Astro 519 HW Solutions #3

1. See back of the book of Rybicki and Lightman.

2. a.) This is a Fresnel diffraction problem. The space ship will block order unity of the light if its radius is twice the Fresnel scale or $\sqrt{\lambda d/\pi}$. Thus, to only block out 1% the radius for $d = 10$ a.u. and taking a typical optical wavelength of $\lambda = 5000$Å:

$$r_{\rm ship} \sim 0.1\sqrt{\lambda d/2\pi} \sim 35 \text{ meter}. \tag{1}$$

Some instead used the Fraunhoffer diffraction limit (i.e. the Airy disk). The Fresnel limit holds more generally, but the Fraunhoffer limit is more restrictive. However, it does hold for the problem at hand as can be checked by evaluating $Q = r_{\rm ship}^2/[d\lambda]$ and checking that $Q \ll 1$ (which is the Fraunhoffer limit). For this problem, $Q \sim 0.1$. Those who used the Fraunhoffer formula found a few times larger $r_{\rm ship}$ than the above, which is probably a bit more accurate.

The exact result can also be obtained by integrating the Fresnel diffraction formula over an aperture and solving for the size.

b.) The spatial resolution of their telescope on earth is

$$\Delta x = \frac{\lambda r_{es}}{d} = 100 \text{ km}. \tag{2}$$

This is much greater than the scale at which turbulence diffracts optical light, which we worked out to be a few cm in class. Thus, the telescope smears out many different phase errors and so the image is not really distorted by structure on the Fresnel scale.

c.) Length contraction makes it so that the 100 light years between the earth and alien planet contracts to 1 light year in the frame of the ship since it has $\gamma = 100$. (The rest frame of the planets is essentially the CMB rest frame.) The earth is traveling towards the ship at almost the speed of light in this frame, meaning that 1 year passes in the frame of the ship before it reaches the earth.

d.) Equation 4.97b in R&L relates the rest frame power per unit solid angle $dP'/d\Omega'$ to this in any frame $dP/d\Omega$:

$$\frac{dP}{d\Omega} = \frac{1}{\gamma^4(1-\beta\mu)^4}\frac{dP'}{d\Omega'} \tag{3}$$

We are interested in the case $\mu = \pm 1$. The emitted power for the isotropic case ($\frac{dP'}{d\Omega'} = P/4\pi$) is

$$\frac{dP}{d\Omega} = \frac{(1\mp\beta)^2}{(1\pm\beta)^2}\frac{P}{4\pi} \tag{4}$$

Thus, because $\beta = 1 - 10^{-4}$ for $\gamma = 100$, the power is enhanced by a factor of 10^8 in one direction and decreased by a factor of 10^8 in the other relative to the isotropic power that is radiated in the frame of the ship. The flux is related to the power by

$$F = \frac{dP}{dA} = \frac{1}{d^2}\frac{dP}{d\Omega}, \tag{5}$$

where $d = 50$light years and noting that $dA = d^2 d\Omega$.

3. a.) The polarization will be perpendicular to the dashed line as this is the direction the electrons must giggle on average because of the symmetry of the problem (just using that the E field is perpendicular to the direction of propagation). However, the scattering will not be 100% polarized because of the finite size of the sun (which is almost as big angularly as the moon) and because some of the electrons are in front of or behind the sun and so while they giggle in the plane perpendicular to the direction that the radiation is propagating, this plane is not orthogonal to P (hence contributing radiation from both polarizations). Only if the sun were a point source and all the electrons were in a plane perpendicular to us, would the polarization be 100%. (Barring dust, as Kolby pointed out.)

   It would not be hard to calculate the polarization fraction, but this was not requested.

   Note that (contrary to what many surmised) turbulence would not change the polarization fraction. It is still Thomson scattering. The small turbulent motions do not change its polarization except maybe at the v/c level owing to aberration effects.

   b.) The power scattered at any radius is (noting that $\tau_{\rm Thomson} \ll 1$. THE CORONAL GAS IS NOT COMPTON THICK!!!!)

$$P \equiv \frac{dW}{dV dt} = \overbrace{\sigma_T n_e(r)}^{\text{scattered}} \times \overbrace{\pi B_{\rm sun} \left(\frac{r}{R_{\rm sun}}\right)^{-2}}^{\text{flux}} \tag{6}$$

   where $B_{\rm sun}$ is the sun's brightness and we have approximated Thomson scattering as isotropic – correcting this would result in a small numerical factor. The "scattered" term is the fraction of the flux that is scattered per unit length. Since it is optically thin, the intensity at impact parameter R follows from the radiative transfer equation

$$\begin{aligned} I &= \int dr_{\parallel} \frac{P(\sqrt{r_{\parallel}^2 + R^2})}{4\pi} \qquad (7) \\ &= \frac{\pi}{8} \sigma_T n_e(R) B_{\rm sun} R \left(\frac{R}{R_{\rm sun}}\right)^{-2} \qquad (8) \end{aligned}$$

   noting that the emission coefficient $j = P/[4\pi]$. The last expression assumes that

$$n_e(R) = n_{e,0}(R_{\rm sun}) \left(\frac{R}{R_{\rm sun}}\right)^{-2}.$$

   We want to set this equal to $10^{-9} B_{\rm sun}$ at $R \sim 3R_{\rm sun}$ (eyeballing the location of P in the drawing). This allows us to solve for the electron density

$$n_e(R) = 10^6 \text{ cm}^{-3} \left(\frac{R}{R_{\rm sun}}\right)^{-2}. \tag{9}$$

   Thus, the electron density is about 10^5 cm^{-3} at $R \sim 3R_{\rm sun}$.