Solving Recursive Sequences

A Simple Glance

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What is a recursive sequence?

A recursive sequence is a sequence that its terms are defined by preceding terms. Several initial values $a_0, a_1 \dots$ are given and all following terms are defined.

Examples (Recursive formyura)

$$a_n = a_{n-1}$$

$$a_n = 3a_{n-1} + 2$$

$$a_n = a_{n-1} + a_{n-2}$$

Also for simplicity, we will only discuss linear formulae.

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Case of One Preceding Term

We will discuss sequences with the following recursice formula:

$$a_n = ma_{n-1} + c, m \neq 0$$

In our discussion, we will only focus on the recursive formulae, and not the initial values.
If from the recursive formulae can we find a general form of the general formula with some constants unknown, then the sequence is said to be solved because those constants can be easily found by substituting the initial values and solving simultaneous equations.

Trivial Cases

For the recursive formula $a_n = ma_{n-1} + c\ldots$

when $m = 1$

It becomes an arithmetic sequence. Its general formula is

$$a_n = a_0 + nc.$$

when $c = 0$

It becomes a geometric sequence. Its general formula is

$$a_n = m^n a_0$$

For non-trivial case $a_n = ma_{n-1} + c$

for $m \neq 0, 1$ and $c \neq 0$

If we could find some equivalent relationship like

$$a_n - k = m(a_{n-1} - k)$$

then we can treat $\{a_n - k\}$ as a geometric sequence.
From it we get

$$a_n = ma_{n-1} + (1 - m)k$$

so we let $k = \frac{c}{1-m}$. By solving the geometric sequence $\{a_n - k\}$ we get

$$a_n - k = m^n(a_0 - k)$$

and thus

$$a_n = m^n(a_0 - \frac{c}{1 - m}) + \frac{c}{1 - m}.$$

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Case of Two Preceding Terms

The linear recursive formula involving two preceding terms can be written as

$$a_n = m_1 a_{n-1} + m_2 a_{n-2} + c$$

Here we will discuss the situation where $c = 0$. Similar methods can be used to solve the situation where $c \neq 0$ and $m_1 + m_2 \neq 1$.
Let's start by a famous example.

Fibanacci Sequence

The Fibonacci sequence is defined as $a_0 = 0$, $a_1 = 1$ and with recursive formula $a_n = a_{n-1} + a_{n-2}$ for all $n = 2, 3, 4 \ldots$

It is well known that, for the Fibonacci sequence there exists a closed-form general formula as below.

Binet's Formula

The general formula of Fibonacci sequence is

$$a_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right].$$

Solving $a_n = m_1 a_{n-1} + m_2 a_{n-2}$

Solution

Consider the recursive formula $a_n = m_1 a_{n-1} + m_2 a_{n-2}$. We want to have

$$a_n - \alpha a_{n-1} = \beta(a_{n-1} - \alpha a_{n-2})$$

so that we can regard $\{a_n - \alpha a_{n-1}\}$ as a geometric sequence. By comparing the coefficients,

$$\begin{cases} \alpha + \beta = m_1 \\ -\alpha\beta = m_2 \end{cases}$$

So, α and β are the roots of the equation

$$x^2 - m_1 x - m_2 = 0.$$

Continued

Assuming the roots are distinct, we notice that the roots of the equation $x^2 - m_1 x - m_2 = 0$ namely α and β are symmetric, so we can write two equations:

$$\begin{cases} a_n - \alpha a_{n-1} = \beta(a_{n-1} - \alpha a_{n-2}) \\ a_n - \beta a_{n-1} = \alpha(a_{n-1} - \beta a_{n-2}) \end{cases}$$

By considering the initial values of these two corresponding geometric sequences constants A_1 and A_2, applying the general formula for geometric sequences gives

$$\begin{cases} a_n - \alpha a_{n-1} = A_1 \beta^n \\ a_n - \beta a_{n-1} = A_2 \alpha^n \end{cases}$$

Simplifying give our final formula

$$a_n = B_1 \alpha^n + B_2 \beta^n$$

where B_1 and B_2 are constants to be found.

An Example

Examples (Fibonacci)

From the recursive formula $a_n = a_{n-1} + a_{n-2}$ we first form an equation

$$x^2 - x - 1 = 0.$$

The roots of this equation are $\frac{1 \pm \sqrt{5}}{2}$, so the general formula must be in the form

$$a_n = A_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + A_2 \left(\frac{1-\sqrt{5}}{2} \right)^n.$$

Substituting $a_0 = 0$ and $a_1 = 1$ and solving simultaneous equations directly give *Binet's Formula*.

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A Note on Induction

All general formulae constructed above can be "found" by mathematical induction if you can *guess* the answer from nothing.
An example using induction is included in the exercise.

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Verify the following conjecture by induction

For a recursive sequence with the recursive formula

$$a_n = \sum_{r=1}^{t} m_r a_{n-r},$$

we have the general formula

$$a_n = \sum_{r=1}^{t} C_r \lambda_r^n$$

where $C_1 \ldots C_t$ are constants and $\lambda_1 \ldots \lambda_r$ are distinct roots of the equation

$$x^t - \sum_{r=1}^{t} m_r x^{t-r} = 0.$$

(All roots are distinct assumed)

Extension to Matrices

Problem (3**)

Let a sequence with recursive formula

$$a_n = m_1 a_{n-1} + m_2 a_{n-2}$$

and $m_1^2 + 4m_2 > 0$.
By setting column vector

$$\mathbf{V}_n = \begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix}$$

and considering rewriting the recursive relationship as

$$\mathbf{V}_{n+1} = \mathbf{R}\mathbf{V}_n$$

where $\mathbf{R}$ *is a* 2×2 *matrix, reach the same result.*

Thank You.