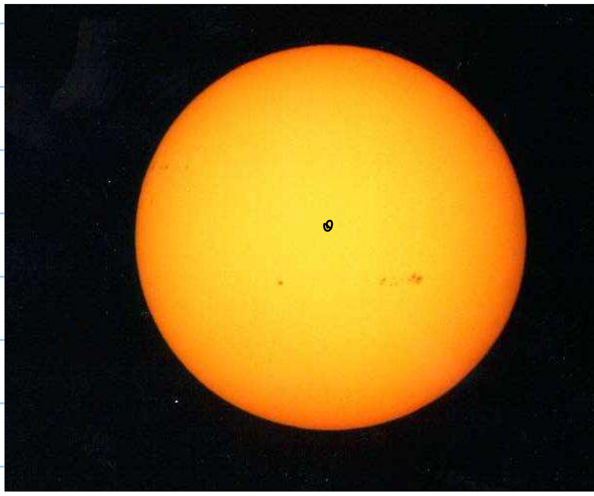
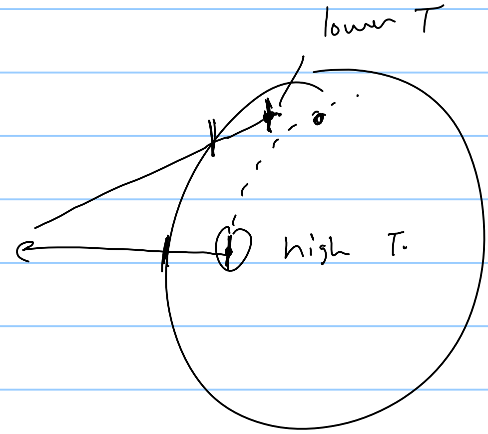
Limb Darkening.

Note Title 1/16/2019





Limb darkening is a general problem radiative transfer. The equation of radiative transfer is:

$$-\frac{1}{k_\lambda \rho}\frac{dI_\lambda}{ds} = I_\lambda - S_\lambda.$$

It gives the intensity I_λ (per unit wavelength) of light traveling through a distance s within some medium.

- $I_\lambda \equiv \frac{\partial I}{\partial \lambda} = \frac{E_\lambda \, d\lambda}{d\lambda \, dt \, dA \cos\theta \, d\Omega}$ → $\sin\theta \, d\theta \, d\phi$
- $S_\lambda \equiv \frac{\partial S}{\partial \lambda}$ → "source function". It describes how photons originally traveling with the beam of light are removed and replaced by photons from the surrounding gas.

  $= \frac{j_\lambda}{k_\lambda}$ → ratio of the emission coefficient to absorption coefficient
- k_λ → opacity.

If $S_\lambda = 0$ (pure absorption)

$$\Rightarrow \quad dI_\lambda = -\kappa_\lambda \rho I_\lambda \, ds. \qquad \left[\kappa_\lambda \rho = \frac{dI_\lambda}{ds} \frac{1}{I_\lambda} \right]$$

Rather than opacity, it is convenient to define the optical depth τ_λ such that

$$\boxed{d\tau_\lambda = -\kappa_\lambda \rho \, ds} \qquad *$$

So $$\tau_\lambda = \int_{\underset{s=0 \text{ surface}}{0}}^{\overset{\text{depth}}{s}} \kappa_\lambda \rho \, ds$$ (taking $\tau_\lambda = 0$ at the surface of the star)

For pure absorption again, in terms of $d\tau_\lambda$:

$$\tau_\lambda = \int_0^s \left(\frac{dI_\lambda}{\cancel{ds}} \right) \frac{1}{I} \cancel{ds} = \int_0^s \frac{dI_\lambda}{I_\lambda}$$

$$= -\ln I_\lambda + \ln \underbrace{I_{\lambda,s}}$$ intensity at surface

$$I_\lambda = I_{\lambda,s} \, e^{-\tau_\lambda}$$

We typically see no deeper into an atmosphere than $\tau_\lambda = 1$.

General Sources of Opacity

- Bound-bound transitions
- Bound-free absorption
  - ↳ photoionization
- free-free
  - ↳ absorption by a free e^-
- Electron scattering

The Source function

In thermodynamic equilibrium, the intensity of radiation is blackbody

$$I_\lambda = B_\lambda$$

If we have equilibrium, $\frac{dI_\lambda}{ds} = 0$.

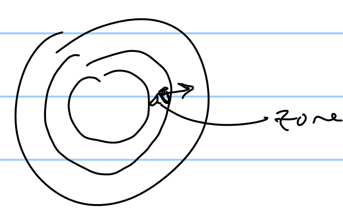
radiative transfer eq. is $S_\lambda = I_\lambda$

$$= B_\lambda.$$

However, a star is <u>not</u> in equilibrium.
But, given that photons (for $\tau_\lambda >> 1$) diffuse upwards via a random walk.

