MODULE 1 | ASSIGNMENT 2
CEPHEID VARIABLE STARS AND RADIATIVE TRANSFER

A. CEPHEID VARIABLE STARS

The stars we've modeled so far – just white drawfs, I guess – have been *static*. Pulsating stars like Cepheid variable stars are more complicated, but there are two simple models we can use to explore them.

1. First, pretend the Cepheid is spherically symmetric and in hydrostatic equilibrium. We'll also assume that the density ρ is constant, even though this is a pretty bad assumption. Show that the pressure is then given by

$$P(r) = \frac{2}{3}\pi G\rho^2(R^2 - r^2),$$

where R is the radius of the star. (You can find the equations of stellar structure back in our notes on how to build a star.)

We can estimate the pulsation period by calculating the time it takes for sound waves to propagate across the diameter of the Cepheid. If the sound speed is given by

$$v_s = \sqrt{\frac{\gamma P}{\rho}}$$

(this is a standard equation assuming an ideal gas; it was derived in Fluid Dynamics last year if you took it then), show that this leads to a period

$$\Pi \approx \sqrt{\frac{3\pi}{2\gamma G\rho}}.$$

If the Cepheid has a mass of 4.5 $M_\odot$ and radius 44.5 $R_\odot$, what is the period? Compare with the period of the famous Cepheid δ Cephei.

2. Now for something a little better (but still simple). Model the Cepheid as a spherically symmetric star consisting of two components: a central point mass (M) surrounded by a single thin shell of mass m and radius R. The interior of the star is filled with a massless gas of pressure P that supports the shell as the gravitational pull of the central mass; see the figure below. This simple model is called the "one-zone" model for Cepheid variable stars.

Since this model isn't in hydrostatic equilibrium, we'll need to modify our equation to include the shell being able to move; the shell's position will be given by

$$m\frac{d^2R}{dt^2} = -\frac{GMm}{R^2} + 4\pi R^2 P$$

(you don't need to derive this equation; you can get from the stellar equations, but it's obviously just $F = ma$). We'll also assume the contraction and expansion of the gas is adiabatic, so that at any two times i and f, we have

$$P_i V_i^\gamma = P_f V_f^\gamma,$$

or, using $V = 4\pi R^3/3$ for a sphere,

$$P_i R_i^{3\gamma} = P_f R_f^{3\gamma}.$$

That completes our model.

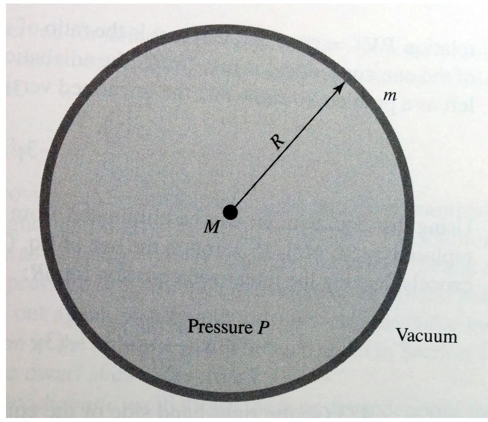

Figure 1: The "one-zone" model for Cepheids.

To begin solving it, we should first nondimensionalize it. First, show that the equilibrium value of the pressure, $P_{\rm eq}$, is given by

$$P_{\rm eq} = \frac{GMm}{4\pi R^4}.$$

Then write

$$\begin{aligned} \eta &= R/R_0 \\ \tau &= t/t_0 \\ \mu_* &= M/M_0 \\ \mu_s &= m/M_0 \\ p &= P/P_0, \end{aligned}$$

where R_0 is one solar radii, M_0 is one solar mass, and P_0 is given by

$$P_0 = \frac{GM_0^2}{4\pi R_0^4}.$$

Show that the equation of motion becomes

$$\frac{d^2\eta}{d\tau^2} = -\frac{\mu_*}{\eta^2} + \frac{\eta^2 p}{\mu_s};$$

what is t_0?

As initial conditions, take $\eta(0) = 44.5$, $d\eta/dt = 0$ at $t = 0$, and take $\mu_* = 4.5$ and $\mu_s = 10^{-5}$. Play with a few different values of $p(0)$ to see what effect it has; to get an idea of what it should be, calculate the equilibrium pressure for the star first, then vary from that.

Solve the equation numerically and plot the motion of the shell. Again, compare with δ Cephei.

B. Monte Carlo Radiative Transfer

Let's continue with the work started in Worksheet for Submodule 1.4. You should now have a working Monte Carlo code that moves a single photon to the top of a plane parallel atmosphere. You'll have to modify it to do it for many photons, but that shouldn't be a problem. It's the analysis afterwards that we're concerned with here.

1. Our first goal is to calculate the limb darkening effect that we discussed (and derived a completely different way) in class. To do this, bin your photons as they come out of the atmosphere into, say, ten bins equally spaced in $\mu = \cos\theta$. Plot it to make sure it makes sense (there should be more photons coming out at small angles, which means around $\mu \sim 1$), but we need to turn this into something we can compare with our other derivation.

   For that, we need to do a couple of things:

   - We want angle on the horizontal axis rather than μ. Make sure you use the angle at the half-way point of each bin rather than the start.
   - Then we have to normalize the intensity similarly to how we did it in class. In this case, though, that means dividing the number of photons in each bin N_i by

     $$2\mu_i N_0/N_\mu,$$

     where N_0 is the total number of photons, N_μ is the number of bins for μ, and μ_i is the value of μ at the centre of each bin. This converts the flux that we calculate into an intensity, normalized to the total intensity overall solid angles.

   Plot your intensity and the theoretical one we did in class on the same plot and compare.

2. We can also bin by the photon's x and y position when it reaches the surface and generate an image of the atmosphere. Try it, but be warned: this isn't actually what would be seen, since the photons are going in different directions, and not all of them would be caught by the detector (which could be placed anywhere). But it's still a good way to see how the photons distribute themselves on the surface after doing their random walk.