Class 2: Radiative transfer (Assignment RL $\leq$ 1.7)

Emission:
As ray travels, energy can be added or subtracted (by emission or absorption)

We define spontaneous emission coefficient as energy emitted per unit time per solid angle per volume per frequency

$$dE = j_\nu \, dt \, d\Omega \, dV \, d\nu$$

$j_\nu \equiv \frac{P_\nu}{4\pi}$ where P_ν is radiated power per volume per frequency (also called the emissivity

Remembering $dE = I_\nu \, dt \, d\Omega \, dA \, d\nu$
& $dV = dA \, ds$

$$\Rightarrow \quad dI_\nu = j_\nu \, ds \quad \Rightarrow \quad I_\nu = \int_0^s j_\nu \, ds$$

i.e. I_ν is increased by $j_\nu ds$

Example: Quasars have luminosity L_ν & number density n. The size of Universe is $\sim R$. Thus, $P_\nu = L_\nu n$

$I_\nu \cong \frac{L_\nu n}{4\pi} R$ This will work! (as long as no absorption)

Absorption

We define an absorption coefficient by

$$dI_\nu = -\alpha_\nu I_\nu \, ds \qquad (1)$$

What are its units?
Why proportional to I_ν?

Positions of absorbers typically can be taken to be uncorrelated

$$\alpha_\nu ds = \sigma N(s, s+ds)$$

where N is the 2D # density in this slice & σ the cross section of the absorbing particle. This holds as long as the slice is so thin s.t. $\sigma N \ll 1$

$\Rightarrow \alpha_\nu = \sigma_\nu n$ where $n = \dfrac{N(s, s+ds)}{ds}$ is 3D number density

Solving eqn. (1) yields

$$I_\nu(s) = I_\nu(0) \exp\left[-\int_0^s ds' \, \alpha_\nu(s')\right]$$

$$= I_\nu(0) \exp[-\tau] \qquad \tau = \int_0^s ds' \, \alpha_\nu(s')$$

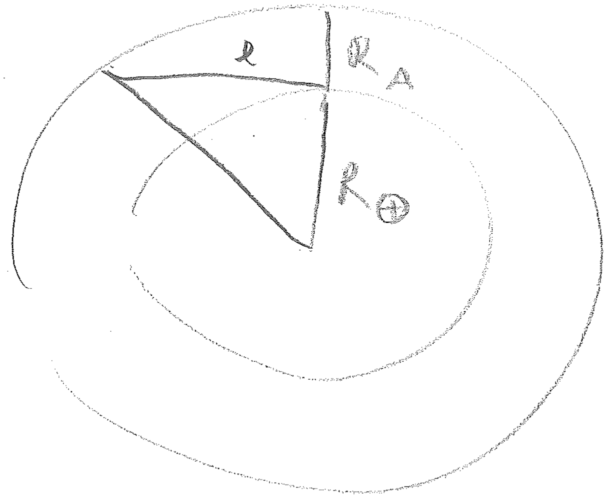
where τ is called the "optical depth."

Ex: Rayliegh Scattering by our atmosphere

Atmosphere mostly N_2: $n_{N_2} = 10^{20}\,\text{cm}^{-3}$

Cross section for Rayleigh Scattering:

$$\sigma_{N_2} = 5\times10^{-27}\,\text{cm}^2 \left(\frac{500\,\text{nm}}{\lambda}\right)^4$$



$$R_A = 10\ \text{km}$$

$$R_\oplus = 6\times10^3\ \text{km}$$

$$l = \sqrt{(R_\oplus + R_A)^2 - R_A^2}$$

$$\approx \sqrt{2 R_\oplus R_A}$$

$$= \sqrt{2\times10\times6\times10^3}$$

$$= 10^{2.5}\ \text{km} = 10^{7.5}\ \text{cm}$$

$$\tau^{\text{sunset}}_{\text{atmos}} = \sigma_{N_2}\, n_{N_2}\, l$$

$$= 10\left(\frac{500\,\text{nm}}{\lambda}\right)^4$$

$$\tau^{\text{zenith}}_{\text{atmos}} = \sigma_{N_2}\, n_{N_2}\, R_A = 0.5\left(\frac{500\,\text{nm}}{\lambda}\right)^4$$

Much more scattering at sunset!!

