Curvature, Geodesics, and Exponential Maps

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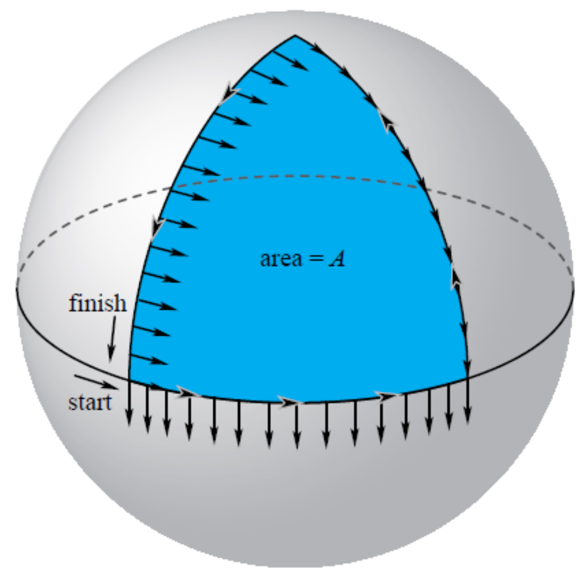
Geometric Phase on a Sphere

- Geometric phase example in Marsden and Ostrowski.
  - Vector locally points in same direction.
  - Orientation is changed as it moves away and back.

- Doing the same thing on the earth?

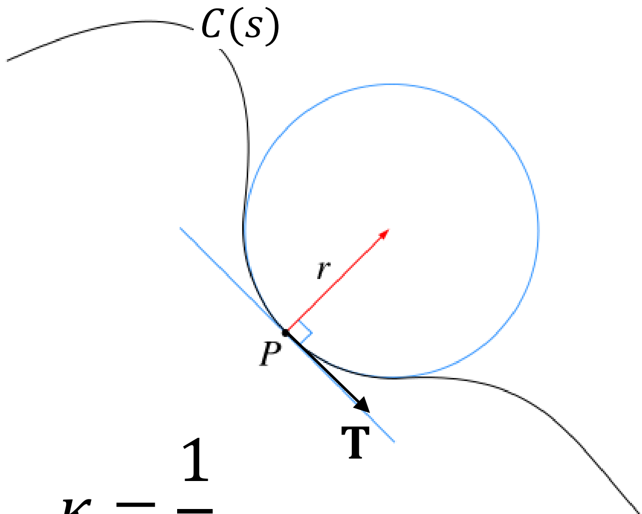
- What if the space were Euclidean?

- What prevents a manifold from *globally* looking Euclidean?



What Is Curvature?

- Simplest 1D manifold—a curve
- ***Curvature*** of the curve at a point P measures how fast we are changing directions at P.
- Two ways of looking at it
  - Draw a circle of curvature.
  - Look at the tangent vector.
- Signed curvature



$$\kappa = \frac{1}{r}$$

$$\kappa = \frac{d\mathbf{T}}{ds}$$

acceleration

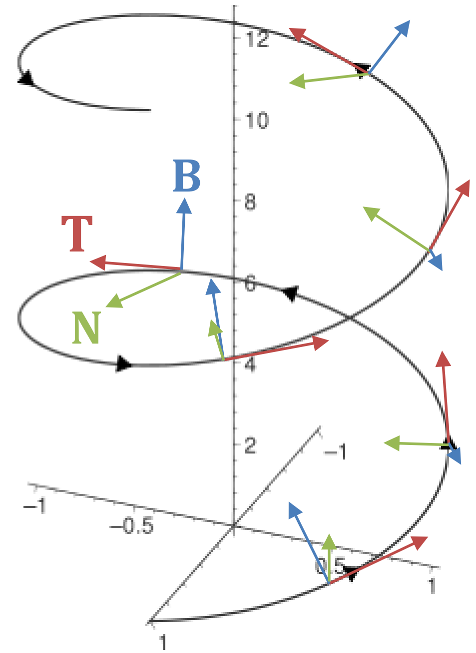
Curves in 3D Space

- Curves embedded in 3D space.
- A helix and circle → constant curvature
  - z component of $\mathbf{T}$ is constant.
- Unit vector from *plane of curvature* has a changing orientation.
- Leads to a reorienting "body frame".

