Lagrangian Hydrodynamics in Planar Symmetry

PHYSICS 598 CPA : Problem Set 5

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1 Finite Difference Equations in Lagrangian Scheme

In planar geometry using the Lagrangian coordinates, the observer follows the fluid element as it moves through space and time. Let r be the spatial coordinate of a fluid element at t=0, while R = R(t,r) = spatial coordinate at any t > 0. We can thus write the Lagrangian difference equations as:

$$\frac{DR}{Dt} = u \tag{1}$$

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial R} = -V_0(\frac{R}{r})^{\alpha-1}\frac{\partial P}{\partial R} \tag{2}$$

$$\frac{D\epsilon}{Dt} = -P\frac{DV}{Dt} \tag{3}$$

where

$$V_0 = \frac{1}{\rho_0}, V = \frac{1}{\rho}$$

so that

$$V = V_0(\frac{R}{r})^{\alpha-1}\frac{\partial R}{\partial r} = V_0\frac{\partial R^\alpha}{\partial r^\alpha} \tag{4}$$

$$\epsilon = f(P, V) = \frac{PV}{\gamma - 1} \tag{5}$$

for perfect gas. To avoid discontinuity in shock waves, we introduce artificial viscosity into FDEs, so the equation becomes:

$$\frac{DR}{Dt} = u \tag{6}$$

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial(P+q)}{\partial R} \tag{7}$$

$$\frac{D\epsilon}{Dt} = -(P+q)\frac{DV}{Dt} \tag{8}$$

where

$$q = \begin{cases} \frac{l^2}{V}(\frac{\partial u}{\partial R})^2 & \frac{\partial u}{\partial R} < 0 \\ 0 & otherwise \end{cases} \tag{9}$$

The parameter l here has the dimension of length, and q has the dimension of ρu^2. In this case, the equations can reproduce jump conditions for a steady shock.

The parameter α is determined by the dimension of our analysis. Assume $V \propto R^p$, we can derive from Eqn.(4) that $p = \alpha$. So in 1D case, we choose $\alpha = 1$ here.

2 Analytical Results

In general for a shock, we have the Rankine-Hugoniot jump conditions:

$$\rho_0 u_0 = \rho_1 u_1 \tag{10}$$

$$P_0 + \rho_0 u_0^2 = P_1 + \rho_1 u_1^2 \tag{11}$$

$$\epsilon_0 + \frac{P_0}{\rho_0} + \frac{u_0^2}{2} = \epsilon_1 + \frac{P_1}{\rho_1} + \frac{u_1^2}{2} \tag{12}$$

Solve the above equations, we get the ratio of the pressure, density and velocity at the front/back of the shock.

$$\frac{P_1}{P_0} = \frac{(\gamma+1)V_0 - (\gamma-1)V_1}{(\gamma+1)V_1 - (\gamma-1)V_0} = \frac{2\rho_0 u_0^2/P_0}{\gamma+1} - \frac{\gamma-1}{\gamma+1} \tag{13}$$

$$\frac{\rho_0}{\rho_1} = \frac{V_1}{V_0} = \frac{(\gamma+1)P_0 + (\gamma-1)P_1}{(\gamma+1)P_1 + (\gamma-1)P_0} = \frac{\gamma-1}{\gamma+1} + \frac{2}{\gamma+1}\frac{\gamma P_0}{\rho_0 u_0^2} \tag{14}$$

So that we know the speed of the fluid behind the shock (which is also the shock speed) can be given by:

$$u_0^2 = \frac{1}{2\rho_0}[(\gamma-1)P_0 + (\gamma+1)P_1] = \frac{c_0^2}{2\gamma}[(\gamma-1) + (\gamma+1)(\frac{P_1}{P_0})] \tag{15}$$

