Lagrangian Hydrodynamics : Spherical Blast Wave

PHYSICS 598 CPA : Problem Set 7

$Ricky\ Chue\ ,\ Yinghe\ L\ddot{u}\ ,\ Wei-Ting\ Liao$

1 Numerical Methods

Similarly to PS5, we use the Lagrangian Finite Difference Equations with pseudo viscosity, written in discretized formula with spatial positions $\{R_j\}$ and time sets $\{t_n\}$:

$$u_j^{n+1} = u_j^n - \Delta t(\frac{V_j^0 + V_{j-1}^0}{2})(\frac{R_j^n}{r_j^n})^{\alpha-1}\frac{P_j^n + q_j^n - P_{j-1}^n - q_{j-1}^n}{\frac{1}{2}(r_{j+1} - r_{j-1})} \tag{1}$$

$$R_j^{n+1} = R_j^n + \Delta t u_j^{n+1} \tag{2}$$

$$V_j^{n+1} = V_j^n \frac{(R_{j+1}^{n+1})^\alpha - (R_j^{n+1})^\alpha}{(r_{j+1})^\alpha - (r_j)^\alpha} \tag{3}$$

$$\epsilon_j^{n+1} = \frac{\epsilon_j^n - (\frac{P_j^n}{2} + q_j^{n+1})(V_j^{n+1} - V_j^n)}{1 + \frac{(\gamma-1)((V_j^{n+1} - V_j^n)}{2V_j^{n+1}}} \tag{4}$$

$$P_j^{n+1} = (\gamma - 1)\frac{\epsilon_j^{n+1}}{V_j^{n+1}} \tag{5}$$

$$q_j^n = \begin{cases} \frac{2a^2}{V_j^n + V_j^{n-1}}(u_{j+1}^n - u_j^n)^2 & (u_{j+1}^n - u_j^n) < 0 \\ 0 & otherwise \end{cases} \tag{6}$$

The time step is given by

$$\Delta t = min[\frac{b(R_{j+1}^n - R_j^n)}{c_j^n}] = min[\frac{b(R_{j+1}^n - R_j^n)}{(\gamma P_j^n/\rho_j^n)^{1/2}}] \tag{7}$$

Here we use a ~ 1.5 and b ~ 0.01. The parameter α is determined by the dimension of our analysis. Assume $V \propto R^p$, we can derive that $p = \alpha$. So in spherical case, we choose $\alpha = 3$ here, and $\gamma = 5/3$.

As a check of computational accuracy, we also computed the total energy, which can be expressed as:

$$E(t) = \int_0^{R_s} 4\pi r^2 dr \left(\varepsilon + \frac{1}{2}\rho u^2\right) = \sum_j 4\pi r_j^2 dr_j \left(\frac{1}{\gamma - 1}P_j^n V_j^n + \frac{1}{2}\frac{(u_j^n)^2}{V_j^n}\right) \tag{8}$$

After normalizing all the physical quantities into non-dimensional values, The initial condition are given by:

$$P_0 = \rho_0 = V_0 = \epsilon_0 = 1,\ u_0 = 0$$

While the boundary conditions are:

$$P_0 = u_0 = \epsilon_0 = 0,\ \rho_0 = V_0 = 1$$

To more convenience, we choose an arbitrary boundary of the sphere to be at the index j=50, so that the first sets of conditions are matched at $0 \leq j \leq 50$, while the second sets are for $j > 50$. The cloud boundary is about R = 2.5 in size.

The analytical result is given by W.Newman (1977), Ap. Sp. Science, 47, 99, which we have computed in PS4.

2 P1: Infinite Gas Cloud at Rest

2.1 Numerical Result

Here we plotted the density, velocity and pressure profile, firstly with respect to the physical radius(figure 1), then with the η parameter in the Sedov solution, where

$$\eta = \frac{r}{R_{sh}}$$

. In figure 1 we choose three different times for comparison. From the figure we can see that there are some wiggles in the plot, this is due to the fact that only in later times, the initial bubble can be approximated as a point source. So to compare the two methods, we also use a later time of evolution, and from figure 2 we find the results match well with each other.

