Class #4

[Reading: Skim
ch 8 + 9 in RL]

warning: advanced reading
You aren't expected to comprehend
all of it.

Absorption & emission coefficients in terms of Einstein Coefficients

For emission coefficient, must make assumptions about the line profile of emitted radiation. Simplest assumption (that generally holds in astrophysics) is that this is the same line profile as absorption, $\phi(\nu)$

$$dE = j_\nu \, dV \, d\Omega \, d\nu \, dt$$

whereas each atom contributes $h\nu$ distributed over 4π

$$dE = \frac{h\nu}{4\pi} \phi(\nu) \, n_2 A_{21} \, dV \, d\Omega \, d\nu \, dt$$

$$\Rightarrow \boxed{j_\nu = \frac{h\nu}{4\pi} n_2 A_{21} \phi_A(\nu)}$$

For absorption coefficient (similarly)

$$dE_{Abs} = \frac{h\nu}{4\pi} n_1 B_{12} \phi(\nu) I_\nu \, dV \, d\Omega \, d\nu \, dt$$

$$\Rightarrow \alpha_\nu = \frac{h\nu}{4\pi} n_1 B_{12} \phi(\nu)$$

If we include stimulated emission this becomes

$$\boxed{\alpha_\nu = \frac{h\nu}{4\pi} \phi_B(\nu) \left(n_1 B_{12} - n_2 B_{21}\right)} = \frac{h\nu}{4\pi} n_1 B_{12} \left(1 - \frac{g_1 n_2}{g_2 n_1}\right) \phi_B(\nu)$$

Thus, the RT equation becomes

$$\frac{dI_\nu}{dt} = -\frac{h\nu}{4\pi} \left(n_1 B_{12} - n_2 B_{21}\right) \phi(\nu) I_\nu + \frac{h\nu}{4\pi} n_2 A_{21} \phi(\nu)$$

Since $S_\nu = \frac{j_\nu}{\alpha_\nu} \Rightarrow S_\nu = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}} = \frac{2h\nu^3}{c^2} \left(\frac{g_2 n_1}{g_1 n_2} - 1\right)^{-1}$

Cases

1) Thermal emission $\frac{n_1}{n_2} = \frac{g_1}{g_2} \exp\left(\frac{h\nu}{kT}\right)$

$$\alpha_\nu = \frac{h\nu}{4\pi} n_1 B_{12} \left[1 - e^{-\frac{h\nu}{kT}}\right] \phi(\nu)$$

$$S_\nu = B_\nu(T)$$

as Kirchoff's law dictates

stimulated emission (important when $h\nu_0 \lesssim kT$)

2) nonthermal emission

$$\frac{n_1}{n_2} \neq \frac{g_1}{g_2} \exp\left[\frac{h\nu}{kT}\right]$$

Affects this term in α_ν & S_ν

3) Masers:
inverted population $\frac{n_1}{g_1} < \frac{n_2}{g_2}$

$\alpha_\nu < 0 \Rightarrow$ intensity increases along ray

Often seen in molecular lines
(I think because complex structure allows more often for inverted states – that & Einstein A coefficient tend to scale as ν^2 & so it is hard to maintain inversion in higher frequency transition i.e. atomic systems)

Common masers: 1.6 GHz OH
22 GHz H_2O

Remark about fermion case & stimulated repulsion.

For classical oscillator

$$\int_0^\infty d\nu\, \sigma(\nu) = \frac{\pi e^2}{mc^2}$$ where m is electron mass

where $\sigma(\nu) = \frac{h\nu_{12}}{4\pi} B_{12}\, \phi(\nu)$ from previous

so

$$B_{12} = \frac{4\pi^2 e^2}{mc^2 h\nu_{12}}$$

For quantum transition we generally define

$$B_{nm} = \frac{4\pi^2 e^2}{mc^2 h\nu_{nm}} f_{nm}$$ where n, m are states

where f_{nm} is the "oscillator strength" that embodies all the quantum mechanics.
[I'm ignoring degeneracy factors - generally gf most important]

For allowed transition, $f_{1m} \sim \frac{1}{m^3}$
if m is radial quantum number.
Generally look up oscillator strength in a book. Once you have oscillator strength, don't need to know anything else about transition as can calculate all Einstein coefficients & hence all radiative properties

