Benjamin Ye

CS/CNS/EE 156a: Learning Systems (Fall 2023)

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**Homework 4**

|  |  |
| --- | --- |
| **Problem** | **Answer** |
| 1 | [d] |
| 2 | [d] |
| 3 | [c] |
| 4 | [e] |
| 5 | [b] |
| 6 | [a] |
| 7 | [b] |
| 8 | [c] |
| 9 | [b] |
| 10 | [e] |

**Generalization Error**

In Problems 1–3, we look at generalization bounds numerically. For , use the simple approximate bound for the growth function .

1. For an with , if you want confidence that your generalization error is at most , what is the closest numerical approximation of the sample size that the VC generalization bound predicts?

**Answer: [d] 460,000**

The VC generalization bound states that for any tolerance ,

with probability . With , the bound becomes

Plugging in , , and , and solving for ,

1. There are a few bounds on the generalization error , all holding with probability at least . Fix and and plot these bounds as a function of . Which bound is the smallest for very large , say ? Note that [c] and [d] are implicit bounds in .

**Answer: [d] Devroye:**

1. For the same values of and of Problem 2, but for small , say , which bound is the smallest?

**Answer: [c] Parrondo and van den Broek:**

(The figure, sample program output, and Python 3 source code are on the following pages. )

**Figure**

A graph of different colored lines

Description automatically generated

**Sample program output**

[Homework 4 Problems 2–3]

N Vapnik–Chervonenkis Rademacher Parrondo–van den Broek Devroye

10000.0 0.632175 0.331309 0.223698 **0.215228**

5.0 13.828161 7.048777 **5.101362** 5.593126

(The Python 3 source code is on the following page. )

**Python 3 source code**

import matplotlib as mpl

import matplotlib.pyplot as plt

import numpy as np

import pandas as pd

from scipy import optimize

mpl.rcParams.update(

    {

        "axes.labelsize": 14,

        "figure.autolayout": True,

        "figure.figsize": (4.875, 3.65625),

        "font.size": 12,

        "legend.columnspacing": 1,

        "legend.edgecolor": "1",

        "legend.framealpha": 0,

        "legend.fontsize": 12,

        "legend.handlelength": 1.25,

        "legend.labelspacing": 0.25,

        "xtick.labelsize": 12,

        "ytick.labelsize": 12,

        "text.usetex": True

    }

)

def vapnik\_chervonenkis\_bound(m\_H, N, delta):

    return np.sqrt(8 \* np.log(4 \* m\_H(2 \* N) / delta) / N)

def rademacher\_bound(m\_H, N, delta):

    return (np.sqrt(2 \* np.log(2 \* N \* m\_H(N)) / N)

            + np.sqrt(2 \* np.log(1 / delta) / N) + 1 / N)

def parrondo\_van\_den\_broek\_bound(m\_H, N, delta, ub=10.0):

    return np.vectorize(

        lambda N: optimize.root\_scalar(

            lambda eps: np.sqrt((2 \* eps + np.log(6 \* m\_H(2 \* N) / delta)) / N)

                        - eps,

            bracket=(0.0, ub), method="toms748"

        ).root

    )(N)

def devroye\_bound(m\_H, N, delta, ub=10.0):

    return np.vectorize(

        lambda N: optimize.root\_scalar(

            lambda eps, N: np.sqrt(

                (4 \* eps \* (1 + eps) + np.log(4 / delta) + np.log(m\_H(N \*\* 2)))

                / (2 \* N)

            ) - eps,

            args=N,

            bracket=(0.0, ub),

            method="toms748"

        ).root

    )(N)

if \_\_name\_\_ == "\_\_main\_\_":

    d\_vc = 50

    delta = 0.05

    m\_H = lambda N: N \*\* d\_vc

    Ns = np.arange(3, 10\_001, dtype=np.longdouble)

    bounds = {

        "Vapnik–Chervonenkis": vapnik\_chervonenkis\_bound(m\_H, Ns, delta),

        "Rademacher": rademacher\_bound(m\_H, Ns, delta),

        "Parrondo–van den Broek": parrondo\_van\_den\_broek\_bound(m\_H, Ns, delta),

        "Devroye": devroye\_bound(m\_H, Ns, delta)