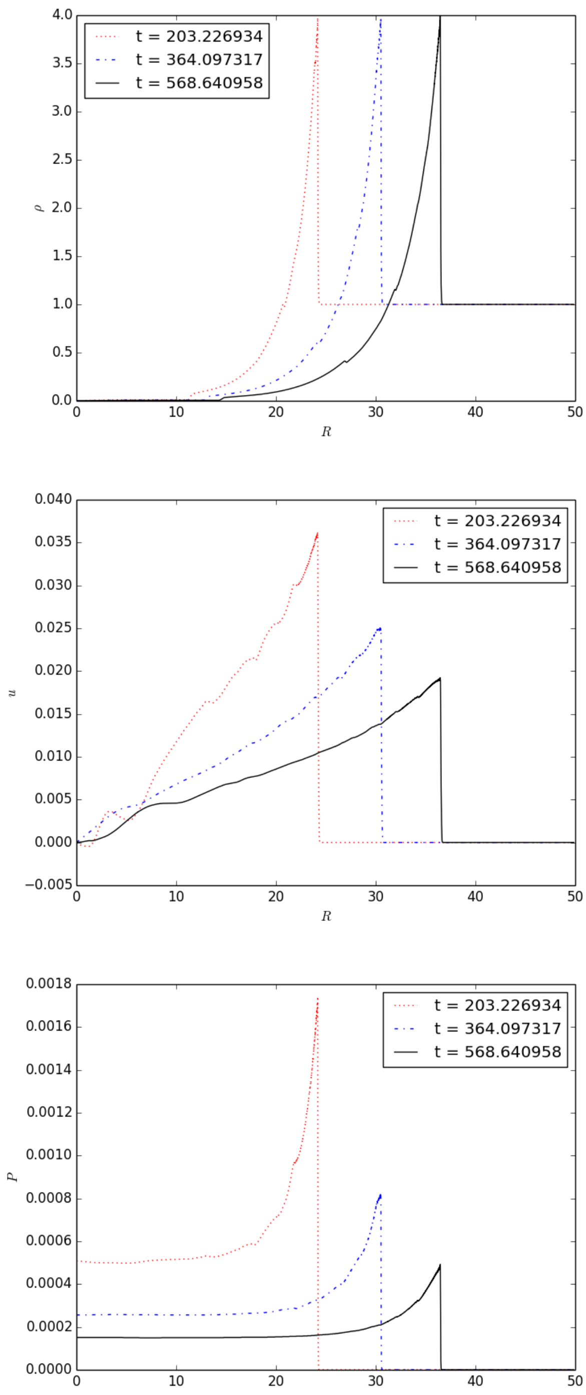

Figure 1: Density, velocity and pressure profile of the spherical cloud.