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

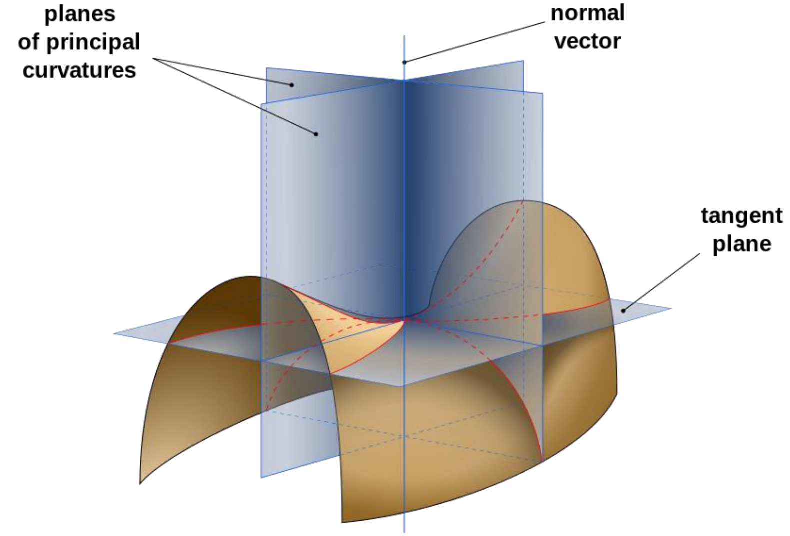
- The *torsion* of a 3D curve tells us how fast the binormal vector is reorienting.

