Geometry of Locomotion
Chapter 2.1

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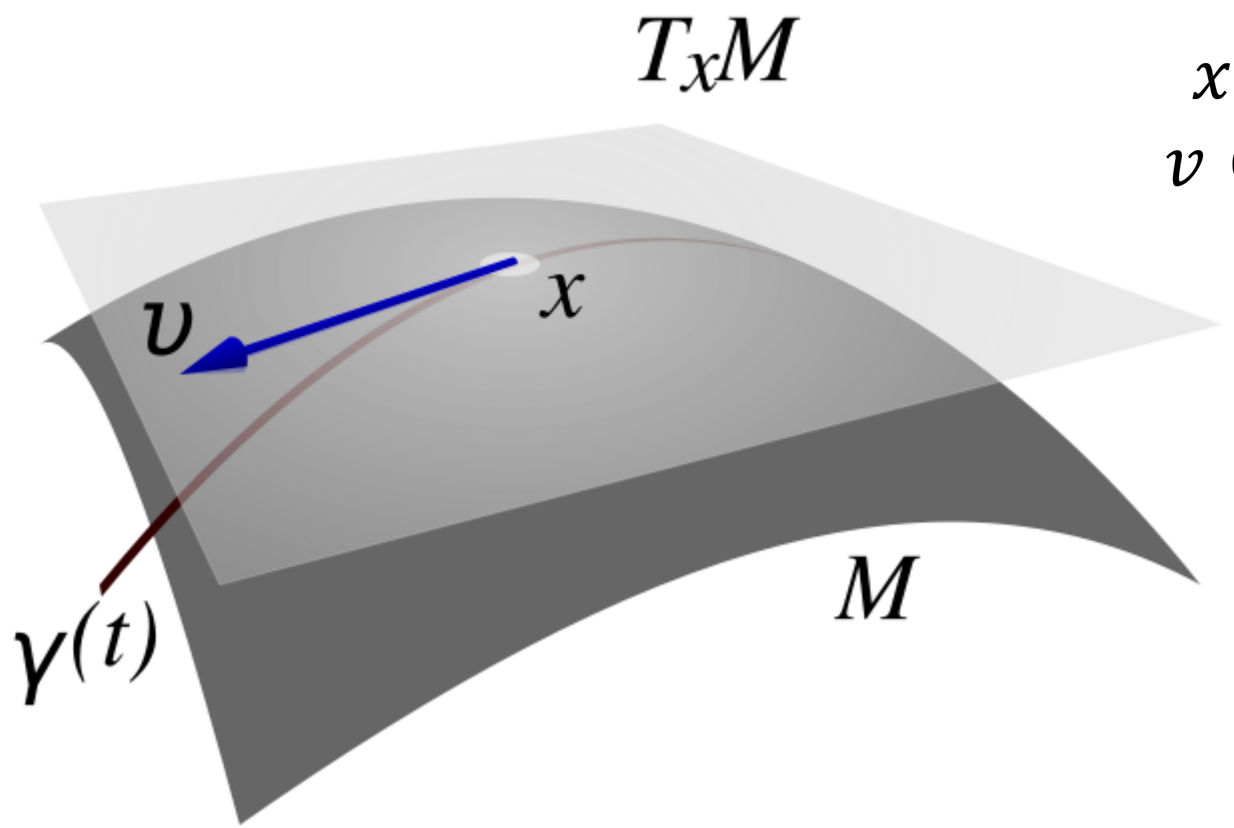
Chapter Highlights

- Tangent Spaces (from configuration mfds)
  - Extrinsic vs. intrinsic bases
  - Same vectors
- Lie Algebra (from Lie Groups)
- Group Derivatives
- Multiplicative Calculus

