. (a) We are given that the rate of Ly$\alpha$ production is

$$\begin{aligned}&\left(4\times10^{-13}\ \mathrm{cm}^3/\mathrm{s}\right)\left(10^4\ \mathrm{cm}^{-3}\right)^2\left(\tfrac{4}{3}\pi\ \mathrm{pc}^3\right)(0.66)\\ =&\,3\times10^{51}\ \mathrm{s}^{-1}.\end{aligned}$$

These photons can be scattered, but not destroyed, and so the symmetry of the problem means that they are still evenly divided over a kpc sphere. So, the Ly$\alpha$ flux here is

$$\frac{3\times10^{51}\ \mathrm{s}^{-1}}{4\pi\ \mathrm{kpc}^2}=3\times10^7\ \mathrm{cm}^{-2}\ \mathrm{s}^{-1}.$$

(b) If we assume that the ionization fraction is in a steady state, then since all ionizing photons are being absorbed, the number of ionizing photons has to equal the recombination rate (the same rate found above, but without the factor of 0.66):

$$N=5\times10^{51}\,\frac{\text{ionizing photons}}{\text{second}}.$$

A (perfect blackbody) star of surface area $A$ and temperature $T$ has a luminosity of $L=A\sigma T^4$. The *specific* luminosity, meanwhile, is

$$\begin{aligned}L_\nu&=\pi AB_\nu\\&=\pi A\frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}.\end{aligned}$$

The outgoing specific photon flux is then $N_\nu=L_\nu/(h\nu)$. Defining $\nu_0=13.7\ \mathrm{eV}/h\approx3.3\times10^{15}$ Hz., the minimum frequency to ionize Hydrogen, we therefore have

$$\begin{aligned}N&=\int_{\nu_0}^{\infty}N_\nu\,\mathrm{d}\nu\\&=\pi A\int_{\nu_0}^{\infty}\frac{2\nu^2}{c^2}\frac{1}{e^{h\nu/kT}-1}\,\mathrm{d}\nu\\&=\frac{\pi L}{\sigma T^4}\int_{\nu_0}^{\infty}\frac{2\nu^2}{c^2}\frac{1}{e^{h\nu/kT}-1}\,\mathrm{d}\nu.\end{aligned}$$

Solving for total luminosity,

$$\begin{aligned}L&=\frac{N\sigma T^4}{\pi\int_{\nu_0}^{\infty}\frac{2\nu^2}{c^2}\frac{1}{e^{h\nu/kT}-1}\,\mathrm{d}\nu}\\&=3\times10^{41}\ \mathrm{erg/s}.\end{aligned}$$

This is about 8 orders of magnitude larger than the Sun's luminosity, an order of magnitude larger than the brightest O-Stars. Furthermore, this is above the Eddington luminosity of a 100 $M_\odot$ star (which has $L_E=1.3\times10^{40}\mathrm{erg\ s}^{-1}$), roughly the highest mass POPII star such that the radiation pressure from this luminosity would blow the star apart. It is equal to the Eddington luminosity of a 3000 $M_\odot$ star, but such a star does not exist.

(c) Applying the equation

$$\frac{\mathrm{d}T_b}{\mathrm{d}\tau_\nu}=-T_b+T$$

to our situation, where $T$ is constant inside the HII bubble and $T_b=0$ for *incoming* rays at the outside of the bubble, we get

$$T_b=T(1-e^{-\tau_\nu}).$$

So, we need to find the optical depth to the center of the bubble. Using $\alpha_\nu = j_\nu/B_\nu$, $j_\nu = P_\nu/4\pi$, and the Rayleigh-Jeans limit $B_{\rm RJ} = 2\nu^2 kT/c^2$, we get

$$\alpha_\nu = \frac{c^2 P_\nu}{8\pi\nu^2 kT}.$$

In the R-J limit, we get $P_\nu \approx 6.8 \times 10^{-38} n^2 T^{-1/2}$ erg s$^{-1}$ cm$^{-3}$ Hz$^{-1}$, and so

$$\alpha_\nu = 6.8 \times 10^{-38} \frac{c^2 n^2}{8\pi\nu^2 k T^{3/2}}.$$

The optical depth is then

$$\begin{aligned}\tau_\nu &= \alpha_\nu \cdot 1 \text{ pc} \\ &= 5 \times 10^{18} \nu^{-2}.\end{aligned}$$

Plugging into

$$T_b = T(1 - e^{-\tau_\nu})$$

gives temperature brightness.