$$\tau = -\mathbf{N} \cdot \mathbf{B}'$$



$\mathbf{T}$: tangent
$\mathbf{N}$: normal
$\mathbf{B}$: binormal

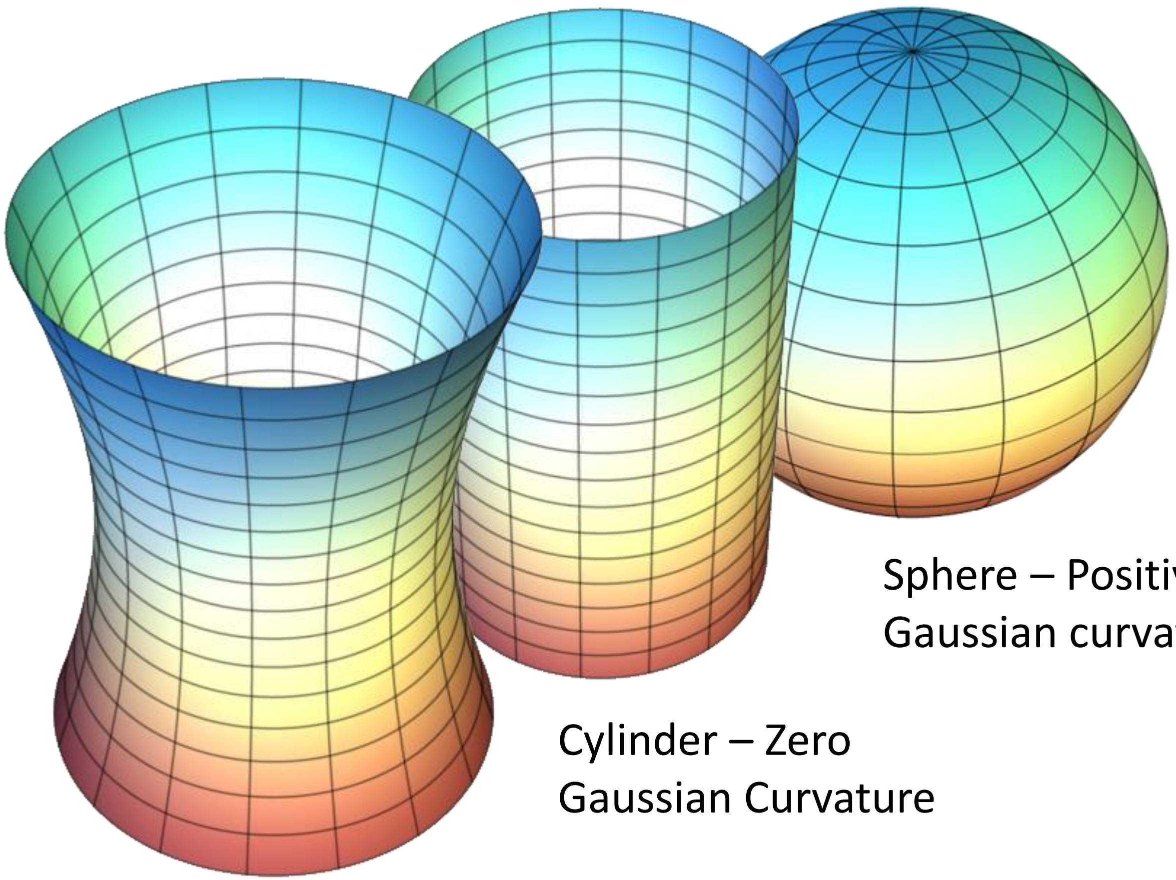
Surfaces: Gaussian Curvature



- 2D surface embedded in 3D.
- Infinitely many curves through P.
- Are just intersections of normal planes with the surface.
- Each curve has *normal curvature* k_n in their resp. planes.
  - Using change in tangent vectors as before.
- Max and min $k_n \rightarrow$ *principal curvatures*, called k_1 and k_2.
- ***Gaussian curvature*** is a single measure for a surface point.

$$K = k_1 k_2$$

Examples



Sphere – Positive, constant
Gaussian curvature

Cylinder – Zero
Gaussian Curvature

Hyperboloid – Negative
Gaussian curvature

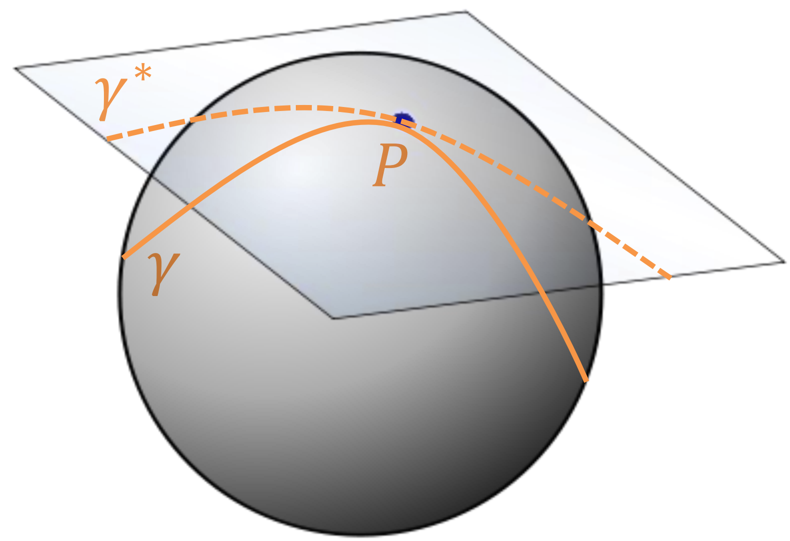
Geodesic Curvature

- Normal curvature of γ at P was defined by the tangent vectors along γ (or embed γ in normal plane).

- Now consider the tangent plane at P.

- Can locally project γ onto tangent plane at P as γ^*.
  - Why can we do this?

- Curvature of γ^* at P is the ***geodesic curvature*** k_g.
  - Sort of like a 1st-order measure.