$\Rightarrow$ photons are effectively confined to a limited volume, a region at nearly constant temperature.

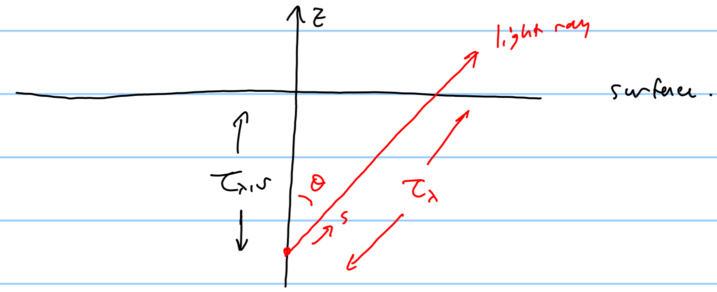
$\Rightarrow$ In LTE,
$S_\lambda = B_\lambda$.



Local thermodynamic equilibrium. (LTE)

Plane - Parallel Atmosphere

Assume the atmosphere of the sun is thin compared to the size of the star.
$\Rightarrow$ treat it as a plane parallel slab:



Define the "vertical optical depth":

$$\tau_{\lambda,v}(z) \equiv \int_z^0 k_\lambda \rho \, dz$$

Radiative transfer eq. $\quad -\frac{1}{k_\lambda \rho} \frac{dI_\lambda}{ds} = I_\lambda - S_\lambda$

in terms of τ_λ

$$\frac{dI_\lambda}{d\tau_\lambda} = I_\lambda - S_\lambda$$

using $\tau_\lambda = \frac{\tau_{\lambda,v}}{\cos\theta}$

$$\cos\theta \frac{dI_\lambda}{d\tau_{\lambda,v}} = I_\lambda - S_\lambda$$

Grey Atmosphere

Assume k doesn't depend on wavelength.
Integrate over all wavelengths to get:

$$\cos\theta \frac{dI}{d\tau_v} = I - S.$$

Integrate over all solid angles:

$$\frac{d}{d\tau_\nu} \underbrace{\int I \cos\theta \, d\Omega}_{F_{rad}} = \underbrace{\int I \, d\Omega}_{4\pi \langle I \rangle} - \int \overset{LTE}{S} \, d\Omega$$

$$- S \int d\Omega \rightarrow 4\pi$$

F_{rad} radiative flux.

↳ $F_{rad} = \sigma T_e^4$ (T_e → "effective temperature")

↳ Stefan-Boltzmann Constant

$$\Rightarrow \frac{d F_{rad}}{d\tau_\nu} = 0. \quad \text{(same everywhere!)}$$

$$\Rightarrow \quad 4\pi \langle I \rangle - 4\pi S = 0$$

$$\Rightarrow \quad \langle I \rangle = S$$

What is the mean intensity?

$$\langle I \rangle = \frac{3\sigma}{4\pi} T_e^4 \left(\tau_\nu + \frac{2}{3} \right)$$

↳ from the Eddington approximation (look it up!)

Limb Darkening

Gray atmosphere → $\frac{dI}{d\tau} = I - S$ (not τ_ν)

Now, multiply by $e^{-\tau}$

$$e^{-\tau} \frac{dI}{d\tau} = I e^{-\tau} - S e^{-\tau}$$

$$\Rightarrow \frac{d}{d\tau}\left(e^{-\tau} I\right) = - S e^{-\tau}$$

Integrate from some point O inside the atmosphere, where $\tau = \tau_0$ and $I = I_0$, to the surface, where $\tau = 0$ and $I = I$

$$\Rightarrow \quad I - \cancel{I_0 e^{-\tau_0}} = -\int_{\tau_0}^{0} S e^{-\tau} d\tau$$

Now, back to vertical optical depth:

$$\tau \to \tau_v \sec\theta$$

and take the initial position of light rays to be at $\tau_v = \infty$.

$$\Rightarrow \quad I = \int_0^\infty S e^{-\tau_v \sec\theta} \sec\theta \, d\tau_v$$

Now use $S = \langle I \rangle$

$$= \frac{3\sigma}{4\pi} T_e^4 \left(\tau_v + \frac{2}{3}\right)$$

and integrate:

$$I = a + b\cos\theta \quad , \quad a = \frac{\sigma}{2\pi} T_e^4$$

$$b = \frac{3\sigma}{4\pi} T_e^4$$

One last thing: it's customary to normalize this w.r.t. the total intensity:

$$I_{tot} = \int I \, d\Omega = \int_0^{2\pi} \int_0^{\pi} (a + b\cos\theta) \sin\theta \, d\theta \, d\phi$$

$$= 2a$$

So $$\boxed{\frac{I}{I_{tot}} = \frac{1}{2} + \frac{3}{4}\cos\theta.}$$