Astro 519 HW Solutions #1

1. (a) For a homogeneous medium, optical depth is proportional to physical depth. We are given that a skewer that goes through a physical depth H of the disk has optical depth τ_ν. So, a skewer that comes in at an angle θ from the normal, passing through a physical depth of $H/\cos\theta$, will have an optical depth of $\tau_\nu/\cos\theta$.

   We also have that, for a homogeneous source function S_ν, with $I_\nu = 0$ before entering the source,

$$I_\nu = S_\nu(1 - e^{-\tau_\nu}).$$

   In the limit of infinite optical depth, the disk is an ideal blackbody, and also we get $I_\nu = S_\nu$. So, S_ν must be the Planck function. We therefore have

$$\boxed{I_\nu = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1}\left(1 - \exp\left(-\tau_\nu/\cos\theta\right)\right)}$$

   (b) Assuming that we want flux through a surface that is facing the disk (as opposed to a surface with a normal vector parallel to the disk axis, which would introduce an extra $\cos\theta$), we just need to multiply the I_ν above by the solid angle subtended by the disk. The physical area of the disk is πR^2, but the area "on the sky" would only be $\pi R^2\cos\theta$, because the disk is seen at an angle. We therefore have

$$\Omega = \frac{\pi R^2\cos\theta}{d^2},$$

   and so

$$\boxed{F_\nu = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1}\left[1 - \exp\left(-\tau_\nu/\cos\theta\right)\right]\left(\pi R^2\cos\theta/d^2\right)}$$

   (c) In the optically thick limit, $\tau_\nu \to \infty$, and so we just get

$$\boxed{F_\nu = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1}\left(\pi R^2\cos\theta/d^2\right)}$$

   In the optically thin limit, $\tau_\nu \ll 1$, and so $e^{-\tau_\nu/\cos\theta} \approx 1 - \tau_\nu/\cos\theta$. This gives

$$\boxed{F_\nu = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1}\left(\pi R^2\tau_\nu/d^2\right)}$$

2. (a) If we let $\tau_0 \gg 1$ be the depth at some point deep in the star, we can calculate the accumulation of intensity along a skewer from that point to the observer. Things may look a little backwards, because "farther along the skewer" is a *lower* "depth" as defined here.

$$\begin{aligned}
I_\nu(\tau) &= I_\nu(\tau_0)e^{\tau-\tau_0} + \int_{\tau_0}^{\tau}(a_\nu + b_\nu\tau'_\nu)e^{-\tau'_\nu}\left(-\,\mathrm{d}\tau'_\nu\right)\\
I_\nu(0) &= I_\nu(\tau_0)e^{-\tau_0} + a_\nu + b_\nu - (a_\nu + b_\nu + b_\nu\tau_0)e^{-\tau_0}.
\end{aligned}$$

Since the center of the star is effectively at infinite optical depth, we are free to send $\tau_0 \to \infty$, and we recover a surface intensity of

$$\boxed{I_\nu(0) = a_\nu + b_\nu}$$

which is equal to the source function at $\tau = 1$.

(b) We care about the source function at an optical depth of one, which means we care about the *temperature* at an optical depth of one. At line center, the optical depth is higher for a given physical depth, and so $\tau = 1$ at a shallower point in the star. So,

- If the star's temperature **decreases** with increasing radial coordinate, then a shallower point corresponds to a **lower** temperature, and so **there is less intensity at line center**.
- If the star's temperature **increases** with increasing radial coordinate, then a shallower point corresponds to a **higher** temperature, and so **there is more intensity at line center**.

3. The fraction of the total luminosity L that is incident on a differential area is

$$\frac{\mathrm{d}A\cos\theta_0}{4\pi D^2}.$$

A fraction a of this is reflected back, with equal intensity I in all outgoing directions. The power per unit surface area of the planet, integrated over the hemisphere, is

$$\int_0^{\pi/2} (2\pi\sin\theta) I\cos\theta \,\mathrm{d}\theta = \pi I.$$

Equating this to the known power above, we get

$$I_\nu = \frac{La\cos\theta_0}{4\pi^2 D^2}.$$

As seen from Earth, a differential surface area $\mathrm{d}A$ on the other planet subtends a solid angle of $\mathrm{d}A\cos\theta_1/d^2$, and so we have

$$\boxed{\mathrm{d}F = \frac{La\cos\theta_0\cos\theta_1\,\mathrm{d}A}{4\pi^2 D^2 d^2}}$$

The total flux received on Earth will depend on the relative positions of the star, planet, and Earth. We can also set up spherical coordinates centered on the planet itself, with the star at $\theta = 0$ and the Earth at $(\theta, \phi) = (\psi, 0)$, where ψ is the angle between the star and the Earth as seen from the planet. From spherical geometry, we have that the angle between the directions (θ_1, ϕ_1) and (θ_2, ϕ_2) is given by

$$\cos^{-1}\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\Delta\phi\right).$$

Applying this to the present case, we get that, for a piece of surface at (θ, ϕ), assuming a perfectly spherical surface,

$$\begin{aligned}\cos\theta_0 &= \cos\theta,\\ \cos\theta_1 &= \cos\theta\cos\psi + \sin\theta\sin\psi\cos\phi.\end{aligned}$$