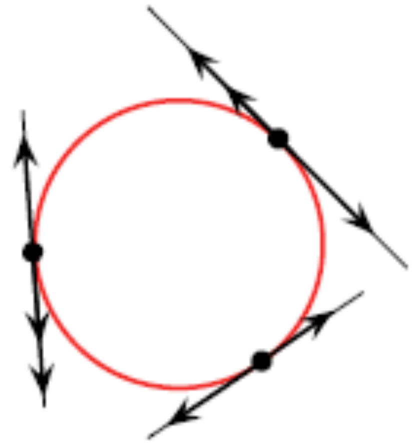
Intuition – Velocity Vectors

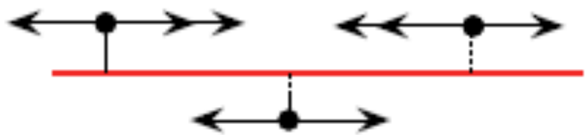


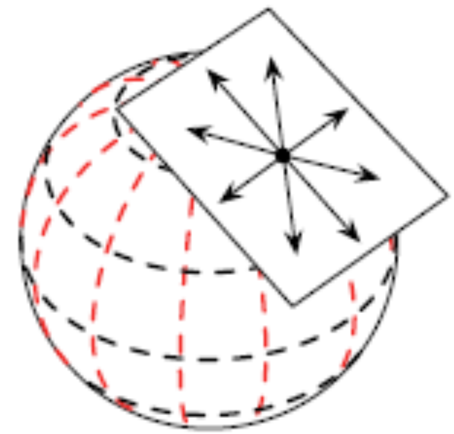
$x \in M$
$v \in T_xM$

contains all possible differential changes in the configuration from point q.

Intuition - Linearization







Tangent bundle is union of
all tan spaces

$$TM = \bigcup_{x \in M} T_x M$$

Parameterization of Tan Spaces

- Tangent spaces are abstract spaces onto themselves.
- We visualize them connected to a manifold at a point where they tangantiate the mfd
- To do calculations in a tan space, we assign a basis (not necessarily unique) that is inhereted from the parameterization of the mfd itself

$$u_i = \left.\frac{\partial q}{\partial q_i}\right|_q$$

u_i represents the infinitesimal change in configuration corresponding to an infinitesimal change in the parameter q_i

$$dq = \sum_i u_i \, dq^i$$

Any differential change in the system configuration is the sum of their basis weighted by their parameter values

$$dq = (dq^1, dq^2, \ldots, dq^n).$$

Taking basis vectors to be implicit

Velocity Vectors

Velocities are tangent vectors in which the magnitudes represent differential displacements over time, rather than simple differential displacements.

The velocities use the same set of differential bases

$$\dot{q} = \frac{dq}{dt} = \sum_i \frac{\partial q}{\partial q_i}\frac{dq^i}{dt} = \sum_i u_i\,\dot{q}^i$$

$$\dot{q} = (\dot{q}^1, \dot{q}^2, \ldots, \dot{q}^n)$$

Vectors – Ask Questions to class

Definition	Meaning
A quantity with magnitude and direction	Directed quantity – what we learn in school growing up
An element of a vector space	Means by which we can add them and multiply to scale them
An n-tuple (or list) of values	Weighted sum of basis vectors whose values in the tupple identifies each vector with respect to the basis Provides a convenient means for performing computations with them – linear xformations for example

Vector Spaces

Vector spaces. A *vector space* is a set V of objects called *vectors*, associated with a set S of objects called *scalars* and four operations:

1. *Vector addition*, in which two vectors are combined to generate a third,
2. *Scalar addition*, in which two scalars are summed to generate another scalar,
3. *Scalar multiplication*, in which the product of two scalars is another scalar,
4. *Scalar multiplication of vectors*, in which a scalar is combined with a vector to produce another vector.

These sets and operations must satisfy the following conditions:

1. The set V must form a commutative group[a] under vector addition.
2. The set S must form a commutative field[b] under the scalar addition and multiplication operations.
3. Scalar multiplication of vectors must have the following properties for $v, w \in V$ and $a, b \in S$:
    a. **Closure**: the product of a scalar with a vector is always also a vector: $av \in V$
    b. **Distributivity I**: Multiplying a scalar by the sum of two vectors is equivalent to multiplying the scalar by each of the two vectors and summing the result: $a(v + w) = av + aw$
    c. **Distributivity II**: Summing two scalars and multiplying them by a vector is equivalent to multiplying each of them by the vector and summing the result: $(a + b)v = av + bv$
    d. **Associativity**: Multiplying two scalars together and then by a vector is equivalent to sequentially multiplying the scalars by the vector: $(ab)v = a(bv)$
    e. **Identity**: The multiplicative identity element of S is also the identity for scalar multiplication: $\mathbf{1}_S v = v$

Bases

Bases. A *basis* U on a vector space V is a set of vectors $u_i \in V$ that *minimally span* the space. Spanning V means that any vector v in the space can be expressed as a weighted sum of the basis vectors,

$$v = \sum_i u_i \, v^i, \tag{2.i}$$

in which the *components* of the vector, the coefficients v^i that are multiplied by the corresponding basis elements u_i, are scalars from the space's associated field. Minimally spanning V means that each vector is identified by a *unique* sum of basis vectors, and thus by a unique set of components. If the scalar field associated with V is the set of real numbers $\mathbb{R}$, then each vector is uniquely identified with a point in $\mathbb{R}^n$, giving rise to the notion of a vector as a "list of numbers."

