A Problem That Changed My Mind About Mathematics

StephenwithaPhD Follow
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In this article, I want to discuss one of the earliest problems which completely changed my viewpoint of mathematics and pushed me to pursue a career in it.

This problem is from the 2009 British Mathematical Olympiad which I first came across when attending a problem session ran by my University's mathematics society. The problem is the following:

Find all nonnegative integers a and b such that

$$\sqrt{a} + \sqrt{b} = \sqrt{2009}.$$

Before looking at one possible way of solving this problem, which requires nothing more than school-level arithmetic, I want to explain why I like this problem, and problems of this nature, so much. Growing up, I loved detective/mystery novels, tv shows, movies, whatever. Being able to construct a solution based on some scattered information seemed almost like a superpower to me.

Looking back now, it seems obvious to me that I would have an interest in mathematics, but of course, I didn't know that at the time. I just thought mathematics was there to do a job and that was it. Whether it was for calculating your taxes, working out the tensile strength of building materials, or following your baking recipe, I understood mathematics had a purpose, but it almost seemed it was just about knowing the right formula and applying it. Yawn! For some people, this is still how they see it. I hope this problem can at least justify why my mindset began to change. For me now, I see every math question as a puzzle. A mystery to be solved and studying mathematics is just a way of fine-tuning my detective abilities and adding to my tool kit.

Let's begin our solution to the problem. The first step most people would take is to square both sides of the equation. This is something we are always told to do when dealing with square roots. It's certainly what I tried. Let's do this with the equation as it is, so we have

$$\left(\sqrt{a}+\sqrt{b}\right)^2=\left(\sqrt{2009}\right)^2 \quad \Longrightarrow \quad a+2\sqrt{ab}+b=2009.$$

Ok, so this is a little nicer, maybe. We have reduced the number of square roots, which is good, but we have also made an unintentional problem for ourselves.
This is the first thing this problem taught me. Sometimes, before jumping in and just applying a method that seems right, ask yourself:
(i) Is this indeed the best method you know for dealing with this problem?
(ii) If this is the best method you know, is the problem set up in the best way for you to apply it?

Let me elaborate here. There is nothing wrong with what we have done. However, in squaring the equation, in the form it is was given, we have added in the variable √(ab). This has made things harder for us since we have now unintentionally muddled together the information given by the variables a and b.

What if instead we first rearranged our equation as

$$\sqrt{a}=\sqrt{2009}-\sqrt{b}.$$

Then we can again square both sides, only this time we get

$$\left(\sqrt{a}\right)^2=\left(\sqrt{2009}-\sqrt{b}\right)^2 \quad \Longrightarrow \quad a=2009-2\sqrt{2009b}+b.$$

Notice that this time we have again reduced the number of square roots we had, but we have additionally kept the variables a and b separate. This may seem like a simple difference but it makes all the difference with how we can proceed. We now make note of the fact that all terms of our new equation are clearly integers, except potentially 2 √(2009b).

Here comes the second thing I learned from this problem, deductive reasoning. Since we know that adding or subtracting two integers always results in an integer, and since we can re-write our equation as

$$2\sqrt{2009b} = 2009 - a + b,$$

we can deduce that 2 √(2009b) must indeed be an integer. This is a huge clue!

There are very few numbers for which taking the square root results in an integer (in fact, if you were to choose a number at random it would have a probability of 0 of having this property, this fact requires a little more explanation).

So how can we use this? Firstly, we will try to break this down a bit further. Think of it as distilling your information of the “excess stuff” that’s no good to you right now. We first notice that

$$2009 = 49 \times 41 = 7^2 \times 41.$$

Thus we can say that

$$\sqrt{2009b} = \sqrt{(7^2)(41b)} = 7\sqrt{41b}.$$

So, since √(2009b) is an integer, we must have that √(41b) is an integer. Thus we have distilled our clue down to a more clear piece of information.

Now that we have distilled our information to its clearest form, we next unpackage it. In particular, we ask ourselves, what does saying "√(41b) is an integer" actually mean. It means that we must have $41b=c^2$ for some integer c. But, the √(41) is not an integer, so this only makes sense if $b=41d^2$ for some integer d, since this allows us to write $41b=(41d)^2$. There we have it, our final, most telling piece of information. The nonnegative number b must be of the form

$$b = 41d^2$$

for some integer d.

The third thing I learned from this problem: Don't waste your time! What I mean by this is the following, there was nothing special about moving the √(b) across the equals sign in our original equation, we could just have easily started with the equation

$$\left(\sqrt{b}\right)^2 = \left(\sqrt{2009} - \sqrt{a}\right)^2$$

and everything would follow in the exact same way. In particular, we would arrive at a similar conclusion that $a=41e^2$ for some integer e. We call this a symmetric argument and really, all we are saying is that since it is clear that everything will work the exact same, if we were to replace the variable b with the variable a, we are not willing to write the argument down again and instead we will just skip to the conclusion.

So altogether we now have the following information (clues):

$$a = 41e^2 \quad and \quad b = 41d^2, \quad \text{where } d \text{ and } e \text{ are integers.}$$

We can in fact conclude that since a and b were nonnegative integers, d and e are also nonnegative integers.

So, going back to our original equation, we are going to fill in these clues and try to get out something new that we couldn't before. To me, this is like when the detective returns to a suspect, who originally supposedly knew nothing of use, but now the detective has evidence and the suspect begins to crack.

Filling back into the original equation we have the following:

$$\begin{aligned}
&\sqrt{a}+\sqrt{b}=\sqrt{2009}\\
&\implies \sqrt{41e^2}+\sqrt{41d^2}=\sqrt{2009}=\sqrt{49\times 41}\\
&\implies \sqrt{41}e+\sqrt{41}d=7\sqrt{41}\\
&\implies e+d=7.
\end{aligned}$$

This is certainly a much easier equation to deal with. There are only a very small number of solutions to this equation for d and e. In particular, we can have

$$d = 0 \quad \text{and} \quad e = 7$$

$$d = 1 \quad \text{and} \quad e = 6$$

$$d = 2 \quad \text{and} \quad e = 5$$

$$d = 3 \quad \text{and} \quad e = 4$$

$$d = 4 \quad \text{and} \quad e = 3$$

$$d = 5 \quad \text{and} \quad e = 2$$

$$d = 6 \quad \text{and} \quad e = 1$$

$$d = 7 \quad \text{and} \quad e = 0.$$

From this, and recalling that $a=41e^2$ and $b=41d^2$, we get our full set of possible solutions

$$a = 2009 \quad \text{and} \quad b = 0$$
$$a = 1476 \quad \text{and} \quad b = 41$$
$$a = 1025 \quad \text{and} \quad b = 164$$
$$a = 656 \quad \text{and} \quad b = 369$$
$$a = 369 \quad \text{and} \quad b = 656$$
$$a = 164 \quad \text{and} \quad b = 1025$$
$$a = 41 \quad \text{and} \quad b = 1476$$
$$a = 0 \quad \text{and} \quad b = 2009.$$

There you have it, another mystery solved by detective mathematics (*I really need to come ou with a better name)*.