$$\forall k \in \mathbb{N}, P(X > k+1) = S_{k+1} = \int_{x_1+\cdots+x_k+x_{k+1}<1,\, x_i \geq 0} 1\mathrm{d}x_1 \cdots \mathrm{d}x_k \mathrm{d}x_{k+1}$$

$$\forall k \in \mathbb{N}^*, S_{k+1} = \int_{x_{k+1}=0}^{x_{k+1}=1^-} \left(\int_{x_1+\cdots+x_k<1-x_{k+1},\, x_i \geq 0} 1\mathrm{d}x_1 \cdots \mathrm{d}x_k \right) \mathrm{d}x_{k+1}$$

par k changements de variables $u_i = \dfrac{1}{1-x_{k+1}} x_i$ on obtient

$$\forall k \in \mathbb{N}^*, S_{k+1} = \int_{x_{k+1}=0}^{x_{k+1}=1^-} \left(\int_{u_1+\cdots u_k<1,\, u_i \geq 0} (1-x_{k+1})^k \,\mathrm{d}u_1 \cdots \mathrm{d}u_k \right) \mathrm{d}x_{k+1}$$

$$\forall k \in \mathbb{N}^*, S_{k+1} = S_k \int_{x_{k+1}=0}^{x_{k+1}=1^-} (1-x_{k+1})^k \,\mathrm{d}x_{k+1}$$

$$\forall k \in \mathbb{N}^*, S_{k+1} = \frac{S_k}{k+1}$$

or $S_1 = 1$ donc par récurrence,

$$\forall k \in \mathbb{N}^*, S_k = \frac{1}{k!}$$

$$\forall k \in \mathbb{N}^*, P(X \leq k) = 1 - \frac{1}{k!}$$

ainsi

$$\forall k \in \mathbb{N}^*, P(X = k) = P(X \leq k+1) - P(X \leq k) = \frac{k-1}{k!}$$

d'où

$$E(X) = \sum_{k=2}^{+\infty} kP(X=k) = \sum_{k=2}^{+\infty} \frac{1}{(k-2)!} = \sum_{k=0}^{+\infty} \frac{1}{k!}$$

La conjecture est démontrée, $E(X) = \mathrm{e}$.