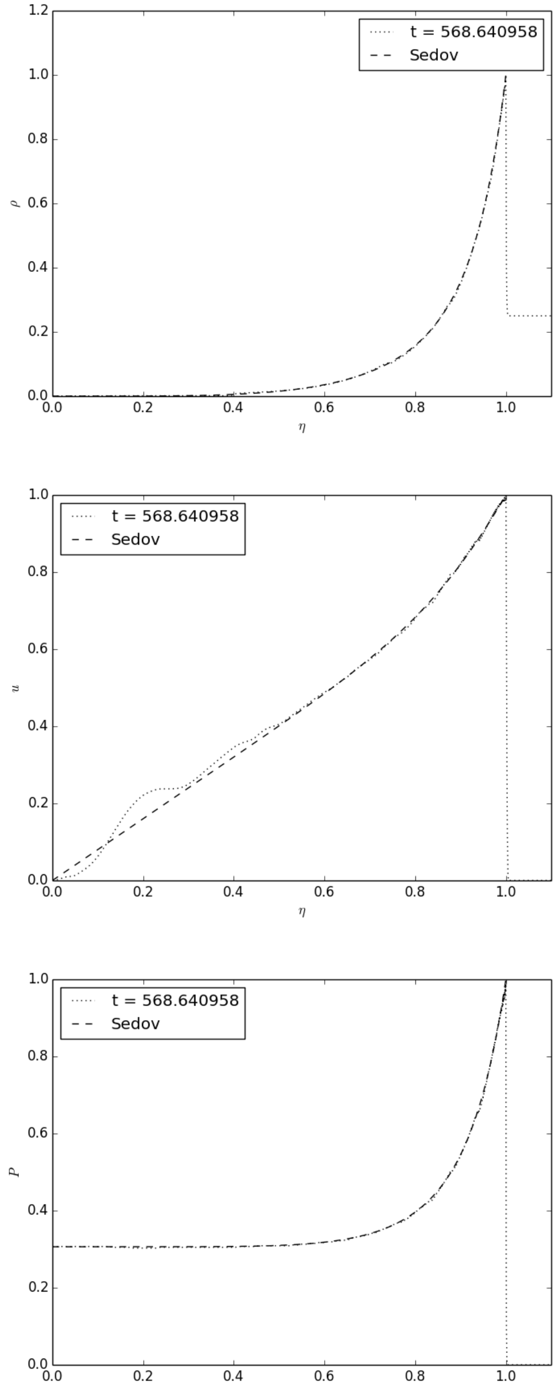

Figure 2: Late solution of density, velocity and pressure profile of the spherical cloud, compared with the Sedov solution.

2.2 Energy Check

In Figure 3, we plotted the total energy of the system based on Eqn (8). The total energy is mostly conserved, the numerical error is within 1 percent.

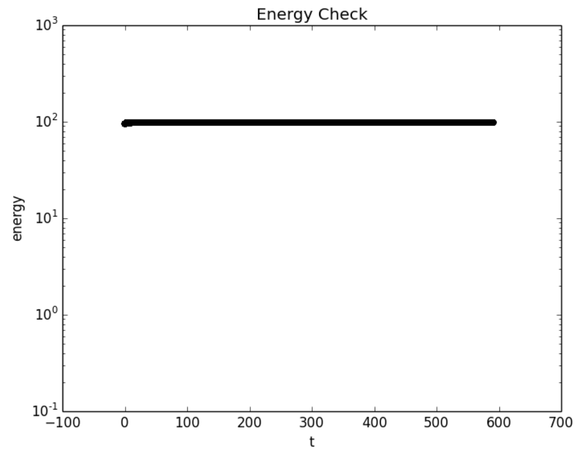

Figure 3: The total energy check of the system in Part 1.

2.3 Shock

We determine the position of the shock as the place where the pressure reaches maximum. In figure 4 we plot the shock position and propagation velocity with time, both numerically and analytically :

$$
\begin{aligned}
R_{\text{shock}} &= \xi_0 \left(\frac{Et^2}{\rho_0} \right)^{1/5} \\
u_{\text{shock}} &= \frac{2}{5} \xi_0 \left(\frac{E}{\rho_0 t^3} \right)^{1/5}
\end{aligned} \tag{9}
$$

For spherical adiabatic cloud ($\gamma = 5/3$), $\xi_0 = 1.17$, which has been solved before. The results are also in great match.