Examples HI Lyα ~~1216 Å~~ L4

$$A_{Ly\alpha} = 4.7 \times 10^8 \, s^{-1} \quad (A_{Ly\alpha}^{-1} \approx 2 ns)$$

Tends to be nonthermal w/ all electrons in ground (collisions not fast enough to thermalize). Possible exception: stars

$$t_{coll} \sim \frac{l_{mfp}}{v} \sim \frac{1}{vn\sigma} \qquad \sigma \sim 10^{-16} \, cm^2 \quad [\pi a_0^2]$$
$$v \sim 10 \, km/s \left(\frac{kT}{10^4 K}\right)^{\frac{1}{2}}$$

For $t_{coll} = A^{-1}$

$$n = \frac{A_{Ly\alpha}}{v\sigma} = \frac{10^9 \, s^{-1}}{10^6 \frac{cm}{s} \times 10^{-16} \, cm^2} \left(\frac{10^4}{kT}\right)^{\frac{1}{2}} = 10^{19} \left(\frac{kT}{10^4}\right)^{\frac{1}{2}} cm^{-3}$$

This is a rather high density, comparable to Earth's atmosphere

Thus, all electrons are in ground state or ionized

$$\tau_\nu = \int \alpha_\nu(s) ds = \int ds \frac{h\nu_{Ly\alpha}}{4\pi} n_{HI} B \phi\Big(\nu\big(\underbrace{1 + \frac{1}{c}\frac{dV}{ds} s}_{\text{Doppler shift}}\big)\Big)$$

where $\frac{dV}{ds}$ is the velocity gradient across the region (so ν is Doppler shifting). Thus,

$$\tau_\nu \approx \frac{h\nu_{Ly\alpha}}{4\pi} n_{HI} B_{Ly\alpha} \int ds \, \phi\left(\nu\left(1 + \frac{1}{c}\frac{dV}{ds} s\right)\right)$$

$$\approx \frac{hc}{4\pi \frac{dV}{ds}} n_{HI} B_{Ly\alpha}$$

we can relate $B_{Ly\alpha} = \frac{c^2}{2h\nu^3} A_{Ly\alpha}$

Now we are set

We need to plug in n & $\frac{dv}{ds}$ for system. One system is IGM & Lyα forest in this case

$$\frac{dv}{ds} = H \qquad n = \frac{Y_H \Omega_b \rho_{crit}}{m_p} \approx 2\times 10^{-7} (1+z)^3 \, cm^{-3}$$

This yields famous Gunn-Peterson formula

$$\tau_{Ly\alpha} = 10^5 x_{HI} \left(\frac{1+z}{5}\right)^{\frac{3}{2}}$$

Unless $x_{HI} \ll 1$, there is no transmission.

In late 1960s this was used to deduce that $z \sim 2$ IGM is highly ionized.

Today we've used this to show $z_{rei} > 6$.

HI in the galactic disk

$$\frac{dV}{dz} = \frac{100\ km/s}{10\ kpc} \qquad \bar{n} = 1\ cm^{-3}$$

$h\nu \approx 0.07\,K$ T is set by multitude of processes, generally ---

$$\frac{h\nu}{kT} \ll 1 \qquad n_1 \hat{=} \frac{g_1}{g_1+g_2} n_H$$

$$\tau_{21cm} = \frac{hc}{4\pi \frac{dV}{ds}} \left(\frac{g_1}{g_1+g_2}\right) n_H \frac{c^2}{2h\nu_{21}^3} A_{21} \frac{h\nu_{21}}{kT}$$

$$A_{21} = 2.9 \times 10^{-15}\,s \qquad (A_{21}^{-1} = 10\,Myr)$$

Plug in values, will get τ_{21cm} from disk. This is used to study galaxy.