Bases can also be defined as sets of vectors with the following properties, which are equivalent to the minimal spanning condition:

1. **Number of basis vectors**: There are exactly as many basis vectors in B as there are dimensions in V.
2. **Linear independence**: No basis vector in B may be expressible as a weighted sum of the other basis vectors.

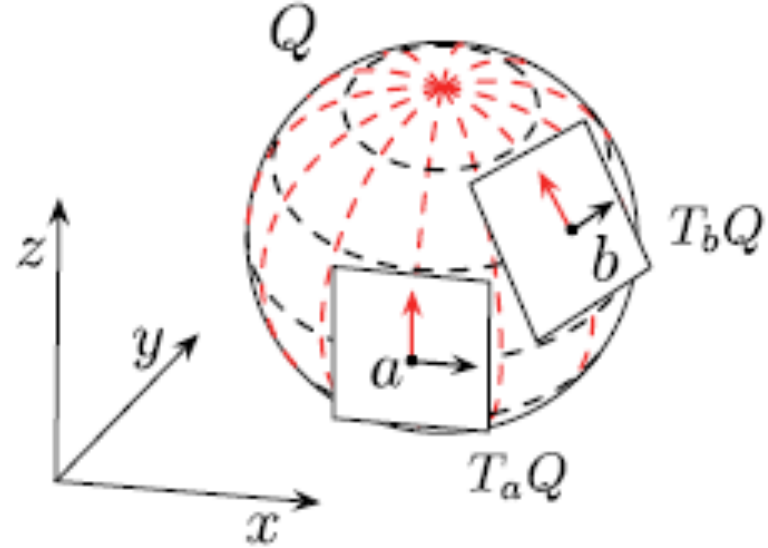
Extrinsic vs. Intrinsic

$$q = (x, y, z) =$$
$$= (\cos\theta\cos\phi,\ \sin\theta\cos\phi,\ \sin\phi),$$

$$u_{1,q} = \left.\frac{\partial q}{\partial \theta}\right|_{\theta,\phi} \quad \text{and} \quad u_{2,q} = \left.\frac{\partial q}{\partial \phi}\right|_{\theta,\phi}$$

$$u_{1,q} = \begin{bmatrix} -\cos(\phi)\sin(\theta) \\ \cos(\phi)\cos(\theta) \\ 0 \end{bmatrix}$$

$$u_{2,q} = \begin{bmatrix} -\cos(\theta)\sin(\phi) \\ -\sin(\phi)\sin(\theta) \\ \cos(\phi) \end{bmatrix}$$



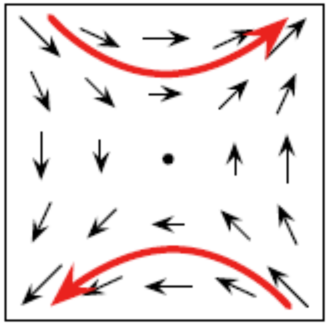
$$u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

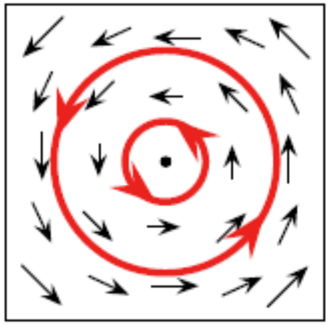
Intrinsic

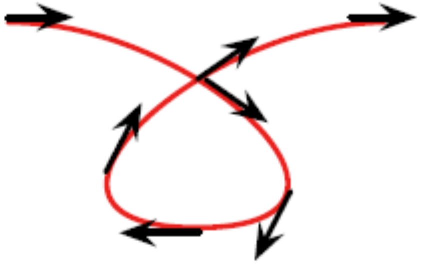
Extrinsic representations relate the velocity vectors back to the motion on the manifold and allow for additional operations, such as applying Lie group actions directly to vectors tangent to the group