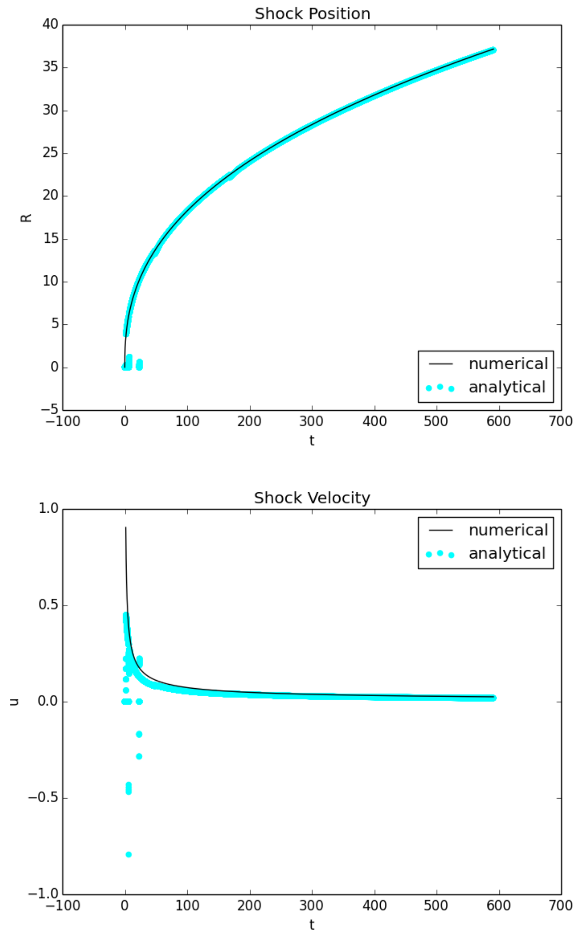

Figure 4: Shock position and shock velocity vs. time. Numerical solution is consistent with analytical solution.

3 P2: Infinite Gas Cloud with Initial Velocity

Here the initial conditions are the same with Problem 1 except that

$$u_0 = \frac{r}{R_0} V_0,$$

when $r \leq R_0$, also since the cloud is cold, we know initially that

$$P = 0$$

Here to avoid dividing by zero, we choose P to be a small value instead, so we use P = 0.001.

V_0 is obtained by requiring that the total energy of the cloud be the same as the energy in Part (1). So we can calculate V_0 as follows. First, the total energy from

Part (1) can be expressed as:

$$E_0 = \int_0^{R_0} 4\pi r^2 dr \varepsilon_0 = \int_0^{R_0} 4\pi r^2 dr \left(\frac{P_0 V_0}{\gamma - 1}\right) = \frac{4\pi R_0^3}{3}\left(\frac{P_0 V_0}{\gamma - 1}\right)$$

While in the case of Part (2):

$$E_0 = \int_0^{R_0} 4\pi r^2 dr \left(\frac{1}{2}\rho_0 v(r)^2\right) = 2\pi\rho_0 \int_0^{R_0} r^2 dr \ v(r)^2$$
$$= 2\pi\rho_0 \left(\frac{V_0}{R_0}\right)^2 \int_0^{R_0} r^4 dr = 2\pi\rho_0 \left(\frac{V_0}{R_0}\right)^2 \frac{R_0^5}{5} = \frac{2}{5}\pi\rho_0 V_0^2 R_0^3$$

Therefore,

$$\begin{aligned} V_0 &= \sqrt{\frac{10}{3}\frac{P_0}{(\gamma - 1)\rho_0}} \\ &= \sqrt{\frac{10}{3}\frac{1}{\gamma - 1}} = \sqrt{5} \end{aligned}$$

since $P_0 = 1 = \rho_0$.

The density, velocity and pressure profile, taken at three different time steps are as follows:

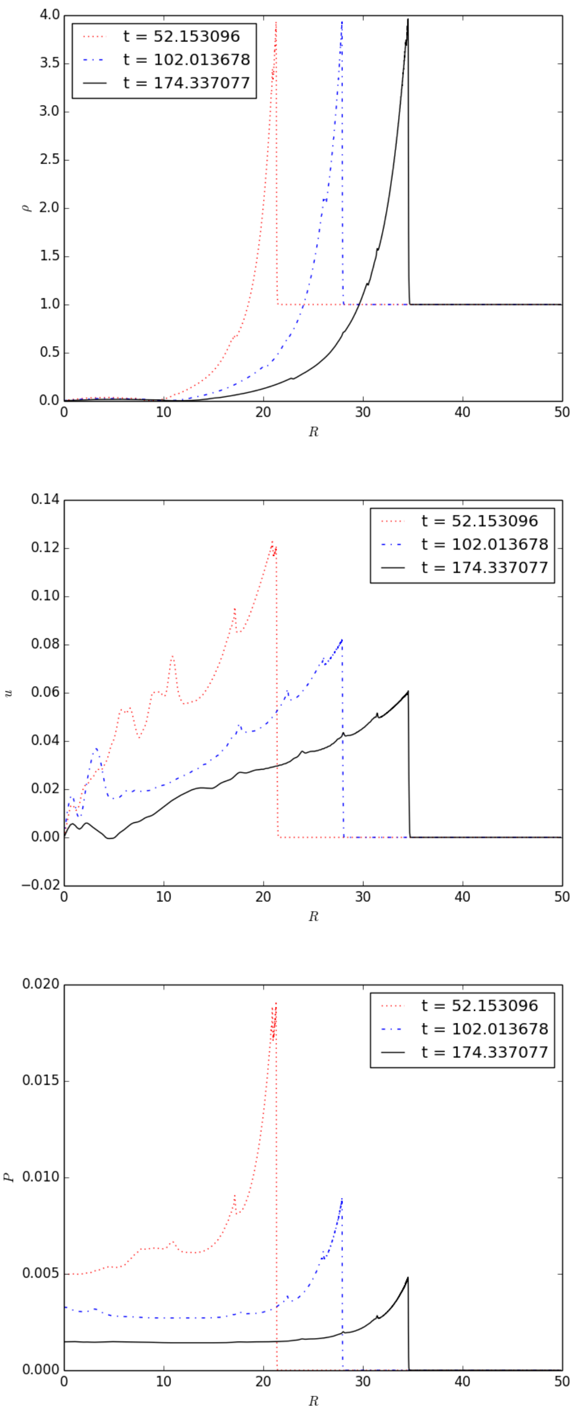

Figure 5: Density, velocity and pressure file of the spherical cloud with initial velocity.

Although the initial condition in Part 1 and Part 2 are different, if we evolve for a significant amount of time when the cloud can be approximated as a point source, the Sedov solution can then apply. In Figure6, we normalize the profiles to relationship between density, velocity,pressure and the η values, and compare with the Sedov solution. As expected, the result tends to match well with the Sedov results, especially in late times.

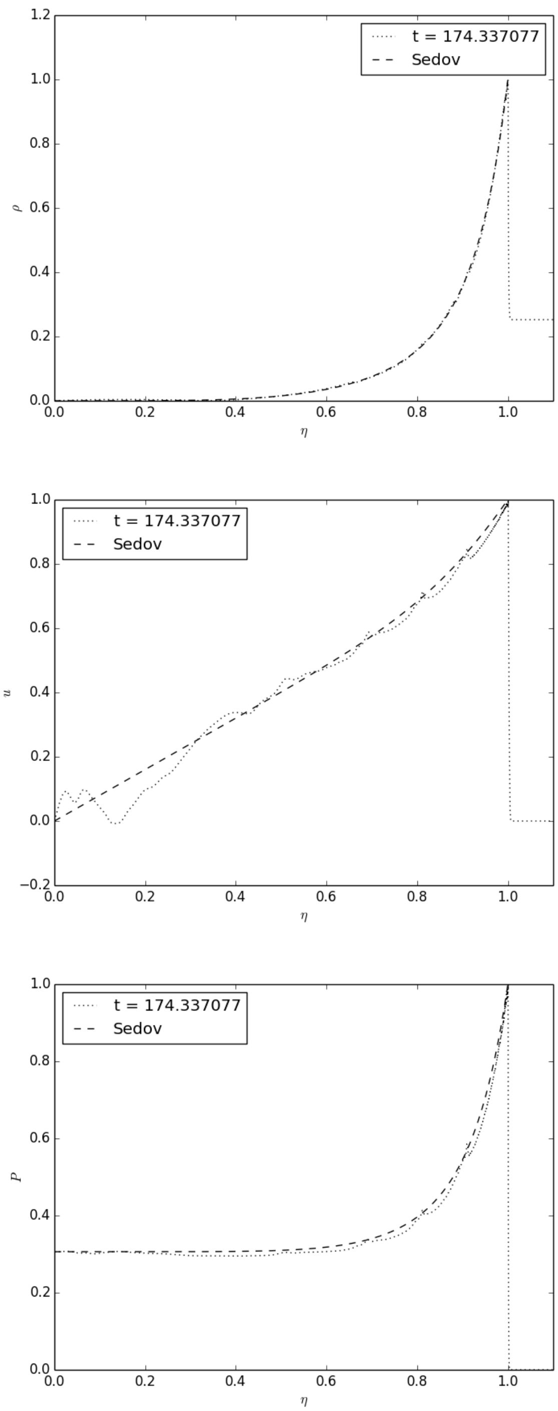

Figure 6: Normalized density, velocity and pressure file of the spherical cloud with initial velocity, with comparison to the Sedov solution.

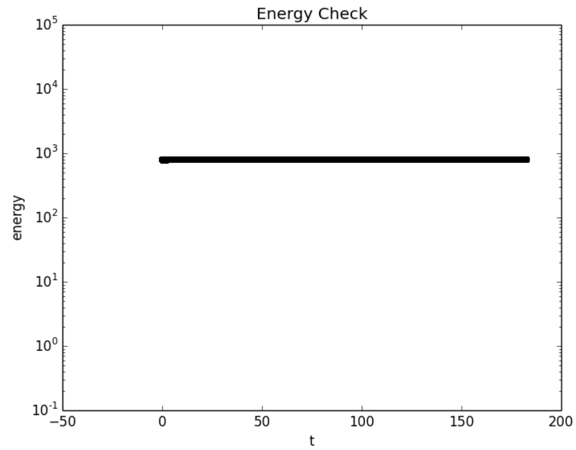

Figure 7: The total energy check of the system in Part 2.

The energy is also conserved, served as a proper check for the numerical accuracy of our program.

4 P3: Vacuum

In this case , the initial/boundary condition becomes For $0 \leq j \leq 50$,

$$P_0 = 1 = \rho_0,\ u_0 = 0$$

For $j > 50$,

$$P_0 = 0 = u_0 = \rho_0$$

To avoid dividing by zero, we use $V_0 = 10^5$ here. The results are shown as follows.

In part 3, the outer density is zero. The core undergoes free expansion, the Sedov solution is no longer applicable.