[or early universe ($z \hat{=} 10-100$).]

$$I_0 = I_{CMB}$$

$$I = I_{CMB}\, e^{-\tau_{21cm}} + B(T)\left(1 - e^{-\tau_{21cm}}\right)$$

$\tau_{21cm} \ll 1$ generally

$$\simeq I_{CMB}(1-\tau_{21cm}) + B(T)\,\tau_{21cm}$$

or in brightness temperature (as $h\nu \ll T_{CMB}, T$

$$T_b = T_{CMB}^{(z)} + \left(T - T_{CMB}\right)\tau_{21cm}$$

$$= T_{CMB}^{(0)} + 20\,mK \left(\frac{T - T_{CMB}^{(z)}}{\ldots}\right)\left(\frac{1+z}{\ldots}\right)^{\frac{1}{2}}$$

Line broadening mechanisms

Doppler broadening [thermal; generally good assumption here]

Atoms moving in gas w/ velocity

$v \simeq \sqrt{\frac{2kT}{M}}$ where $M = A m_p$ & A is atomic #

$\simeq 12 \left(\frac{T}{A\, 10^4 K}\right)^{\frac{1}{2}}$ km/s

An atom will absorb at frequency ν_0 in rest frame. For atom moving at v_z this becomes $\nu = \nu_0 (1 + \frac{v_z}{c})$ using Doppler shift formula. An ensemble of atoms w/ Maxwellian distribution will absorb as

$$P(v_z)\, dv_z \propto \exp\left[-\frac{M v_z^2}{2kT}\right] dv_z$$

$$\propto \exp\left[-\frac{M c^2 (\nu - \nu_0)^2}{2 \nu_0^2 k T}\right] \frac{c\, d\nu}{\nu}$$

Normalizing to unity to yield line profile function

$$\phi(\nu) = \frac{1}{\Delta\nu_D \sqrt{\pi}} \exp\left[\frac{-(\nu-\nu_0)^2}{\Delta\nu_D^2}\right]$$

where the Doppler width is $\Delta\nu_D \equiv \frac{\nu_0}{c}\sqrt{\frac{2kT}{A m_p}}$

One can use our relation between oscillator strength & B to calculate optical depth at line center $\sigma_{\nu_0} = 1.2 \times 10^{14} \lambda_{0,1\text{Å}} \sqrt{\frac{A}{T}}\, f_{nn'}$ cm^2

Natural Broadening (from quantum mechanical uncertainty)

- Produces large damping wings. Seen for HI if $N_{HI} \gtrsim 10^{19}\ cm^2$ (these are called DLAs)
- Because I can't know exact time photon was emit $P_{em}^{n \to n'} = e^{-A_{nn'} t}$, I cannot know its energy exactly
- One finds

$$\phi(\nu) = \frac{\gamma_{nn'}/4\pi^2}{(\nu-\nu_0)^2 + \left(\frac{\gamma_{nn'}}{4\pi}\right)^2} \qquad \gamma_{nn'} = \sum_{m<n} A_{nm} + \sum_{m<n'} A_{n'm}$$

Lorentian Profile!!

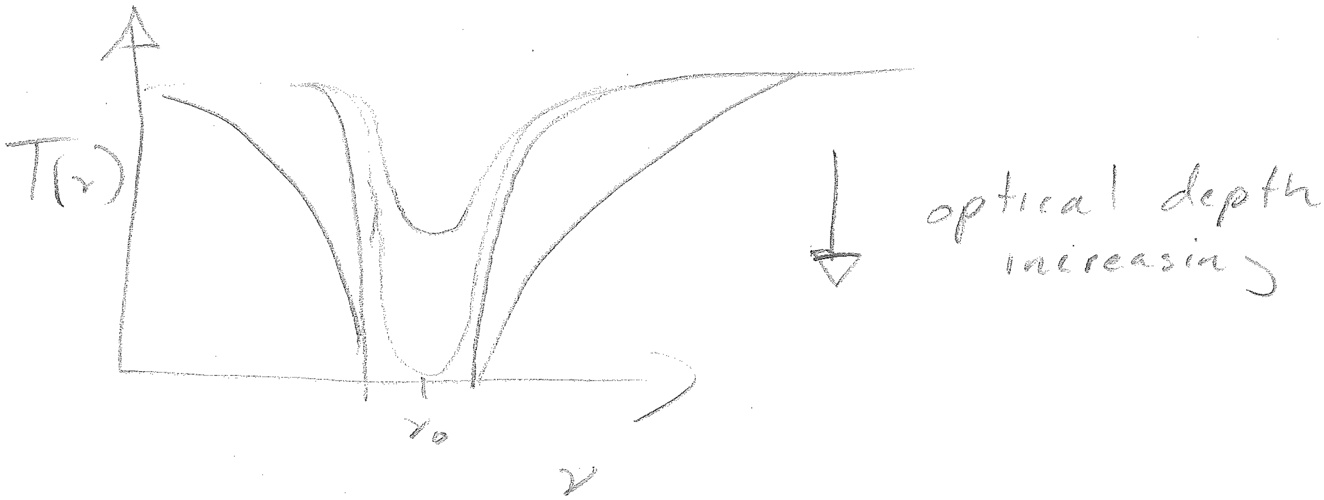
Collisional broadening (need high densities: stars/planets)