However, we have to beware of the fact that not all parts of the planet face the star, and that not all of the star-facing parts of the planet face the Earth. It's easier to handle the former case (only integrate up to $\theta_{\max} = \pi/2$), but the latter is a bit more annoying. If we define a "masking" function

$$\mathcal{M}(x) \equiv \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases}$$

then we can write

$$\mathrm{d}F = \frac{La}{4\pi^2 D^2 d^2} \cos\theta \cdot \mathcal{M}\left(\cos\theta\cos\psi + \sin\theta\sin\psi\cos\phi\right)\,\mathrm{d}A$$

which gives, for a planet of radius R,

$$\boxed{F(\psi) = \frac{LaR^2}{4\pi^2 D^2 d^2}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi/2}\mathcal{M}\left(\cos\theta\cos\psi + \sin\theta\cos\phi\sin\psi\right)\cos\theta\sin\theta\;\mathrm{d}\theta}$$

4. (a) The total amount of "covered" area (counting multiplicity) is NP, and so the optical depth is just

$$\boxed{\tau = \frac{NP}{A}}$$

(b) The opacity of thrown pizza slices is exactly the same as the opacity of particles with some cross section for an incoming ray. We have seen that the latter problem is solved by the radiative transfer equation. To see this, imagine the pizza slices thrown at different times are put at some height, s, indexed by the time they were thrown and uniform to some $s_{\max}$, but maintaining there $x-y$ position on the ground. This indexing does not change the visibility of the ground when viewed from above $s_{\max}$. The probability of transmission is the solution to

$$\frac{dP}{ds} = -\alpha P, \tag{1}$$

where the probability of being absorbed P is the analog for intensity and $\alpha = NP/(As_{\max})$ is the analog to the absorption coefficient in our familiar problem. This equation has the solution adhering to the boundary conditions $P(s=0)=1$ of

$$\boxed{P = e^{-\alpha s_{\max}} = e^{-NP/A} = e^{-\tau}.} \tag{2}$$

Another solution that yields insights but not exactly what the problem was looking for: The probability of a given point on the ground being under a given slice of pizza is P/A, and so the probability that it *isn't* is $1-P/A$. Since the slices are stated to be independent, we have the probability

$$p_0 = (1-P/A)^N$$

for a given point not to be covered at all. This is therefore the fraction of the floor that is uncovered.

In the limit of a very large number of slices (and very small P/A), it's useful to rewrite this as

$$(1-\tau/N)^N \xrightarrow{N\to\infty} e^{-\tau}$$

which matches our notion of optical depth. Plugging back in, in this limit,

$$\boxed{p_0 = e^{-NP/A}}$$

(c) In the same high-N, low-P/A limit, we can apply the Poisson distribution, which in this case is of the form

$$\mathrm{Pr}(k) = \frac{\tau^k e^{-\tau}}{k!}$$

where $\mathrm{Pr}(k)$ is the probability of a given spot on the floor being covered by exactly k slices of pizza. It's nice to note that, for $k=0$, we recover p_0 from above. For $k=2$, we get

$$\boxed{\mathrm{Pr}(k=2) = \tfrac{1}{2}\tau^2 e^{-\tau}}$$

5. (a) By Planck's law,

$$\begin{aligned}
I_\nu &= \frac{2h\nu^3/c^2}{e^{h\nu/kT}-1} \\
\frac{\mathrm{d}I_\nu}{\mathrm{d}\nu} &= \left(\frac{3}{\nu} - \frac{h}{kT}\frac{e^{h\nu/kT}}{e^{h\nu/kT}-1}\right) I_\nu \qquad \text{(factoring } I_\nu \text{ out of product rule)} \\
&= \left(\frac{3}{\nu} - \frac{h/kT}{1-e^{-h\nu/kT}}\right) I_\nu = 0 \qquad \text{(at peak frequency, } \nu_0\text{)} \\
\frac{3}{\nu_0} &= \frac{h/kT}{1-e^{-h\nu_0/kT}}.
\end{aligned}$$

If we let $x \equiv h\nu_0/kT$, this is of the form

$$\frac{x}{1-e^{-x}} = 3,$$

which has a single (valid) solution at $x \approx 2.82$. So,

$$\begin{aligned}
\nu_0 &\approx 2.82 kT_{\mathrm{CMB}}/h \\
&\approx 58.8\ \mathrm{GHz} \cdot \frac{T_{\mathrm{CMB}}}{1\ \mathrm{K}} \\
&\boxed{\approx 160\ \mathrm{GHz}}
\end{aligned}$$

The relationship between frequency and wavelength is $\nu = c/\lambda$, so

$$\begin{aligned}
I_\lambda &= \left|\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\right| I_\nu \\
&= \frac{c}{\lambda^2} I_\nu \\
&= \frac{\nu^2}{c} I_\nu.
\end{aligned}$$