Intrinsic representations have trivial bases, which allows linear mappings between vectors to be encoded as matrices multiplied by lists of coefficients,

Vector Field







A vector field is a (possibly time-varying) assignment of a single vector to each point in a subset of a manifold

$$\mathbf{X} : Q_1 \times \mathbb{R} \to T_q Q$$
$$(q, t) \mapsto v,$$

Vector field on a curve

Velocity fields

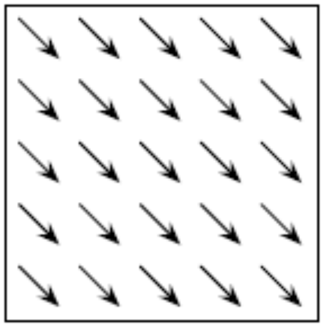
$$\dot{q} = \mathbf{X}(q, t).$$

Integral curves / solutions

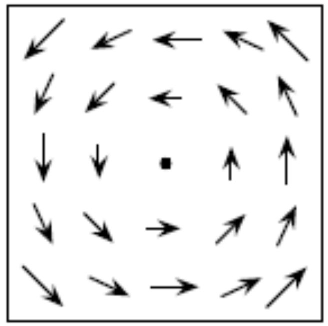
$$\gamma : [0, T] \to Q$$
$$t \mapsto q,$$

$$\dot{\gamma}(t) = \mathbf{X}(\gamma(t), t).$$

Constant Vector field



Same X,Y



Same r,θ

Symmetry: change in x or y has a useful physical meaning that does not depend on the current value of x and y.

Goal: come up with coordinate invariant methods of measureing sameness

Remind everyone that this is a vector field so this is a manifold and a vector from each tangent space in the manifold

Vectors that are Equivalent with respect to Functions

- two vectors are equivalent with respect to a function, the central concept is that they have corresponding effects on the input and output of the function
- $dq \ \equiv_f \ dp$ if $f(q + dq) = f(q) + dp \ \ (p = f(q))$

Two vectors dp and dq are equivalent if you step along dq from the point q and evaluate the function at the new point and that gets you to the same point if you evaluate first f(q)= p and step along the vector dp

- two velocity vectors $\dot{q} \in T_q Q$ and $\dot{p} \in T_p P$ are equivalent with respect to f if the time derivative $\dot{p}$ of f(q) when the configuration is changing at $\dot{q}$

$$\dot{q} \equiv_f \dot{p} \qquad \left.\frac{d}{dt} f(q)\right|_{\dot{q}} = \dot{p}.$$

Vectors that are equivalent to each other under this definition are related to each other by the Jacobian of f.

Differential Maps and Jacobians

Differential Mappings and Jacobians

A function mapping between manifolds A and B,

$$\begin{aligned} f : A &\rightarrow B \\ a &\mapsto b, \end{aligned} \tag{2.v}$$

has a set of associated *differential mappings* (also called *tangent mappings*) between the tangent spaces of A and B, defined so that the output velocity $\dot{b}$ is the rate at which the output is changing, given a known rate at which the input is changing,

$$\begin{aligned} f' : T_aA &\rightarrow T_bB \\ \dot{a} &\mapsto \dot{b} = \tfrac{d}{dt} f(a)\big|_{a,\dot{a}}. \end{aligned} \tag{2.vi}$$

By the chain rule, this derivative decomposes into the product of the derivative of f over the input space and the rate at which the input is changing,

$$f'(\dot{a}) = \left.\frac{\partial f(a)}{\partial a}\right|_a \dot{a}. \tag{2.vii}$$

Jacobians

The derivative term in (2.vii) is the *Jacobian* of f, denoted J. If A or B are multi-dimensional spaces, J includes the derivative of each degree of freedom in the range of f with respect to each component of its domain. In coordinates, these derivatives can be grouped into matrix form as

$$\begin{bmatrix} \dot{b}_1 \\ \vdots \\ \dot{b}_n \end{bmatrix} = \overbrace{\begin{bmatrix} \partial f_1/\partial a_1 & \cdots & \partial f_1/\partial a_m \\ \vdots & \ddots & \vdots \\ \partial f_n/\partial a_1 & \cdots & \partial f_n/\partial a_m \end{bmatrix}}^{J} \begin{bmatrix} \dot{a}_1 \\ \vdots \\ \dot{a}_m \end{bmatrix}. \tag{2.viii}$$

in which each row is the derivative of one component of the output and each column is the derivative with respect to one component of the input.

$$J = \begin{bmatrix} \partial f_1/\partial a_1 & \cdots & \partial f_1/\partial a_m \\ \vdots & \ddots & \vdots \\ \partial f_n/\partial a_1 & \cdots & \partial f_n/\partial a_m \end{bmatrix} \tag{2.ix}$$

expands into a matrix in which each row is the derivative of one component of the output and each column is the derivative with respect to one component of the input.

