DM - BERTHOLET - $\binom{n}{p}$ & Fibo.

(Fibo-1)

GHYS
Conf. Maths. Demain

Street Art
à Gre.

PB1

Partie I $\quad 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89$

1) • $F_5 = 5$, $F_6 = 8 \geq 6$

• Récu. sur $n \geq 7$ via $F_n = F_{n-1} + F_{n-2}$

2) • $(F_n) \nearrow$ via $F_{n+1} - F_n = F_{n-1}$

• (F_n) st $\nearrow$ dès $n \geq 2$

3) Méthode 1

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix}$$

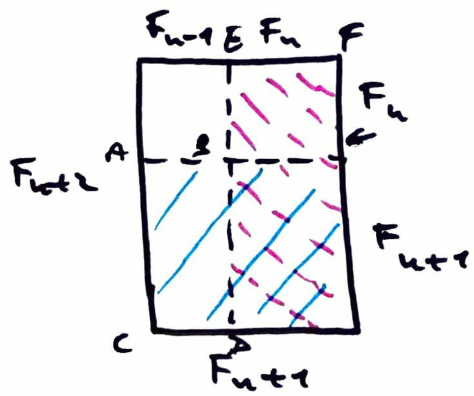
$$\hookrightarrow \begin{pmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

Via det, on a :

$$F_{n+2}F_n - F_{n+1}^2 = -\left(F_{n+1}F_{n-1} - F_n^2\right)$$

... etc

Méthode 2 (vue sur le net)



$$F_{n+1}^2 - F_n F_{n+2}$$
$$= a_{ABCD} - a_{AEFG}$$
$$= F_{n+1}F_{n-1} - F_n^2$$
... etc

$\frac{F_n}{F_{n+1}}$ irréd. ? Evident car $\operatorname{pgcd}(F_{n+1}, F_n) \mid (-1)^n$

Fibo-2

$$G_n = \varphi^n - \frac{(-1)^n}{\varphi^n} = \frac{1}{\varphi^n}\left(\varphi^{2n} - (-1)^n\right)$$

$$\varphi^{n+2}\,(G_n + G_{n+1})$$
$$= \varphi^2\left(\varphi^{2n} - (-1)^n\right) + \varphi\left(\varphi^{2n+2} - (-1)^{n+1}\right)$$
$$= \varphi^{2n+4}\underbrace{\left(\frac{1}{\varphi^2} + \frac{1}{\varphi}\right)}_{\frac{1+\varphi}{\varphi^2}} - (-1)^n(\varphi^2 - \varphi)$$

$$\begin{cases} \varphi^2 = \frac{1}{4}(1 + 5 + 2\sqrt{5}) = \frac{3+\sqrt{5}}{2} \\ 1 + \varphi = \frac{3+\sqrt{5}}{2} \\ \varphi^2 - \varphi = 1 \end{cases}$$

$$\varphi^{n+2}(G_n + G_{n+1}) = \varphi^{2n+4} - (-1)^{n+2}$$

$$G_n + G_{n+1} = G_{n+2}$$

... etc

<u>Partie II</u> (cf. ch. 5?)

6/ $S_n = \sum_0^n (F_{k+2} - F_{k+1})$

$$= \sum_2^{n+2} F_k - \sum_1^{n+1} F_k$$
$$= F_{n+2} - F_1$$

$F_0 + \dots + F_n = F_{n+2} - 1$

Via $F_{k+1} - F_{k-1}$:

$$\sum_0^n F_k = \sum_1^n (F_{k+1} - F_{k-1})$$
$$= \sum_2^{n+1} F_k - \sum_0^{n-1} F_k$$
$$= F_{n+1} + F_n - F_0 - F_1$$
$$= F_{n+2} - 1$$

