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Lifting the exponent

Posted on April 13, 2014 by apgoucher

I overheard mention of a particular problem on a recent British Mathematical Olympiad, namely the following:

> *A number written in base 10 is a string of 3^2013 digit 3s. No other digit appears. Find the highest power of 3 which divides this number.*

Personally, I bemoan such problems that are trivialised by the knowledge of advanced theorems, as it enables competitors to gain an unfair advantage by rote-learning many results rather than demonstrating creative mathematical thought. In this case, the question is trivialised by a rather elegant but little-known lemma called *lifting the exponent.*

So, what does the lemma state? Firstly, we establish the following definition:

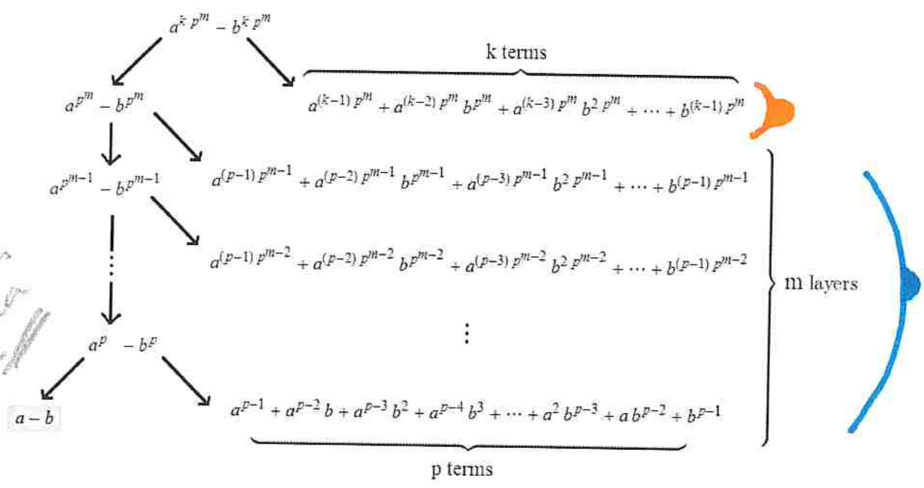
Definition: the *p-adic valuation v_p(n)* of an integer *n* to be the highest power of *p* which divides *n*. For example, *v*_2(40) = 3, since 2^3 divides 40 but 2^4 does not.

Then the lemma is as follows:

Theorem (lifting the exponent): Let *p* be an odd prime, and *a* and *b* integers such that neither *a* nor *b* is divisible by *p*, but *p* divides their difference *a* – *b*. Then *v_p(a^n* – *b^n)* = *v_p(a* – *b)* + v_*p(n)*.

Why is this true? The idea is that we factorise *a^n* – *b^n* like so, where *n* = *k p^m* and *k* is not divisible by *p*:

Joli ! Mais pas simple sinon [illegible]



The factors in the final factorisation are highlighted. It is clear that it is sufficient to prove that the yellow factor is coprime to *p* (which is easy, since all of the terms are congruent modulo *p* and are non-zero) and each of the blue factors are divisible by *p* (easy for the same reason) but not by *p*², as we shall prove:

- If *a* and *b* are congruent modulo *p*², this is again trivial for the same reason as before.
- Otherwise, we have to actually rely on the property that *p* is odd (if *p* = 2, we need *a* and *b* to be congruent modulo 4 rather than modulo 2). We let *x* and *y* be equal to *a^p^i* and *b^p^i*, respectively, for the obvious value of *i*, such that the factor is of the form:

$\Gamma = x^{(p-1)} + x^{(p-2)} y + x^{(p-3)} y^2 + \ldots + y^{(p-1)}$

Then we set $y = x + lp$ for some integer l (which we can do, since y and x are clearly congruent modulo p). Expand the factor Γ to produce a sum of binomial expansions; we can ignore all terms of order p^2 and higher since we're only interested in the residue modulo p^2. This gives the following expression:

$$px^{p-1} + \frac{1}{2}p^2(p-1)x^{p-1}$$

The rightmost term vanishes, leaving something that is clearly not divisible by p^2. Consequently, the proof is complete and the result follows immediately.

Zsigmondy's theorem

Another useful fact concerning $a^n - b^n$ is this: except in a few exceptional cases, it has a new prime factor p that does not occur in any of $a - b$, $a^2 - b^2$, $a^3 - b^3$, ..., $a^{(n-1)} - b^{(n-1)}$. The exceptions to the rule are the following:

- **$a = 2$, $b = 1$, $n = 6$:** we have $2^6 - 1^6 = 63$, whose prime factors are 3 and 7, which occur in $2^2 - 1^2$ and $2^3 - 1^3$, respectively.
- **$a + b$ is a power of 2, and $n = 2$:** then $a^2 - b^2 = (a + b)(a - b)$. The first factor is a power of 2 (so no new primes there), and the second factor is itself the previous term in the sequence (so, by definition, no new primes there either).

A related statement about Fibonacci numbers (where the integers a and b are replaced with irrational algebraic integers, and an extra factor of $\sqrt{5}$ slips in) is known as Carmichael's theorem. Zsigmondy's theorem and Carmichael's theorem can be mutually generalised to other Lucas sequences.

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