Introduction to plasma dynamics - GRB project
II. Further developments

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1 Theory: Relativistic, electrostatic two-stream instability

The nonrelativistic calculations in part I should have brought you to the following final results: dispersion relation

$$1 = \frac{\omega_p^2}{(\omega - kv_0)^2} + \frac{\omega_p^2}{(\omega + kv_0)^2}, \tag{1}$$

with a condition on $k < \sqrt{2}\omega_p/v_0$; growth rate as a function of k

$$\omega(k) = \sqrt{1 + \frac{k^2 v_0^2}{\omega_p^2} - \sqrt{1 + 4\frac{k^2 v_0^2}{\omega_p^2}}}, \tag{2}$$

with maximum value equal to $\omega_{\max} = \omega_p/2$ found at $k_{\max} = \sqrt{3}\omega_p/(2v_0)$.

We now want to push forward and consider the relativistic case. The initial state is exactly the same as before, but now consider the relativistic momentum equation

$$m_s \frac{\partial v_s \gamma_s}{\partial t} + m_s v_s \frac{\partial v_s \gamma_s}{\partial x} = q_s E, \tag{3}$$

where $\gamma = (1 - v^2)^{1/2}$ is the Lorentz factor. Here, v is in units of the speed of light c. What you are asked to do is the following:

- Follow the same procedure as before for linearising Euler's and Poisson's equations with the relativistic correction. In doing so, use a linearisation on $(v\gamma)$ rather than on v only; hence, introduce a perturbation $v_s\gamma_s \to \widetilde{v_s\gamma_s}|_{v_s = v_{s,0}}$, using the standard expansion around x_0 for a function $\tilde{f}(x)|_{x=x_0} = f(x_0) + f'(x_0)\delta x$ (here, remember that your displacement δx is the perturbation $v_{s,1}$). In the final linearised equations, no $\gamma_{s,1}$ term should appear; only $\gamma_{s,0}$.

- Assume a wave-like behaviour of the perturbed quantities and write the final dispersion relation for $\omega(k)$. Give the condition on k that allows for imaginary ω, and solve the dispersion relation, identifying the maximum growth rate. Is it larger or smaller than that obtained for nonrelativistic plasmas?

If you successfully solved the tasks in part I, you should be able to follow the exact same procedure and arrive to the results of these new tasks.

2 Numerics: Your first electrostatic PiC simulation

Now that you have a working proof-of-concept code, that can at least model Maxwellian plasmas, it is time to try a simulation of an actual interesting physical phenomenon. For doing this, however, we need first some modifications, and then a suitable initial setup. Follow the instrucitons below carefully:

- We want to have the freedom to use as many particle species as needed. Each species is determined primarily by a different q/m, v_{th}, and ρ_0 set of values. Additionally, you could also have a different *nppc* for different species. Here we are going to keep things as simple as possible: we will use 2 species, namely electron and positrons, with the same background density $\rho_0 = 1$, same thermal

spread v_{th}, and same number of particles per cell *nppc*. The only difference will be in q/m: you can set electrons to $q/m = -1$, and positrons to $q/m = -1$. Note that as a consequence, the macroparticle charge $q = \rho_0/(q/m)\Delta x$ will also have opposite sign. In your code, it is best to create one array of size $nppcN_g$ for each particle species, and then initialise each species (by setting position and velocity) separately before the beginning of the time loop.

- We now want to initialise our electrons and positrons such that a two-stream instability will take place. For this to happen, we need to consider a few things:
  - First, we need to set up two counterstreaming beams in space. We are free to decide that each beam is constituted by half of the total electrons + half of the total positrons. We can set up each beam as uniformly distributed across the domain: hence, for each particle species, you must distribute half of its particles uniformly along the domain, and then distribute the remaining half uniformly again. Think about this and try to do it in the most efficient way.
  - We have to decide upon an initial velocity for the counter-streaming beams. Choose $v_0 = 0.01$ for now. Such a drift velocity must be added to the particle velocity after initialising them with a Maxwellian distribution: $v_p = v_{th} + v_0$. Remember that the beams travel in opposite directions: one with speed v_0, the other with speed $-v_0$. Initialise each species accordingly.
  - With this choice for the distribution of particles in the two beams, we have made sure that each beams has $\omega_p = 1$ in code units (check this). We now have to decide on the length of the domain L. Look up the "Theory" section above. You can see that imaginary solutions of the dispersion relation are allowed only if $k < \sqrt{2}\omega_p/v_0$. These are the ones that represent instabilities; hence, if such a range of k is not represented in our simulation, we will not see any instability arising. In order to make sure that we see an instability, consider that the maximum wavelength of any wave in our simulation will be equal to L (i.e. the whole domain). The corresponding wavenumber is $k = 2\pi/L$. Imposing the condition on k gives $L > \sqrt{2}\pi v_0/\omega_p$. Therefore, you have to set a domain length of at least this size (but preferably appreciably larger).
- Before running the simulation, we need appropriate diagnostic tools. In particular, we want to monitor the electric field energy, the kinetic energy, and the maximum velocity reached during the simulation. Create suitable vectors of size N_t and make sure that, at the end of each time step (after the field update), you compute and store the following quantities:
  - Electric energy $e_E = \sum_g E_g^2 \Delta x/2$;
  - Kinetic energy $e_K = \sum_p m_p v_p^2/2$;
  - Maximum velocity $v_{\max} = \max(v_p)$.
- At the end of the simulation, you want to plot the evolution of these quantities in time. You will know that you did things correctly if you see the electric field energy growing in time. In particular, the growth should exhibit an initial linear (in logarithmic scale) behaviour, and then a final "steady-state" where no further secular growth is observed.

Follow each point carefully, and do not hesitate to contact me in case of doubts.