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The Fokker-Planck equation for weakly anisotropic functions

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Fun algebra

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I. EXPANSION OF THE FOKKER-PLANCK EQUATION

The Fokker-Planck equation can be written as in Eq. (7-55) in Ref.[1]:

$$\frac{1}{Y}\left(\frac{\delta f}{\delta t}\right) = 4\pi\frac{m}{M}Ff + \left(\frac{M-m}{M+m}\right)\overrightarrow{\nabla}\mathbb{H}(F)\cdot\overrightarrow{\nabla}f + \frac{\overrightarrow{\nabla}\overrightarrow{\nabla}\mathbb{G}(F) : \overrightarrow{\nabla}\overrightarrow{\nabla}f}{2}$$

where F is the distribution function and M the mass of the scatterer. The constant Y is given from Eq. (7-44c):

$$Y = 4\pi\left(\frac{ZZ'e^2}{m}\right)^2 \ln\Lambda$$

and $\mathbb{H}, \mathbb{G}$ the Rosenbluth potentials Eqs. (7-45) – (7-46) are integral operators for F. We expand the distribution functions of F, f to an isotropic part F_0, f_0 and an anisotropic perturbation $\widetilde{F}_a, \widetilde{f}_a$: $F = F_0 + \widetilde{F}_a, f = f_0 + \widetilde{f}_a$.

The zeroth order equation is Eq. (7-67):

$$\frac{1}{Y}\left(\frac{\delta f_0}{\delta t}\right) = 4\pi\frac{m}{M}F_0 f_0 + \left(\frac{M-m}{M+m}\right)\overrightarrow{\nabla}\mathbb{H}(F_0)\cdot\overrightarrow{\nabla}f_0 + \frac{\overrightarrow{\nabla}\overrightarrow{\nabla}\mathbb{G}(F_0) : \overrightarrow{\nabla}\overrightarrow{\nabla}f_0}{2} \tag{1}$$

The first order equation becomes:

$$\begin{aligned}\frac{1}{Y}\left(\frac{\delta \widetilde{f}_a}{\delta t}\right) &= 4\pi\frac{m}{M}[F_0\widetilde{f}_a + f_0\widetilde{F}_a] + \left(\frac{M-m}{M+m}\right)[\overrightarrow{\nabla}\mathbb{H}(F_0)\cdot\overrightarrow{\nabla}\widetilde{f}_a + \overrightarrow{\nabla}f_0\cdot\overrightarrow{\nabla}\mathbb{H}(\widetilde{F}_a)] \\ &\quad + \frac{\overrightarrow{\nabla}\overrightarrow{\nabla}\mathbb{G}(F_0) : \overrightarrow{\nabla}\overrightarrow{\nabla}\widetilde{f}_a}{2} + \frac{\overrightarrow{\nabla}\overrightarrow{\nabla}f_0 : \overrightarrow{\nabla}\overrightarrow{\nabla}\mathbb{G}(\widetilde{F}_a)}{2}\end{aligned} \tag{2}$$

II. SPHERICAL HARMONICS

The natural choice for this problem is the spherical harmonics, where the zeroth harmonic corresponds to the isotropic part of the distribution. There's more than one equivalent ways of writing the expansion.

$$\begin{aligned} Y_{\ell ms} &= P_\ell^m(\cos\theta)(\delta_{0s}\cos m\phi + \delta_{1s}\sin m\phi), \qquad f = f_{\ell ms}Y_{\ell ms} \qquad \text{with the Einstein summation convention.} \\ f &= f_0 + \overrightarrow{f}_1\cdot\frac{\overrightarrow{v}}{v} + \overleftrightarrow{f}_2 : \frac{\overrightarrow{v}\,\overrightarrow{v}}{v^2} + \ldots \\ f &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} f_{\ell m}(v)P_\ell^{|m|}(\cos\theta)e^{im\phi} \end{aligned} \tag{3}$$

And identically for the scatterer (F). The same expansions can be used for the Rosenbluth potentials. To simplify the expressions of the potentials Shkarofsky defines the integrals:

$$\begin{aligned}\overleftrightarrow{I}^{\,i}_{j} &= \frac{4\pi}{v^j}\int_0^v \overleftrightarrow{F}_i V^{j+2}dV \\ \overleftrightarrow{J}^{\,i}_{j} &= \frac{4\pi}{v^j}\int_v^\infty \overleftrightarrow{F}_i V^{j+2}dV\end{aligned}$$

These tensor expressions are then multiplied by the tensor $\overrightarrow{v}^\ell/v^\ell$ to yield the scalar Rosenbluth potentials. Here we instead define the scalar integrals:

$$I_j^{\ell m} = \frac{4\pi}{v^j}\int_0^v F_{\ell m} V^{j+2} dV, \qquad \frac{\partial}{\partial v} I_j^{\ell m} = 4\pi v^2 F_{\ell m} - \frac{j}{v} I_j^{\ell m} \tag{4}$$

$$J_j^{\ell m} = \frac{4\pi}{v^j}\int_v^\infty F_{\ell m} V^{j+2} dV, \qquad \frac{\partial}{\partial v} J_j^{\ell m} = -4\pi v^2 F_{\ell m} - \frac{j}{v} J_j^{\ell m} \tag{5}$$

and the integrals in Shkarofsky can be recovered by summation: $\overleftrightarrow{I}_j^\ell = \sum_{m=-\ell}^{\ell} I_j^{\ell m} P_\ell^{|m|}(\cos\theta)e^{im\phi}$, $\overleftrightarrow{J}_j^\ell = \sum_{m=-\ell}^{\ell} J_j^{\ell m} P_\ell^{|m|}(\cos\theta)e^{im\phi}$. We may now write the expressions for the Rosenbluth potentials in either of these two alternative ways:

$$\begin{aligned}\mathbb{H}(F) &= \frac{M+m}{Mv}\sum_{\ell=0}^{\infty}\left(\overleftrightarrow{I}_\ell^\ell + \overleftrightarrow{J}_{-1-\ell}^\ell\right)(\cdot\ell)\frac{\overrightarrow{v}^\ell}{(2\ell+1)v^\ell} \\ &\equiv \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\underbrace{P_\ell^{|m|}(\cos\theta)e^{im\phi}}_{Y_{\ell m}} \times \underbrace{\left\{\frac{M+m}{M}\frac{1/v}{2\ell+1}\left[I_\ell^{\ell m} + J_{-1-\ell}^{\ell m}\right]\right\}}_{\mathbb{H}_{\ell m}(v)}\end{aligned} \tag{6}$$