Atom isn't in isolation. The energy of transition is changed by passing particles

$$\Delta E_u - \Delta E_e = h\Delta\nu = \frac{C}{r^p}$$

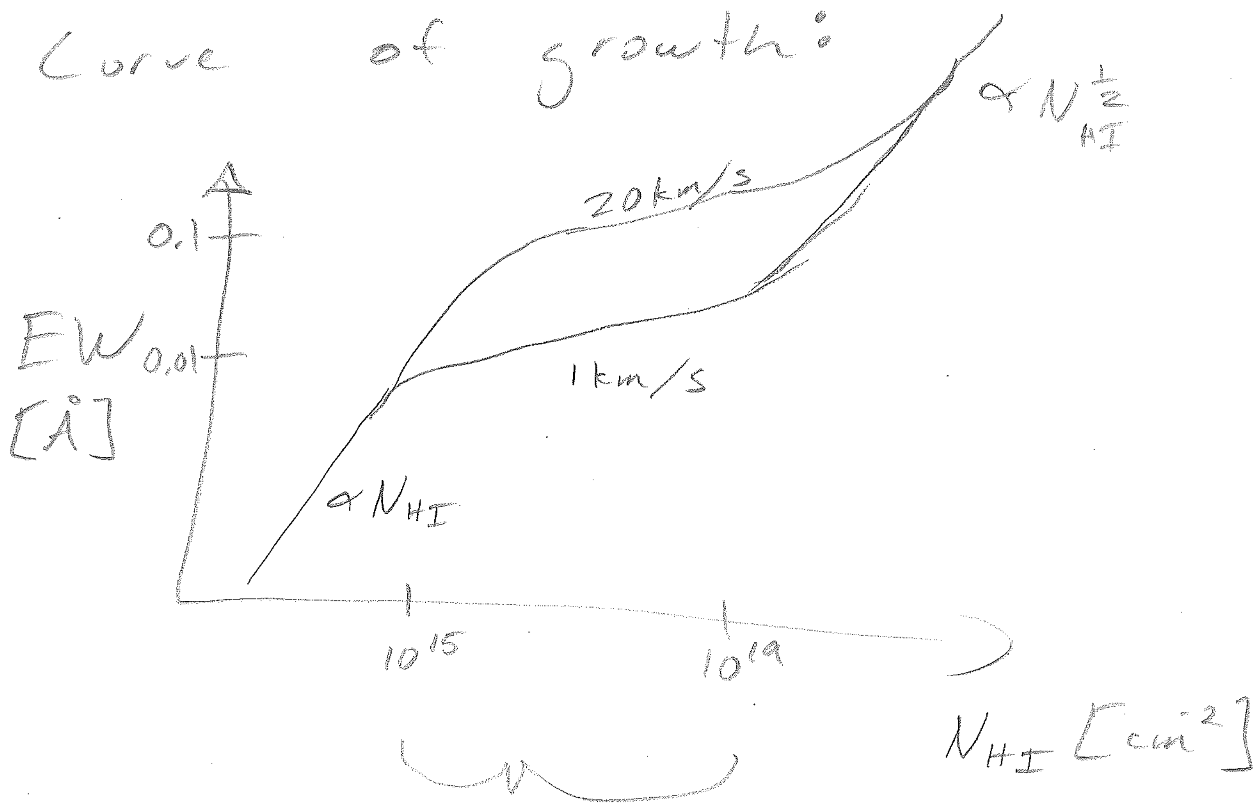
p = 2 "linear stark" - passing electron
Results in a Lorentzian profile.
Broadens Balmer lines in main sequence stars, white dwarfs

p = 3 "resonance" - passing atom
Important in stars w/ no free electrons



equivalent width = $\int d\lambda \, T(\lambda)$
[EW]

Curve of growth:



Hard to measure N_{HI} from EW over this range.