Solving Recursive Sequences

the Homogeneous Linear case

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Fibanacci Sequence

The Fibonacci sequence is defined as $a_0 = 0$, $a_1 = 1$ and with recursive formula $a_n = a_{n-1} + a_{n-2}$ for all $n = 2, 3, 4 \ldots$

It is well known that, for the Fibonacci sequence there exists a closed-form general formula as below.

Binet's Formula

The general formula of Fibonacci sequence is

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

First Insight

Can we find a closed-form general formula to any *homogeneous linear* recursive sequence, probably involving more backward terms and some parameters on these terms?

Problem

Can we find a general formula to the following recursion such that

$$a_n = \sum_{r=1}^{t} m_r a_{n-r}$$

for all $n \in \mathbb{N} \geq t$ with $a_0 \ldots a_{t-1}$ given?

Let we start with a simpler case where $t = 2$.

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Solving the special case where $t = 2$

Solution

Consider the recursive formula $a_n = m_1 a_{n-1} + m_2 a_{n-2}$. We want to have

$$a_n - \alpha a_{n-1} = \beta(a_{n-1} - \alpha a_{n-2})$$

so that we can regard $\{a_n - \alpha a_{n-1}\}$ as a geometric sequence. By comparing the coefficients,

$$\begin{cases} \alpha + \beta = m_1 \\ -\alpha\beta = m_2 \end{cases}$$

So, α and β are the roots of the equation

$$x^2 - m_1 x - m_2 = 0.$$

Continued

Assuming the roots are distinct, we notice that the roots of the equation $x^2 - m_1 x - m_2 = 0$ namely α and β are symmetric, so we can write two equations:

$$\begin{cases} a_n - \alpha a_{n-1} = \beta(a_{n-1} - \alpha a_{n-2}) \\ a_n - \beta a_{n-1} = \alpha(a_{n-1} - \beta a_{n-2}) \end{cases}$$

By considering the initial values of these two corresponding geometric sequences constants A_1 and A_2, applying the general formula for geometric sequences gives

$$\begin{cases} a_n - \alpha a_{n-1} = A_1 \beta^n \\ a_n - \beta a_{n-1} = A_2 \alpha^n \end{cases}$$

Simplifying give our final formula

$$a_n = B_1 \alpha^n + B_2 \beta^n$$

where B_1 and B_2 are constants to be found.

An Example

Examples (Fibonacci)

From the recursive formula $a_n = a_{n-1} + a_{n-2}$ we first form an equation

$$x^2 - x - 1 = 0.$$

The roots of this equation are $\frac{1 \pm \sqrt{5}}{2}$, so the general formula must be in the form

$$a_n = A_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + A_2 \left(\frac{1-\sqrt{5}}{2} \right)^n .$$

Substituting $a_0 = 0$ and $a_1 = 1$ and solving simultaneous equations directly give *Binet's Formula*.

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Conjecture

For a recursive sequence with the recursive formula

$$a_n = \sum_{r=1}^{t} m_r a_{n-r},$$

we have the general formula

$$a_n = \sum_{r=1}^{t} C_r \lambda_r^n$$

where $C_1 \dots C_t$ are constants and $\lambda_1 \dots \lambda_r$ are distinct roots of the equation

$$x^t - \sum_{r=1}^{t} m_r x^{t-r} = 0.$$

(All roots are distinct assumed)

Concepts

We will mainly borrow a concept called *generating function*.

Definition (Generating function)

For any sequence $a_0, a_1, a_2 \ldots$, the *generating function* of this sequence is a polynomial defined by

$$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum a_r x^r.$$

Here x is an arbitrary variable. The function can be finite or infinite.

Concepts

Definition (Characteristic equation)

For a linear recursive formula

$$a_n = \sum_{r=1}^{t} m_r a_{n-r}$$

its *characteristic equation* is

$$x^t - \sum_{r=1}^{t} m_r x^{t-r} = 0.$$

One can see that this definition is just an abbreviation of the equation we mentioned several times before. In our conjecture $\lambda_1 \ldots \lambda_r$ are distinct roots of the characteristic equation.

Key Idea

Our goal is to solve the general formula of the coefficients of the generating function $G(x)$. In our case, the generating function is infinite, so it is not easy to directly solve for the general formula.
If we could find a finite function, called $R(x)$, such that

$$G(x) \cdot R(x) = f(x)$$

where $f(x)$ is a finite polynomial, then the coefficients of $G(x)$ *could* be solved using Taylor expansion.
We claim explicitly there exists such $R(x)$.

Claim

Let

$$R(x) = 1 - \sum_{r=1}^{t} m_r x^r,$$

then $f(x)$ is a finite polynomial with degree $< t$.

Proof.

For all $\varepsilon \in \mathbb{N} \geq t$ we have the term x^ε in $f(x)$ calculated as

$$1 \cdot a_\varepsilon x^\varepsilon - \sum_{r=1}^{t} (m_r x^r \cdot a_{\varepsilon - r} x^{\varepsilon - r}) = \left(a_\varepsilon - \sum_{r=1}^{t} m_r a_{\varepsilon - r} \right) x^\varepsilon$$

which is zero according to the recursive formula. □

To a Final Result

Hence we have found that, the generating function of the recursive sequence can be represented by a quotient of two finite polynomials, i.e.

$$G(x) = \frac{f(x)}{R(x)} = \frac{f(x)}{1 - \sum_{r=1}^{t} m_r x^r}.$$

Examples (Fibonacci)

The generating function of Fibonacci sequence is

$$G(x) = 0 + x + x^2 + \cdots = \frac{x}{1 - x - x^2}.$$

Final Result

By decomposing it into partial fractions and then implementing Taylor(binomial) expansion, one can find the form of the general formula of a homogeneous linear recursive sequence. Simplifying process is omitted here.

Alert

When using Taylor(binomial) expansion, it is necessary to consider whether the series is convergent. What is the range of values of x that the expansion is garanteed to be convergent? This is left as an exercise.

Finally we have found that our conjecture is true. The result is shown again on the next slide.

Theorem (General formula of homogeneous linear recursion)

For a recursive sequence with the recursive formula

$$a_n = \sum_{r=1}^{t} m_r a_{n-r},$$

we have the general formula

$$a_n = \sum_{r=1}^{t} C_r \lambda_r^n$$

where $C_1 \dots C_t$ *are constants and* $\lambda_1 \dots \lambda_r$ *are distinct roots of its characteristic equation. It is assumed that all roots are distinct.*

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Exercise

Exercise

Find the range of values of x that the Taylor(binomial) expansion in our method is garanteed to be convergent.

Removing the Limitation on the Characteristic Equation

Problem (1*)

Using a similar approach, find the general formula of any homogeneous linear recursive requence where its characteristic equation may has duplicate roots.

Extensions

Problem (2**)

Prove for any recursive sequence with the recursive formula

$$a_n = \sum_{r=1}^{t} m_r a_{n-r} + P(n)$$

where $P(n)$ is a polynomial on n and

$$\sum_{r=1}^{t} m_r \neq 1$$

there exists a calculabe general formula.
Hint: start with $P(n) = c$ where c is a constant.

Extensions

Problem (3**)

By setting column vector

$$\mathbf{V}_n = (a_{n-t}, a_{n-t+1} \dots a_n)^T$$

and considering rewriting the recursive relationship as

$$\mathbf{V}_{n+1} = \mathbf{R}\mathbf{V}_n$$

where $\mathbf{R}$ *is a* $t \times t$ *matrix, reach the same result as the case when all roots of the characteristic equation are distinct.*
Then, explain why its characteristic equation is defined in that way.

Thank You.