Similarly, we also have

$$u_1^2 = \frac{c_0^2}{2\gamma}\frac{[(\gamma+1) + (\gamma-1)(\frac{P_1}{P_0})]^2}{(\gamma-1) + (\gamma+1)(\frac{P_1}{P_0})} \tag{16}$$

$$u_0 - U_{piston} = u_1 \tag{17}$$

3 Numerical Methods

Now discretize the system into many cells in space and time. Consider a staggered spatial coordinate mesh with spatial positions $\{R_j\}$ and time sets $\{t_n\}$, for

$\forall j$, we can evolve the above equations in time. So if we go from element n to n+1, we can rewrite the equations as:

$$u_j^{n+1} = u_j^n - \Delta t(\frac{V_j^0 + V_{j-1}^0}{2})(\frac{R_j^n}{r_j^n})^{\alpha-1}\frac{P_j^n + q_j^n - P_{j-1}^n - q_{j-1}^n}{\frac{1}{2}(r_{j+1} - r_{j-1})} \tag{18}$$

$$R_j^{n+1} = R_j^n + \Delta t u_j^{n+1} \tag{19}$$

$$V_j^{n+1} = V_j^n \frac{(R_{j+1}^{n+1})^\alpha - (R_j^{n+1})^\alpha}{(r_{j+1})^\alpha - (r_j)^\alpha} \tag{20}$$

$$\epsilon_j^{n+1} = \epsilon_j^n - (\frac{P_j^n + P_j^{n+1}}{2} + q_j^{n+1})(V_j^{n+1} - V_j^n) \tag{21}$$

$$P_j^{n+1} = (\gamma - 1)\frac{\epsilon_j^{n+1}}{V_j^{n+1}} \tag{22}$$

$$q_j^n = \begin{cases} \frac{2a^2}{V_j^n + V_j^{n-1}}(u_{j+1}^n - u_j^n)^2 & (u_{j+1}^n - u_j^n) < 0 \\ 0 & otherwise \end{cases} \tag{23}$$

Here we have replaced l by $a\Delta x$, where a $\sim$ 1.5 - 2.0 . Note that here ρ, P and q are actually centered on the half-grid points in space and time. So we solve Eqns.(18) to (23) numerically, with j indexing from 0 to j_{max}, and n from 0 to n_{max}.

The time step is given by

$$\Delta t = min[\frac{b(R_{j+1}^n - R_j^n)}{c_j^n}] = min[\frac{b(R_{j+1}^n - R_j^n)}{(\gamma P_j^n/\rho_j^n)^{1/2}}] \tag{24}$$

Here b $\sim$ 0.2 for initial value . Depending on the requirements, we adjust b so as to control the time step size.

To simplify our computation, we first normalize the equations to a dimensionless form:

$$\rho \to \frac{\rho}{\rho_0}, P \to \frac{P}{P_0}, u \to \frac{u}{u_0}$$

$$\Delta t \to \frac{\Delta t}{t_0}, R \to \frac{R}{u_0 t_0}, \Delta x \to \frac{\Delta x}{u_0 t_0}$$

Here t_0 is arbitrary, so we choose it to be 1. Now we have initial conditions:

$$P_j^0 = V_j^0 = 1$$

$$R_j^0 = j\Delta x$$

$$\epsilon_j^0 = \frac{P_j^0 V_j^0}{\gamma - 1} = \frac{1}{\gamma - 1}$$

$$u_j^0 = \begin{cases} -U_{piston}/u_0 & j = 0 \\ 0 & j = 1, 2 ... j_{max} \end{cases}$$

As well as boundary conditions:

$$\rho^n_{j_{max}} = V^n_{j_{max}} = 1$$

$$u^n_0 = -U_{piston}/u_0$$

for all n.