$$\begin{aligned}\mathbb{G}(F) &= v\sum_{\ell=0}^{\infty}\left(\frac{\overleftrightarrow{I}_{\ell+2}^\ell + \overleftrightarrow{J}_{-1-\ell}^\ell}{2\ell+3} - \frac{\overleftrightarrow{I}_\ell^\ell + \overleftrightarrow{J}_{1-\ell}^\ell}{2\ell-1}\right)(\cdot\ell)\frac{\overrightarrow{v}^\ell}{(2\ell+1)v^\ell} \\ &\equiv \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\underbrace{P_\ell^{|m|}(\cos\theta)e^{im\phi}}_{Y_{\ell m}} \times \underbrace{\left\{\frac{v}{2\ell+1}\left[\frac{I_{\ell+2}^{\ell m} + J_{-1-\ell}^{\ell m}}{2\ell+3} - \frac{I_\ell^{\ell m} + J_{1-\ell}^{\ell m}}{2\ell-1}\right]\right\}}_{\mathbb{G}_{\ell m}(v)}\end{aligned} \tag{7}$$

We may now rewrite the linearized Fokker-Planck equation as:

$$\begin{aligned}\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\left\{\frac{1}{Y}\left(\frac{\delta f_{\ell m}Y_{\ell m}}{\delta t}\right)\right. &= 4\pi\frac{m}{M}[F_0 f_{\ell m}Y_{\ell m} + f_0 F_{\ell m}Y_{\ell m}] + \left(\frac{M-m}{M+m}\right)[\overrightarrow{\nabla}\mathbb{H}_0\cdot\overrightarrow{\nabla}f_{\ell m}Y_{\ell m} + \overrightarrow{\nabla}f_0\cdot\overrightarrow{\nabla}\mathbb{H}_{\ell m}Y_{\ell m}] \\ &\left. + \frac{\overrightarrow{\nabla}\overrightarrow{\nabla}\mathbb{G}_0 : \overrightarrow{\nabla}\overrightarrow{\nabla}f_{\ell m}Y_{\ell m}}{2} + \frac{\overrightarrow{\nabla}\overrightarrow{\nabla}f_0 : \overrightarrow{\nabla}\overrightarrow{\nabla}\mathbb{G}_{\ell m}Y_{\ell m}}{2}\right\}\end{aligned} \tag{8}$$

where the summation can be dropped, however the $Y_{\ell m}$ cannot be cancelled until the effect of the operators on it has been determined.

III. EQUATIONS FOR THE ISOTROPIC PART OF THE DISTRIBUTION

The zeroth order equation has been solved in its entirety in Ref.[1]. Below we quote the following results

$$\overrightarrow{\nabla}f_0 = \frac{1}{v}\frac{\partial f_0}{\partial v}\overrightarrow{v} \tag{9}$$

$$\overrightarrow{\nabla}\overrightarrow{\nabla}f_0 = \left(\frac{\partial^2 f_0}{\partial v^2} - \frac{1}{v}\frac{\partial f_0}{\partial v}\right)\frac{\overrightarrow{v}\,\overrightarrow{v}}{v^2} + \frac{1}{v}\frac{\partial f_0}{\partial v}\overleftrightarrow{I} \tag{10}$$

$$\overrightarrow{\nabla}\mathbb{H}_0 = -\frac{(M+m)I_0^0}{Mv^3}\overrightarrow{v} \tag{11}$$

$$\overrightarrow{\nabla}\overrightarrow{\nabla}\mathbb{G}_0 = \frac{I_2^0 - I_0^0}{v^3}\overrightarrow{v}\,\overrightarrow{v} + \frac{-I_2^0 + 2J_{-1}^0 + 3I_0^0}{3v}\overleftrightarrow{I} \tag{12}$$

And the zeroth order equation becomes:

$$\frac{1}{Y}\left(\frac{\delta f}{\delta t}\right) = \frac{1}{3v}\frac{\partial^2 f_0}{\partial v^2}(I_2^0 + J_{-1}^0) + \frac{1}{3v^2}\frac{\partial f_0}{\partial v}\left(2J_{-1}^0 - I_2^0 + 3I_0^0\frac{m}{M}\right) + 4\pi\frac{m}{M}f_0F_0 \tag{13}$$

$$= \frac{1}{3v^2}\frac{\partial}{\partial v}\left[f_0\frac{3m}{M}I_0^0 + v(I_2^0 + J_{-1}^0)\frac{\partial f_0}{\partial v}\right] \tag{14}$$

IV. OPERATORS

For the first order equation we first note:

$$\begin{aligned}
\overrightarrow{v}\,\overrightarrow{v} : \overrightarrow{\nabla}\overrightarrow{\nabla} &= v^2 \frac{\partial^2}{\partial v^2} \\
\overleftrightarrow{I} : \overrightarrow{\nabla}\overrightarrow{\nabla} &= \nabla^2
\end{aligned}$$

so we obtain:

$$\begin{aligned}
\overrightarrow{\nabla} f_0 \cdot \overrightarrow{\nabla} &= \frac{\partial f_0}{\partial v}\frac{\partial}{\partial v} \\
\overrightarrow{\nabla}\overrightarrow{\nabla} f_0 : \overrightarrow{\nabla}\overrightarrow{\nabla} &= \left(\frac{\partial^2 f_0}{\partial v^2} - \frac{1}{v}\frac{\partial f_0}{\partial v}\right)\frac{\partial^2}{\partial v^2} + \frac{1}{v}\frac{\partial f_0}{\partial v}\nabla^2 \\
\overrightarrow{\nabla}\mathbb{H}_0 \cdot \overrightarrow{\nabla} &= -\frac{(M+m)I_0^0}{Mv^2}\frac{\partial}{\partial v} \\
\overrightarrow{\nabla}\overrightarrow{\nabla}\mathbb{G}_0 : \overrightarrow{\nabla}\overrightarrow{\nabla} &= \frac{I_2^0 - I_0^0}{v}\frac{\partial^2}{\partial v^2} + \frac{-I_2^0 + 2J_{-1}^0 + 3I_0^0}{3v}\nabla^2
\end{aligned}$$

The only operator that affects the spherical harmonics is ∇^2 where we can write:

$$\nabla^2 Y_{\ell m} f_{\ell m} = \left[\frac{\partial^2}{\partial v^2} + \frac{2}{v}\frac{\partial}{\partial v} - \frac{\ell(\ell+1)}{v^2}\right] Y_{\ell m} f_{\ell m}$$

