Geometry of Locomotion
Chapter 2.2 – Velocities in Lie groups

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Differential and Infinitesmal

- Euclidean Space allows difference of two points to form a vector
- An infinitesimal element is one that is close to the identity. All such elements form a Euclidean space –OR- a linearization is the “closest” Euclidean space about a pint
- A difference between (either) such elements and the identity forms a vector, which as its length approaches zero is identified with a vector in the tangent space.
- We call these vectors differential when perhaps the terms should be infinitesimally differential

Quick Intuition:
Velocity Relative to Current Configuration

- Rate at which bank account groups
- It is not specified in terms of absolute rate of growth
- It is specified in a % of the current balance
- Call this a proportional rate, as opposed to the absolute rate in which money is added

- Vehicle direction is easily specified in terms of left, right, forward and back, a relative measure, as opposed to an absolute one

Lie Groups and Algebras

- These notions of velocities are captured by Lie Groups and Algebras
- Ask: What is a Lie Group
  - Group
  - Underlying structure forms a manifold and hence has a tangent bundle. Can you say vector field?
- Lie Algebra are infinitesimal actions on the group. Are these actions members of the group or it tan space?

Sophus Lie

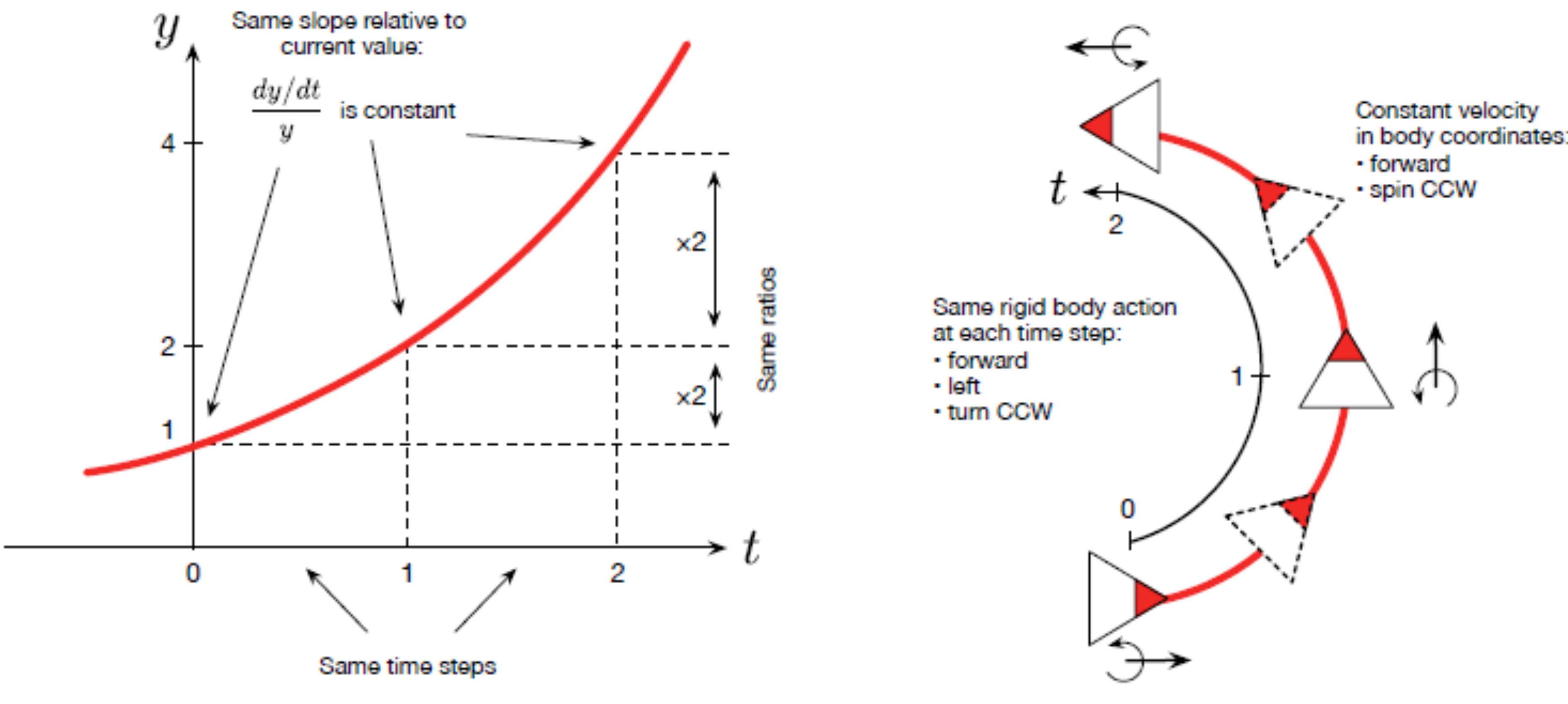
- 17 December 1842 – 18 February 1899
- continuous symmetry and applied it to the study of geometry and differential equations.
- Arrested for being falsly accused of being a German spy during the Franco-Prussian war. This made him famous in Normay
- Inventor of Lie Groups and Algebras

Core Concept

each tangent vector dq on the manifold corresponds to an infinitesimal group action g_δ,

each velocity vector $\dot{q} = \frac{dq}{dt}$ correspond to the rate at which infinitesimal group actions are being applied to the configuration, $\overset{\circ}{g} = g_\delta / dt$

Groupwise Velocity Examples



(a) (b)

velocity of the system relative to its current configuration, and a system moving at a constant $\mathring{g}$ experiences the same group displacement g over each unit time,

$\mathfrak{g}$ Lie Algebra is Linearization of G

- Since $\mathfrak{g}$ is a linearization, it can be identified easily with T_eG
- Since identity can be anywhere, $\mathfrak{g}$ can be identified by every T_gG
- The way in which the Lie algebra propagates out along the tangent bundle is itself closely tied to the group structure, and may be handled in two ways:
  - intrinsically, where all velocities that are equivalent under the Jacobian of the group action correspond to the same element of g;
  - extrinsically, where transforming the basis of g by g gives its representation at TgG.

Lifted Action

- derivatives of the group actions
- Relate velocities (expressed in the manifold's parameter bases) that represent the same groupwise velocities across different locations in the manifold

Groupwise Velocity Definitions and Notation

$$g_\Delta = g_1^{-1} \circ g_2$$

The core idea behind groupwise definitions of velocity is that displacements on a Lie group are fundamentally characterized by group actions

$$g_\delta = g^{-1}(t) \circ g(t + \delta t)$$

group action separating two configurations an infinitesimal times tep apart

$$\mathring{g} = \lim_{\delta t \to 0} \frac{g_\delta - \mathbf{e}}{\delta t}$$
$$= \lim_{\delta t \to 0} \frac{g^{-1}(t) \circ g(t + \delta t) - \mathbf{e}}{\delta t}$$

that velocities on the group are fundamentally characterized by the rate at which group actions are being applied to the configuration

Note the e is subtracted out leaving the difference between g_\delta and e behind which becomes the tan vector as \delta t goes to zero

Note the “-” is a create a vector and not subtr
Compare with normal derivative