How Build a Star, Part 2

4 main equations:

① $$\frac{dP}{dr} = -\frac{G M_r \rho}{r^2}$$, M_r = Mass inside a radius r.

② $$\frac{dM_r}{dr} = 4\pi r^2 \rho$$

③ $$\frac{dL_r}{dr} = 4\pi r^2 \rho \epsilon$$

④ $$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\bar{\kappa}\rho}{T^3}\frac{L_r}{4\pi r^2} \quad \text{(radiation)}$$

$$= -\left(1 - \frac{1}{\gamma}\right)\frac{\mu m_H}{k}\frac{G M_r}{r^2} \quad \text{(adiabatic Convection)}$$

Luminosity

Luminosity → energy emitted per second (power).

ϵ → total energy released per kg per s. by all nuclear reactions.

Then $dL = \epsilon\, dm$

⇒ spherically symmetric → $dm \to \rho dV = 4\pi \rho r^2 dr$

$$\frac{dL_r}{dr} = 4\pi r^2 \rho \epsilon$$

$\Rightarrow$ $L_r \rightarrow$ luminosity due to all the energy generated within the star's interior out to a radius r.

What is ϵ?

$$\epsilon = \epsilon_{nuclear}$$

$$\boxed{\epsilon = \epsilon_{pp} + \epsilon_{CNO} + \epsilon_{3\alpha}}$$

where

- $\epsilon_{pp} \rightarrow$ proton-proton chain
  $\rightarrow$ turns 4 H nuclei into ^4He (and some e^+, ν_e, γ)

$$\epsilon_{pp} = 0.241 \rho X^2 f_{pp} \psi_{pp} C_{pp} T_6^{-2/3} e^{-33.80 T_6^{-1/3}} \quad W/kg.$$

- $T_6 = \dfrac{T}{10^6 K}$
- X - hydrogen mass fraction $\left(X = \dfrac{\text{total mass of H}}{\text{total mass of gas}}\right)$
- Y - helium mass fraction
- Z - metal mass fraction (C, N, O, etc.)
- $f_{pp} \simeq 1$ "pp chain screening factor"
- $\psi_{pp} = 1 + 1.412 \times 10^8 \left[\dfrac{1}{X} - 1\right] e^{-49.98 T_6^{-1/3}}$

  $\Rightarrow$ Correction factor that accounts for the simultaneous occurrence of different pp chains.

- $C_{pp} \rightarrow$ Some other corrections.

$$= 1 + 0.0123 T_6^{1/3} + 0.0109 T_6^{2/3} + 0.000938 T_6$$

- CNO cycle

$$\epsilon_{CNO} = 8.67 \times 10^{20} \rho X X_{CNO} C_{CNO} T_6^{-2/3} e^{-152.28 T_6^{-1/3}} \text{ W/kg.}$$

where : $X_{CNO} \rightarrow$ mass fraction of C, N, O.

$= Z/2$ (assumption).

$C_{CNO} \rightarrow$ corrective term

$$= 1 + 0.0027 T_6^{1/3} - 0.00778 T_6^{2/3} - 0.000149 T_6.$$

- Triple-α process $\rightarrow$ burns helium

$$\epsilon_{3\alpha} = 50.9 \rho^2 Y^3 T_8^{-3} f_{3\alpha} e^{-44.027 T_8^{-1}} \text{ W/kg.}$$

where $T_8 = T/10^8 K$

$f_{3\alpha} \approxeq 1.$

ENERGY TRANSPORT

Energy produced in the core is transported outwards.

- Radiative transport: start with

$$\cos\theta \frac{dI}{d\tau_\nu} = I - S$$

If we multiply by $\cos\theta$ and integrate over $d\Omega = \sin\theta d\theta d\phi$ we get, for a spherical system:

$$\frac{dP_{rad}}{dr} = -\frac{\bar{\kappa} \rho}{c} F_{rad}$$

$$P_{rad} = \frac{1}{c} \int I \cos^2\theta \, d\Omega \quad \text{radiation pressure}$$

$$F_{rad} = \int I \cos\theta \, d\Omega \quad \text{radiative flux.}$$

Opacities: $\boxed{\bar{\kappa} = \bar{\kappa}_{bf} + \bar{\kappa}_{ff} + \bar{\kappa}_{es} + \bar{\kappa}_{H^-}}$

- Bound - free processes.

