Polynomials as polygons

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Recently I came upon an old Russian periodical "Messenger of Experimental Physics and Elementary Mathematics", that was published in 1886–1917. The magazine is available in the digital form [9]. The editors goal, in my translation from somewhat old-fashioned Russian, was as follows:

> ... Our magazine is intended primarily, but not exclusively, for the young men who study in our educational institutions, and therefore it will foremost strive to satisfy, in the realm of Physics and Mathematics, the urge to expand one's mental horizons which particularly strongly claims its rights in the juvenile age, always manifesting itself among young students as an irresistible desire to learn more than the official curriculum demands...

Browsing through the issues of this magazine, I discovered for myself a curious approach to finding real roots of real polynomials in one variable, Lill's method [8]. In spite of having been described in the popular literature [7, 4, 3], it remains a lesser known gem.

M. E. Lill (1830–1900) was an Austrian military engineer who published his method of solving polynomial equations in 1867 [5] and later presented it at the Vienna World Exposition in 1873. W. H. Bixby described Lill's method in the privately published pamphlet [1], see also [2]. One of the reason to revisit Lill's method is that it is easily implemented on a computer; in particular, the illustration in this note were created with the interactive geometry software Cinderella [10].

Let $p(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$ be a real polynomial. The first step of Lill's method is to represent $p(x)$ by a planar polygonal line whose edges have alternating horizontal and vertical directions.

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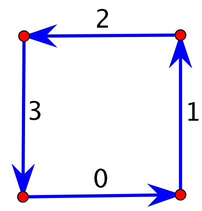

Figure 1: The positive direction convention.

One starts from the origin in the horizontal direction, going distance $|a_0|$, then turns 90° and goes distance $|a_1|$, etc. The positive direction for the coefficient a_k depends on the residue of k mod 4 and is depicted in Figure 1. See Figure 2 for the polygonal representation of the polynomial $x^3 - 2x^2 - 5x + 6$.

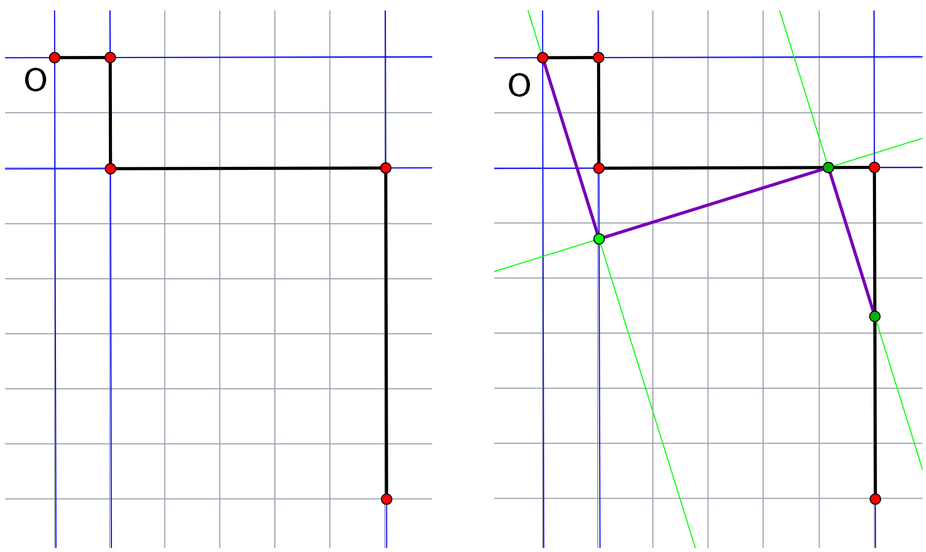

Figure 2: The polynomial $x^3 - 2x^2 - 5x + 6$ and a billiard trajectory.

Now we shall play a non-conventional billiards game. The usual billiard reflection law is that the angle of incidence equals the angle of reflection; in our game, the billiard trajectory turns 90° at the impact point.

Shoot the billiard ball from the origin until it meets the vertical line containing a_1, make a 90° turn and go straight until the intersection with the horizontal line containing a_2, make make a 90° turn, and so on. The last point of the trajectory is the intersection with the line containing a_n, see Figure 2.

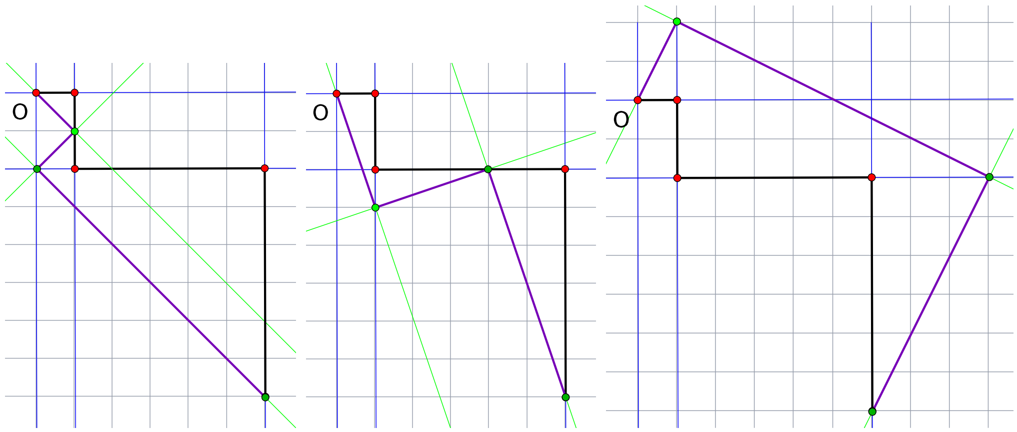

Figure 3: The three roots, $1, 3, -2$, of the polynomial $x^3 - 2x^2 - 5x + 6$.