    }

    df = pd.DataFrame(columns=["N"] + [b for b in bounds.keys()])

    \_, ax = plt.subplots()

    for label, bound in bounds.items():

        ax.plot(Ns, bound, label=label)

    ax.set\_yscale("log")

    ylim = ax.get\_ylim()

    for N in (10\_000, 5):

        index, = np.where(Ns == N)[0]

        df.loc[len(df)] = (N, \*[bound[index] for bound in bounds.values()])

        ax.plot((N, N), ylim, "k:")

    ax.legend(title="Generalization bound")

    ax.set\_xlabel("$N$")

    ax.set\_xscale("log")

    ax.set\_ylabel("$\epsilon$")

    ax.set\_ylim(ylim)

    plt.show()

    print(f"\n[Homework 4 Problems 2–3]\n{df.to\_string(index=False)}")

**Bias and Variance**

Consider the case where the target function is given by and the input probability distribution is uniform on . Assume that the training set has only two examples (picked independently), and that the learning algorithm produces the hypothesis that minimizes the mean squared error on the examples.

1. Assume the learning model consists of all hypotheses of the form . What is the expected value of the hypothesis produced by the learning algorithm (expected value with respect to the data set)? Express your as , and round to two decimal digits only, the match *exactly* to one of the following answers.

**Answer: [e] None of the above**

1. What is the closest value to the bias in this case?

**Answer: [b] 0.3**

1. What is the closest value to the variance in this case?

**Answer: [a] 0.2**

1. Now, let’s change . Which of the following learning models has the least expected value of out-of-sample error?

**Answer: [b] Hypotheses of the form**

Simulation results show that

* for [a] , the out-of-sample error is (which matches from Example 2.8 in the *Learning from Data* textbook),
* for [b] , the out-of-sample error is (from Problem 6),
* for [c] , the out-of-sample error is (which agrees with from Example 2.8 in the *Learning from Data* textbook), and
* for [d] and [e] , the out-of-sample errors are multiple orders of magnitude greater than those for hypotheses [a] through [c]. This is not surprising since the variances are expected to be much higher due to the increased complexity of the forms of the hypotheses compared to those for [a] through [c].

(The derivations, sample program output, and Python 3 source code are on the following pages. )

**Derivations**

For the hypothesis with form and a sample size of 2, the mean squared error (MSE) is

By minimizing the MSE with respect to , we obtain the optimal value of for a pair of points:

In the simulations, ten million pairs of and are randomly generated, the optimized values are determined using the formula above, and is found by averaging the values.

Then, the bias and variance are evaluated on a new set of ten million pairs using

For the four other hypotheses with linear and quadratic forms in Problem 7, their MSE, optimal values for (and ), bias, and variance can be calculated in a fashion like that above and are given by

* [a]
* [c]
* [d]
* [e]

(The sample program output and Python 3 source code are on the following pages. )

**Sample program output**

[Homework 4 Problems 4–7]

choice h(x) g(x) bias variance

[a] b 0.00 0.499970 2.499281e-01

[b] ax **1.43x** **0.270684** **2.365877e-01**

[c] ax+b 0.78x+0.00 0.206688 1.676365e+00

[d] ax^2 14.63x^2 43.338501 2.449373e+07

[e] ax^2+b 21.95x^2-0.68 87.382128 4.026515e+07

Note that the bias and variance for [d] and [e] fluctuate wildly but are consistently much greater than those for [a] through [c].

**Python 3 source code**

import numpy as np

import pandas as pd

def generate\_data(

        N, f, d=2, lb=-1.0, ub=1.0, \*, bias=False, rng=None, seed=None):

    if rng is None:

        rng = np.random.default\_rng(seed)

    x = rng.uniform(lb, ub, (N, d))

    if bias:

        x = np.hstack((np.ones((N, 1)), x))

    return x, f(x)

if \_\_name\_\_ == "\_\_main\_\_":

    n\_runs = 10\_000\_000

    f = lambda x: np.sin(np.pi \* x)

    hs = {

        # h(x): ((a(x\_in, y\_in), b(x\_in, y\_in)),

        #        bias(x\_out, y\_out, a\_avg, b\_avg),

        #        var(x\_out, a, a\_avg, b, b\_avg),

        #        fmt(a\_avg, b\_avg))

        "b": (

            lambda x, y: (None, (y[::2] + y[1::2]) / 2),

            lambda xt, yt, ah, bh: ((bh - yt) \*\* 2).mean(),

            lambda xt, a, ah, b, bh: ((np.tile(b, (2, 1)) - bh) \*\* 2).mean(),

            lambda ah, bh: f"{bh:.2f}"