7/ $I_n = \sum_0^n (F_{2k+2} - F_{2k})$

$$= \sum_1^{n+1} F_{2k} - \sum_0^n F_{2k}$$
$$= F_{2n+2} - F_0$$

8/ $P_n = S_{2n+1} - I_n$

$$= F_{2n+3} - 1 - F_{2n+2} = \dots$$

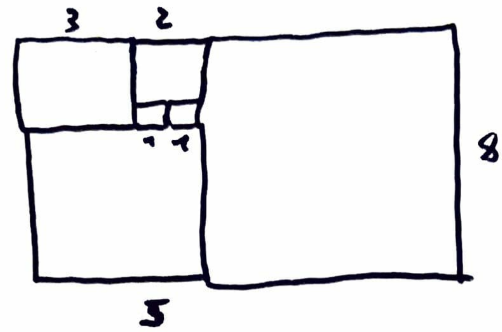
(Fibo-3

9/ $\sum_0^n F_{k+1}^2 = \sum_0^n (F_{k+2}F_{k+1} - F_{k+1}F_k)$

$= \sum_1^{n+1} F_{k+1}F_k - \sum_0^n F_{k+1}F_k$

$= F_{n+1}F_n - F_1F_0$

<u>ou</u>



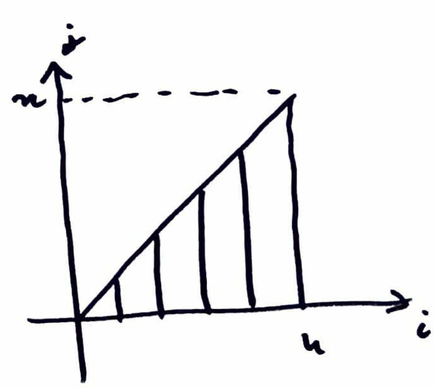
10/ • $D_n = \sum_{i=0}^n \sum_{j=0}^i F_j$

$= \sum_{i=0}^n (F_{i+2} - 1)$

$= \sum_{i=2}^{n+2} F_i - n - 1 - F_0 - F_1$

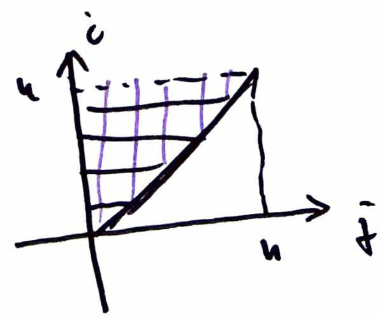
$= F_{n+4} - n - 2$

• $D_n = \sum_{i=0}^n (n-i+1)F_i$



$D_n = \sum_{j=0}^n \sum_{i=j}^n F_j$

$= \sum_{j=0}^n (n+1-j)F_j$



• $E_n = ?$

$$D_n = (n+1)\sum_0^n F_j - \sum_0^n j F_j$$

$$D_n = (n+1)(F_{n+2} - 1) - E_n$$

... etc

Partie III (où est en 12 ?)

Pourquoi ?

On peut observer que occu. de 6 arrive sur $\binom{a}{b-1}$ et $\binom{a-1}{b}$?

À creuser !

12) a) $\binom{a}{b-1} / \binom{a-1}{b}$

$$= \frac{a!}{(b-1)!\,(a-b+1)!} \times \frac{b!\,(a-b-1)!}{(a-1)!}$$

$$= \frac{a\,b}{(a-b)(a-b+1)}$$

b) $\binom{a}{b-1} = \binom{a-1}{b}$

$\Leftrightarrow \quad ab = (a-b)(a-b+1) \quad \boxed{c}$

$a = F_k F_{k+1}, \; b = F_k F_{k-1}$

$a - b = F_k^2$

$\boxed{c}$

$\Leftrightarrow \quad F_k^2 F_{k-1} F_{k+1} = F_k^2 (F_k^2 + 1)$

$\Leftrightarrow \quad F_k^2 + 1 = F_{k-1} F_{k+1}$

$\Leftrightarrow \quad F_k^2 - F_{k-1} F_{k+1} = -1 = (-1)^{k-1}$

$\Leftrightarrow \quad k \in 2\mathbb{N}$ Joli !

& utile...

<u>Ex</u> : $a = F_4 F_5 = 15$, $b = F_4 F_3 = 6$

$\binom{a}{b-1} = \binom{15}{5} = 3003$

$\binom{a-1}{b} = \binom{14}{6} = 3003$

occu. ≥ 6 dans ces cas !

