$$\underbrace{\sum_{k=0}^{n} 2^k = 2^{n+1}-1}_{} \Longrightarrow \sum_{k=0}^{n} q^k = \frac{q^{n+1}-1}{q-1} \quad (q \neq 1)$$

Identité non démontrée ici!

① $\forall m \in \mathbb{N}_{\geq 2}, \quad \displaystyle\sum_{k=0}^{n} 2^{mk} = \frac{2^{m(n+1)}-1}{2^m-1}$.

Démo.

On pose : • $S_n = \displaystyle\sum_{k=0}^{n} 2^k$

• $S_n^m = \displaystyle\sum_{k=0}^{n} 2^{mk}$ (donc $S_n^1 = S_n$)

$$\begin{aligned}
S_n^m &= 1 + 2^m + 2^{2m} + \dots + 2^{(n-1)m} + 2^{nm} \\
&= S_{nm} - (2+2^2+\dots+2^{m-1}) \\
&\quad - (2^{m+1}+2^{m+2}+\dots+2^{2m-1}) \\
&\quad - \dots \\
&\quad - (2^{(n-1)m+1}+2^{(n-1)m+2}+\dots+2^{nm-1}) \\
&= S_{nm} - 2(1+2+\dots+2^{m-2}) \\
&\qquad \times (1+2^m+\dots+2^{(n-1)m}) \\
S_n^m &= S_{nm} - 2S_{m-2} \times (S_n^m - 2^{nm})
\end{aligned}$$

On a alors :

$$\begin{aligned}
(1+2S_{m-2})S_n^m &= S_{nm} + 2^{nm+1} S_{m-2} \\
(1+2(2^{m-1}-1))S_n^m &= 2^{nm+1}-1 \\
&\quad + 2^{nm+1}(2^{m-1}-1) \\
(2^m-1)S_n^m &= 2^{(n+1)m}-1
\end{aligned}$$

Bingo!

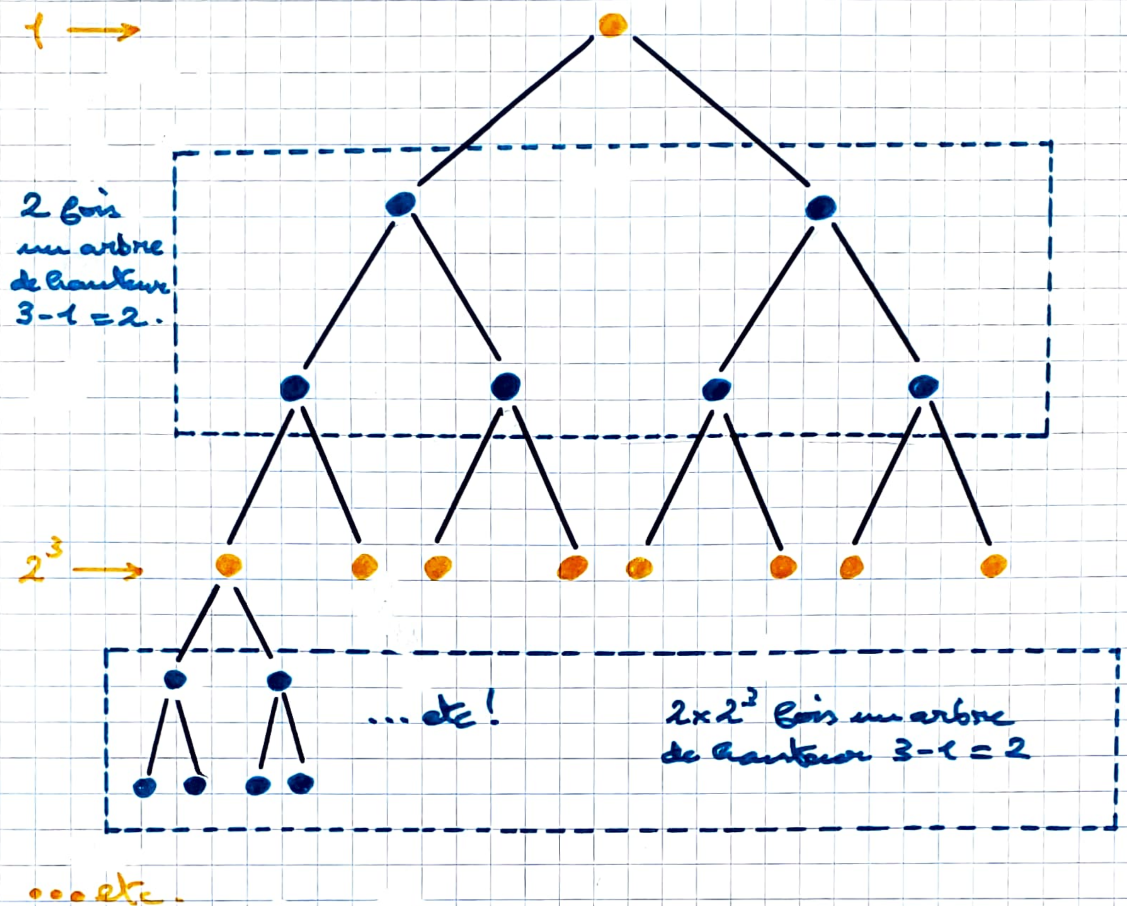
② Passage à $q \neq 1$ quelconque!

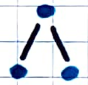
Démo.

$(q-1)\cdot\displaystyle\sum_{k=0}^{n} q^k = q^{n+1}-1$ admet une infinité de solutions, les 2^m pour $m \in \mathbb{N}^*$.

Rebingo!

③ Prenons du recul via un arbre (ici $m = 3$).



Voir : 

On obtient aisément :

$$S_{(n+1)m-1} = S_n^m + 2 \times S_{m-2} \times S_n^m$$

$$2^{(n+1)m} - 1 = S_n^m \left(1 + 2\left(2^{m-1} - 1\right)\right)$$

$$2^{(n+1)m} - 1 = S_n^m \left(2^m - 1\right)$$

Bien plus simple !