As indicated, we use $\gamma = 5/3$ for equation of state. During evolution of time steps, Eqns. (22)(23) are coupled to each other. So we rewrite them as:

$$P^{n+1}_j = \frac{P^n_j(V^{n+1}_j - \frac{\gamma+1}{\gamma-1}V^n_j) + q^{n+1}_j(V^{n+1}_j - V^n_j)}{V^n_j - \frac{\gamma+1}{\gamma-1}V^{n+1}_j} \tag{25}$$

$$\epsilon^{n+1}_j = \frac{\epsilon^n_j - (\frac{P^n_j}{2} + q^{n+1}_j)(V^{n+1}_j - V^n_j)}{1 + \frac{(\gamma-1)((V^{n+1}_j - V^n_j)}{2V^{n+1}_j}} \tag{26}$$

4 Results

4.1 Rarefaction

In this case the piston is **withdrawn** from the fluid with (a) $U = (1/10)U_{max}$ and (b) $U = 10U_{max}$, where $U_{max} = (2/(\gamma-1))C_0$, $C_0 = (\gamma P_0/\rho_0)^{1/2}$. So we can write the initial condition of velocity field as:

$$u^0_j = \begin{cases} -k\frac{2\sqrt{\gamma}}{\gamma-1} & j = 0 \\ 0 & j = 1, 2...j_{max} \end{cases}$$

Here k = 1/10 (case a) or 10(case b). The density, pressure and velocity profiles (density, pressure and velocity with respect to the spatial coordinate R) are plotted in Fig.1 and Fig.2 .

The non-dimensional analytical solutions are as following. First, the region with $R > \sqrt{\gamma}t$ is left untouched. Therefore, flow properties in the untouched region are the same as the initial condition. Flow in the transition region has the following analytical solutions:

$$\begin{aligned} u = -\frac{2\sqrt{\gamma}}{\gamma+1}\left[1 - \frac{R}{t}\right] \quad , \quad c = \sqrt{\gamma} + \frac{\gamma-1}{2}u \equiv c(u) \ , \\ \rho = (c/\sqrt{\gamma})^{2/(\gamma-1)} \equiv \rho(u) \quad , \quad P = \rho^\gamma \equiv P(u) \ . \end{aligned} \tag{27}$$

And, for $R < \left[\sqrt{\gamma} - U_p \times (\gamma+1)/2\right] \times t$, flow will have constant properties as

$$u = -U_p \ , \quad c = c(-U_p) \ , \quad \rho = \rho(-U_p) \ , \quad P = P(-U_p) \ . \tag{28}$$