        ),

        "ax": (

            lambda x, y: (

                (x[::2] \* y[::2] + x[1::2] \* y[1::2])

                / (x[::2] \*\* 2 + x[1::2] \*\* 2),

                None

            ),

            lambda xt, yt, ah, bh: ((ah \* xt - yt) \*\* 2).mean(),

            lambda xt, a, ah, b, bh: (

                ((np.tile(a, (2, 1)) - ah) \* xt) \*\* 2

            ).mean(),

            lambda ah, bh: f"{ah:.2f}x"

        ),

        "ax+b": (

            lambda x, y: (

                (y[::2] - y[1::2]) / (x[::2] - x[1::2]),

                (x[::2] \* y[1::2] - x[1::2] \* y[::2]) / (x[::2] - x[1::2])

            ),

            lambda xt, yt, ah, bh: ((ah \* xt + bh - yt) \*\* 2).mean(),

            lambda xt, a, ah, b, bh: (

                ((np.tile(a, (2, 1)) - ah) \* xt + np.tile(b, (2, 1)) - bh) \*\* 2

            ).mean(),

            lambda ah, bh: f"{ah:.2f}x{'+' if bh >= 0 else ''}{bh:.2f}"

        ),

        "ax^2": (

            lambda x, y: (

                (x[::2] \* y[::2] + x[1::2] \* y[1::2])

                / (x[::2] \*\* 3 + x[1::2] \*\* 3),

                None

            ),

            lambda xt, yt, ah, bh: ((ah \* xt \*\* 2 - yt) \*\* 2).mean(),

            lambda xt, a, ah, b, bh: (

                ((np.tile(a, (2, 1)) - ah) \* xt \*\* 2) \*\* 2

            ).mean(),

            lambda ah, bh: f"{ah:.2f}x^2"

        ),

        "ax^2+b": (

            lambda x, y: (

                (y[::2] - y[1::2]) / (x[::2] \*\* 2 - x[1::2] \*\* 2),

                (x[::2] \*\* 2 \* y[1::2] - x[1::2] \*\* 2 \* y[::2])

                / (x[::2] \*\* 2 - x[1::2] \*\* 2)

            ),

            lambda xt, yt, ah, bh: ((ah \* xt \*\* 2 + bh - yt) \*\* 2).mean(),

            lambda xt, a, ah, b, bh: (

                ((np.tile(a, (2, 1)) - ah) \* xt \*\* 2

                 + np.tile(b, (2, 1)) - bh) \*\* 2

            ).mean(),

            lambda ah, bh: f"{ah:.2f}x^2{'+' if bh >= 0 else ''}{bh:.2f}"

        )

    }

    x\_train, y\_train = generate\_data(2 \* n\_runs, f, 1)

    x\_test, y\_test = generate\_data(2 \* n\_runs, f, 1)

    df = pd.DataFrame(columns=["choice", "h(x)", "g(x)", "bias", "variance"])

    for i, (h, (f\_ab, f\_bias, f\_var, fmt)) in enumerate(hs.items()):

        as\_, bs = f\_ab(x\_train, y\_train)

        a\_avg = None if as\_ is None else as\_.mean()

        b\_avg = None if bs is None else bs.mean()

        bias = f\_bias(x\_test, y\_test, a\_avg, b\_avg)

        var = f\_var(x\_test, as\_, a\_avg, bs, b\_avg)

        df.loc[len(df)] = (f"[{chr(97 + i)}]", h, fmt(a\_avg, b\_avg), bias, var)

    print(f"\n[Homework 4 Problems 4–7]\n{df.to\_string(index=False)}")

**VC Dimension**

1. Let be an integer and assume that . What is the VC dimension of a hypothesis set whose growth function for all satisfies ? Recall that when .

