Solving Linear and Transformable Non-linear Recursive Sequences using Generating Functions

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Overview

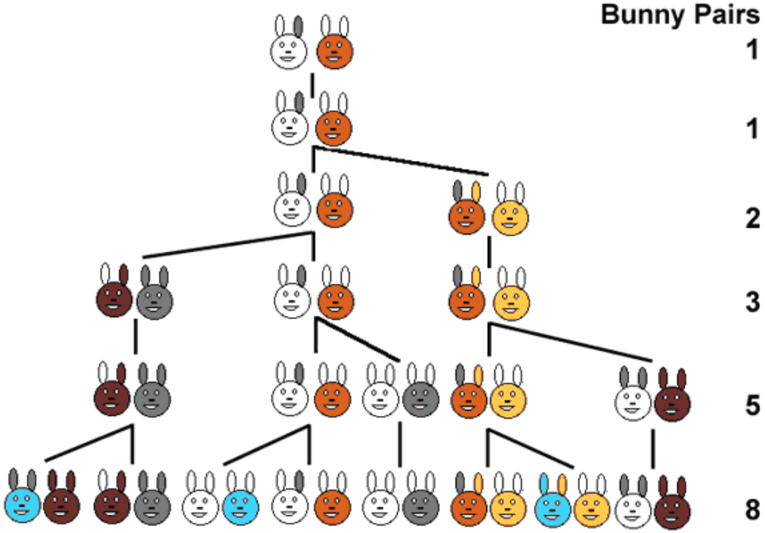
1. Introduction
2. Solving linear recursions
3. Translating non-linear recursions into linear ones
4. Conclusion and summary

1.1 Why did I choose this topic?

- The Fibonacci sequence
  - How is its formula derived?
  - Why there is irrational numbers?

- How about $a_n = m_1 a_{n-1} + m_2 a_{n-2}$?
  - Solved

- More generally?



$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

1.2 Mathematical terminologies

- *Linear recursion* order t
  - $a_n = m_1 a_{n-1} + m_2 a_{n-2} + \cdots + m_t a_{n-t} + c, m_t \neq 0. \begin{cases} c = 0 \text{ Homogeneous} \\ c \neq 0 \text{ Non-Homogeneous} \end{cases}$
- *Generating function* for $a_0, a_1, a_2, a_3, \ldots a_n, \ldots$
  - $G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$
- *Characteristic equation*
  - $x^t = m_1 x^{t-1} + m_2 x^{t-2} + \cdots + m_t x^{t-t}$
  - $\Leftrightarrow x^t - m_1 x^{t-1} - m_2 x^{t-2} - \cdots - m_t = 0$
  - Distinct non-zero roots: $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_j$, with multiplicity $p_1, p_2, p_3, \ldots p_j$

2.1 Idea to solve linear recursion

- The *recurrence function* $R(x)$
  - So that $G(x) \cdot R(x) = f(x)$ is finite
  - $G(x) = \frac{f(x)}{R(x)}$
  - Then Taylor expansion
- For order t linear recursion
  - $\underline{R(x) = 1 - m_1 x - m_2 x^2 - \cdots - m_t x^t}$
  - $f(x)$ is a finite polynomial
  - Taylor expansion of *generating function* of other form

- Example: Fibonacci sequence
- $a_n = a_{n-1} + a_{n-2}$
- $\Leftrightarrow a_n - a_{n-1} - a_{n-2} = 0$
- $G(x) = 0 + x + x^2 + 2x^3 + \cdots$
- $R(x) = 1 - x - x^2$
- $f(x) = 0 + x + 0x^2 + 0x^3 + \cdots$
- Hence $G(x) = \frac{x}{1-x-x^2}$.