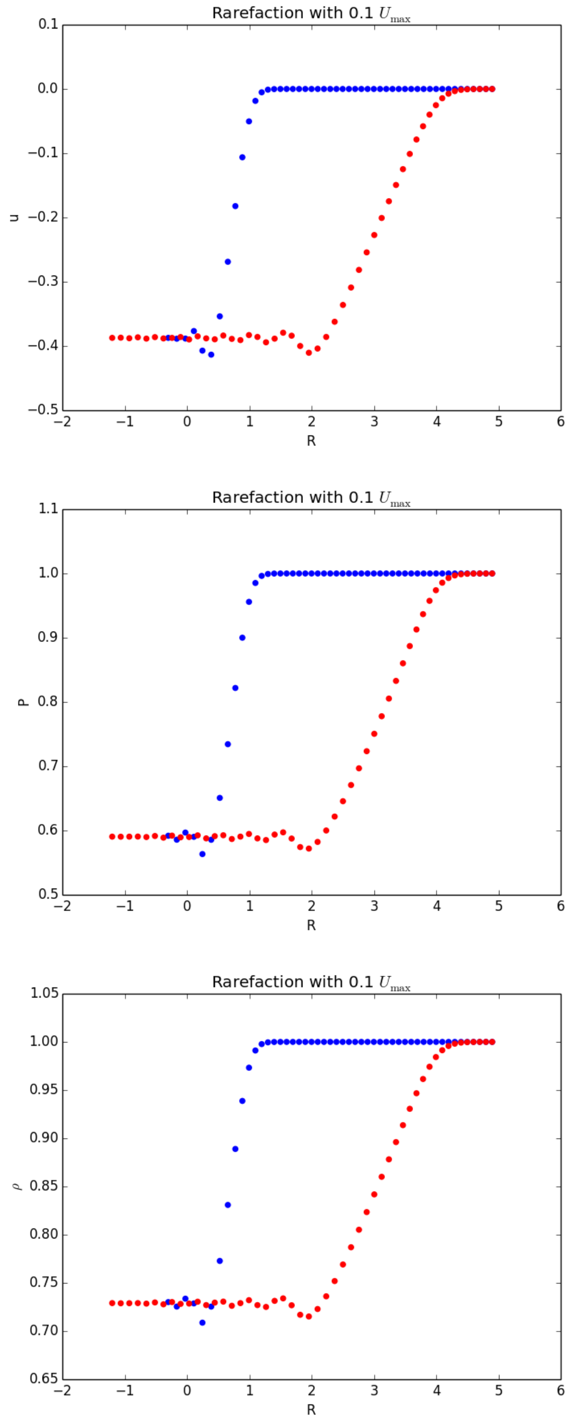

Figure 1: Rarefaction with 0.1 $U_{\max}$. The two curves show how the wave is transported to the right with time $t = 0.7746$ and $t = 3.098$.

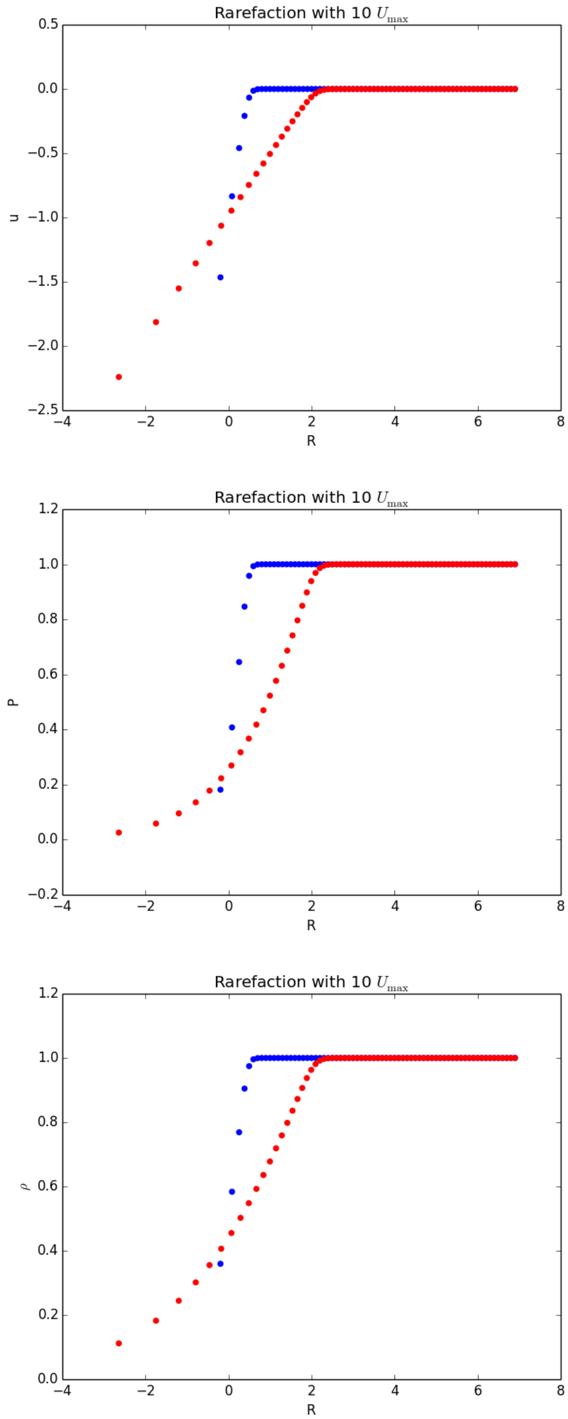

Figure 2: Rarefaction with 10 $U_{\max}$. The two curves show how the wave is transported to the right with time.

