1 Calcul de $\displaystyle\int \frac{1}{(1-x)\sqrt{1-x^2}}\,\mathrm{d}x$

1.1 Méthode 1 (pb de signes)

$$
\begin{aligned}
&\int \frac{1}{(1-x)\sqrt{1-x^2}}\,\mathrm{d}x\\
&= \int \frac{1+x}{(1-x^2)^{1.5}}\,\mathrm{d}x\\
&= \int \frac{1}{(1-x^2)^{1.5}}\,\mathrm{d}x + \int \frac{x}{(1-x^2)^{1.5}}\,\mathrm{d}x && \color{blue}{x = \sin\theta}\\
&= \int \frac{\cos\theta}{(1-\sin^2\theta)^{1.5}}\,\mathrm{d}\theta + \frac{1}{(1-x^2)^{0.5}}\\
&= \int \frac{1}{\cos^2\theta}\,\mathrm{d}\theta + \frac{1}{\sqrt{1-x^2}}\\
&= \tan\theta + \frac{1}{\sqrt{1-x^2}} && \color{blue}{x = \sin\theta}\\
&= \tan(\operatorname{asin} x) + \frac{1}{\sqrt{1-x^2}}
\end{aligned}
$$

1.2 Méthode 2 (pb de signes)

$$
\begin{aligned}
&\int \frac{1}{(1-x)\sqrt{1-x^2}}\,\mathrm{d}x && \color{blue}{x = \cos\theta}\\
&= \int \frac{1}{\cos\theta - 1}\,\mathrm{d}\theta && \color{blue}{\cos\theta = \frac{1-t^2}{1+t^2}\ \textit{avec}\ t = \tan\left(\frac{\theta}{2}\right)\ \textit{et}\ \mathrm{d}t = (1+t^2)\cdot\frac{\mathrm{d}\theta}{2}}\\
&= \int \frac{1+t^2}{-2t^2}\cdot\frac{2\,\mathrm{d}t}{1+t^2}\\
&= \int \frac{-1}{t^2}\,\mathrm{d}t\\
&= \frac{1}{t} && \color{blue}{t = \tan\left(\frac{\theta}{2}\right)\ \textit{et}\ x = \cos\theta}\\
&= \frac{1}{\tan\left(\frac{\operatorname{acos} x}{2}\right)}
\end{aligned}
$$

1.3 Méthode 3

Changeons de point de vue et tentons notre chance avec $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$.
On choisit $u(x) = \sqrt{1-x^2}$ de sorte que l'on a :

$$\begin{aligned} & u'(x)v(x) - u(x)v'(x) \\ = & \frac{-x}{\sqrt{1-x^2}} v(x) - \sqrt{1-x^2}\, v'(x) \\ = & \frac{-xv(x) - (1-x^2)v'(x)}{\sqrt{1-x^2}} \end{aligned}$$

Essayons $v(x) = 1 - x$. Ceci nous donne :

$$\begin{aligned} & -xv(x) - (1-x^2)v'(x) \\ = & -x(1-x) + (1-x^2) \\ = & -x + 1 \end{aligned}$$

Nos choix aboutissent...

$$\begin{aligned} & \left(\frac{u}{v}\right)'(x) \\ = & \frac{-x+1}{(1-x)^2\sqrt{1-x^2}} \\ = & \frac{1}{(1-x)\sqrt{1-x^2}} \end{aligned}$$

Finalement l'audace a payé.

$$\begin{aligned} & \int \frac{1}{(1-x)\sqrt{1-x^2}}\,\mathrm{d}x \\ = & \frac{\sqrt{1-x^2}}{1-x} \end{aligned}$$

1.4 Méthode 4

$$\int \frac{1}{(1-x)\sqrt{1-x^2}}\,\mathrm{d}x$$

$$= \int \frac{1+x}{(1-x^2)^{1.5}}\,\mathrm{d}x$$

$\binom{-1.5}{k} = \frac{1}{k!}\cdot\prod_{i=0}^{k-1}(-1.5-i)$

$$= \int (1+x)\cdot\sum_{k\geq 0}\binom{-1.5}{k}(-x^2)^k\,\mathrm{d}x$$

$n/2$: quotient de la division euclidienne de n par 2.