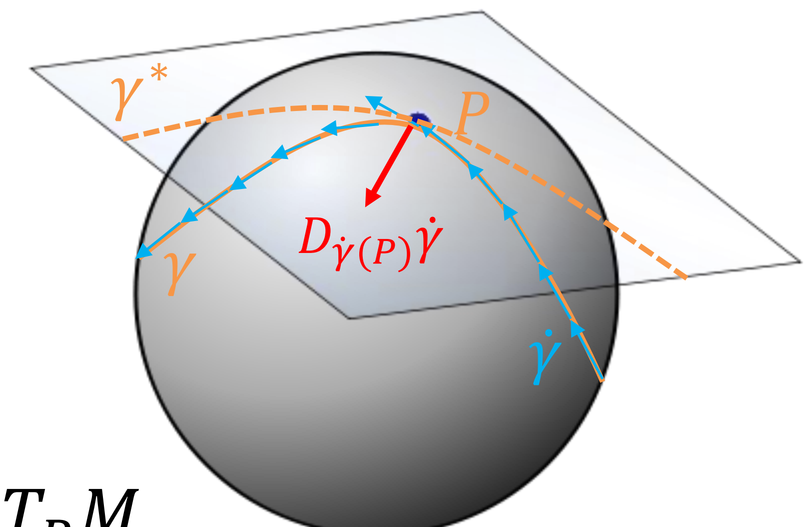
The Covariant Derivative

- Projection of both curve and its tangent vectors into a tangent space.
- Tangent vectors make up a velocity vector field $\dot{\gamma}$ along γ.
- Infinitesimal vector change around $P \rightarrow$ ***covariant derivative***.
- Geodesic curvature at P is magnitude of this vector.

$$D_{\dot{\gamma}(P)}\dot{\gamma} \in T_P M$$

Tangent vector Vector field

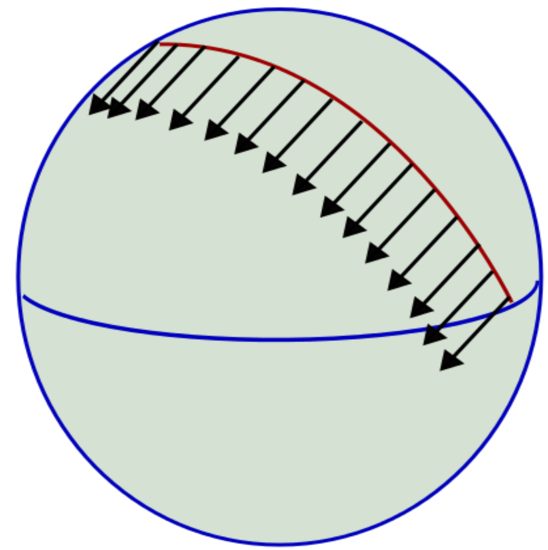
- Alternatively, find vector difference first, then project.



- More generally, the covariant derivative of a vector field X at P on a manifold M in a direction $Y \in T_P M$ measures fast X is changing, projected into $T_P M$.