So we can substitute for the operators above:

$$\begin{aligned}
\overrightarrow{\nabla} f_0 \cdot \overrightarrow{\nabla} &= \frac{\partial f_0}{\partial v}\frac{\partial}{\partial v} \\
\overrightarrow{\nabla}\overrightarrow{\nabla} f_0 : \overrightarrow{\nabla}\overrightarrow{\nabla} &= \left(\frac{\partial^2 f_0}{\partial v^2} - \frac{1}{v}\frac{\partial f_0}{\partial v}\right)\frac{\partial^2}{\partial v^2} + \frac{1}{v}\frac{\partial f_0}{\partial v}\left[\frac{\partial^2}{\partial v^2} + \frac{2}{v}\frac{\partial}{\partial v} - \frac{\ell(\ell+1)}{v^2}\right] \\
&= \frac{\partial^2 f_0}{\partial v^2}\frac{\partial^2}{\partial v^2} + \frac{2}{v^2}\frac{\partial f_0}{\partial v}\frac{\partial}{\partial v} - \frac{\ell(\ell+1)}{v^3}\frac{\partial f_0}{\partial v} \\
\overrightarrow{\nabla}\mathbb{H}_0 \cdot \overrightarrow{\nabla} &= -\frac{(M+m)I_0^0}{Mv^2}\frac{\partial}{\partial v} \\
\overrightarrow{\nabla}\overrightarrow{\nabla}\mathbb{G}_0 : \overrightarrow{\nabla}\overrightarrow{\nabla} &= \frac{I_2^0 - I_0^0}{v}\frac{\partial^2}{\partial v^2} + \frac{-I_2^0 + 2J_{-1}^0 + 3I_0^0}{3v}\left[\frac{\partial^2}{\partial v^2} + \frac{2}{v}\frac{\partial}{\partial v} - \frac{\ell(\ell+1)}{v^2}\right] \\
&= \frac{2(I_2^0 + J_{-1}^0)}{3v}\frac{\partial^2}{\partial v^2} + \frac{2(-I_2^0 + 2J_{-1}^0 + 3I_0^0)}{3v^2}\frac{\partial}{\partial v} - \ell(\ell+1) \times \frac{(-I_2^0 + 2J_{-1}^0 + 3I_0^0)}{3v^3}
\end{aligned}$$

And finally, multiplying with the appropriate factors:

$$\begin{aligned}
\left(\frac{M-m}{M+m}\right)\overrightarrow{\nabla} f_0 \cdot \overrightarrow{\nabla} &= \left(\frac{M-m}{M+m}\right)\frac{\partial f_0}{\partial v}\frac{\partial}{\partial v} \\
\frac{\overrightarrow{\nabla}\overrightarrow{\nabla} f_0 : \overrightarrow{\nabla}\overrightarrow{\nabla}}{2} &= \frac{1}{2}\frac{\partial^2 f_0}{\partial v^2}\frac{\partial^2}{\partial v^2} + \frac{1}{v^2}\frac{\partial f_0}{\partial v}\frac{\partial}{\partial v} - \frac{\ell(\ell+1)}{2v^3}\frac{\partial f_0}{\partial v} \\
\left(\frac{M-m}{M+m}\right)\overrightarrow{\nabla}\mathbb{H}_0 \cdot \overrightarrow{\nabla} &= -\frac{(M-m)I_0^0}{Mv^2}\frac{\partial}{\partial v} \\
\frac{\overrightarrow{\nabla}\overrightarrow{\nabla}\mathbb{G}_0 : \overrightarrow{\nabla}\overrightarrow{\nabla}}{2} &= \frac{(I_2^0 + J_{-1}^0)}{3v}\frac{\partial^2}{\partial v^2} + \frac{(-I_2^0 + 2J_{-1}^0 + 3I_0^0)}{3v^2}\frac{\partial}{\partial v} - \frac{\ell(\ell+1)}{2} \times \frac{(-I_2^0 + 2J_{-1}^0 + 3I_0^0)}{3v^3}
\end{aligned}$$

V. DERIVATIVES OF THE ROSENBLUTH POTENTIALS

A. Derivatives of $\mathbb{H}_{\ell m}$

$$\begin{aligned}\frac{\partial}{\partial \upsilon}\mathbb{H}_{\ell m} &= \frac{M+m}{M}\frac{\partial}{\partial \upsilon}\left\{\frac{1/\upsilon}{2\ell+1}\left[I_\ell^{\ell m}+J_{-1-\ell}^{\ell m}\right]\right\} \\ &= \frac{M+m}{M}\frac{1/\upsilon}{2\ell+1}\left\{-\frac{1}{\upsilon}\left[I_\ell^{\ell m}+J_{-1-\ell}^{\ell m}\right]+\left[(I_\ell^{\ell m})'+(J_{-1-\ell}^{\ell m})'\right]\right\}\end{aligned}$$

For the second term in this equation:

$$\left.\begin{aligned}(I_\ell^{\ell m})' = 4\pi\upsilon^2 F_{\ell m} - \tfrac{\ell}{\upsilon}I_\ell^{\ell m} \\ (J_{-1-\ell}^{\ell m})' = -4\pi\upsilon^2 F_{\ell m} - \tfrac{-\ell-1}{\upsilon}J_{-\ell-1}^{\ell m}\end{aligned}\right\} \Rightarrow \upsilon\Big[(I_{\ell+2}^{\ell m})'+(J_{-1-\ell}^{\ell m})'\Big] = -\ell I_\ell^{\ell m}+(\ell+1)J_{-\ell-1}^{\ell m}$$

Substituting above:

$$\begin{aligned}\frac{\partial}{\partial \upsilon}\mathbb{H}_{\ell m} &= \frac{M+m}{M}\frac{1/\upsilon^2}{2\ell+1}\left\{-\left[I_\ell^{\ell m}+J_{-1-\ell}^{\ell m}\right]+\left[-\ell I_\ell^{\ell m}+(\ell+1)J_{-1-\ell}^{\ell m}\right]\right\} \\ &= -\frac{1}{\upsilon^2}\frac{M+m}{M}\frac{1}{2\ell+1}\left[(\ell+1)I_\ell^{\ell m}-\ell J_{-1-\ell}^{\ell m}\right]\end{aligned} \tag{15}$$

B. Derivatives of $\mathbb{G}_{\ell m}$

For the first derivative:

$$\begin{aligned}\frac{\partial}{\partial \upsilon}\mathbb{G}_{\ell m} &= \frac{\partial}{\partial \upsilon}\left\{\frac{\upsilon}{2\ell+1}\left[\frac{I_{\ell+2}^{\ell m}+J_{-1-\ell}^{\ell m}}{2\ell+3}-\frac{I_\ell^{\ell m}+J_{1-\ell}^{\ell m}}{2\ell-1}\right]\right\} \\ &= \frac{1}{2\ell+1}\left[\frac{I_{\ell+2}^{\ell m}+J_{-1-\ell}^{\ell m}}{2\ell+3}-\frac{I_\ell^{\ell m}+J_{1-\ell}^{\ell m}}{2\ell-1}\right]+\frac{\upsilon}{2\ell+1}\left[\frac{(I_{\ell+2}^{\ell m})'+(J_{-1-\ell}^{\ell m})'}{2\ell+3}-\frac{(I_\ell^{\ell m})'+(J_{1-\ell}^{\ell m})'}{2\ell-1}\right]\end{aligned}$$

For the second term in this equation:

$$\left.\begin{aligned}(I_{\ell+2}^{\ell m})' = 4\pi\upsilon^2 F_{\ell m} - \tfrac{\ell+2}{\upsilon}I_{\ell+2}^{\ell m} \\ (J_{-1-\ell}^{\ell m})' = -4\pi\upsilon^2 F_{\ell m} - \tfrac{-\ell-1}{\upsilon}J_{-\ell-1}^{\ell m}\end{aligned}\right\} \Rightarrow \upsilon\Big[(I_{\ell+2}^{\ell m})'+(J_{-1-\ell}^{\ell m})'\Big] = -(\ell+2)I_{\ell+2}^{\ell m}+(\ell+1)J_{-\ell-1}^{\ell m}$$

$$\left.\begin{aligned}(I_\ell^{\ell m})' = 4\pi\upsilon^2 F_{\ell m} - \tfrac{\ell}{\upsilon}I_\ell^{\ell m} \\ (J_{1-\ell}^{\ell m})' = -4\pi\upsilon^2 F_{\ell m} - \tfrac{(1-\ell)}{\upsilon}J_{1-\ell}^{\ell m}\end{aligned}\right\} \Rightarrow \upsilon\Big[(I_\ell^{\ell m})'+(J_{1-\ell}^{\ell m})'\Big] = -\ell I_\ell^{\ell m}+(\ell-1)J_{1-\ell}^{\ell m}$$

We then have:

$$\begin{aligned}\frac{\partial}{\partial \upsilon}\mathbb{G}_{\ell m} &= \frac{1}{2\ell+1}\left[\frac{I_{\ell+2}^{\ell m}+J_{-1-\ell}^{\ell m}}{2\ell+3}-\frac{I_\ell^{\ell m}+J_{1-\ell}^{\ell m}}{2\ell-1}+\frac{-(\ell+2)I_{\ell+2}^{\ell m}+(\ell+1)J_{-\ell-1}^{\ell m}}{2\ell+3}-\frac{-\ell I_\ell^{\ell m}+(\ell-1)J_{1-\ell}^{\ell m}}{2\ell-1}\right] \\ &= \frac{1}{2\ell+1}\left[\frac{-(\ell+1)I_{\ell+2}^{\ell m}+(\ell+2)J_{-\ell-1}^{\ell m}}{2\ell+3}-\frac{(1-\ell)I_\ell^{\ell m}+\ell J_{1-\ell}^{\ell m}}{2\ell-1}\right]\end{aligned} \tag{16}$$

For the second derivative:

$$\frac{\partial^2}{\partial \upsilon^2}\mathbb{G}_{\ell m} = \frac{1}{2\ell+1}\left[\frac{-(\ell+1)(I_{\ell+2}^{\ell m})'+(\ell+2)(J_{-\ell-1}^{\ell m})'}{2\ell+3}-\frac{(1-\ell)(I_\ell^{\ell m})'+\ell(J_{1-\ell}^{\ell m})'}{2\ell-1}\right]$$

Where:

$$\begin{aligned}\upsilon\Big[-(\ell+1)(I_{\ell+2}^{\ell m})'+(\ell+2)(J_{-1-\ell}^{\ell m})'\Big] &= -(2\ell+3)4\pi\upsilon^3 F_{\ell m}+(\ell+1)(\ell+2)[I_{\ell+2}^{\ell m}+J_{-\ell-1}^{\ell m}] \\ \upsilon\Big[(1-\ell)(I_\ell^{\ell m})'+\ell(J_{1-\ell}^{\ell m})'\Big] &= -(2\ell-1)4\pi\upsilon^3 F_{\ell m}+(\ell-1)\ell[I_\ell^{\ell m}+J_{1-\ell}^{\ell m}]\end{aligned}$$

Substituting these two equations it is obvious that the terms containing $F_{\ell m}$ cancel. We then obtain:

$$\frac{\partial^2}{\partial \upsilon^2}\mathbb{G}_{\ell m} = \frac{1/\upsilon}{2\ell+1}\left[\frac{(\ell+1)(\ell+2)}{2\ell+3}\Big(I_{\ell+2}^{\ell m}+J_{-\ell-1}^{\ell m}\Big)-\frac{(\ell-1)\ell}{2\ell-1}\Big(I_\ell^{\ell m}+J_{1-\ell}^{\ell m}\Big)\right] \tag{17}$$

VI. FINAL EQUATIONS

Applying the operators on $Y_{\ell m}f_{\ell m}, Y_{\ell m}\mathbb{H}_{\ell m}, Y_{\ell m}\mathbb{G}_{\ell m}$ and using the derivatives above we obtain:

$$\left(\frac{M-m}{M+m}\right)\overrightarrow{\nabla} f_0 \cdot \overrightarrow{\nabla} Y_{\ell m}\mathbb{H}_{\ell m} = \left(\frac{M-m}{M+m}\right)\frac{\partial f_0}{\partial v}\frac{\partial}{\partial v}Y_{\ell m}\mathbb{H}_{\ell m}$$

$$= -\left(\frac{M-m}{Mv^2}\right)\frac{\partial f_0}{\partial v} \times \left\{\frac{\ell+1}{2\ell+1}I_{\ell}^{\ell m} - \frac{\ell}{2\ell+1}J_{-1-\ell}^{\ell m}\right\} \times Y_{\ell m} \tag{18}$$

$$\frac{\overrightarrow{\nabla}\overrightarrow{\nabla} f_0 : \overrightarrow{\nabla}\overrightarrow{\nabla}}{2} Y_{\ell m}\mathbb{G}_{\ell m} = \frac{1}{2}\frac{\partial^2 f_0}{\partial v^2}\frac{\partial^2}{\partial v^2}Y_{\ell m}\mathbb{G}_{\ell m} + \frac{1}{v^2}\frac{\partial f_0}{\partial v}\frac{\partial}{\partial v}Y_{\ell m}\mathbb{G}_{\ell m} - \frac{\ell(\ell+1)}{2v^3}\frac{\partial f_0}{\partial v}Y_{\ell m}\mathbb{G}_{\ell m}$$

$$= \frac{1}{2v}\frac{\partial^2 f_0}{\partial v^2}Y_{\ell m}\left[\frac{(\ell+1)(\ell+2)}{(2\ell+1)(2\ell+3)}\left(I_{\ell+2}^{\ell m} + J_{-\ell-1}^{\ell m}\right) - \frac{(\ell-1)\ell}{(2\ell+1)(2\ell-1)}\left(I_{\ell}^{\ell m} + J_{1-\ell}^{\ell m}\right)\right]$$

$$+\frac{1}{v^2}\frac{\partial f_0}{\partial v}Y_{\ell m}\left[\frac{-(\ell+1)I_{\ell+2}^{\ell m} + (\ell+2)J_{-\ell-1}^{\ell m}}{(2\ell+1)(2\ell+3)} - \frac{(1-\ell)I_{\ell}^{\ell m} + \ell J_{1-\ell}^{\ell m}}{(2\ell+1)(2\ell-1)}\right]$$

$$-\frac{\ell(\ell+1)}{2v^2}\frac{\partial f_0}{\partial v}Y_{\ell m}\left[\frac{I_{\ell+2}^{\ell m} + J_{-1-\ell}^{\ell m}}{(2\ell+1)(2\ell+3)} - \frac{I_{\ell}^{\ell m} + J_{1-\ell}^{\ell m}}{(2\ell+1)(2\ell-1)}\right] \tag{19}$$

$$\left(\frac{M-m}{M+m}\right)\overrightarrow{\nabla}\mathbb{H}_0 \cdot \overrightarrow{\nabla} f_{\ell m}Y_{\ell m} = -\left(\frac{M-m}{Mv^2}\right)I_0^0\frac{\partial f_{\ell m}}{\partial v} \times Y_{\ell m} \tag{20}$$

$$\frac{\overrightarrow{\nabla}\overrightarrow{\nabla}\mathbb{G}_0 : \overrightarrow{\nabla}\overrightarrow{\nabla}}{2} f_{\ell m}Y_{\ell m} =$$

$$= Y_{\ell m}\left\{\frac{(I_2^0 + J_{-1}^0)}{3v}\frac{\partial^2 f_{\ell m}}{\partial v^2} + \frac{(-I_2^0 + 2J_{-1}^0 + 3I_0^0)}{3v^2}\frac{\partial f_{\ell m}}{\partial v} - \frac{\ell(\ell+1)}{2} \times \frac{(-I_2^0 + 2J_{-1}^0 + 3I_0^0)}{3v^3}f_{\ell m}\right\} \tag{21}$$

We may now substitute Eq. (18)-(21) into the linearized Fokker-Planck equation:

$$\begin{aligned}
\frac{1}{Y}\left(\frac{\delta f_{\ell m}}{\delta t}\right) \;=\;& 4\pi\frac{m}{M}[F_0 f_{\ell m} + f_0 F_{\ell m}] + \\
& -\frac{(M-m)}{Mv^2}\left\{\frac{\partial f_0}{\partial v}\left[\frac{\ell+1}{2\ell+1}I_{\ell}^{\ell m} - \frac{\ell}{2\ell+1}J_{-1-\ell}^{\ell m}\right] + I_0^0\frac{\partial f_{\ell m}}{\partial v}\right\} \\
& +\frac{(I_2^0 + J_{-1}^0)}{3v}\frac{\partial^2 f_{\ell m}}{\partial v^2} + \frac{(-I_2^0 + 2J_{-1}^0 + 3I_0^0)}{3v^2}\frac{\partial f_{\ell m}}{\partial v} \\
& -\frac{\ell(\ell+1)}{2} \times \frac{(-I_2^0 + 2J_{-1}^0 + 3I_0^0)}{3v^3}f_{\ell m} \\
& +\frac{1}{2v}\frac{\partial^2 f_0}{\partial v^2}\left[\frac{(\ell+1)(\ell+2)}{(2\ell+1)(2\ell+3)}\left(I_{\ell+2}^{\ell m} + J_{-\ell-1}^{\ell m}\right) - \frac{(\ell-1)\ell}{(2\ell+1)(2\ell-1)}\left(I_{\ell}^{\ell m} + J_{1-\ell}^{\ell m}\right)\right] \\
& +\frac{1}{v^2}\frac{\partial f_0}{\partial v}\left[\frac{-\ell(\ell+1)/2 - (\ell+1)}{(2\ell+1)(2\ell+3)}I_{\ell+2}^{\ell m} + \frac{-\ell(\ell+1)/2 + (\ell+2)}{(2\ell+1)(2\ell+3)}J_{-\ell-1}^{\ell m}\right. \\
& \qquad \left. + \frac{\ell(\ell+1)/2 + (\ell-1)}{(2\ell+1)(2\ell-1)}I_{\ell}^{\ell m} + \frac{\ell(\ell+1)/2 - \ell}{(2\ell+1)(2\ell-1)}J_{1-\ell}^{\ell m}\right]
\end{aligned} \tag{22}$$

Notice that for $\ell = 1$ this equation reduces to Eq. (7-75) in Shkarofsky. For $\ell = 2$ it reduces to Eq. (7-77) in the same reference.

VII. OBSERVATIONS

A. Observations on Eq. (22)

Let us consider each line in Eq. (22) separately:

- For the first two terms:

$$4\pi\frac{m}{M}[F_0 f_{\ell m} + f_0 F_{\ell m}]$$

we note that for $\ell > 0$ the nonzero harmonics vanish in the vicinity of $\upsilon \simeq 0$. However, it is exactly in this region where the collisional effects are expected to be most pronounced. This term can become important if the scatterer (e.g. electrons) has much smaller mass than the scattering particles (e.g. ions).

- The second line in Eq. (22) identifies the additional terms that need be included if the scatterer has different mass than the scattering particle. This term will also become most important for the scattering of massive particles off of much lighter ones. This is to be expected since this term comes from the Rosenbluth potential $\mathbb{H}$ that is associated with the friction coefficient.

- The third line (along with the first term in the first line), yields the same results as the right-hand-side of the equation below (see Ref.[7]):

$$\left(\frac{\delta f_{\ell m}}{\delta t}\right) = \frac{1}{\upsilon^2}\frac{\partial}{\partial \upsilon}\left[\upsilon^2 \frac{D_1(f_0)}{2}\frac{\partial}{\partial \upsilon} f_\ell^m + \upsilon^2 C(f_0) f_\ell^m\right]$$

- This last term is the only term that is not of order $O(1)$ with respect to the order of the harmonic ℓ. It is in fact of order $O(\ell^2)$ and it is clearly the only term that matters for high order harmonics. This is the major term for scattering of electrons off of cold ions. To see that let us assume cold ions in which case $F(\upsilon)$ shall be substituted by the appropriate spherical delta function and all integrals other than I_0^0 vanish. The equation for $\ell > 0$ then becomes:

$$\frac{1}{Y_{ei}}\left(\frac{\delta f_{\ell m}}{\delta t}\right)_{ei} \simeq -\frac{\ell(\ell+1)}{2} \times \frac{n_i}{\upsilon^3} f_{\ell m}$$

  Where the approximate sign comes about from the fact that we are neglecting the effect of $F_0 f_{\ell m}$, that is to say F_0 is not *really* a delta function.

- The only major comment concerning the term with $\frac{\partial^2 f_0}{\partial \upsilon^2}$ is that this term can be substantial for low order harmonics, but it can be dropped for higher order ones.

- The same comment as above holds for the last two lines as well, with the additional caveat that $\partial f_0 / \partial \upsilon$ is small in the region where collisions are important (that is for small υ).

B. Numerical method

A numerical method may now be suggested:

- We first solve the nonlinear equation for the isotropic part of the distribution f_0 and advance to the timestep $i + 1$.

- We use this $f_0^{(i+1)}$ to calculate the parameters for the linearlized Fokker-Planck equation. The resulting arrays can be inverted to evaluate $f_{\ell m}^{(i+1)}$.

- Because the matrix will be diagonally dominant the Gauss-Seidel method is a reasonable choice. Notice that the Gauss-Seidel algorithm doe not parallelize wewll but we are not concerned with making it parallel, so it is ok.

- For high order harmonics no matrices need to be inverted as the only term that matters is the *scattering term*, and we can therefore just include the damping rate for these harmonics.

VIII. NUMERICAL METHOD FOR ELECTRON-ELECTRON COLLISIONS

For electron-electron collisions we can set $M = m$ and $F_0 = f_0, F_{\ell m} = f_{\ell m}$. Finally, let us denote $f \equiv f_0, I_j \equiv I_j^0, J_j \equiv J_j^0$ and $\boldsymbol{f} = f^{\ell m}, \boldsymbol{I}_j \equiv I_j^{\ell m}, \boldsymbol{J}_j \equiv J_j^{\ell m}$ so as to simplify the notation. We then have

$$
\begin{aligned}
\frac{1}{Y}\left(\frac{\delta \boldsymbol{f}}{\delta t}\right) = {} & 8\pi f \boldsymbol{f} + \\
& + \frac{(I_2 + J_{-1})}{3v}\frac{\partial^2 \boldsymbol{f}}{\partial v^2} + \frac{(-I_2 + 2J_{-1} + 3I_0)}{3v^2}\frac{\partial \boldsymbol{f}}{\partial v} \\
& - \frac{\ell(\ell+1)}{2} \times \frac{(-I_2 + 2J_{-1} + 3I_0)}{3v^3}\boldsymbol{f} \\
& \frac{1}{2v}\frac{\partial^2 f}{\partial v^2}\left[\frac{(\ell+1)(\ell+2)}{(2\ell+1)(2\ell+3)}\Big(\boldsymbol{I}_{\ell+2} + \boldsymbol{J}_{-\ell-1}\Big) - \frac{(\ell-1)\ell}{(2\ell+1)(2\ell-1)}\Big(\boldsymbol{I}_{\ell} + \boldsymbol{J}_{1-\ell}\Big)\right] \\
& + \frac{1}{v^2}\frac{\partial f}{\partial v}\left[\frac{-\ell(\ell+1)/2 - (\ell+1)}{(2\ell+1)(2\ell+3)}\boldsymbol{I}_{\ell+2} + \frac{-\ell(\ell+1)/2 + (\ell+2)}{(2\ell+1)(2\ell+3)}\boldsymbol{J}_{-\ell-1}\right. \\
& \quad \left. + \frac{\ell(\ell+1)/2 + (\ell-1)}{(2\ell+1)(2\ell-1)}\boldsymbol{I}_{\ell} + \frac{\ell(\ell+1)/2 - \ell}{(2\ell+1)(2\ell-1)}\boldsymbol{J}_{1-\ell}\right]
\end{aligned}
$$

We shall define the following functions of ℓ:

$$
\begin{aligned}
A_1[\ell] &= \frac{(\ell+1)(\ell+2)}{(2\ell+1)(2\ell+3)} \\
A_2[\ell] &= -\frac{(\ell-1)\ell}{(2\ell+1)(2\ell-1)} \qquad \text{where: } A_2[1] = 0 \\
B_1[\ell] &= \frac{-\ell(\ell+1)/2 - (\ell+1)}{(2\ell+1)(2\ell+3)} \\
B_2[\ell] &= \frac{-\ell(\ell+1)/2 + (\ell+2)}{(2\ell+1)(2\ell+3)} \\
B_3[\ell] &= \frac{\ell(\ell+1)/2 + (\ell-1)}{(2\ell+1)(2\ell-1)} \\
B_4[\ell] &= \frac{\ell(\ell+1)/2 - \ell}{(2\ell+1)(2\ell-1)} \qquad \text{where: } B_4[1] = 0
\end{aligned}
$$

So we can write:

$$
\begin{aligned}
\frac{1}{Y}\left(\frac{\delta \boldsymbol{f}}{\delta t}\right) = {} & 8\pi f \boldsymbol{f} + \frac{(I_2 + J_{-1})}{3v}\frac{\partial^2 \boldsymbol{f}}{\partial v^2} + \frac{(-I_2 + 2J_{-1} + 3I_0)}{3v^2}\frac{\partial \boldsymbol{f}}{\partial v} \\
& - \frac{\ell(\ell+1)}{2} \times \frac{(-I_2 + 2J_{-1} + 3I_0)}{3v^3}\boldsymbol{f} \\
& + \frac{1}{2v}\frac{\partial^2 f}{\partial v^2}\left[A_1\Big(\boldsymbol{I}_{\ell+2} + \boldsymbol{J}_{-\ell-1}\Big) + A_2\Big(\boldsymbol{I}_{\ell} + \boldsymbol{J}_{1-\ell}\Big)\right] + \frac{1}{v^2}\frac{\partial f}{\partial v}\Big[B_1\boldsymbol{I}_{\ell+2} + B_2\boldsymbol{J}_{-\ell-1} + B_3\boldsymbol{I}_{\ell} + B_4\boldsymbol{J}_{1-\ell}\Big]
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{Y}\left(\frac{\delta \boldsymbol{f}}{\delta t}\right) = {} & 8\pi f \boldsymbol{f} + \frac{(I_2 + J_{-1})}{3v}\frac{\partial^2 \boldsymbol{f}}{\partial v^2} + \frac{(-I_2 + 2J_{-1} + 3I_0)}{3v^2}\frac{\partial \boldsymbol{f}}{\partial v} \\
& - \frac{\ell(\ell+1)}{2} \times \frac{(-I_2 + 2J_{-1} + 3I_0)}{3v^3}\boldsymbol{f} \\
& + \left[A_1\frac{1}{2v}\frac{\partial^2 f}{\partial v^2} + B_1\frac{1}{v^2}\frac{\partial f}{\partial v}\right] \times \boldsymbol{I}_{\ell+2} \\
& + \left[A_1\frac{1}{2v}\frac{\partial^2 f}{\partial v^2} + B_2\frac{1}{v^2}\frac{\partial f}{\partial v}\right] \times \boldsymbol{J}_{-\ell-1} \\
& + \left[A_2\frac{1}{2v}\frac{\partial^2 f}{\partial v^2} + B_3\frac{1}{v^2}\frac{\partial f}{\partial v}\right] \times \boldsymbol{I}_{\ell} \\
& + \left[A_2\frac{1}{2v}\frac{\partial^2 f}{\partial v^2} + B_4\frac{1}{v^2}\frac{\partial f}{\partial v}\right] \times \boldsymbol{J}_{1-\ell}
\end{aligned}
$$

A. Standard Tridiagonal Terms

In this subsection we will discuss the inclusion of the standard tridiagonal terms to the matrix.
Where we define $\Delta_n = (\upsilon_{n+1} - \upsilon_{n-1})/2$ and $\Delta_{n+\frac{1}{2}} = \upsilon_{n+1} - \upsilon_n$:

$$\begin{aligned}
\frac{\partial \boldsymbol{f}}{\partial \upsilon} &= \frac{\boldsymbol{f}_{n+1} - \boldsymbol{f}_{n-1}}{\upsilon_{n+1} - \upsilon_{n-1}} \\
&= \frac{1}{2\Delta_n}\boldsymbol{f}_{n+1} - \frac{1}{2\Delta_n}\boldsymbol{f}_{n-1} \\
\frac{\partial^2 \boldsymbol{f}}{\partial \upsilon^2} &= \frac{1}{\Delta_n}\left[\frac{\boldsymbol{f}_{n+1} - \boldsymbol{f}_n}{\upsilon_{n+1} - \upsilon_n} - \frac{\boldsymbol{f}_n - \boldsymbol{f}_{n-1}}{\upsilon_n - \upsilon_{n-1}}\right] \\
&= \frac{1}{\Delta_n \Delta_{n+\frac{1}{2}}}\boldsymbol{f}_{n+1} + \left(-\frac{1}{\Delta_n \Delta_{n+\frac{1}{2}}} - \frac{1}{\Delta_n \Delta_{n-\frac{1}{2}}}\right)\boldsymbol{f}_n + \frac{1}{\Delta_n \Delta_{n-\frac{1}{2}}}\boldsymbol{f}_{n-1}
\end{aligned}$$

So that for the tridiagonal terms one obtains:

$$\begin{aligned}
\text{Tridiagonal}[n] &= \left[\frac{(-I_2 + 2J_{-1} + 3I_0)}{3\upsilon^2}\left(-\frac{1}{2\Delta_n}\right) + \frac{(I_2 + J_{-1})}{3\upsilon}\left(\frac{1}{\Delta_n \Delta_{n-\frac{1}{2}}}\right)\right] \times \boldsymbol{f}_{n-1} \\
&+ \left[8\pi f + \frac{(I_2 + J_{-1})}{3\upsilon}\left(-\frac{1}{\Delta_n \Delta_{n+\frac{1}{2}}} - \frac{1}{\Delta_n \Delta_{n-\frac{1}{2}}}\right) - \frac{\ell(\ell+1)}{2} \times \frac{(-I_2 + 2J_{-1} + 3I_0)}{3\upsilon^3}\right] \times \boldsymbol{f}_n \\
&+ \left[\frac{(-I_2 + 2J_{-1} + 3I_0)}{3\upsilon^2}\left(\frac{1}{2\Delta_n}\right) + \frac{(I_2 + J_{-1})}{3\upsilon}\left(\frac{1}{\Delta_n \Delta_{n+\frac{1}{2}}}\right)\right] \times \boldsymbol{f}_{n+1}
\end{aligned}$$

Note that near the zeroth cell the only harmonic that can be nonzero is that for $\ell = 1$:

$$\begin{aligned}
\text{For } \ell = 1 : f(\upsilon \simeq 0) = f(\upsilon_1)\frac{\upsilon}{\upsilon_1} \;&\Rightarrow\; f(\upsilon_0) = f(\upsilon_1)\frac{\upsilon_0}{\upsilon_1} \\
\left.\frac{\partial f}{\partial \upsilon}\right|_{\upsilon \simeq 0} = \frac{f(\upsilon_1)}{\upsilon_1} \;&\Rightarrow\; \left.\frac{\partial f}{\partial \upsilon}\right|_{\upsilon_0} = \frac{f(\upsilon_1)}{\upsilon_1} = \frac{f(\upsilon_0)}{\upsilon_0} \\
\left.\frac{\partial^2 f}{\partial \upsilon^2}\right|_{\upsilon \simeq 0} = 0 &
\end{aligned}$$

Hence at cell "0" only the diagonal term survives:

$$\begin{aligned}
\text{Tridiagonal}[\ell = 1, n = 0] &= \left[8\pi f + \frac{(-I_2 + 2J_{-1} + 3I_0)}{3\upsilon^2}\left(\frac{1}{\upsilon_0}\right) - \frac{\ell(\ell+1)}{2} \times \frac{(-I_2 + 2J_{-1} + 3I_0)}{3\upsilon^3}\right] \times \boldsymbol{f}_0 \\
&= \left[8\pi f_0\right] \times \boldsymbol{f}_0
\end{aligned}$$

Where obviously there is no harm in using the same tridiagonal term for high order harmonics since in that case $8\pi f_0$ is zero anyway. The contribution of the ions must of course be also included.

B. Non-tridiagonal terms

The $\boldsymbol{I}$ integrals can be expressed as:

$$
\begin{aligned}
\boldsymbol{I}_{\ell+2}(\upsilon_n) &= \frac{4\pi}{\upsilon_n^{\ell+2}} \sum_{\kappa=1}^{n} \frac{1}{2}\Big(\boldsymbol{f}_\kappa \upsilon_\kappa^{\ell+4} + \boldsymbol{f}_{\kappa-1}\upsilon_{\kappa-1}^{\ell+4}\Big)\Delta_{\kappa-\frac{1}{2}} \\
&= \frac{4\pi}{\upsilon_n^{\ell+2}}\left(\frac{1}{2}\boldsymbol{f}_0\upsilon_0^{\ell+4}\Delta_{\frac{1}{2}} + \sum_{\kappa=1}^{n-1}\boldsymbol{f}_\kappa\upsilon_\kappa^{\ell+4}\Delta_\kappa + \frac{1}{2}\boldsymbol{f}_n\upsilon_n^{\ell+4}\Delta_{n-\frac{1}{2}}\right) \\
&= \frac{4\pi}{\upsilon_n^{\ell+2}}\left[\frac{1}{2}\upsilon_0^{\ell+4}\Delta_{\frac{1}{2}}, \upsilon_\kappa^{\ell+4}\Delta_\kappa, \frac{1}{2}\upsilon_n^{\ell+4}\Delta_{n-\frac{1}{2}}, 0\right]\cdot\boldsymbol{f} \\
&= 4\pi\left[\frac{1}{2}\left(\frac{\upsilon_0}{\upsilon_n}\right)^{\ell+2}\upsilon_0^2\Delta_{\frac{1}{2}}, \left(\frac{\upsilon_\kappa}{\upsilon_n}\right)^{\ell+2}\upsilon_\kappa^2\Delta_\kappa, \frac{1}{2}\upsilon_n^2\Delta_{n-\frac{1}{2}}, 0\right]\cdot\boldsymbol{f} \\
\boldsymbol{I}_\ell(\upsilon_n) &= 4\pi\left[\frac{1}{2}\left(\frac{\upsilon_0}{\upsilon_n}\right)^{\ell}\upsilon_0^2\Delta_{\frac{1}{2}}, \left(\frac{\upsilon_\kappa}{\upsilon_n}\right)^{\ell}\upsilon_\kappa^2\Delta_\kappa, \frac{1}{2}\upsilon_n^2\Delta_{n-\frac{1}{2}}, 0\right]\cdot\boldsymbol{f}
\end{aligned}
$$

The $\boldsymbol{J}$ integrals can be expressed as:

$$
\begin{aligned}
\boldsymbol{J}_{-\ell-1}(\upsilon_n) &= 4\pi\times\frac{1}{\upsilon_n^{-\ell-1}}\sum_{\kappa=n+1}^{\mathcal{N}}\frac{1}{2}\Big(\boldsymbol{f}_\kappa\upsilon_\kappa^{-\ell+1} + \boldsymbol{f}_{\kappa-1}\upsilon_{\kappa-1}^{-\ell+1}\Big)\Delta_{\kappa-\frac{1}{2}} \\
&= 4\pi\upsilon_n^{\ell+1}\left(\frac{1}{2}\frac{\boldsymbol{f}_n}{\upsilon_n^{\ell+1}}\upsilon_n^2\Delta_{n+\frac{1}{2}} + \sum_{\kappa=n+1}^{\mathcal{N}-1}\frac{\boldsymbol{f}_\kappa}{\upsilon_\kappa^{\ell+1}}\upsilon_\kappa^2\Delta_\kappa + \frac{1}{2}\frac{\boldsymbol{f}_\mathcal{N}}{\upsilon_\mathcal{N}^{\ell+1}}\upsilon_\mathcal{N}^2\Delta_{\mathcal{N}-\frac{1}{2}}\right) \\
&= 4\pi\left[0, \frac{1}{2}\upsilon_n^2\Delta_{n+\frac{1}{2}}, \left(\frac{\upsilon_\kappa}{\upsilon_n}\right)^{-\ell-1}\upsilon_\kappa^2\Delta_\kappa, \frac{1}{2}\left(\frac{\upsilon_\mathcal{N}}{\upsilon_n}\right)^{-\ell-1}\upsilon_\mathcal{N}^2\Delta_{\mathcal{N}-\frac{1}{2}}\right]\cdot\boldsymbol{f} \\
\boldsymbol{J}_{1-\ell}(\upsilon_n) &= 4\pi\left[0, \frac{1}{2}\upsilon_n^2\Delta_{n+\frac{1}{2}}, \left(\frac{\upsilon_\kappa}{\upsilon_n}\right)^{1-\ell}\upsilon_\kappa^2\Delta_\kappa, \frac{1}{2}\left(\frac{\upsilon_\mathcal{N}}{\upsilon_n}\right)^{1-\ell}\upsilon_\mathcal{N}^2\Delta_{\mathcal{N}-\frac{1}{2}}\right]\cdot\boldsymbol{f}
\end{aligned}
$$

It is worth noting that the diagonal terms can be calculated irrespective ℓ, they are $\frac{1}{2}\upsilon_n^2\Delta_{n-\frac{1}{2}}$ for $\boldsymbol{I}$ and .

C. Computational Approach

- Write and test a Gauss-Seidel Algorithm. Put it in a separate file. Specifically call with a lower triangular
- Test the Gauss-Seidel Algorithm for a tridiagonal array
- Create the tridiagonal array as above and import to the Gauss-Seidel algorithm to see what it does. Use the damped result as an initial guess. Notice that it does not seem a big deal to recalculate the tridiagonal terms for each ℓ. So all elements of the array will be recalculated for different ℓ (but not m) this should be fine...
- Try to get the Spitzer Conductivity to see how that works out
- Include the non-tridiagonal Array to see how this works out

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