$$= \int \sum_{n\geq 0}\binom{-1.5}{n/2}\cdot(-1)^{n/2}x^n\,\mathrm{d}x$$

Choix de la constante pour la suite...

$$= 1 + \sum_{n\geq 0}\binom{-1.5}{n/2}\cdot(-1)^{n/2}\frac{x^{n+1}}{n+1}$$

Distinguons deux cas pour étudier les coefficients devant les x^{n+1}.

1. $n = 2k$ i.e. $n+1 = 2k+1$

$$\binom{-1.5}{n/2}\cdot\frac{(-1)^{n/2}}{n+1}$$

$$= \binom{-1.5}{k}\cdot\frac{(-1)^k}{2k+1}$$

$$= \frac{1}{k!}\cdot\prod_{i=0}^{k-1}(-1.5-i)\cdot\frac{(-1)^k}{2k+1}$$

$j = i+1$

$$= \frac{1}{k!}\cdot\prod_{j=1}^{k}(-0.5-j)\cdot\frac{(-1)^k}{2k+1}$$

$\frac{-0.5-k}{2k+1} = -0.5$

$$= \frac{1}{k!}\cdot\prod_{j=0}^{k-1}(-0.5-j)\cdot(-1)^k$$

$$= \binom{-0.5}{k}\cdot(-1)^k$$

Nous avons un coefficient simple de $x^{2k+1} = x\cdot x^{2k}$.

2. $n = 2k+1$ i.e. $n+1 = 2(k+1)$. Cherchons un coefficient simple de $x^{n+1} = x^{2(k+1)}$.

$$\begin{aligned}
&\binom{-1.5}{n/2} \cdot \frac{(-1)^{n/2}}{n+1} \\
&= \binom{-1.5}{k} \cdot \frac{(-1)^k}{2k+2} \\
&= \frac{1}{k!} \cdot \prod_{i=0}^{k-1} (-1.5 - i) \cdot \frac{(-1)^k}{2k+2} \\
&= \frac{1}{k!} \cdot \prod_{j=1}^{k} (-0.5 - j) \cdot \frac{(-1)^k}{2(k+1)} \qquad j = i+1 \\
&= \frac{1}{(k+1)!} \cdot \prod_{j=0}^{k} (-0.5 - j) \cdot (-1)^{k+1} \\
&= \binom{-0.5}{k+1} \cdot (-1)^{k+1}
\end{aligned}$$

Finalement nous obtenons :

$$\begin{aligned}
&\int \frac{1}{(1-x)\sqrt{1-x^2}} \,\mathrm{d}x \\
&= 1 + \sum_{n \geq 0} \binom{-1.5}{n/2} \cdot (-1)^{n/2} \frac{x^{n+1}}{n+1} \\
&= 1 + \sum_{k \geq 0} \binom{-1.5}{k} \cdot (-1)^k \frac{x^{2k+2}}{2k+2} + \sum_{k \geq 0} \binom{-1.5}{k} \cdot (-1)^k \frac{x^{2k+1}}{2k+1} \\
&= 1 + \sum_{k \geq 0} \binom{-0.5}{k+1} \cdot (-1)^{k+1} (x^2)^{k+1} + \sum_{k \geq 0} \binom{-0.5}{k} \cdot (-1)^k \cdot x \cdot (x^2)^k \\
&= \sum_{j \geq 0} \binom{-0.5}{j} \cdot (-x^2)^j + x \cdot \sum_{k \geq 0} \binom{-0.5}{k} \cdot (-x^2)^k \\
&= (1+x)(1-x^2)^{-0.5} \\
&= \frac{1+x}{\sqrt{1-x^2}}
\end{aligned}$$