$$\bar{\kappa}_{bf} = 4.34\times10^{21}\left(\frac{g_{bf}}{t}\right) Z(1+X)\frac{\rho}{T^{3.5}} \quad m^2/kg.$$

where: • g_{bf} "Gaunt factor"
• t "guillotine factor" $\Big\} \; \frac{t}{g_{bf}} = 0.708\left[\rho(1+X)\right]^{1/5}$

- free - free process

$$\bar{\kappa}_{ff} = 3.68\times10^{18}\, g_{ff}\,(1-Z)(1+X)\frac{\rho}{T^{3.5}} \quad m^2/kg.$$

$g_{ff} \gtrsim 1$

- electron scattering

$$\bar{\kappa}_{es} = 0.02\,(1+X) \quad m^2/kg$$

- H^- ions can play a role in opacity, if:

$3000k \le T \le 6000\ k$

$10^{-7}\ kg/m^3 \le \rho \le 10^{-2}\ kg/m^3$

$0.001 < Z < 0.03.$

Then

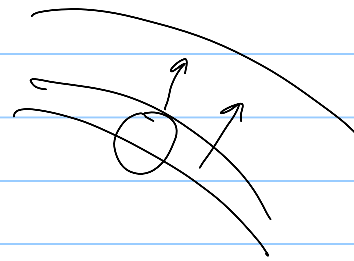
$$\bar{\kappa}_{H^-} = 7.9\times10^{-34}\left(\frac{Z}{0.02}\right)\rho^{1/2}\,T^9 \quad m^2/kg.$$

Back to: $\frac{dP_{rad}}{dr} = -\frac{\bar{\kappa}\rho}{c}F_{rad}$

- Use $P_{rad} = \frac{1}{3}aT^4$ $\quad (a = 7.565767\times10^{-16}\ J\,m^{-3}\,K^{-4})$
- and $F_{rad} = \frac{L_r}{4\pi r^2}$ for a sphere, then we get equation (4).

- Energy transport by Convection

Consider a bubble of gas rising and expanding adiabatically.



From the ideal gas law,

$$P = \frac{\rho k T}{\mu m_H}$$

($\mu \rightarrow$ mean molecular weight of the gas)

$$\mu = \left[2X + \frac{3}{4}Y + \frac{1}{2}Z\right]^{-1}$$

($m_H \rightarrow$ mass of H)

Take the derivative:

$$\frac{dP}{dr} = \frac{P}{\rho}\frac{d\rho}{dr} + \frac{P}{T}\frac{dT}{dr}$$

Adiabatic process $\rightarrow$ $P = k\rho^{\gamma}$, $\gamma = 5/3$ (for an ideal gas)

Differentiate and rewrite:

$$\frac{dP}{dr} = \gamma \frac{P}{\rho}\frac{d\rho}{dr}$$

$$\frac{dT}{dr} = \left(1 - \frac{1}{\gamma}\right)\frac{T}{\rho}\frac{dP}{dr}$$

Use eq. (1) for dP/dr and the ideal gas law one more time:

$$\frac{dT}{dr} = -\left(1 - \frac{1}{\gamma}\right)\frac{\mu m_H}{k}\frac{G M_r}{r^2} \qquad \star.$$

What determines whether the energy is transported by radiative processes or by convection?

$\Rightarrow$ how steep the temperature gradient is.

If $\left|\frac{dT}{dr}\right| > \left|\frac{dT}{dr}\right|_{ad}$ the Convection takes over.

$\left|\frac{dT}{dr}\right|$ → actual temperature gradient

$\left|\frac{dT}{dr}\right|_{ad}$ → gradient if Convection present

Write this differently:

$$\boxed{\frac{T}{P}\frac{dP}{dT} = \frac{d\ln P}{d\ln T} < \frac{\gamma}{\gamma-1}}$$ for convection to occur.

One more thing: an equation of state $P(\rho)$.

$$\rho = \frac{P_{gas}\,\mu m_H}{kT}$$

but photons have a pressure too:

$$P_{rad} = \frac{1}{3} a T^4$$

$$\Rightarrow P = P_{tot} = P_{gas} + P_{rad}.$$

$$\Rightarrow \boxed{\rho = \left(P - \frac{1}{3} a T^4\right)\frac{\mu m_H}{kT}}$$

Boundary Conditions:

Centre: $\left.\begin{aligned} M_r &\to 0 \\ L_r &\to 0 \end{aligned}\right\}$ as $r \to 0$

Surface: $\left.\begin{aligned} T &\to 0 \\ P &\to 0 \\ \rho &\to 0 \end{aligned}\right\}$ as $r \to R_*$.