**Answer: [c]**

The growth function is

When , the combination term in the right-hand side of the growth function becomes zero. By starting with and recursively multiplying by 2 for each increment in , the growth function simplifies to

Now, let’s consider the case when . For , the growth function is

Since is less than , the break point is and the VC dimension is .

1. For hypothesis sets with finite, positive VC dimensions (same input space ), some of the following bounds are correct and some are not. Which, among the correct ones, is the tightest bound (the smallest range of values) on the VC dimension of the *intersection* of the sets ? (The VC dimension of an empty set or a singleton set is taken as zero.)

**Answer: [b]**

Since the intersection of the sets can be an empty or singleton set, the lower bound must be zero.

There is some flexibility in the upper bound. By definition, a hypothesis set or intersection set with a VC dimension of can shatter data sets with points.

The loosest possible upper bound is since it will always be greater than the number of points the intersection set able to shatter, even if for all hypothesis sets.

A tighter upper bound is since it is the number of points that the most “flexible” hypothesis set can shatter. This would happen to be the tightest upper bound only when all hypothesis sets share the same , i.e., .

By the same logic, the tightest upper bound must be for the general case because all hypothesis sets in the intersection set should, at a minimum, be able to shatter data sets with that many points.

1. For hypothesis sets with finite, positive VC dimensions (same input space ), some of the following bounds are correct and some are not. Which, among the correct ones, is the tightest bound (the smallest range of values) on the VC dimension of the *union* of the sets ?

**Answer: [e]**

Following the train of thought used in Problem 9, the lower bound of the union set is since it is always possible to shatter that many points by taking the most “flexible” hypothesis set, which has the highest VC dimension.

One approach to find the upper bound is to determine the break point (and consequently, the VC dimension) using the growth function of the union set of two hypothesis sets and work recursively from that/use mathematical induction.

For any two hypothesis sets and in the union set, the growth function is bounded by

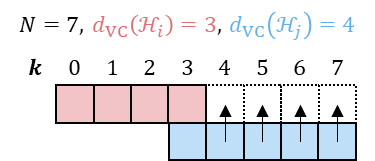
since it cannot be more than the sum of the individual growth functions (limiting case where two subsets of a data set with VC dimensions of and , respectively, are classified correctly exactly and only by the two hypothesis sets).

Rewriting the second term using the equality (from a Pascal’s triangle perspective) and applying a change of variables ,

Then, the break point can be determined by finding the that gives , where is some arbitrary positive function.

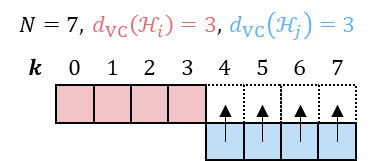
The first summation becomes equal to when the contributions from to is added to it.

* When , the second summation has more than enough terms to “transfer” to the first summation to change the upper bound of the latter to . After this “move”, the second summation still has remaining terms, making it a positive term (thus not satisfying the conditions for the term). An example with is shown in the schematic below, where each box is a combination term, and the top and bottom rows represent the first and second summations.

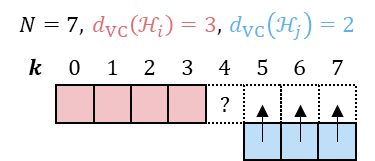


If or increases, there will be more boxes on the second row left over and the second summation stays a positive term. Therefore, also holds.

* When , the second summation has just enough terms that, when combined with the first summation, gives .



* When , the second summation does not have enough terms to “complete” the first summation. An example is shown below for only, but the same scenario arises when .



As such, the missing terms come from outside the second summation and must be subtracted away in the end, i.e., , where , and and are the smaller and larger, respectively, of . Therefore, the break point is , and the corresponding VC dimension is .

This means that the VC dimension of the union set of two hypothesis sets is bounded by

From the logic and schematics on the previous page, it is obvious that for each additional hypothesis set , the number of boxes (and consequently, the VC dimension) increases by .

For example, the upper bound on the VC dimension for a union set with three hypothesis sets is

By generalizing for a union set with hypothesis sets, the upper bound on the VC dimension can be shown to be