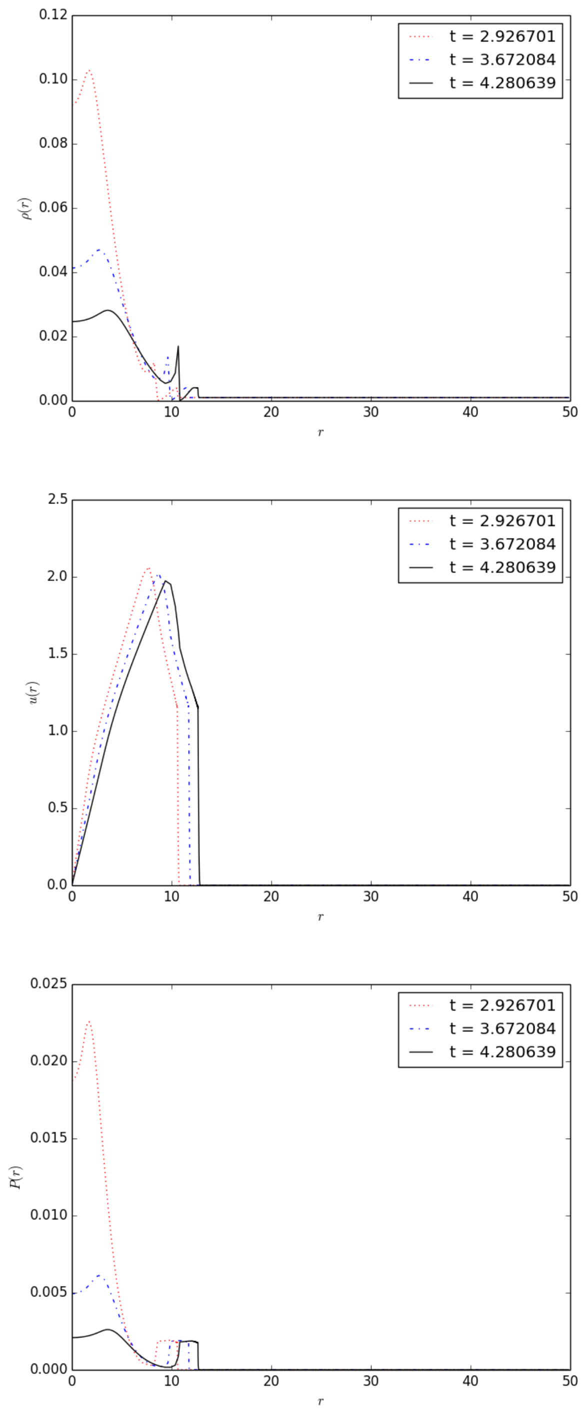

Figure 8: Density, velocity and pressure file of the spherical cloud in Part 3.

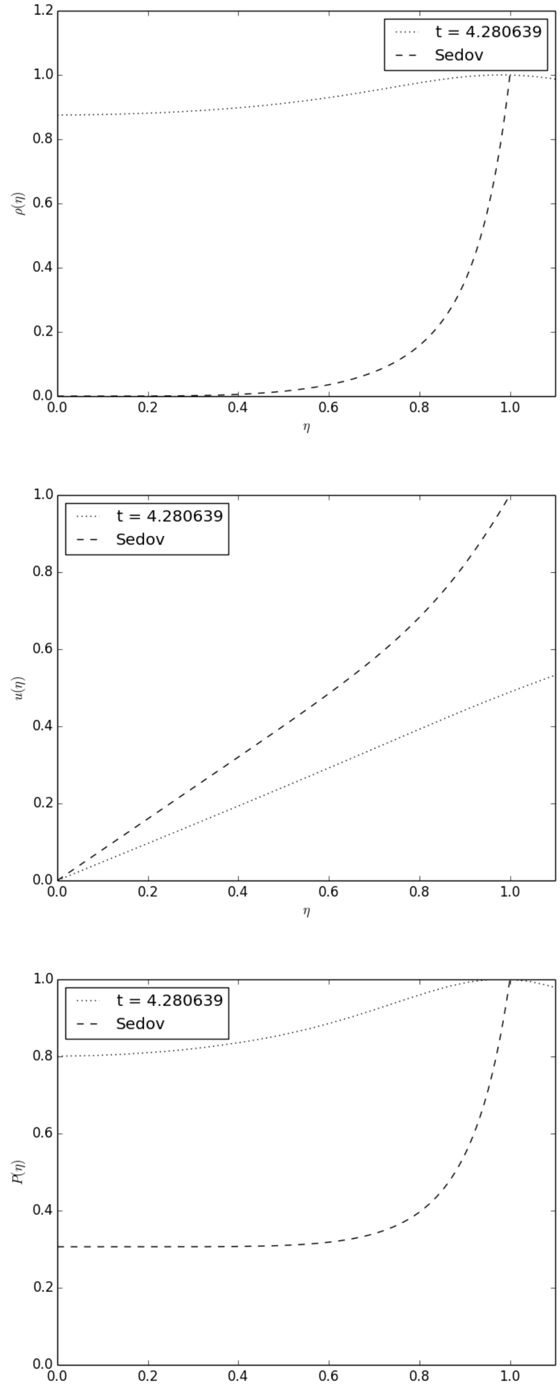

Figure 9: Density, velocity and pressure file of the spherical cloud, compared to the Sedov solution. The numerical solution does not meet the Sedov solution.