Jacobians

$$\begin{tikzcd} a \arrow[r, "d/dt"] \arrow[d, "f"'] & \dot{a} \arrow[d, "f'"] \\ b \arrow[r, "d/dt"'] & \dot{b} \end{tikzcd}$$

$$\dot{b}(t) = \frac{d}{dt} f(a(t)) = f'(\dot{a}(t)) = J\dot{\alpha}(t),$$

Interpretations

. **Mapping between the intrinsic and extrinsic forms of a vector.** Given a parameterization function[a] ϕ, its inverse maps from the parameters to the manifold,

$$\phi^{-1} : \{q_i\} \mapsto q. \tag{2.xi}$$

Each column of the Jacobian of this function matches the corresponding basis vector from (2.1),

$$J_{\phi^{-1}} = \left[\frac{\partial \phi^{-1}}{\partial q_1} \quad \cdots \quad \frac{\partial \phi^{-1}}{\partial q_n}\right]\Bigg|_{\{q_i\}} = \left[\frac{\partial q}{\partial q_1} \quad \cdots \quad \frac{\partial q}{\partial q_n}\right] = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix}, \tag{2.xii}$$

and thus provides the mapping from the intrinsic representation (the set of dq_i or $\dot{q}_i$ coefficients) to the full extrinsic form of the vector.

2. **Change-of-basis operations linked to changes in parameterization.** Given a transition map[b] $\tau_{a,b} = \phi_a^{-1}\phi_b$ that reparameterizes a space as

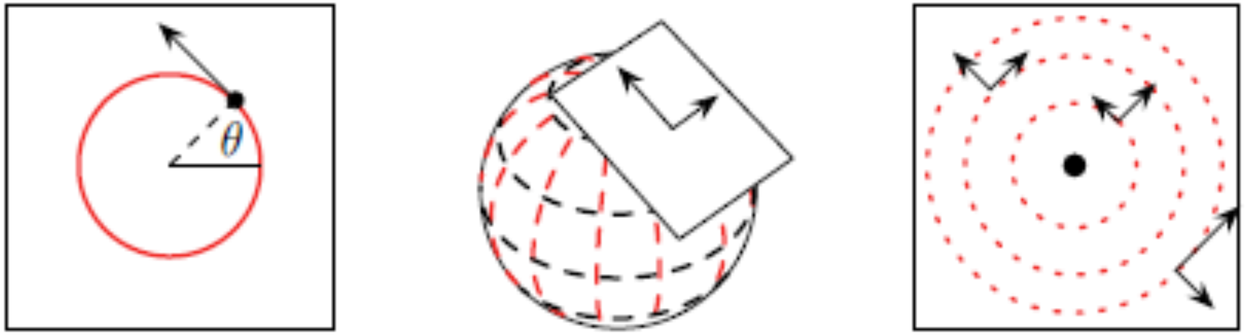
$$\tau_{a,b} : \{q_{i,a}\} \mapsto \{q_{i,b}\}, \tag{2.xiii}$$

the Jacobian of this function maps vectors from their representation in the first parameter-induced basis to their representation in the second parameter-induced basis,

$$J\tau_{a,b} : \{dq_{i,a}\} \mapsto \{dq_{i,b}\}. \tag{2.xiv}$$

3. **Identifying vectors represent the "same" motion.** Given a function $f : Q \to Q$ mapping between points in a single space, two vectors related by J_f correspond to a *single* action that commutes with the function.