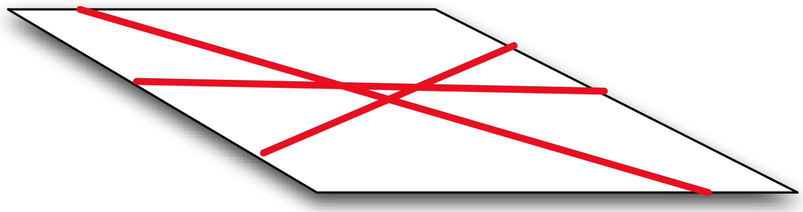
$$D_Y X \in T_P M$$

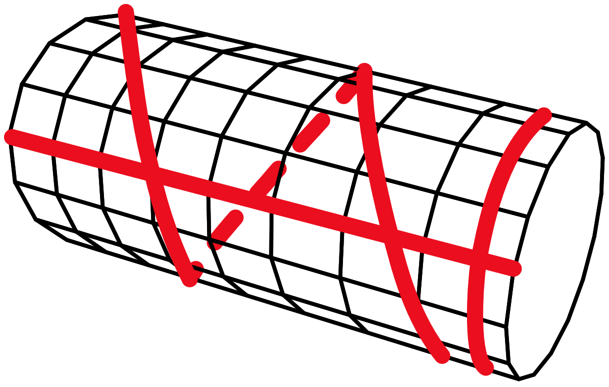
Parallel Transport and Geodesics

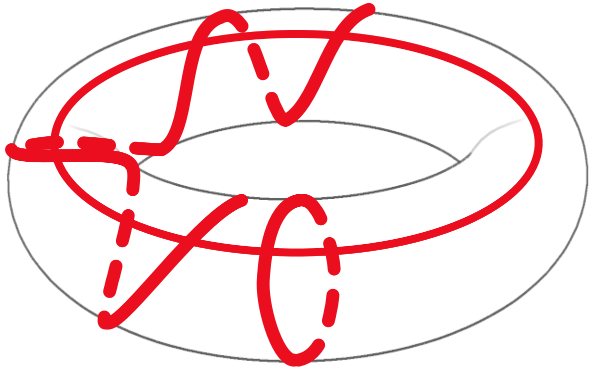


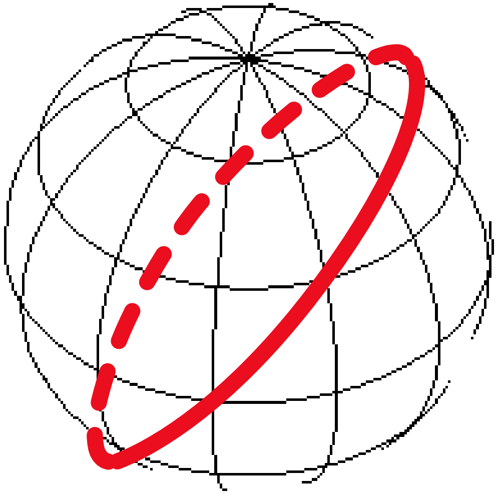
- A vector field X is *parallel* along curve γ if $D_{\dot{\gamma}} X = 0$ everywhere along γ.
- Given γ and vector x, the ***parallel transport*** of x along γ is the vector field X, such that X is parallel along γ.
- A curve γ is a ***geodesic*** on a manifold M if
  - γ has zero geodesic curvature everywhere.
  - the projection into the tangent space anywhere along γ is a straight line.
  - its velocity vector field $\dot{\gamma}$ is parallel along itself.
  - γ parallel transports its own velocity vector, i.e. the parallel transport of the velocity vector at any point $P \in \gamma$ is the velocity vector field $\dot{\gamma}$.

Examples









Riemannian Manifolds

- Distances on manifolds are specified by *metrics*.

$$g_P : T_P M \times T_P M \to \mathbf{R}$$

- Properties of metrics on manifolds
  - Bilinearity. $g_p(aU_p + bV_p, Y_p) = a g_p(U_p, Y_p) + b g_p(V_p, Y_p)$
  - Symmetry. $g_p(X_p, Y_p) = g_p(Y_p, X_p)$
  - Nondegeneracy. For every $X_p \neq 0$ there exists Y_p s.t. $g_p(X_p, Y_p) \neq 0$.

- If $g_P(X(P), Y(P))$ is a smooth mapping from M to $\mathbf{R}$ for differentiable vector fields X, Y on M, then g_P is a *Riemannian metric* and M is a ***Riemannian manifold***.
  - Curve lengths along M can be found by integrating g.