…1/a/– f/ Immédiat!

Fibo-5

8/ $\binom{78}{2} = \dfrac{78\times 77}{2}$

$= 39 \times 7 \times 11$

$= 3003$ ...

$40 \times 7 = 280$

$39 \times 7 = 273$

273×11

$$\begin{array}{r} 273 \\ 273 \\ \hline 3003 \end{array}$$

14/

1

1 2 1 0

1 3 3 1

1 4 6 4 1

0 1 5 ...

1 6 ...

• = • + •

Joli aussi!

1

1 1

1 2 1

1 3 3 1

Donc Σ ligne

$= \square + \circ$

$= 2\,\Sigma$ ligne précédente

$$D_n = \sum_{k=0}^{n-1} \binom{n-1-k}{k}$$ → n-ième diago.

$$D_{n+2} = \sum_{0}^{n+1} \binom{n+1-k}{k}$$

$$\binom{N}{K} + \binom{N}{K+1} = \binom{N+1}{K+1}$$

$$= \sum_{1}^{n+1} \binom{n+1-k}{k} + \binom{n+1}{0}$$

$$= \sum_{0}^{n} \binom{n-k}{k+1} + 1$$

$$= \sum_{0}^{n} \binom{n-k-1}{k} + \sum_{0}^{n} \binom{n-k-1}{k+1} + 1$$

$$= D_n + \binom{-1}{0} + \sum_{1}^{n+1} \binom{n-k}{k} + 1$$

$$= D_n + D_{n+1} + \binom{-1}{n+1} \underbrace{- \binom{n}{0} + 1}_{0}$$

$$= D_n + D_{n+1}$$

Fibo.7

On a aussi :

$$\begin{array}{ccc} & (a+b)^n & \\ a(a+b)^{n-1} & & b(a+b)^{n-1} \\ \vdots & & \vdots \end{array}$$

Donc, $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

Mieux :

$$\begin{array}{ccc} & (fg)^n = ((fg)')^{n-1} & \\ (f'g)^{n-1} & & (fg')^{n-1} \\ \vdots & & \vdots \end{array}$$

Donc, $(fg)^n = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$

Élémentaire
& Joli !
(revoir Blog via opérateurs
sur bases ∞ cas joli aussi)

$\boxed{\text{OU}}$

Via $\underbrace{\sum_0^\infty \frac{F_n}{n!} x^n}$ (et $\sum_0^\infty F_n x^n$)