The goal of the game is to hit the terminal point of the polygonal line representing the polynomial. The real roots of the polynomial $p(x)$ are the numbers $-k$, where k are the slopes of the winning billiard shots. See Figures 3 and 4.

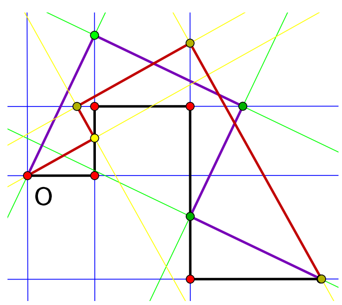

Figure 4: A polynomial of degree four and its two real roots.

The proof is disappointingly easy, especially in the case when all coefficients are positive. Let k be the slope of a winning billiard shot, and $x_0 = -k$. Consider Figure 5.

One has:

$$OA_0 = a_0,\ A_0B_1 = ka_0,\ B_1A_1 = a_1 - ka_0,\ A_1B_2 = k(a_1 - ka_0),$$
$$B_2A_2 = a_2 - k(a_1 - ka_0),\ A_2B_3 = k(a_2 - k(a_1 - ka_0)) = a_3,$$

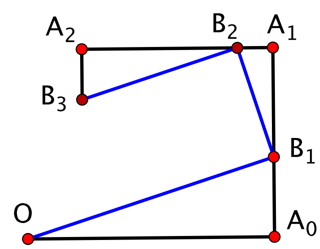

Figure 5: Proof of Lill's method (polynomial of degree three with positive coefficients).

hence,

$$a_3 + x_0(a_2 + x_0(a_1 + x_0a_0)) = 0, \tag{1}$$

that is, x_0 is a root of the polynomial $a_0x^3 + a_1x^2 + a_2x + a_3 = 0$. The reader has recognized in (1) Horner's rule for calculating polynomials.

We leave it to the reader to convince herself that the same argument works for any degree and an arbitrary combination of the signs of coefficients.

Furthermore, the billiard trajectory, corresponding to a root x_0 of a polynomial $p(x)$, is again a right-angled polygonal line. Rotate it to make the first side horizontal, and we again deal with a polynomial, of degree one less. Not surprisingly, up to scaling, this polynomial is $p(x)/(x - x_0)$.

Let us explain this on the same example, Figure 5. Let $\alpha = \angle A_0OB_1$, and let

$$OB_1 = b_0,\ B_1B_2 = b_1,\ B_2B_3 = b_2.$$

Then

$$a_0 = b_0\cos\alpha,\ a_1 = b_1\cos\alpha + b_0\sin\alpha,\ a_2 = b_2\cos\alpha + b_1\sin\alpha,\ a_3 = b_2\sin\alpha,$$

and hence

$$\begin{aligned} a_0x^3 + a_1x^2 + a_2x + a_3 &= (x\cos\alpha + \sin\alpha)\ (b_0x^2 + b_1x + b_2) \\ &= \cos\alpha\ (x - x_0)\ (b_0x^2 + b_1x + b_2). \end{aligned}$$

Thus one can apply Lill's method iteratively.

And what about complex roots? A version of Lill's method was described in [6], a sequel to the original paper. The answer is less elegant: one still has a sequence of similar triangles, but their vertices are do not lie on the polygonal

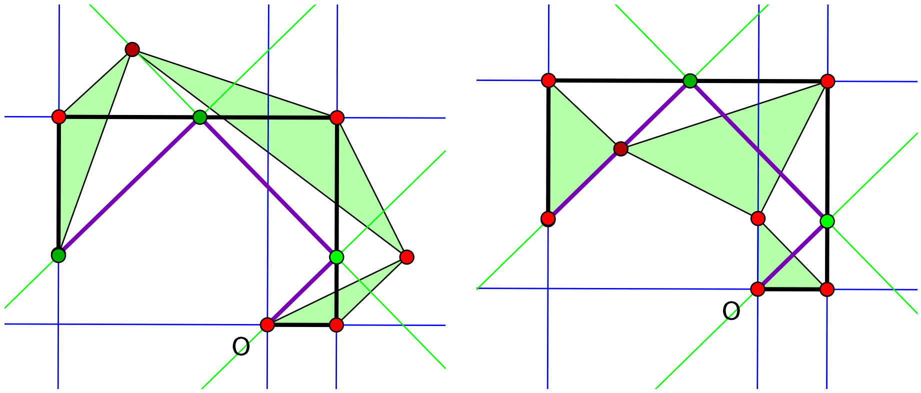

Figure 6: Two complex conjugate roots $-1 \pm i$ of the polynomial $x^3 + 3x^2 + 4x + 2$.

line that describes the polygon: the distance from this line is proportional to the imaginary part of the root. See Figure 6.

Let us conclude with another example, depicting $-\sqrt[10]{2}$, see Figure 7.

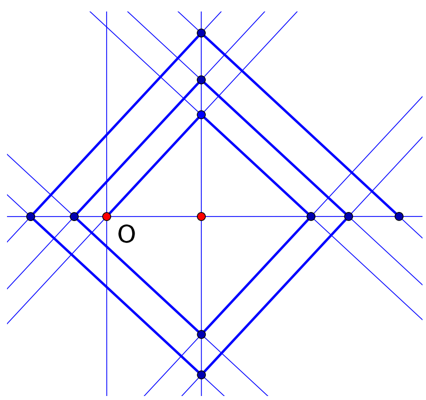

Figure 7: A root of the polynomial $x^{10} - 2$.

References

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[9] `http://vofem.ru` (in Russian).

[10] `http://www.cinderella.de/tiki-index.php`.