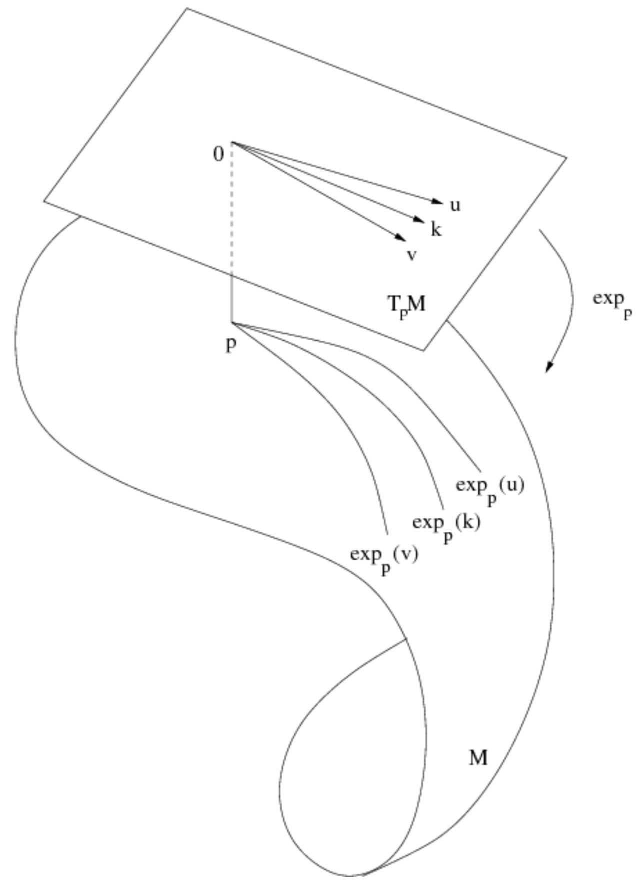
Exponential Maps

- Uniqueness: On a Riemannian manifold, given point $P \in M$ and velocity vector $X \in T_P M$, there is a *unique geodesic* γ around P s.t. $\gamma'(P) = X$.

- The ***exponential map*** starts at $\gamma_X(0) = P$ and runs along the geodesic with velocity X to $\gamma_X(1) \in M$.

$$\exp_P(X) = \gamma_X(1)$$
$$\exp_P: D \to M, \text{where } D \subset T_P M$$

- Distance traveled from P to $\gamma_X(1)$ is $|X|$, measured by the metric.



Domain of exp in general is only a subset of $T_P M$, since γ_X may not always be defined in the entire interval $[0, 1]$.

Exponential Maps on Lie Groups

- Lie groups have manifold structure -> can define exp maps.

- For a Lie group G and Lie algebra element $X \in \mathfrak{g}$, there exists a unique smooth homeomorphism
$$\gamma_X : \mathbf{R} \to G \text{ s.t. } \gamma_X(0) = e \text{ and } \dot{\gamma}_X(0) = X.$$
  – Can define "geodesic curves" from the identity.

- Exponential map takes elements in the Lie algebra $\mathfrak{g}$ and maps them along γ to an element of the group.
$$\exp : \mathfrak{g} \to G \qquad \exp(X) = \gamma_X(1)$$

Example: $\mathfrak{so}(3)$ and SO(3)

Matrix Lie groups: $$\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k$$

Lie algebra for SO(n): $$\mathfrak{so}(n) = \{\widehat{\omega} \in \mathbf{R}^{n\times n} \mid \widehat{\omega}^T = -\widehat{\omega}\}$$

Exp map for SO(3): $$\exp(\widehat{\omega}\theta) = I + \widehat{\omega}\sin\theta + \widehat{\omega}^2(1-\cos\theta) = R(\omega,\theta)$$

- Since every $\widehat{\omega} \in \mathfrak{so}(3)$ is isomorphic to $\omega \in \mathbf{R}^3$, the components $\omega\theta$ serve as the ***exponential coordinates*** (angle-axis representation) for $R(\omega,\theta)$.

- The *logarithm map* is an "inverse" to the exponential map for matrix Lie groups.
- Extracting angle/axis from rotation matrix.

$$\log R = \begin{cases} 0, & \theta = 0 \\ \dfrac{\theta}{2\sin\theta}(R - R^T), & \theta \neq 0 \end{cases}$$