4.2 Compression

Now we have the piston **plunged into** the fluid with the velocity (a) $U = (1/10)C_0$ and (b) $U = 10C_0$. Different from the rarefaction case, we just need to change the boundary condition of velocity, with other parameters staying the same:

$$u_0^n = U_{piston}/u_0 = k\sqrt{\gamma}$$

. The density, pressure and velocity profiles (density, pressure and velocity with respect to the spatial coordinate R) are plotted in Fig.3 and Fig.4 .

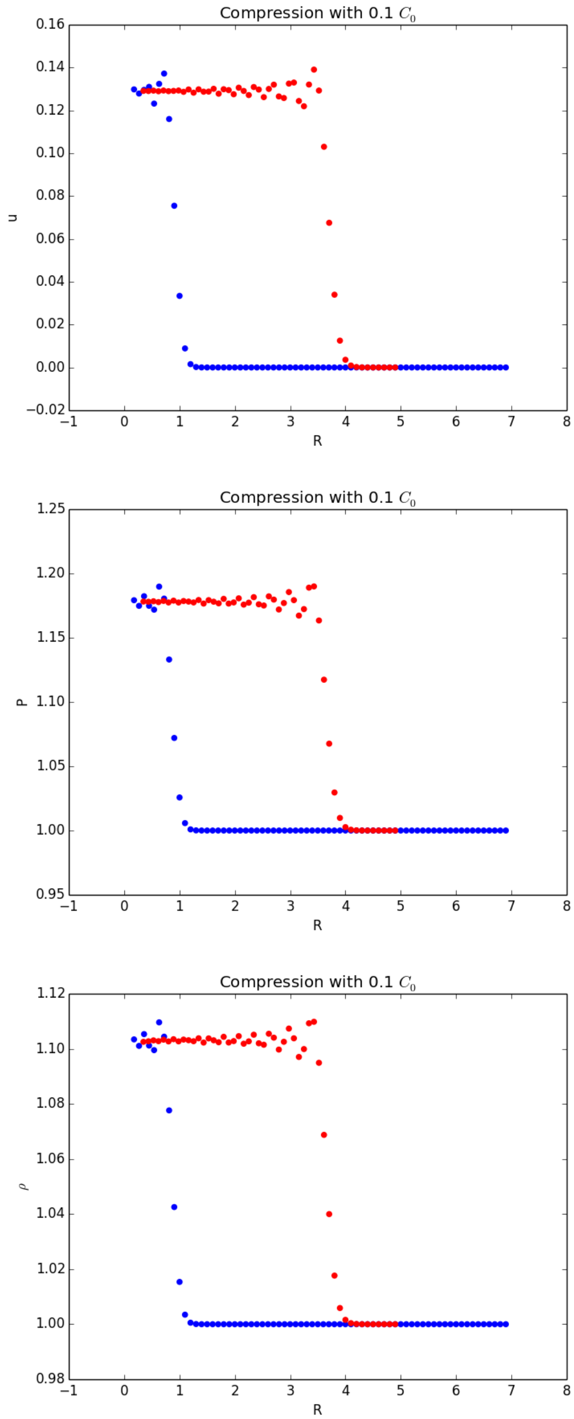

Figure 3: Compression with 0.1 C_0. The two curves show how the wave is transported to the right with time.