Combining absorption & emission, the RT equation becomes

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu$$

We can also write this eqn as function of optical depth rather than distance since

$d\tau = \alpha_\nu(s)\,ds \implies$

$$\frac{dI_\nu}{d\tau} = -I_\nu + S_\nu \qquad (2)$$

where $S_\nu \equiv \frac{j_\nu}{\alpha_\nu}$ is called the "source function"

We want a general sol^n to this eqn.

Define $X = I_\nu e^{\tau_\nu}$ $\quad Y = S_\nu e^{\tau_\nu}$

$$\frac{dI_\nu}{d\tau_\nu} = \frac{dX}{d\tau_\nu} e^{-\tau_\nu} - X e^{-\tau_\nu}$$

$\implies$ eqn (2) becomes

$$\frac{dX}{d\tau} e^{-\tau} \cancel{- X e^{-\tau}} = \cancel{-X e^{-\tau_\nu}} + Y e^{-\tau_\nu}$$

$$\implies \frac{dX}{d\tau_\nu} = Y$$

$$X = X(0) + \int_0^{\tau_\nu} d\tau_\nu' \, Y(\tau_\nu')$$

$$\boxed{I_\nu = I_\nu(0) e^{-\tau_\nu} + \int_0^{\tau} d\tau' \, e^{-(\tau_\nu - \tau_\nu')} S_\nu(\tau_\nu')}$$

This eqn. is easily interpretted as absorbed initial I_ν + source-diminished absorption. LS

For case of constant source function this is easily solved.

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + S_\nu(1-e^{-\tau_\nu})$$

$$= S_\nu + e^{-\tau_\nu}(I_\nu(0) - S_\nu)$$

Thus, as $\tau \to \infty$ $I_\nu(\tau_\nu) \to S_\nu$

Source function is important!

For thermal processes, we will see that S_ν is the Planck black body formula!

Mean free path

Typically defined as distance photon travels before experiencing $\tau_\nu = 1$.

Note that the mean optical depth a photon experiences is 1:

$$\langle \tau_\nu \rangle = \int_0^\infty d\tau_\nu \, \tau_\nu e^{-\tau_\nu} = 1$$

For homogeneous medium

$$\tau_\nu = \alpha_\nu s_{MFP} = 1 \Rightarrow \boxed{s_{MFP} = \frac{1}{\alpha_\nu} = \frac{1}{n\sigma_\nu}}$$

Radiative transfer algorithms.

Want to solve RT eqn

$$\frac{d}{dt} I(\nu, t, \vec{x}, \hat{n}) = \frac{c}{4\pi} P(\nu, t, \vec{x}, \hat{n}) - c\alpha(\nu, t, \vec{x}, \hat{n}) I(\nu, t, \vec{x}, \hat{n})$$

Same as $\frac{d}{ds}$, traveling w/ element

This is a 6D + time problem. That's very high dimensionality!!

Methods to solve:

Monte-carlo – randomly populate phase space distribution + follow rays. (End up following a fraction of all rays)
Use: scattering problems w/ low optical depth regions

Moment methods – take moments of (1) w.r.t. angle, + truncate w/ some closure approximation the heirachy (each eqn connects n^{th} moment to $n+1^{th}$)
– holds for high optical depth where radiation field becomes highly isotropic + so can be approximated by lowest moments
uses: often, even if approximation is dubious

Ray tracing – follow all rays (often impossible)
Uses: Few sources, broadband absorption problems (so few frequencies have to be followed)
– short charachteristics – rays only move across 1 cell + merge into other rays