(d) H$\alpha$ emission is a function of $n^2$ integrated along the line of sight since it is a recombination line and the recombination rate scales as $n^2$. (As with Ly$\alpha$, some fraction of recombinations result in H$\alpha$ photons.) This assumes the line is optically thin, but for $10^4$K you can show that the fraction in the 2S lower state is essentially zero and so optically thin is a great approximation.

In the microwave where the CMB is observed ($\sim$ 100 GHz), our expression for Bremsstrahlung has $\tau_\nu \ll 1$, and so we get

$$\begin{aligned}T_b &\approx T\tau_\nu \\ &\propto \int_{\rm cloud} ds\, n^2 T^{-1/2}.\end{aligned}$$

Thus, the intensity will be proportional to $n^2 T^{-1/2}$ integrated along the line of sight. The temperature is roughly known because photoionization results in about $10^4$K. Thus H$\alpha$ can be used as a measure of $n^2$. In fact, the line integral of $n^2$ is referred to in astronomy as the "emission measure" because most emission processes scale with density as $n^2$. Measuring one emission measure allows you to predict the emission of other processes.

3. We are given:

$$\tilde{E}_T(\nu) = \int_{t_i - T/2}^{t_i + T/2} \mathrm{d}t\, e^{2\pi i \nu (t - t_i)} E(t).$$

(a) i. Since $\Delta t = T$, the bandwidth must satisfy

$$\Delta\nu > \frac{1}{2\pi T}.$$

ii. The window function $W(t)$ is some constant $C$ for $t_i - T/2 \le t \le t_i + T/2$ and 0 otherwise. So, our definition of $\tilde{E}_T$ could just as easily be written as

$$\begin{aligned}\tilde{E}_T(\nu) &= \int_{-\infty}^{\infty} \mathrm{d}t\, e^{2\pi i \nu (t - t_i)} E(t) W(t) \\ &= (2\pi)^2 \int \mathrm{d}\nu'\, \tilde{E}_\infty(\nu') \tilde{W}(\nu - \nu').\end{aligned}$$

The transform of the window function is

$$\begin{aligned}
\tilde{W}(\nu) &= \int_{-\infty}^{\infty} \mathrm{d}t\, e^{2\pi i\nu(t-t_i)} W(t) \\
&= C \int_{t_i-T/2}^{t_i+T/2} \mathrm{d}t\, e^{2\pi i\nu(t-t_i)} \\
&= C \int_{-T/2}^{T/2} e^{2\pi i\nu u}\, \mathrm{d}u \\
&= C\, \frac{e^{2\pi i\nu u}}{2\pi i\nu}\Bigg|_{-T/2}^{T/2} \\
&= \frac{C}{2\pi i\nu}\left[2i\sin(\pi\nu T)\right] \\
&= \frac{C\sin(\pi\nu T)}{\pi\nu}.
\end{aligned}$$

Since we want $\tilde{W}(0) = CT = 1$, we need $C = 1/T$, and so

$$\tilde{W}(\nu) = \frac{\sin(\pi\nu T)}{\pi\nu T}.$$

The full width half maximum of the window function is given by $2(x/\pi T)$, where $\sin(x) = x/2$. This occurs at $x \approx 1.895$, giving

$$\begin{aligned}
\mathrm{FWHM} &= \frac{3.79}{\pi T} \\
\Delta\nu &= \frac{1.21}{T}.
\end{aligned}$$

(b) The energy density of an electromagnetic field is $u = E^2/4\pi$ (in cgs units), giving a flux of

$$F = \frac{cE^2}{4\pi}.$$

If we're averaging over the intervals given above, we get

$$F(t_i) = \frac{c}{4\pi T}\int_{t_i-T/2}^{t_i+T/2} E^2(t)\,\mathrm{d}t.$$

Unless $\nu T \gg 1$ for all frequencies of interest, specific flux gets awkward, because different frequencies wind up coupling to each other, and so you get cross-terms when you square and integrate the Fourier series of the electric field, i.e., not all of the energy content can be uniquely assigned to a specific frequency mode (in line with the uncertainty principle). However, ignoring this, we can write

$$F_\nu(t_i) = \frac{c}{2\pi T}|\tilde{E}_T(\nu)|^2$$

(where the factor of 2 shows up because the signal is real).

(c) Having a dipole length $L \gtrsim \lambda$ would cause problems because multiple oscillations of the field would occur. While this was not expected, having $L \ll \lambda$ would reduce sensitivity as only a faction of the voltage across the dipole would be induced as if it were in resonance such that $L \approx \lambda$. Generally half-wave dipoles with $L = \lambda/2$ are optimal.