Examples



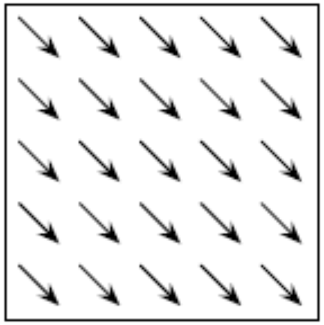
- Embed S^1 $$f_1 : \theta \mapsto \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \implies J_1 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}.$$

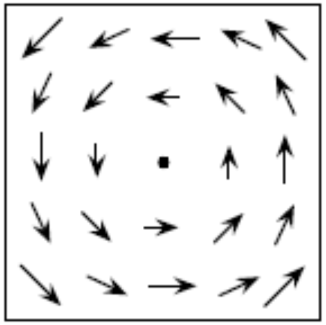
- Embed S^2 $$f_2 : \theta \mapsto \begin{bmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ \sin\phi \end{bmatrix} \implies J_2 = \begin{bmatrix} -\cos(\phi)\sin(\theta) & -\cos(\theta)\sin(\phi) \\ \cos(\phi)\cos(\theta) & -\sin(\phi)\sin(\theta) \\ 0 & \cos(\phi) \end{bmatrix}$$

- Embed $R^1 \times S^1$ $$f_3 : (r,\, \theta) \mapsto \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} r \implies J_3 = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

Sameness Returned

the vectors are invariant with respect to translation.





$$f_1 : \mathbb{R}^2 \to \mathbb{R}^2$$
$$(x, y) \mapsto (x + a, y + b).$$

$$f_2 : \mathbb{R} \times \mathbb{S} \to \mathbb{R} \times \mathbb{S}$$
$$(r, \theta) \mapsto (r + \rho, \theta + \phi).$$

$$J_1 = \begin{bmatrix} \frac{\partial(x+a)}{\partial x} & \frac{\partial(x+a)}{\partial y} \\ \frac{\partial(y+b)}{\partial x} & \frac{\partial(y+b)}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$J_2 = \begin{bmatrix} \frac{\partial(r+\rho)}{\partial r} & \frac{\partial(r+\rho)}{\partial \theta} \\ \frac{\partial(\theta+\phi)}{\partial r} & \frac{\partial(\theta+\phi)}{\partial \theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Another Example

Cartesian representation of a polar translation which extends the original (x, y) vector by a length ρ, then rotates the new vector by an angle ϕ

$$f_3 : \mathbb{R}^2 \to \mathbb{R}^2 \tag{2.29}$$

$$(x, y) \mapsto R(\phi) \begin{bmatrix} x + \frac{x}{\sqrt{(x^2+y^2)}}\rho \\ y + \frac{y}{\sqrt{(x^2+y^2)}}\rho \end{bmatrix}, \tag{2.30}$$

Taking the Jacobian of this function results in the expression

$$J_3 = R(\phi) \begin{bmatrix} \frac{\partial(1+\rho(x^2+y^2)^{-\frac{1}{2}})x}{\partial x} & \frac{\partial(1+\rho(x^2+y^2)^{-\frac{1}{2}})x}{\partial y} \\ \frac{\partial(1+\rho(x^2+y^2)^{-\frac{1}{2}})y}{\partial x} & \frac{\partial(1+\rho(x^2+y^2)^{-\frac{1}{2}})y}{\partial y} \end{bmatrix} \tag{2.31}$$

$$= R(\phi) \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{\rho}{(x^2+y^2)^{\frac{3}{2}}} \begin{bmatrix} y^2 & -xy \\ -xy & x^2 \end{bmatrix} \right). \tag{2.32}$$

Although this expression looks complicated, breaking it down into components highlights the ways in which f_3' transforms vectors into their equivalents in different tangent spaces:

1. If the distance from the origin does not change ($\rho = 0$), then the Jacobian is a simple rotation of the vector by the same angle as was used to move to the new point.
2. For initial positions along a principal axis of the parameterization (e.g., along the x axis with $y = 0$) , the Jacobian simplifies to scaling the transverse component of the velocity (here, the y component) proportionally to the new distance of the point from the origin,

$$J_3|_{y=0} = R(\phi) \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 + \frac{\rho}{x} \end{bmatrix} \right), \tag{2.33}$$

   capturing the idea that a larger translational velocity is required to produce a given rotational velocity when further away from the center of rotation.
3. For initial (x, y) positions not on a principal axis, the terms multiplied by ρ in (2.32) generalize the transverse-scaling principle in (2.33) to arbitrary angles.

Key insight

the Jacobian provides a structured way to lift information from mappings on a manifold into equivalence relationships between vectors in different tangent spaces, without relying on any particular choice of coordinates.