MODULE 1 | ASSIGNMENT 1
WHITE DWARF STARS

A. A SIMPLE TOY MODEL

If we assume that a white dwarf star consists of nonrelativistic degenerate electron gas,[1], the equation of state is

$$p_{\mathrm{nr}} = \frac{h^2}{20m_e}\left(\frac{3}{\pi}\right)^{2/3}\frac{1}{(2m_p)^{5/3}}\rho^{5/3} = k_{\mathrm{nr}}\rho^{5/3}.$$

The constant k_{nr}, in cgs units, takes on the value $k_{\mathrm{nr}} = 3.166 \times 10^{12}$. This is a polytropic equation of state, meaning the white dwarf is described by the Lane-Emden equation, with $n = 1.5$.

Here's a series of questions to answer about our model.

1. Set up and solve the Lane-Emden equation numerically, using the Runge-Kutta fourth order method, for the white dwarf. The only parameter for the dimensionless Lane-Emden model is n, which is 1.5 always for our toy model. So you really only need to do this once. Plot $\varrho(\eta)$, and identify the surface η_s. Calculate the quantity, which I'll call a dimensionless mass, given by

$$m = \int_0^{\eta_s} \varrho^n \eta^2 d\eta.$$

2. Now we need to work in real numbers. First, we want a white dwarf that's exactly 1.0 $M_\odot$. What value of ρ_c does this require? What is the radius r_s of the star, in solar radii? Plot the density $\rho(r)$ in convenient units. More generally, how does the mass of the star depend on the central density ρ_c? How does the radius r_s depend on ρ_c? Finally, how does the radius depend on the mass? (These last three questions can be answered without running any code; I'm looking for proportionality relationships.)

B. A DIFFERENT MODEL

It turns out that we're missing a whole bunch of things that would modify our equation of state $p(\rho)$. We're neglecting relativistic effects, Columb energy of an ion lattice, Thomas-Fermi deviations from a uniform distribution of electrons, the exchange energy between electrons, and the spin-spin interactions between electrons. Of these, though, the biggest problem is the lack of relativistic degeneracy pressure. Adding it in is complicated, so let's try another simple toy model – a white dwarf that is purely relativistic. In that case, the equation of state is

$$p_{\mathrm{r}} = \frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3}\frac{1}{(2m_p)^{4/3}}\rho^{4/3} = k_{\mathrm{r}}\rho^{4/3}.$$

This time, $k_{\mathrm{r}} = 4.936 \times 10^{14}$ in cgs units.

Once again, we have a polytrope, this time of index $n = 3$.

1. Set up and solve the Lane-Emden equation numerically, using the Runge-Kutta fourth order method, for the white dwarf with $n = 3$. Plot $\varrho(\eta)$, and identify the surface η_s. Calculate the "dimensionless mass" m for the star.

[1] Not a bad approximation, necessarily, except that it might be relativistic, contain various ions, have a high temperature with thermal pressure, ... well, maybe it's not so great. But onwards!

2. Now we need to work in real numbers. First, we want a white dwarf that has a radius of exactly 8890 km. What value of ρ_c does this require? What is the mass M of the star, in solar masses? Plot the density $\rho(r)$ in convenient units. More generally, how does the mass of the star depend on the central density ρ_c? How does the radius r_s depend on ρ_c? Finally, how does the radius depend on the mass?

C. A Better Model

In reality, white dwarfs can be both nonrelativistic and relativistic. We need an equation of state that combines the two above. One simple way to do this (there are more complicated ways, but this gives us a good approximation) is to define a characteristic density ρ_0 where the two pressures are equal. This leads to

$$\rho_0 = \left(\frac{k_{\rm r}}{k_{\rm nr}}\right) = 3.789 \times 10^6 \text{ g cm}^{-3}.$$

We'll then combine the equations of state in the following way:

$$p = \frac{p_{\rm nr} p_{\rm r}}{\sqrt{p_{\rm nr}^2 + p_{\rm r}^2}} = \frac{k_{\rm nr}\rho^{5/3}}{\sqrt{1 + (\rho/\rho_0)^{2/3}}}.$$

This equation of state is *not* a polytrope, and things get more complicated.

1. Go back before we derived the Lane-Emden equation to (in page 2 in our notes, or equation 10.108 in Carroll and Ostlie)

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G\rho.$$

   Now let $\rho(r) = \rho_0 \varrho(r)$ (note that this is *not* the central density as before) and $r = \lambda\eta$ and show that the equation becomes

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left[\frac{\eta^2}{\varrho}\frac{d(\varrho^{5/3}(1+\varrho^{2/3})^{-1/2}}{d\eta}\right] = -\varrho.$$

2. This is …not a nice looking equation. But we're going to solve it numerically, anyway, so next show that it can be written as two first order coupled differential equations,

$$\begin{aligned}\frac{d\varrho}{d\eta} &= \sigma \\ \frac{d\sigma}{d\eta} &= -\frac{2}{\eta}\sigma - \frac{A(\varrho)}{B(\varrho)}\sigma^2 - \frac{1}{B(\varrho)}\varrho,\end{aligned}$$

   where $A(\varrho)$ and $B(\varrho)$ are the rather complicated equations

$$\begin{aligned}A(x) &= -(5/9)x^{-4/3}(1+x^{2/3})^{-1/2} - (2/3)x^{-2/3}(1+x^{2/3})^{-3/2} + (1/3)(1+x^{2/3})^{-5/2}, \\ B(x) &= (5/3)x^{-1/3}(1+x^{2/3})^{-1/2} - (1/3)x^{1/3}(1+x^{2/3})^{-3/2}.\end{aligned}$$

3. Now we're ready to solve it, but we need our boundary conditions. They're the usual:

$$\varrho(0) = \varrho_c, \quad \frac{d\varrho}{d\eta} = 0 \text{ at } \eta = 0.$$

   Note, though, that ϱ_c isn't equal to one, but rather the value of the central density in units of ρ_0. Regardless, as a first test case, try solving the equations with $\varrho_c = 1$. Stop the integration when the density drops below 0.1% or so of ϱ_c (rather than waiting for it to go negative, which might lead to some numerical difficulties). Plot $\rho(r)$, identify the radius of the surface, and calculate the total mass of your white dwarf.

4. Once your test case is done, we're going to generate a bunch of models of different central densities. Do about 25 models, with ρ_cs that range from 10^4 g/cm^3 to 10^{12} g cm^3. Then:

   (a) Make a plot of the masses of your white dwarf models (in $M_\odot$) versus the logarithm of their central density. Identify the maximum stable mass of a white dwarf.

   (b) Make a plot of the radii of your models (in units of $R_\odot/100$) versus the logarithm of their central density.

   (c) Make a plot of the radii of your models versus their mass. Be sure to compare your results with the results from parts A and B above.

   (d) This last plot, of the radii versus mass, is a famous result in white dwarf physics. Is our model any good? If you have time and desire, take a look at a few references like Nalezyty and Madej (2004) or Panei et al. (2000), or search up some of your own.