Modeling Structural and Magnetic Field Stability in Magnetars with Applications to Continuous Gravitational Wave Production

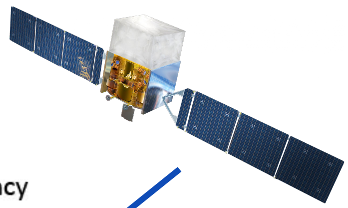
Sam Frederick

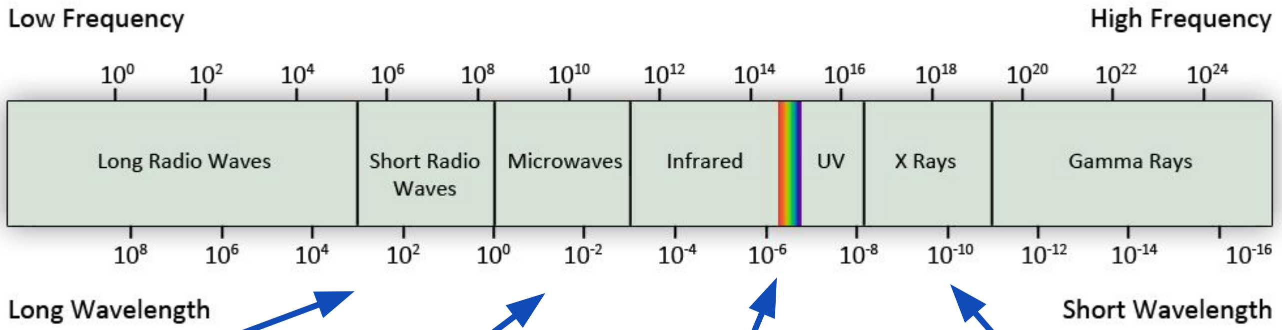
Ways of Observing

- ➔ Astronomy & Astrophysics: an ***observational*** *science*
- ➔ **Light**: The electromagnetic spectrum

Fermi Telescope

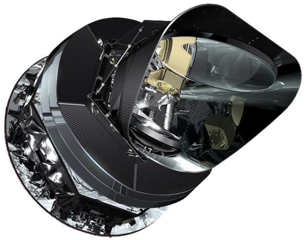
Image credit: NASA





Very Large Array (VLA)

Image credit: NRAO

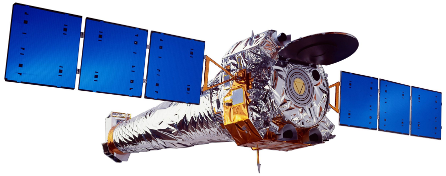


Planck Telescope

Image credit: NASA

Hubble Telescope

Image credit: NASA



Chandra Telescope

Image credit: NASA

Ways of Observing

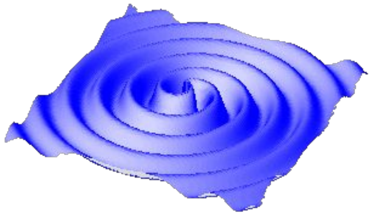


Image credit: Caltech

- ➔ **Gravitational Waves** (GWs)
- ➔ Predicted by general relativity (1915), confirmed by LIGO (2015)
  - ◆ Linearized Weak-Field Theory: $g_{\mu\nu} = \eta_{\mu\nu} + \boxed{h_{\mu\nu}}$
  - ◆ Wave equation: $\Box h_{\mu\nu} = 0$
  - ◆ Plane wave solutions: $h_{\mu\nu} = \epsilon_{\mu\nu} e^{ik_\rho x^\rho}$
  - ◆ Transverse propagation at the the speed of light
  - ◆ A changing *gravitational quadrupole moment* produces gravitational radiation

Linear perturbation to Minkowski (flat) spacetime

Research Goal

- Compute amplitude for continuous gravitational waves produced by magnetars via computational simulation:

    1. Model stellar structure and magnetic field, confirm stability of model.

    2. Evolve initial state and quantify stellar deformation due to magnetic field.

    3. Compute amplitude for continuous gravitational wave

Frequency of Gravitational Wave

Amplitude $$h \propto \frac{f^2}{r}\epsilon$$ Ellipticity (Measure of deformation)

Star Distance

Ways of Observing

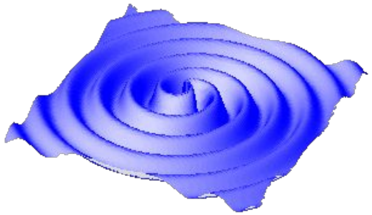


Image credit: Caltech

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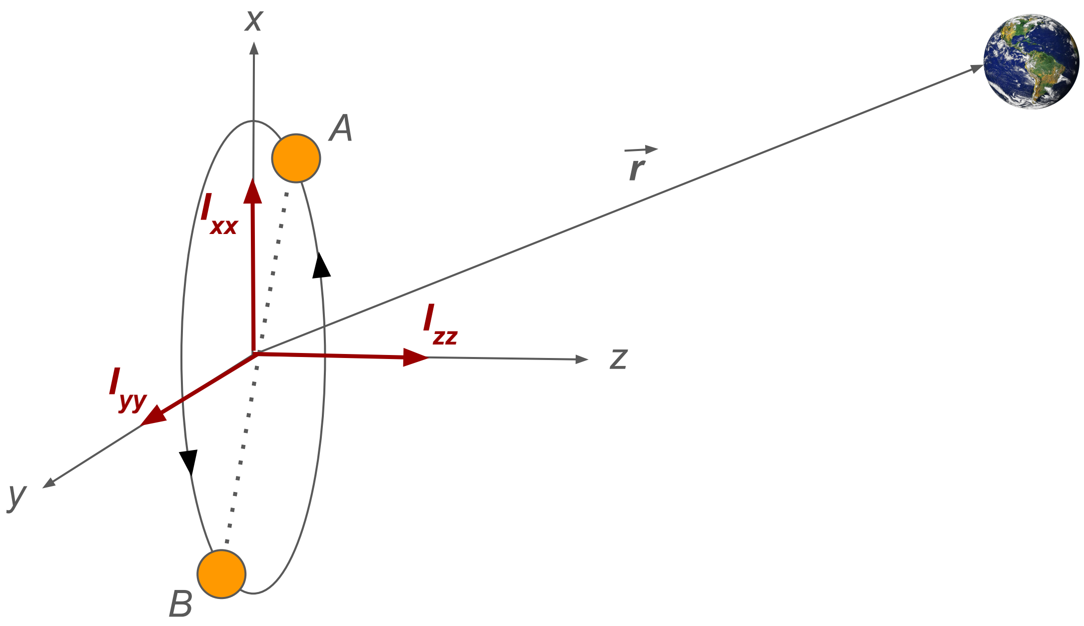
Linear perturbation to Minkowski (flat) spacetime

The Gravitational Quadrupole

	Electromagnetic Radiation	Gravitational Radiation
Monopole Moment	Constant, charged is conserved	Constant, mass is conserved (linear approximation)
Dipole Moment	Non-constant, electric dipole radiation	Constant, angular momentum is conserved
Quadrupole Moment	Non-constant, non-dominating term	Non-constant, gravitational-wave radiation

The Gravitational Quadrupole

- The gravitational quadrupole moment describes the mass distribution along each axis.
- A binary star system will have a non-constant quadrupole moment determined by the second-mass moment inertia tensor, $\boldsymbol{I_{jk}}$.
- Changes in the quadrupole moment produce gravitational waves.



Gravitational Wave Sources

- **Binary "inspiral" mergers (black holes and neutron stars) Detected**
- **Dramatic supernovae explosions**
- **Rapidly rotating stars with rotationally asymmetric deformations**
  - Appear to "wobble" as they rotate.
  - Gravitational Waves emitted as **continuous signals**

My Research

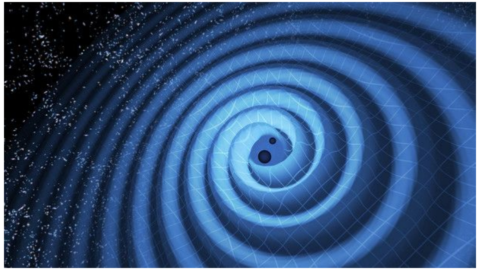
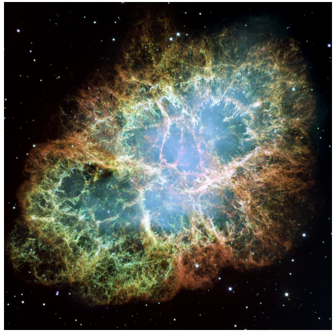


Image credit: LIGO/T. Pyle



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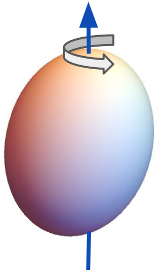
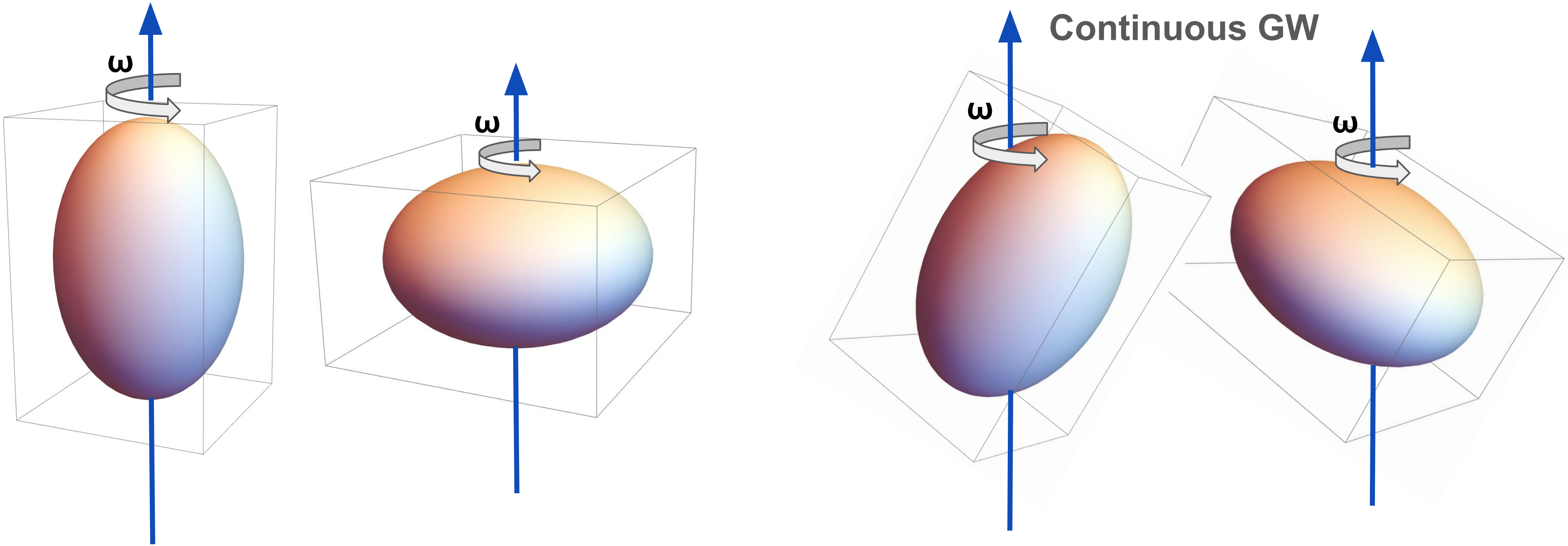


Asymmetric Rotation

Symmetric Rotators: No Continuous GW

Asymmetric Rotators: Wobble, Produce Continuous GW

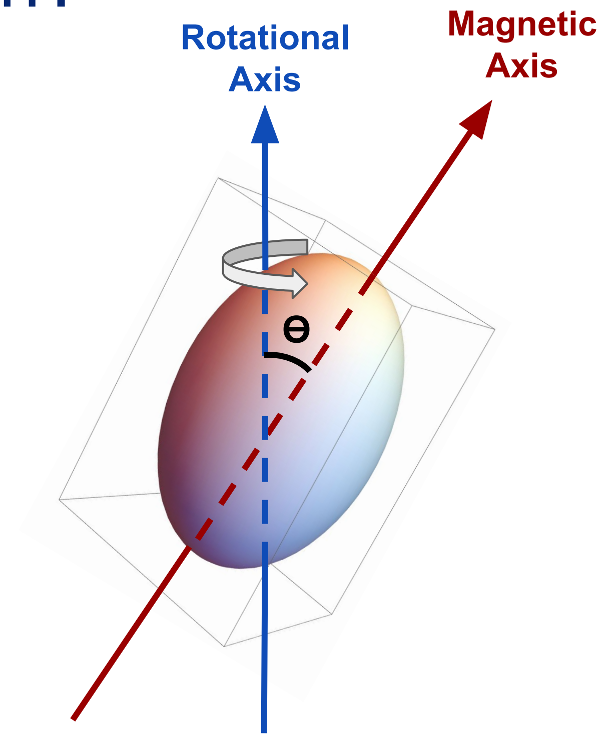


Axis of rotation in blue

Deformations not to scale

What Produces Asymmetric Rotation?

- Strong stellar magnetic fields with central axes not aligned with the rotational axis
- Production of Continuous GWs Require:
    - **Rapid Rotation**
    - **Magnetic Fields ~*1 trillion - 100 trillion times stronger* than Earth's field.**
- **Magnetars provide these conditions.**



Magnetars: Background

Neutron Stars - Remnant core of massive star

- ➔ Composed almost completely of neutrons
- ➔ Core densities ~ 10^{15} g cm^{-3}

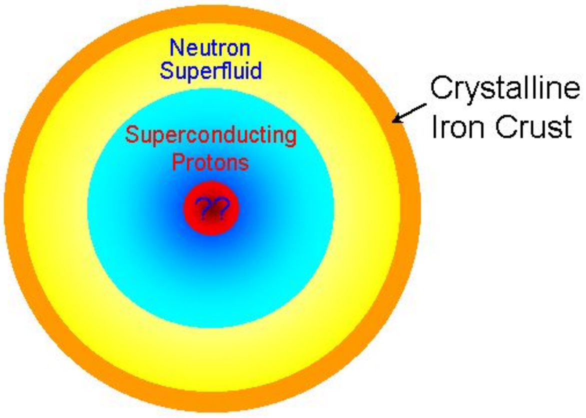


Image credit: Ohio State University

Pulsars - Rapidly rotating neutron star with a strong magnetic field

- ➔ B-field ~ 10^{12} - 10^{13} gauss
- ➔ Rotational period: ~ 0.7 s

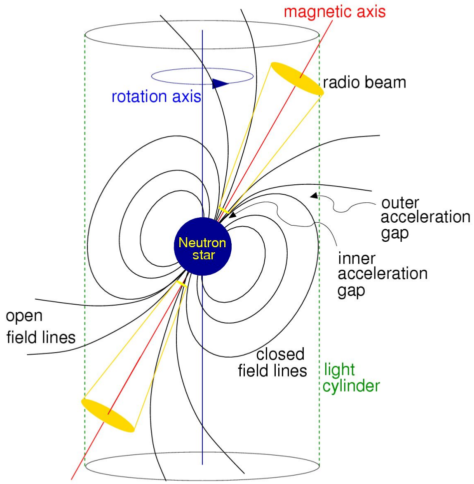


Image credit: NRAO

Magnetars - Highly magnetic pulsars

- ➔ B-field ~ 10^{15} gauss, *strongest in universe*
- ➔ Fields alter structure of star via Lorentz force.

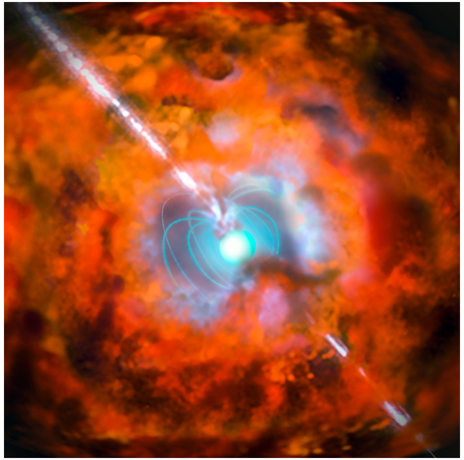
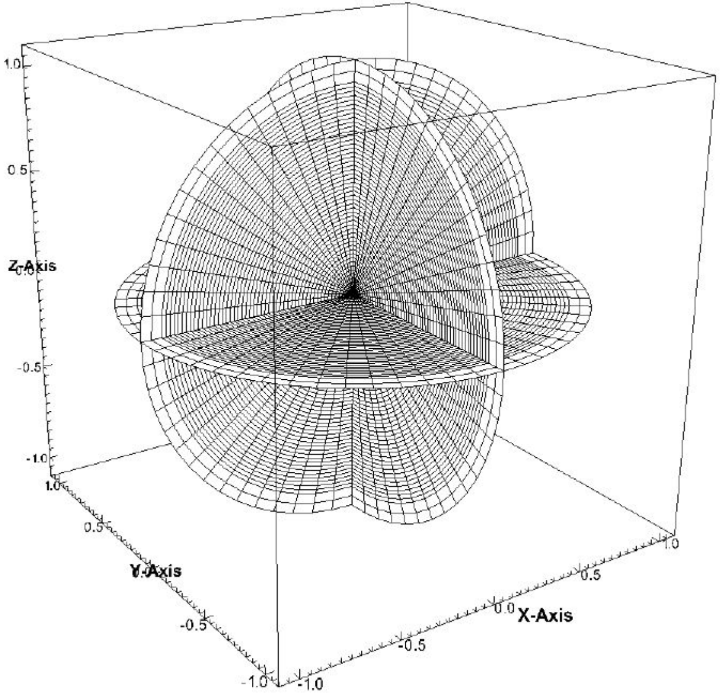


Image credit: ESO

Constructing a Stellar Model: The Computational Domain

- Utilize finite volume grid-mesh:
  - Structural equations discretized over computational domain.
- Each grid cell assigned attributes:
  - Density
  - Pressure
  - Vector-valued Magnetic Field
  - Vector-valued Velocity



Stellar Model: Requirements

➔ Static, Newtonian Structural Equations

- Hydrostatic Equilibrium
- Mass Conservation
- Equation of State
- Gravitational Potential

$\rho(r), P(r)$

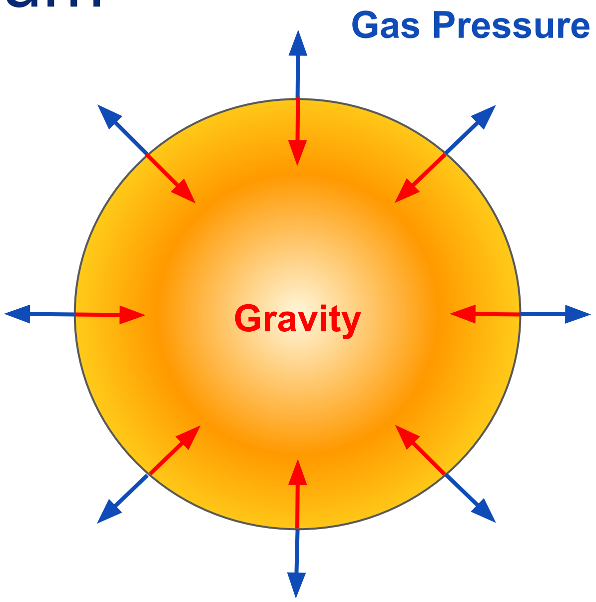
Describe stellar structure

Stellar Model: Hydrostatic Equilibrium

➔ Hydrostatic Equilibrium:

$$\frac{dP}{dr} = -G\frac{M_r \rho}{r^2}$$

- Balance inward force of gravity with pressure gradient.
- M_r , known as the *interior mass*, is the mass interior to the sphere of radius r



Stellar Hydrostatic Equilibrium

Stellar Model: Mass Conservation

➔ Mass Conservation:

$$\frac{dM_r}{dr} = 4\pi r^2 \rho$$

- Describes the change in the interior mass as a function of radius
- Integrate to obtain total mass, for neutron star ~ 1.4 $M_{\odot}$

Determining Radial Expression for Density

➔ Require a radial parameterization of density, $\rho(r)$

- Obtain differential equation for $\frac{d\rho}{dr}$

$$\frac{dP}{dr} = -G\frac{M_r\rho}{r^2}$$

$$\frac{dM_r}{dr} = 4\pi r^2 \rho$$

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dM_r}{dr} \longrightarrow \frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

➔ Express DE fully in terms of density via polytropic equation of state (EOS)

$$P(\rho) = K\rho^\gamma$$

- Determine γ via choice of polytropic index '*n*' $\longrightarrow \gamma = \frac{n+1}{n}$
- The choice of ***n* = 1** suits density profile of neutron stars well.

The Lane-Emden Equation

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho \rightarrow \left(\frac{n+1}{n}\right)\frac{K}{r^2}\frac{d}{dr}\left[r^2\rho^{(1-n)/n}\boxed{\frac{d\rho}{dr}}\right] = -4\pi G\rho$$

$$P(\rho) = K\rho^{\gamma}$$

➔ Under dimensionless parameterizations, arrive at the *Lane-Emden Equation*

$$\rho(r) \equiv \rho_c[\Theta_n(r)]^n$$

$$\alpha_n \equiv \left[(n+1)\left(\frac{K\rho_c^{(1-n)/n}}{4\pi G}\right)\right]^{1/2}$$

$$\xi \equiv \frac{r}{\alpha_n}$$

$$\boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{d\Theta_n}{d\xi}\right] = -(\Theta_n)^n}$$

Structural Equations

➔ For n = 1, $\Theta_{n=1}(\xi) = \frac{\sin \xi}{\xi}$

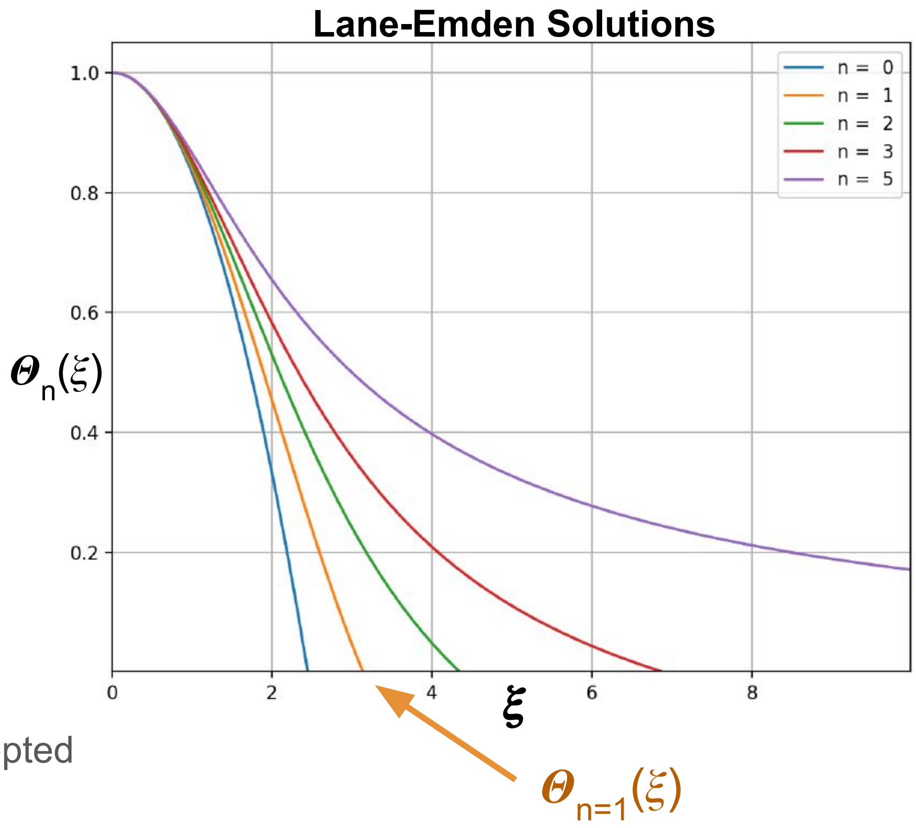
Since $\rho(r) \equiv \rho_c[\Theta_n(r)]^n$ and

$P(\rho) = K\rho^\gamma$,

via substitution,

$$\boxed{\begin{gathered}\rho(r) = \rho_c \frac{\sin(\pi r/R)R}{r\pi} \qquad\qquad r < R \\[2em] P_{n=1}(r) = K\rho(r)^2\end{gathered}}$$

➔ Constants K and ρ_c are determined via accepted stellar parameters for neutron stars.



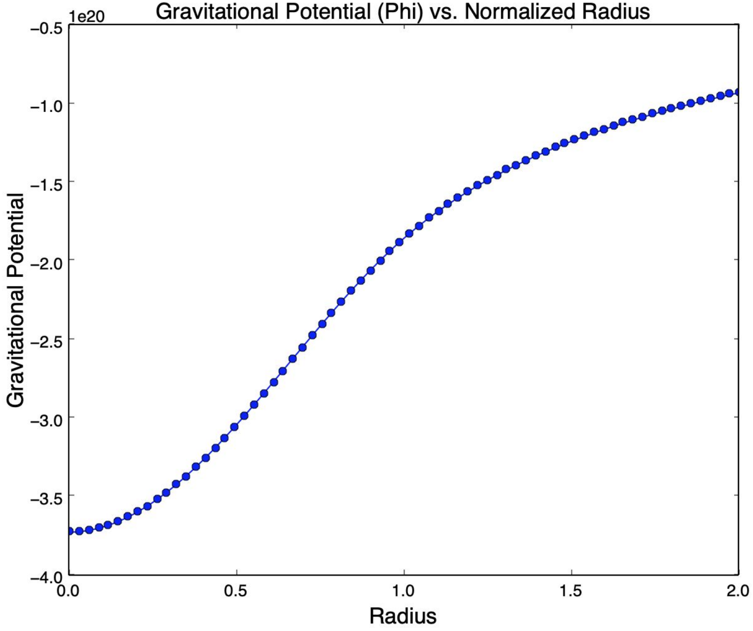
Stellar Model: Gravitational Potential

- Hydrostatic equilibrium requires a balancing force opposing gravity. We seek solutions to the spherically symmetric form of Poisson's equation for gravitational potential energy:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \varphi}{\partial r}\right) = 4\pi G\rho$$

- φ(r) in three equations:
    - φ(r = 0)
    - φ(r) for 0 < r < 1.0
    - φ(r) for r ≥ 1.0

- We require φ(r) to be piecewise continuous across the computational domain:



Magnetic Field Model: Requirements

- **Require stable magnetic field model**
  - **Dynamic Stability: Field retains configuration throughout time**
  - Not all fields have stable configurations.
  - ***Unstable Poloidal Field Configuration***

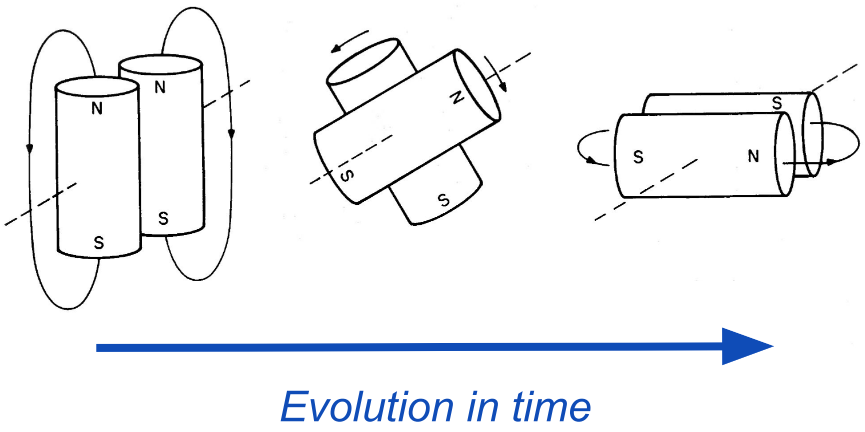


*Image Credit: E G. Flowers and M A. Ruderman. “*Evolution of pulsar magnetic fields*”. In: The Astrophysical Journal 215 (June 1977), pp. 302–310.

Magnetic Field Model: Requirements

- **Stable field configuration**
  - **Poloidal Field** wrapped by **Toroidal Field**
  - Fields stabilize each other; one component affects the other.

Poloidal component stabilized by **toroidal** field

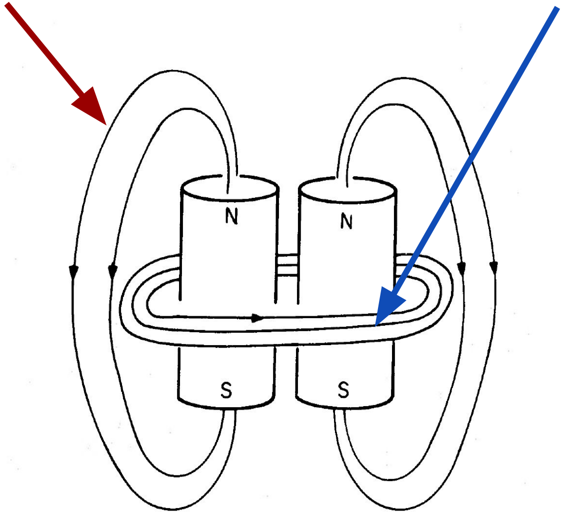


*Image Credit: E G. Flowers and M A. Ruderman. "*Evolution of pulsar magnetic fields*". In: The Astrophysical Journal 215 (June 1977), pp. 302–310.

Magnetic Field Solutions for Barotropic EOS

- Constraints:
  - Hydromagnetic Equilibrium: $$\frac{\nabla P}{\rho} + \nabla\varphi = \frac{(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}}{4\pi\rho} = \frac{\boldsymbol{L}}{4\pi\rho}$$
  - Density Constraint: $$\nabla \times \left[\frac{\boldsymbol{B} \times (\nabla \times \boldsymbol{B})}{\rho}\right] = 0 \qquad \rho(r) = \rho_c \frac{\sin(\pi r/R)R}{r\pi}$$
  - Maxwell's Equations, Divergence Condition: $$\nabla \cdot \boldsymbol{B} = 0$$

Field Expressions

- Construct spherical expressions for magnetic field model specific to *n = 1* polytrope, Haskell *et* al. (2008).
- B_k sets the strength of the magnetic field. We set $B_k = 1 \times 10^{17}$ G such that surface magnetic field strength matches evaluations of magnetar field strength.

B_r B_θ B_φ

$$\boldsymbol{B} = \left(\frac{2A\cos\theta}{r^2}, \frac{-A'\sin\theta}{r}, \frac{\pi\lambda A\sin\theta}{rR} \right)$$

$$A = \frac{B_k R^2}{(\lambda^2-1)^2 y}\left[2\pi\frac{\lambda y\cos(\lambda y)-\sin(\lambda y)}{\pi\lambda\cos(\pi\lambda)-\sin(\pi\lambda)} + \left((1-\lambda^2)y^2-2\right)\sin y + 2y\cos y\right]$$

Research Goal

- Compute amplitude for continuous gravitational waves produced by magnetars via computational simulation:

1. Model stellar structure and magnetic field, confirm stability of model.
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Methods

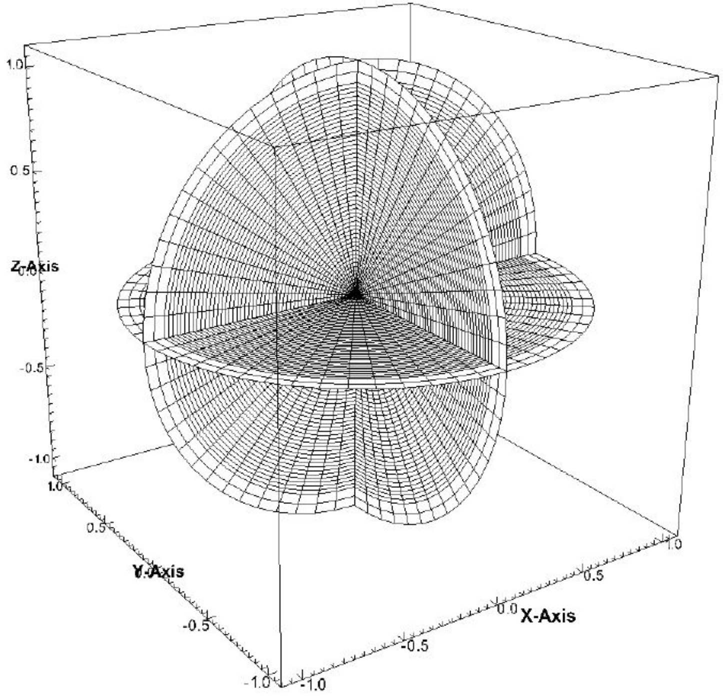
Software:

➔ PLUTO - Computational package for evolving computation

➔ VisIt - Open source visualization and graphical analysis software

The Computational Domain:

- Spherical domain for evolving simulations.
- 50 equally spaced points for $0 \leq r \leq 1.0$, 2 equally spaced points for $1.0 < r \leq 1.1$
- 30 equally spaced points in both the domains of ϴ and φ.



A 3-D cutaway of the computational domain

The Magnetohydrodynamic (MHD) Equations

- Set of conservation (continuity) equations, dictate time evolution of stellar structure and magnetic field.
- Solve MHD equations via finite-difference PDE *Hartman-Lax-Leer* (*HLL-c*) solver.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \, ,$$

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} + \nabla p - \frac{1}{\mu_0} (\nabla \times \boldsymbol{B} \times \boldsymbol{B}) = 0 \, ,$$

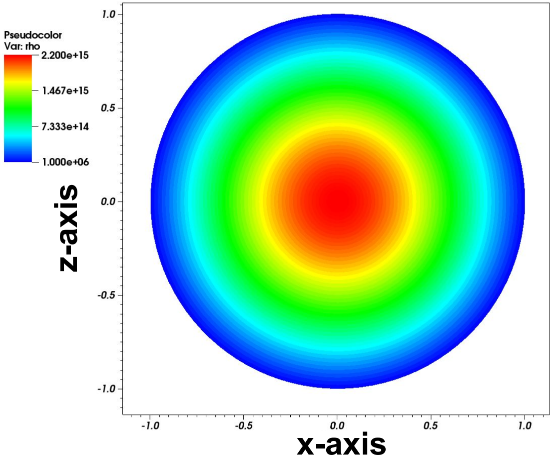
$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0 \, ,$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times (c \boldsymbol{E}) = 0 \, , \qquad \nabla \cdot \boldsymbol{B} = 0 \, .$$

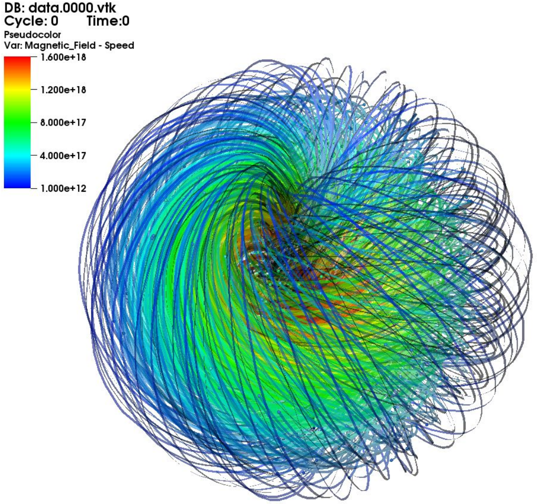
Results: Stability Validation

➔ Structural Stability
  - ◆ Analyze changes in hydrostatic equilibrium (B = 0) density profile under simulation evolution.

➔ Magnetic Field Configuration Stability
  - ◆ Dynamic stability implies large-scale field geometry is preserved under evolution. Analyze field geometry under evolution timescales.

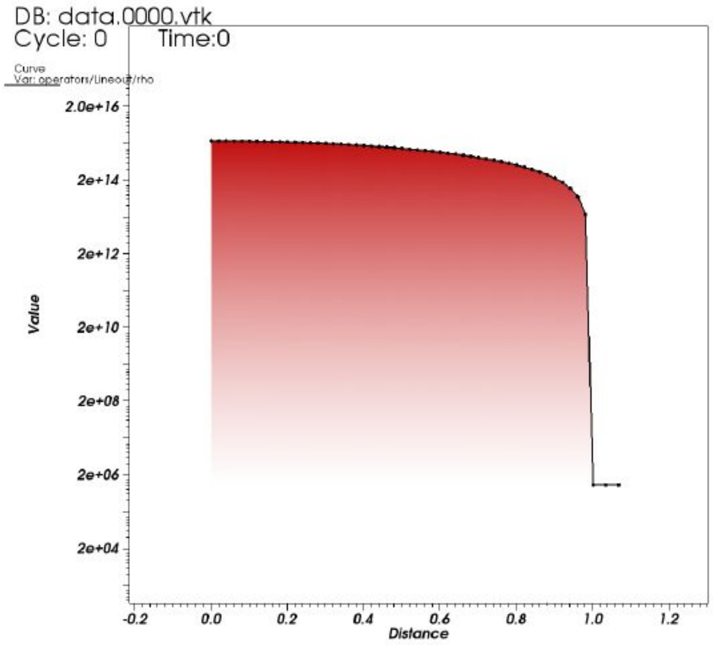


Time → ***Structural Profile Preserved***

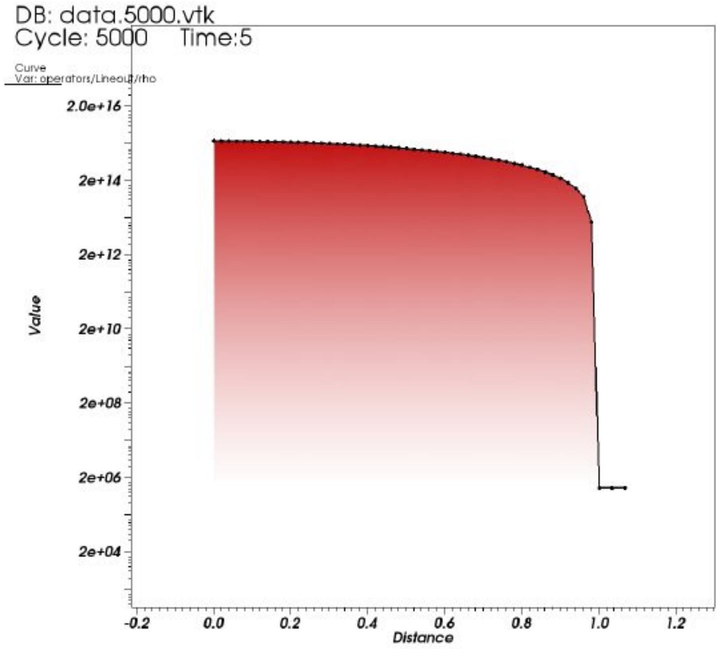


Time → ***Field Geometry Preserved***

Structural Stability



(A) Density profile, t = 0 s



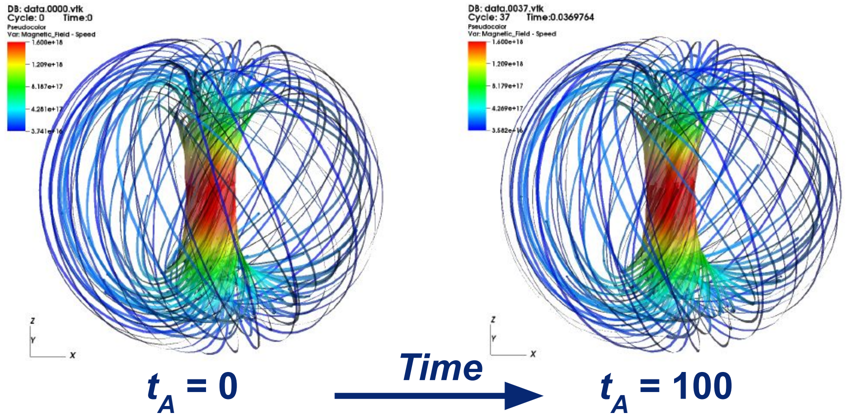
(B) Density profile, t = 5.0 s

Magnetic Field Stability

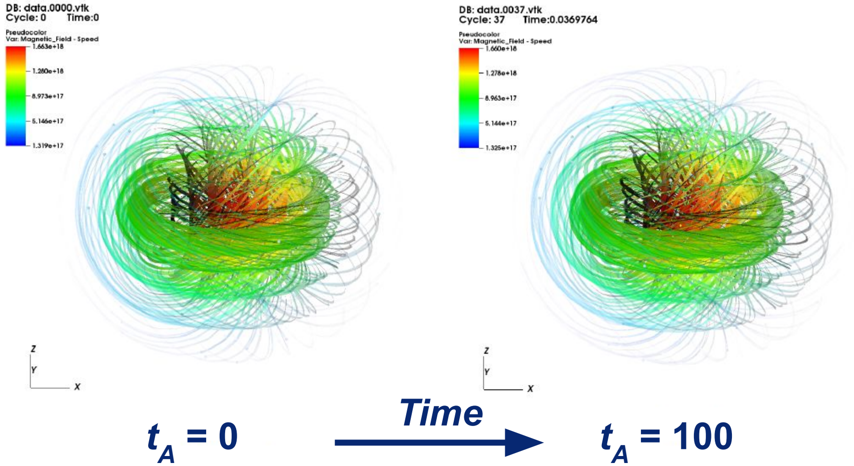
- Base analysis on Alfvén timescales

$$t_A = \frac{d}{v_A} \quad , \quad v_A = \frac{B_0}{\sqrt{\mu_0 \rho_0}}$$

- Alfvén waves result from tension in magnetic field lines, determine geometric evolution of field.
- Compare field geometry for $t_A = 0$ and $t_A = 100$. Stable field configurations should maintain initial geometry.



Poloidal Field Evolution



Toroidal Field Evolution

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Quantifying Stellar Deformation

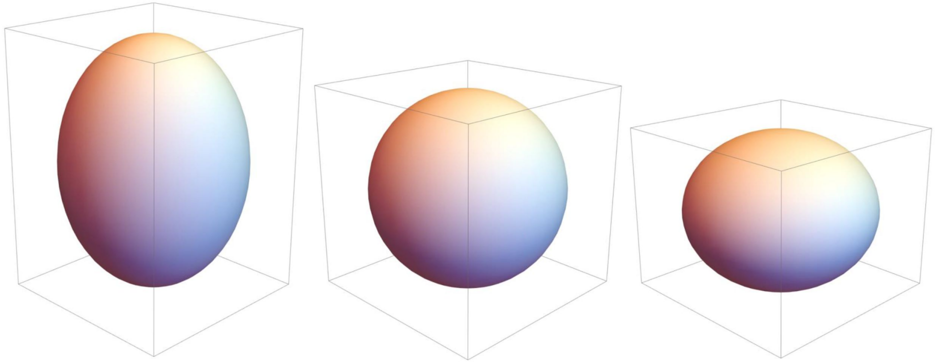
- Stellar Ellipticity

$$\epsilon = \frac{I_{zz} - I_{xx}}{I_0}$$

Moment of inertia for a spherical star

- Inertia Tensor

$$I_{jk} = \int_V \rho(r)\left(r^2\delta_{jk} - x_j x_k\right)dV$$



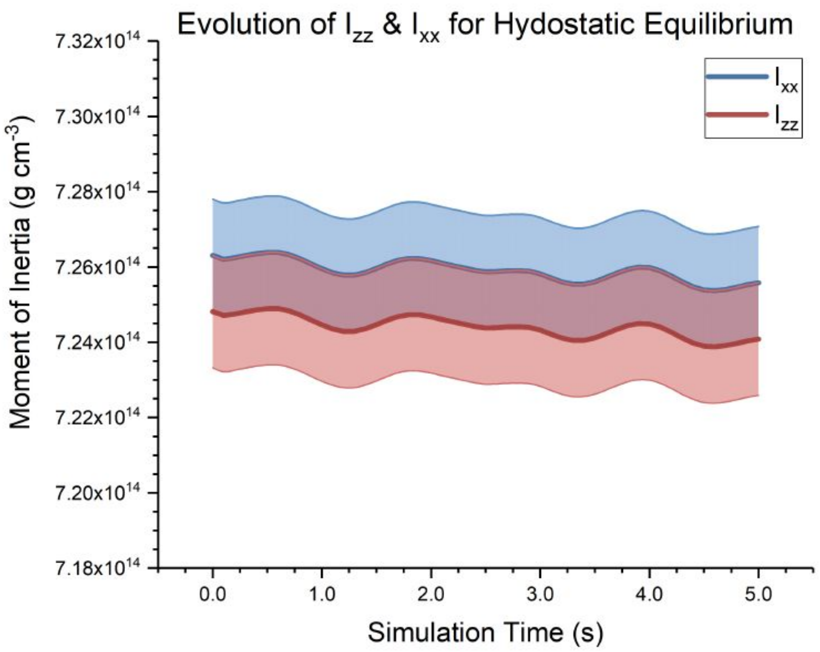
ε < 0, *prolate* star

ε = 0, spherical star

ε > 0, *oblate* star

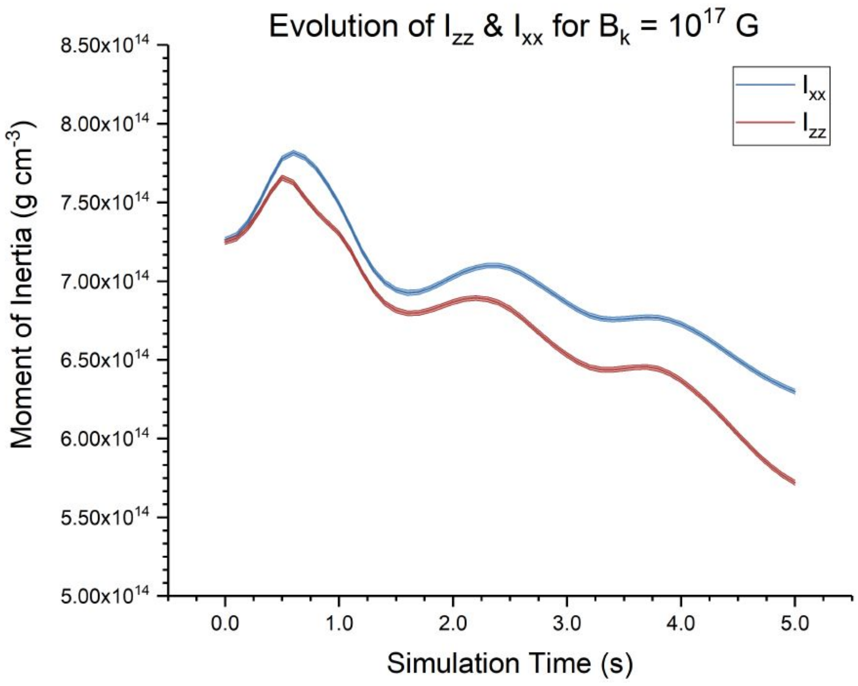
Ellipticity: Hydrostatic Equilibrium

- Null Hypothesis: Absence of B-field causes no stellar deformation, $\varepsilon = 0$.
- Our results strongly support null hypothesis; I_{xx} and I_{zz} overlap within computational error margins.
  - Further confirmation of model stability.



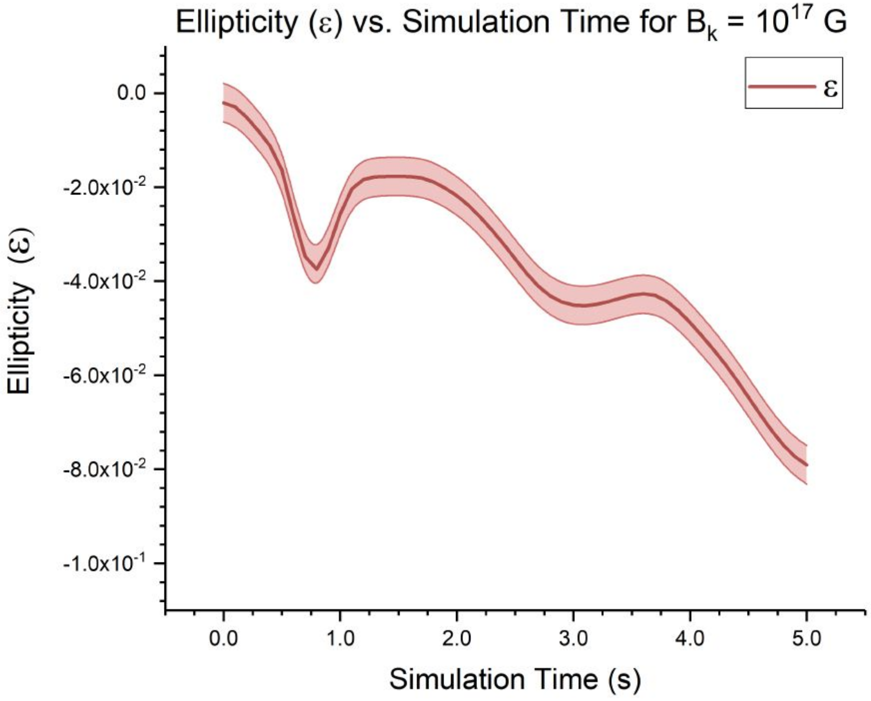
Ellipticity: Simulated Magnetar

- Field strength parameterized whereby surface field strength is of order 10^{15} G.
- Simulation evolved through t = 5.0 s, star becomes increasingly prolate.
- Star exhibits damped oscillatory behavior as it approaches hydromagnetic equilibrium.



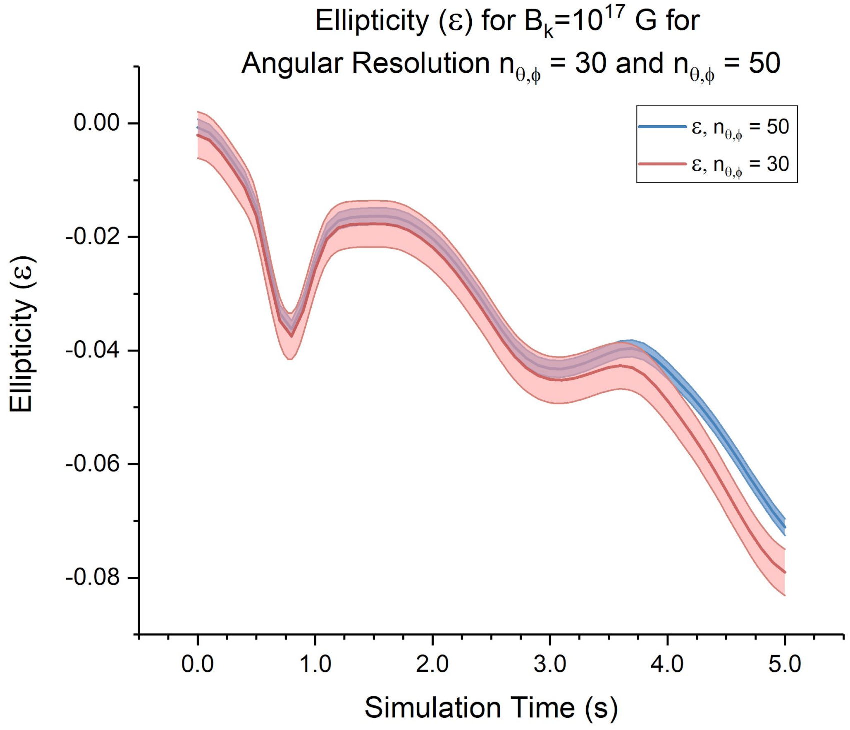
Ellipticity: Simulated Magnetar

- $\varepsilon(t = 5.0\ s) = -7.9 \times 10^{-2}$
- Results suggest equilibrium configuration between stellar structure and field strength lies beyond simulation duration.



Ellipticity: Effect of Angular Resolution

- Angular resolution (dθ and dφ) increased from 30 grid cells to 50 (each hemisphere)
- Strong agreement for t $\lesssim$ 3.5 s, after which solutions diverge.
  - $\varepsilon_{30}(t = 5.0\ s) = -7.9 \times 10^{-2}$
  - $\varepsilon_{50}(t = 5.0\ s) = -7.1 \times 10^{-2}$
- Suggests that lower resolution simulation may overestimate stellar ellipticity.



Ellipticity: Analysis of Results

- We compare our results against published findings for relativistic magnetar models implementing pure-toroidal field structure.
  - Kiuchi and Yoshida (2008), deformation of order $\sim -10^{-1}$. Our results appear to be in qualitative agreement despite newtonian approximations.
  - We implement a mixed magnetic field configuration, direct comparison is ill-advised.
- Longer-duration simulation at high angular resolution required to evaluate stellar ellipticity for equilibrium configuration.

Gravitational Wave Strain Estimates

- Implement parameter values based on observed quantities:
  - $f_{\mathrm{gw}} = 2/P$ (*P* period of stellar rotation, ~ 5 s for magnetars)
    - $f_{\mathrm{gw}} = 0.4$ Hz
  - *d,* stellar distance from Earth. For galactic sources, set average stellar distance of 50,000 ly.

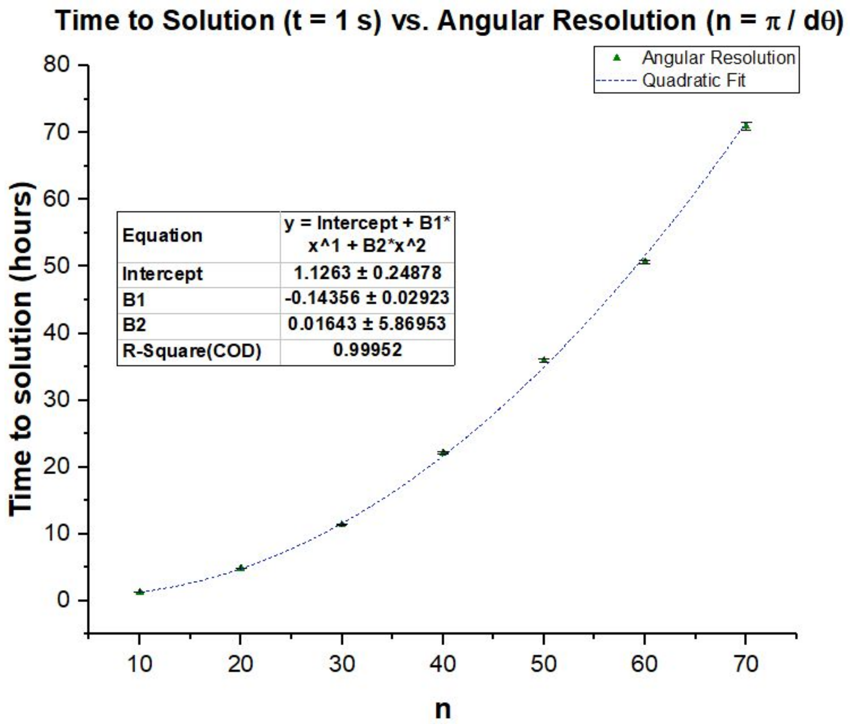
$$h_0 = \frac{4\pi^2 G}{c^4}\frac{I_0 f_{\mathrm{gw}}^2}{d}\epsilon$$

- $\boxed{h_0 \gtrapprox 1.72 \times 10^{-6}}$

- Current gravitational wave detectors are not sensitive to frequencies below ~ 10 Hz; slowly rotating sources can not be presently detected.

Computational Limitations

- Future work likely requires longer simulations at higher resolution.
  - Compute time scales quadratically with increased angular resolution.
    - Compute time to t = 5.0 s, n = 30 (~50 hours)
    - Compute time to t = 5.0 s, n = 50 (~175 hours)
  - Multi-core processing strongly recommended for future study.



Future Work: Improving our Model

- Relativistic Equation of State:
  - Tolman-Oppenheimer-Volkoff (TOV) equation (replaces newtonian formulation of hydrostatic equilibrium:

$$\frac{dP}{dr} = -\frac{Gm(r)}{r^2}\rho\left(1+\frac{P}{\rho c^2}\right)\left(1+\frac{4\pi r^3 P}{m(r)c^2}\right)\left(1-\frac{2Gm(r)}{rc^2}\right)^{-1}$$

  - EOS as a function of many variables:

$$P = P(\rho, T, x_p, \ldots)$$

- As we change our EOS, we must choose a new magnetic field model:
  - Fujisawa & Kisaka (Sept. 2014) via SLy EOS.

Acknowledgements

- I wish to extend appreciation and thanks to my co-mentors Dr. Michelle Kuchera and Dr. Kristen Thompson.
- I also thank former Duke undergraduate-student Emily Kuhn for providing an important starting point in informing my research.
- Many thanks to members of the Physics Department, including both faculty and students . You all have been encouraging, insightful, and supportive.
- Finally, I wish to thank my family for their abiding support in my endeavors.

Why Study Magnetars?

- Potential sources of gravitational waves
  - Verify existence of continuous GW sources
  - Inform, set boundaries for models of neutron star composition.
- Rapidly growing field: Rate of detection dramatically increased following launch of Swift and Fermi space telescopes.
  - First detection in 1979.

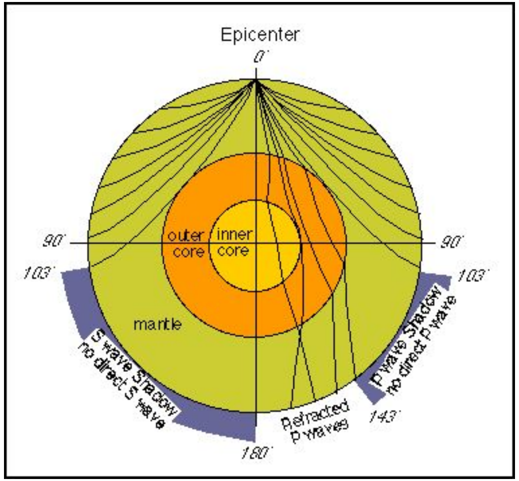


Image Credit: Columbia/Dr. Vic DiVenere

Fermi Gamma-ray Space Telescope: Launched June, 2008

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