This basically just changes the 3 to a 5 in the maximization equation above, i.e.,

$$\begin{aligned}
\frac{y}{1-e^{-y}} &= 5 \\
y &\approx 4.97
\end{aligned}$$

where I replaced x with y to avoid confusion of the two different values. We can now calculate

$$\begin{aligned}
\lambda_0 &= \frac{c}{\nu} \\
&= \frac{ch}{ykT_{\mathrm{CMB}}} \\
&\approx \frac{0.290\ \mathrm{cm}\cdot\mathrm{K}}{T_{\mathrm{CMB}}} \\
&\boxed{\approx 0.106\ \mathrm{cm}}
\end{aligned}$$

To see how these quantities are related, we can note that both depend only on T, and that they rely on T in opposite senses. So, we can take the product,

$$\begin{aligned}
\nu_0\lambda_0 &\approx \left(58.8\ \mathrm{GHz}\cdot\frac{T}{1\ \mathrm{K}}\right)\left(\frac{0.290\ \mathrm{cm}\cdot\mathrm{K}}{T}\right) \\
&\approx 1.70\times 10^{10}\ \mathrm{cm/s} \approx 0.57c.
\end{aligned}$$

Or, comparing x and y, we can see that ν_0 photons have only about 57% of the energy of λ_0 photons.

(b) The energy density from photons in a small frequency band for an (effectively) isotropic source like the CMB is

$$\begin{aligned}\Delta u &= u_\nu \Delta\nu \\ &= \frac{4\pi}{c} J_\nu \Delta\nu\end{aligned}$$

where J_ν is just Planck's formula. The total energy density is then

$$\begin{aligned}u &= \frac{4\pi}{c}\int_0^\infty J_\nu \,\mathrm{d}\nu \\ &= \frac{8\pi h}{c^3}\int_0^\infty \frac{\nu^3}{e^{h\nu/kT}-1}\,\mathrm{d}\nu \\ &= \frac{8\pi h}{c^3}\cdot\frac{\pi^4 k^4 T^4}{15h^4} \\ &= \frac{8\pi^5 (kT)^4}{15(ch)^3} \\ &= 7.55\times 10^{-15}\cdot\left(\frac{T}{1\ \mathrm{K}}\right)^4\ \mathrm{erg/cm^3} \\ &\boxed{= 4.19\times 10^{-13}\ \mathrm{erg/cm^3}} \\ &\boxed{= 0.262\ \mathrm{eV/cm^3}}\end{aligned}$$

for the CMB temperature.

For number density, we just have to divide each differential slice of energy density by the photon energy at the relevant frequency, $h\nu$:

$$\begin{aligned}n &= \frac{4\pi}{c}\int_0^\infty \frac{J_\nu}{h\nu}\,\mathrm{d}\nu \\ &\boxed{= 412\ \mathrm{cm^{-3}}}\end{aligned}$$

(c) Let's follow a specific set of photons as they redshift, and define $N(\nu)\Delta\nu$ to be the number of photons in the frequency range from ν to $\nu+\Delta\nu$ (with $\Delta\nu$ small). Since photons aren't being created or destroyed, we can write

$$N(\nu)\Delta\nu = N'(\nu')\Delta\nu'$$

where the primed values refer to redshift z. Since the frequency of each photon is scaled up as you go back to higher redshifts, the *width* of a given frequency slice is also scaled up, with $\Delta\nu' = (1+z)\Delta\nu$. So, we have

$$N'(\nu') = N(\nu)(1+z)^{-1}.$$

We can now convert this to a number *density*, by dividing by some volume, which is V_0 today and $V_0(1+z)^{-3}$ at redshift z:

$$n(\nu) = \frac{N(\nu)}{V_0}, \qquad n'(\nu') = \frac{N'(\nu')}{V_0(1+z)^{-3}}.$$

Putting the RHS in terms of the LHS,

$$\begin{aligned}n'(\nu') &= \frac{N'(\nu')}{V_0}(1+z)^3 \\ &= \frac{N(\nu)}{V_0}(1+z)^2 \\ &= n(\nu)(1+z)^2.\end{aligned}$$

Multiplying by $h\nu$ to go from number density to energy density,

$$\begin{aligned} u'(\nu') &= n'(\nu')h\nu' \\ &= n(\nu)h\nu(1+z)^3 \\ &= u(\nu)(1+z)^3. \end{aligned}$$

If we let $T \propto (1+z)$, then the denominator in Planck's Law is constant for the neighborhood around a given photon as it redshifts, but the numerator goes as $(1+z)^3$, which is exactly the behavior derived above.

(d) Following the question's approximation that all baryons are protons, we get

$$\begin{aligned} u_B &= m_p n_B = 190 \text{ eV/cm}^3, \\ u_{\text{CMB}} &= 0.262 \text{ eV/cm}^3. \end{aligned}$$

The baryon energy density scales with $(1+z)^3$, while the CMB energy density scales with $(1+z)^4$, so the relative scaling is simply $1+z$. Defining z_0 to be the redshift where the two are equal, we can therefore write

$$1 + z_0 = \frac{u_B}{u_{\text{CMB}}} = 716,$$

but that's way more sig figs than were given for n, so really it's just

$$\boxed{z_0 \approx 700}$$