$\hookrightarrow$ ... $\times e^x$ fera apparaître $\binom{n}{k}$

15) **Récu. num avec m qcq**

- **n = 0**

$$F_{m+2\times 0} = F_m = \binom{0}{0} F_m$$

- **n = 1**

$$F_{m+2\times 1} = F_m + F_{m+1} = \binom{1}{0} F_m + \binom{1}{1} F_{m+1}$$

- **n ≥ 2**

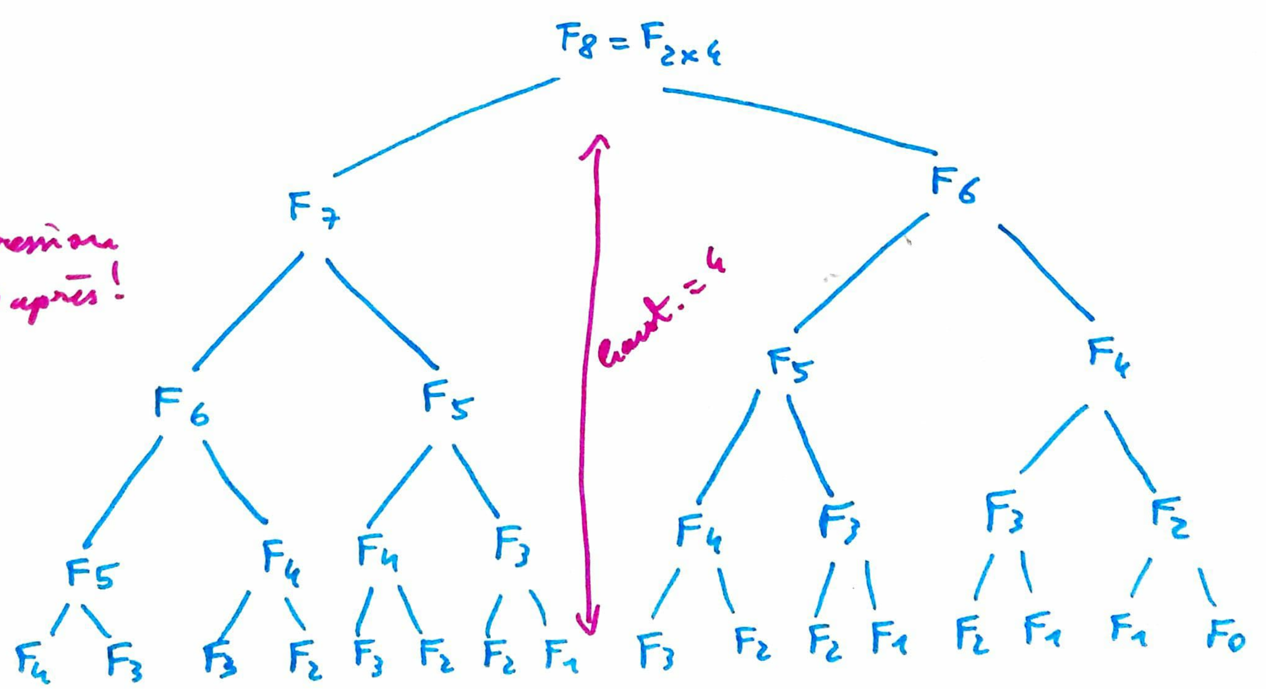
On pose $n = N + 1$

$n \geq 1$ OK

$$F_{m+2n} = F_{m+2N} + F_{m+2N+1}$$
$$= \sum_0^N \binom{N}{k} F_{k+m} + \sum_0^N \binom{N}{k} F_{k+m+1}$$
$$= \binom{N}{0} F_m + \sum_1^N F_{k+m}\left(\binom{N}{k} + \binom{N}{k-1}\right) + \binom{N}{N} F_{N+m+1}$$
$$= \sum_0^{N+1} \binom{N+1}{k} F_{k+m}$$
$$= \sum_0^n \binom{n}{k} F_{k+m}$$

D'où cela vient-il ?

Le jeu des indices et de la rela° de récu. permet de se concentrer sur $F_{2n} = \sum_0^n \binom{n}{k} F_k$.



Digression Top après !

$F_0 : 1$
$F_1 : 4$
$F_2 : 6$
$F_3 : 4$
$F_4 : 1$

Que c'est beau ! On compte alors les chgts de direc° et les $\binom{n}{k}$ apparaissent.

Partie IV

16/ Facile mais Ea mauvais.

17) $n \in \mathbb{N}^*$

- $n=1$: $n = F_1$ ou F_2
- $n=2$: $n = F_3$ pas de choix.

- n geq ≥ 2

  ℓ max tq $F_\ell \leq n$

  Donc $n < F_{\ell+1}$.

  Mq $n - F_\ell \leq F_{\ell-1}$ de sorte qu'une récu. sera ok.

  Sinon $n - F_\ell > F_{\ell-1}$

  $\Rightarrow n > F_{\ell+1}$ contradic°

OK mais sans non consécutivité

Facile via $n - F_\ell < F_{\ell-1}$ car $n - F_\ell \geq F_{\ell-1}$ pas possible.

18/ $$\sum_i F_{\ell_i} \leq \underbrace{\sum_0^{\ell_{i\max}} F_j}_{F_{\ell_{i\max}+2}} - 1$$

$$\sum_i F_{\ell_i} = \sum_j F_{\ell_j}$$ possible uniquem^t^

si $\ell_{i\max} = \ell_{j\max} + 1$ (ou version rec.)

$$F_{\ell+1} - F_\ell = F_{\ell-1}$$ → on descend jusqu'au oracle!