2.2 Solution to homogeneous linear recursion

- Taylor expansion and simplifying gives:

$$a_n = \sum_{s=1}^{j} \left(C_{s,1}\lambda_s^n + C_{s,2} n \lambda_s^n + C_{s,3} n^2 \lambda_s^n + \cdots + C_{s,p_s} n^{p_s - 1} \lambda_s^n \right)$$

- When the roots of the *characteristic equation* are all distinct:

$$a_n = C_1\lambda_1^n + C_2\lambda_2^n + C_3\lambda_3^n + \cdots + C_t\lambda_t^n$$

- Manually find the constants

2.3 Solution to non-homogeneous recursion

- General case
- Use *recurrence function*
- $R(x) = 1 - m_1 x - m_2 x^2 - \cdots - m_t x^t$
- $G(x) \cdot R(x) = q(x) + \frac{cx^t}{1-x}$
- $G(x) = \frac{q(x)+\frac{cx^t}{1-x}}{R(x)} = \frac{(1-x)q(x)+cx^t}{(1-x)R(x)}$
- Manually expand the *generating function* using Taylor series

2.4 Special non-homogeneous recursion

- If $m_1 + m_2 + m_3 + \cdots + m_t \neq 1$
- Translate to *homogeneous linear recursion*
- $b_n = a_n - \frac{c}{1 - m_1 - m_2 - m_3 - \cdots - m_t}$
- *Homogeneous linear recursion* for (b_n)
- $b_n = m_1 b_{n-1} + \cdots + m_t b_{n-t}$

- Example: the recursion

$$a_n = 2a_{n-1} + a_{n-2} + 2$$

$$\Leftrightarrow (a_n + 1) = 2(a_{n-1} + 1) + (a_{n-2} + 1)$$

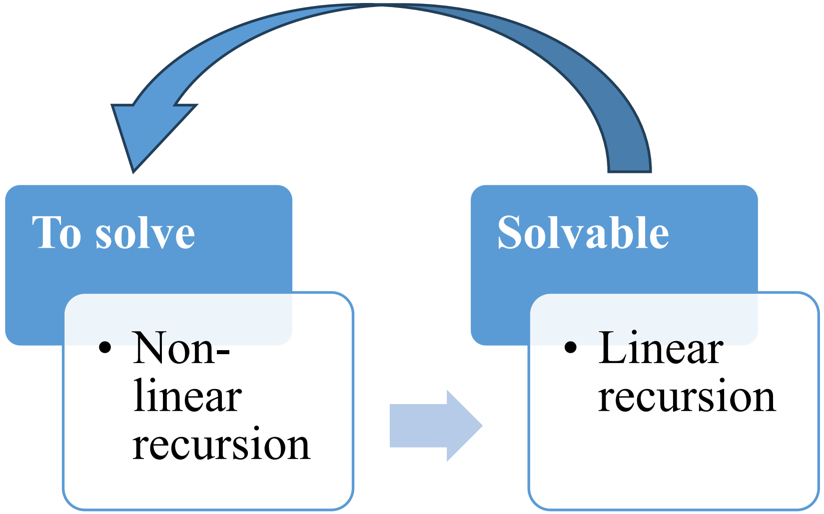
Hence let $b_n = a_n - \frac{2}{1-2-1} = a_n + 1$

Then $b_n = 2b_{n-1} + b_{n-2}$

3.1 Methods to translate non-linear recursion to linear ones

- Three methods:

- Substitution method
- Logarithmic method
- Trigonometric method



3.2 Substitution method

- Involving n, the term number
- "Pack" a_n and n, and letting it be b_n

- Example:
- $na_n = (n-1)a_{n-1} + (n-2)a_{n-2}$
- Letting $b_n = na_n$
- $\Leftrightarrow b_n = b_{n-1} + b_{n-2}$

a_n, n

Forms a linear recursion

- Noteworthy result:
- The recursive sequence $a_n = m_1 a_{n-1} + m_2 a_{n-2} + \cdots + m_t a_{n-t} + dn + c$

must have a caculable general formula given $m_1 + m_2 + m_3 + \cdots + m_t \neq 1$

3.3 Logarithmic method

- Involving multiplications
- Logarithms can turn multiplications into sums.

$$a_n = k a_{n-1} a_{n-2} \dots a_{n-t}$$
$$\Leftrightarrow \ln a_n = \ln a_{n-1} + \ln a_{n-2} + \cdots + \ln a_{n-t} + \ln k$$

- More generally, if $b_n = h(a_n)$

$$b_n = k b_{n-1} b_{n-2} \dots b_{n-t}$$
$$\Leftrightarrow \ln b_n = \ln b_{n-1} + \ln b_{n-2} + \cdots + \ln b_{n-t} + \ln k$$

3.4 Trigonometric method

- Recursive formula in the form of trigonometric identities

- Consider the *domains* and *codomains* of trigonometric functions carefully

- Example:

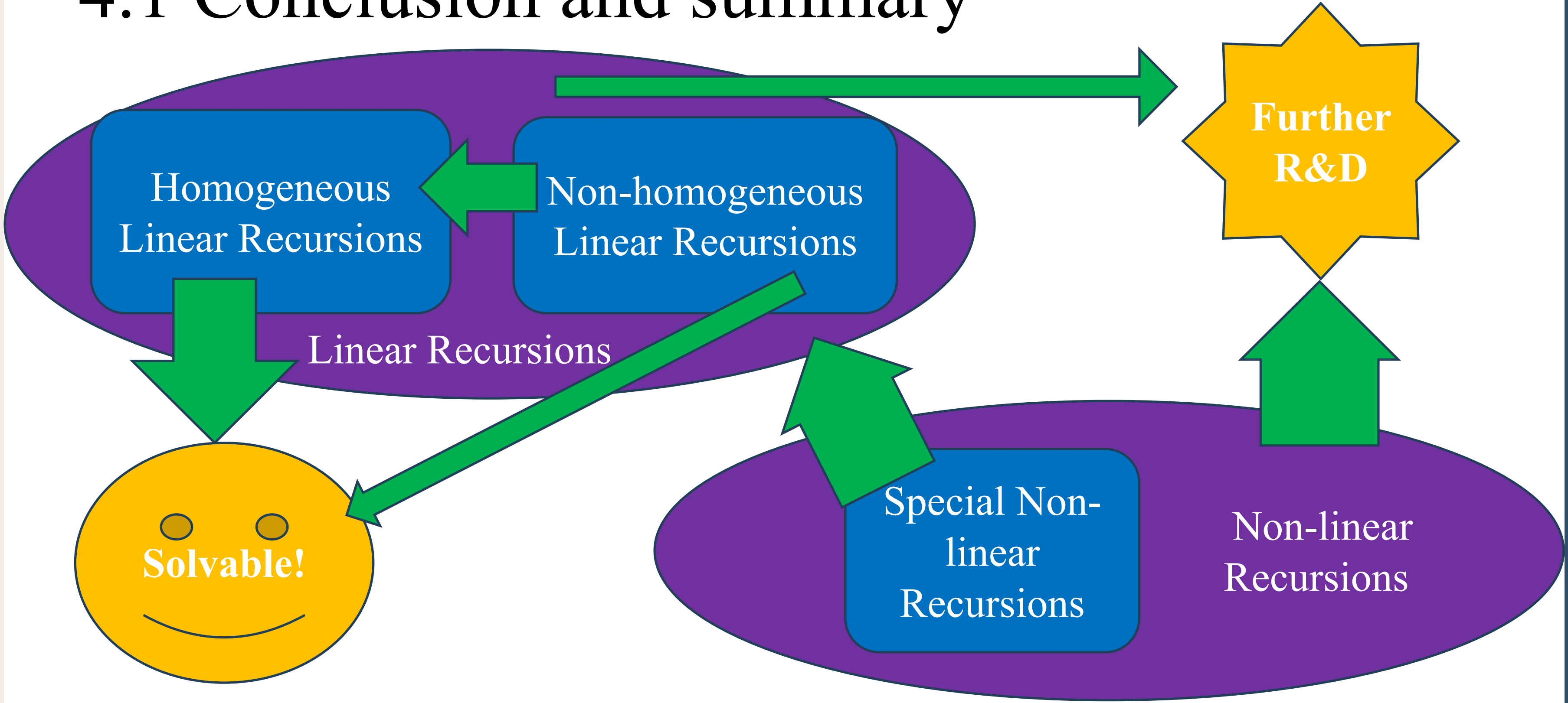
- $a_n = \frac{2a_n}{1-a_n^2} \leftrightarrow \tan b_n = \frac{2\tan b_{n-1}}{1-(\tan b_{n-1})^2} = \tan 2b_{n-1}$

- $a_n = a_{n-1}\sqrt{1-a_{n-2}^2} + \sqrt{1-a_{n-1}^2}\,a_{n-2}$

$\leftrightarrow$

$$\sin b_n = \sin b_{n-1}\cos b_{n-2} + \cos b_{n-1}\sin b_{n-2}$$
$$= \sin(b_{n-1} + b_{n-2})$$

4.1 Conclusion and summary



Thank you for your patience

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