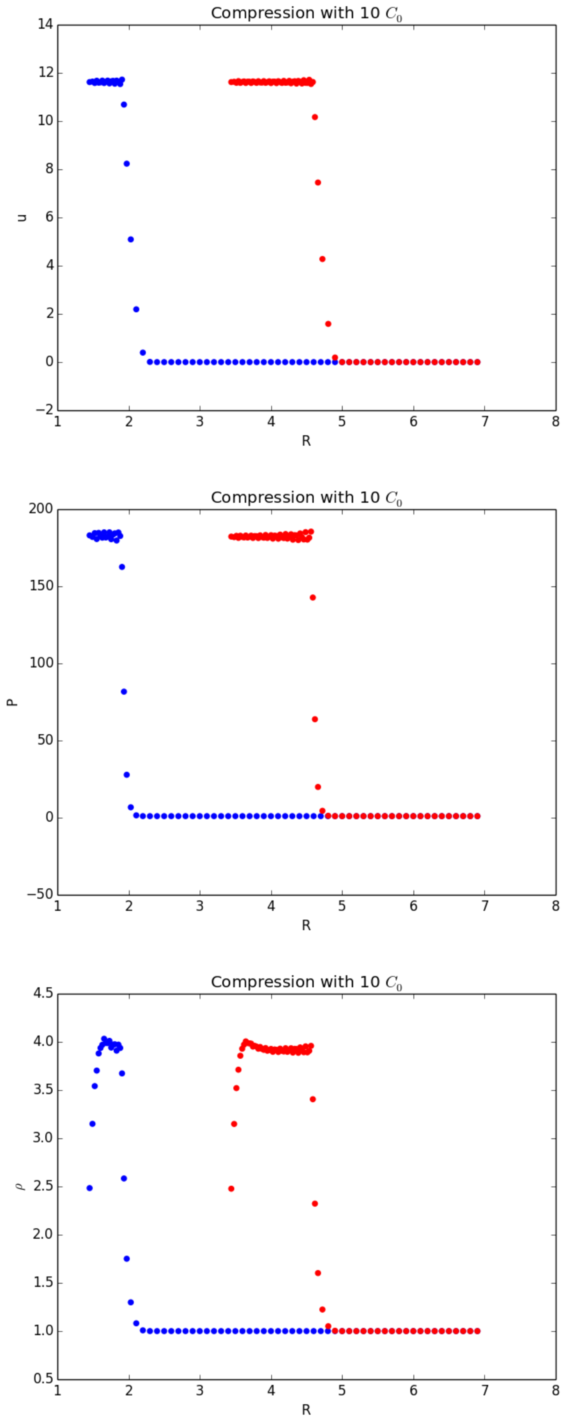

Figure 4: Compression with 10 C_0. The two curves show how the shock is transported to the right with time.

5 Discussion

5.1 Comparing with Analytical Values

In Table 1, we have calculated P_1/P_0, ρ_1/ρ_0, as well as the shock speed u_0 for the analytical method.

Case	P_1/P_0	ρ_1/ρ_0	u_0/c_0
$U = (1/10)U_{max}$			1.57
$U = 10U_{max}$			40
$U = -(1/10)C_0$	1.62	1.33	1.23
$U = -10C_0$	2000	4	40

Table 1. Jump Conditions with Analytical Approach

As a comparison, all the same parameters using the output of numerical calculation are shown in Table 2.

Case	P_1/P_0	ρ_1/ρ_0	u_0/c_0
$U = (1/10)U_{max}$			
$U = 10U_{max}$			
$U = -(1/10)C_0$			
$U = -10C_0$			

Table 1. Jump Condition with Numerical Approach

We can see that in general, the analytical and numerical approaches are in good agreement with each other.

5.2 Conservation of Energy

The total energy is given by integrating the energy of each element:

$$E = \int dr(\epsilon + \frac{1}{2}\rho u^2) = \int dr(\frac{PV}{\gamma - 1} + \frac{1}{2}\rho u^2) \tag{29}$$

Numerically this becomes:

$$E^n = \sum_{j=1}^{j_{max}} \Delta x[\frac{P_j^n V_j^n}{\gamma - 1} + \frac{1}{2}\rho_j^n (u_j^n)^2] \tag{30}$$

Compare this to the PdV work done by the piston, which is

$$W = \sum_{j=1}^{j_{max}} P_j^n V_j^n \tag{31}$$

We have plotted the energy evolution in Fig. 5 - 8.

5.3 Shock Propagation

The propagation speed of the shock is given by:

$$
{M_1}^2 = \frac{1 + \left[\left(\gamma - 1 \right) / 2 \right] {M_0}^2}{\gamma {M_0}^2 - \left(\gamma - 1 \right) / 2}
$$

So we can plot the shock speed U_s with time, which is shown in Fig. 9: