Answer Key for Exam $\boxed{\text{A}}$

1. The 1D Poisson equation
$$-u''(x) = f(x)$$
is discretized using the Python function

```
def discretize(n, frhs):
    from numpy import eye, linspace
    x = linspace(0, 1, n+1)
    A = n**2 * (2*eye(n+1) - eye(n+1, k=-1) - eye(n+1, k=1))
    A[0,0] /= 2
    A[-1,-1] /= 2
    rhs = frhs(x)
    rhs[[0,-1]] = 0
    return x, A, rhs
```

which creates the tridiagonal matrix

$$A = n^2 \begin{bmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix}.$$

(a) What boundary conditions have been implemented?

(b) What is the rank of A?

(c) Are there any requirements on the function `frhs()` for this problem to have a solution?

(d) What is the order of accuracy of this discretization? (Hint: Look at the boundary conditions.)

(e) How can the order of accuracy be improved?

Answer: (a) Examining the first and last rows, we see first order differencing implementing $u' = 0$ at both ends.

(b) The operator has a one dimensional null space consisting of the constants. The rank of the $(n+1) \times (n+1)$ matrix is thus n.

(c) The RHS function sampled at the interior points should sum to zero. If it does, then the equation has infinitely many solutions (see null space above).

(d) The boundary is first order accurate when `frhs()` is smooth, unless it is zero at the endpoints in which case the discretization is second order accurate.

(e) As seen in the example, the same matrix can be obtained via symmetry arguments and using `frhs(x)/2` as RHS. This leads to second order accuracy (the interior order of accuracy).

2. Consider the family of functions

$$\phi(x,\theta) = e^{i\theta x}$$

sampled on the grid $x \in \mathbb{Z}$ (where $\mathbb{Z}$ is the set of all integers).

(a) Show that for any frequency $\theta \notin [-\pi,\pi]$, there is a $\bar{\theta} \in [-\pi,\pi]$ such that

$$\phi(x,\theta) = \phi(x,\bar{\theta}) \quad \text{for all } x \in \mathbb{Z}.$$

We say that the high frequency θ is *aliased* onto the associated low frequency $\bar{\theta}$.

(b) We consider an operation R defined by the stencil

$$R = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix},$$

i.e.,

$$Ru(x) = \frac{1}{4}u(x-1) + \frac{1}{2}u(x) + \frac{1}{4}u(x+1).$$

Compute the symbol $\hat{R}(\theta)$, the function satisfying

$$R\phi(x,\theta) = \hat{R}(\theta)\phi(x,\theta).$$

(c) The operation R is a "restriction" used to transfer from $\mathbb{Z}$ to the coarser grid $2\mathbb{Z}$. This coarser grid is only able to represent frequencies $\bar{\theta} \in [-\pi/2,\pi/2]$. We call a function $\phi(x,\theta)$ on the fine grid "high frequency" if $\pi/2 < |\theta| < \pi$. Upon restriction, high frequencies θ will be aliased onto a frequency $\bar{\theta} = \theta \pm \pi/2$ that is representable on the coarse grid. Use the symbol $\hat{R}(\theta)$ and a sketch or informal argument to show that aliasing is negligible when $\bar{\theta} \to 0$.

Answer: (a) If $\theta \notin [-\pi,\pi]$ there exists an integer k such that $\theta = \bar{\theta} + 2\pi k$ where $\bar{\theta} \in [-\pi,\pi]$. Now for any integer x, we have

$$\phi(x,\theta) = e^{i\theta x} = e^{i(2\pi k+\bar{\theta})x} = e^{i2\pi kx}e^{i\bar{\theta}x} = e^{i\bar{\theta}x} = \phi(x,\bar{\theta}).$$

(b) Let $\bar{\theta} = \theta \pm \pi \in [-\pi/2,\pi/2]$ and compute

$$\begin{aligned}(R\phi)(x,\theta) &= \big(\frac{1}{4}e^{-i\theta} + \frac{1}{2} + \frac{1}{4}e^{i\theta}\big)\phi(x,\theta) \\ &= \underbrace{\frac{1}{2}\big(1+\cos\theta\big)}_{\hat{R}(\theta)}\phi(x,\theta)\end{aligned}$$

(c) Any high frequency θ can be expressed as $\theta = \bar{\theta} \pm \pi$ where $|\bar{\theta}| \le \pi/2$ is the corresponding low frequency onto which θ is aliased. The symbol of the operator is thus

$$\begin{aligned}\hat{R}(\bar{\theta} \pm \pi) &= \frac{1}{2}\big(1+\cos(\bar{\theta}\pm\pi)\big)\phi(x,\bar{\theta}) \\ &= \frac{1}{2}\big(1-\cos\bar{\theta}\big)\phi(x,\bar{\theta})\end{aligned}$$

where we have used the fact that $\phi(x,\theta) = \phi(x,\bar{\theta})$ for $x \in 2\mathbb{Z}$.

3. The weak form of a PDE on domain Ω is: find $u(\mathbf{x}) \in V$ such that

$$\int_\Omega \nabla v \cdot \mathbf{w} u = \int_\Gamma v \mathbf{w} \cdot \hat{\mathbf{n}}, \qquad \forall v(\mathbf{x}) \in V$$

where $\mathbf{w}$ is a smooth vector field, $\Gamma \subset \partial\Omega$ with outward-facing unit normal $\hat{\mathbf{n}}$, and V is a space of smooth functions.

(a) What strong form PDE does the above correspond to? Hint: the Divergence theorem states

$$\int_\Omega \nabla \cdot \mathbf{f} = \int_{\partial\Omega} \mathbf{f} \cdot \hat{\mathbf{n}}$$

and the product rule is

$$\nabla \cdot (\mathbf{f} g) = g \nabla \cdot \mathbf{f} + \mathbf{f} \cdot \nabla g$$

where $\mathbf{f}$ is a smooth vector field and g is a scalar field.

(b) What boundary condition is applied?

(c) Write this PDE in non-divergence form in the special case where the vector field $\mathbf{w}$ is divergence-free: $\nabla \cdot \mathbf{w} = 0$.

Answer: (a) Choosing $\mathbf{f} = \mathbf{w} u$ and $g = v$ in the product rule above, we have

$$\nabla \cdot (v \mathbf{w} u) = v \nabla \cdot \mathbf{w} u + \nabla v \cdot \mathbf{w} u$$

thus

$$\int_\Omega \nabla \cdot (v \mathbf{w} u) = \int_\Omega v \nabla \cdot \mathbf{w} u + \int_\Omega \nabla v \cdot \mathbf{w} u$$

and the divergence theorem yields

$$\int_{\partial\Omega} v \mathbf{w} u \cdot \hat{\mathbf{n}} = \int_\Omega v \nabla \cdot \mathbf{w} u + \int_\Omega \nabla v \cdot \mathbf{w} u.$$

The first term on the right is zero when $\nabla \cdot \mathbf{w} u = 0$, resulting in

$$\begin{aligned} \int_\Omega \nabla v \cdot \mathbf{w} u &= \int_{\partial\Omega} v \mathbf{w} u \cdot \hat{\mathbf{n}} \\ &= \int_\Gamma v \mathbf{w} u \cdot \hat{\mathbf{n}} + \int_{\partial\Omega \setminus \Gamma} v \mathbf{w} u \cdot \hat{\mathbf{n}}. \end{aligned}$$

The first term on the right matches the original problem statement if $u = 1$. The second term can vanish if a $u = 0$ Dirichlet condition is enforced or if no boundary condition is admissible (e.g., an outflow boundary).

(b) The boundary condition on Γ is $u = 1$ (enforced "weakly" via the integral). We would need to look at the function space V to learn what boundary condition is enforced on $\partial\Omega \setminus \Gamma$; it could be $u = 0$ or "no" boundary condition at an outflow boundary.

(c) By the product rule,

$$\nabla \cdot \mathbf{w} u = u \underbrace{\nabla \cdot \mathbf{w}}_{=0} + \mathbf{w} \cdot \nabla u$$

so the equation is $\mathbf{w} \cdot \nabla u = 0$ when $\mathbf{w}$ is divergence free.

4. We can construct a Chebyshev method to solve $u''(x) = 0$ using the generalized Vandermonde matrices

$$V = \left[\psi_0(x) \middle| \psi_1(x) \middle| \cdots \right]$$

and

$$V'' = \left[\psi_0''(x) \middle| \psi_1''(x) \middle| \cdots \right]$$

using the basis functions $\psi(x)$ evaluated at the points x.

(a) Write an expression for the Laplacian matrix $L(x)u(x) \approx u''(x)$ in terms of the matrices V and V''.

(b) For each family of basis functions ψ and spacing of points x, indicate whether V is well-conditioned or ill-conditioned.

i. monomials, `linspace`

ii. Chebyshev polynomials, `linspace`

iii. monomials, `cosspace`

iv. Chebyshev polynomials, `cosspace`

(c) Does the expression you derived for part (a) depend on which polynomial basis is chosen? In exact arithmetic or finite precision?

(d) Given a set of points $\{x_1, \ldots, x_n\}$, define the 2D grid $g_{i,j} = (x_i, x_j)$. What is the complexity (as a function of n) to solve with the Laplacian defined on the grid $\mathbf{g}$?

Answer: (a) The Laplacian can be written $L = V''V^{-1}$.

(b) The matrix V is ill-conditioned in all cases except (iv).

(c) $L = V''V^{-1}$ is the same regardless of basis when working in exact arithmetic. The computation is unstable (for high order) in finite precision.

(d) There are n^2 grid points and the Laplacian is dense, therefore it costs $(n^2)^3 = n^6$ to factor. Using Kronecker products, it can be done by factoring the 1D operator L in n^3 operations, and then requires $O(n^3$ operations to apply.