Geometry of Locomotion
Chapter 1.3

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Chapter 1 Key Ideas

- Configuration Space
  - Configuration Manifold
  - Configuration Group
- Rigid Body
- Degrees of Freedom

Rigid Body

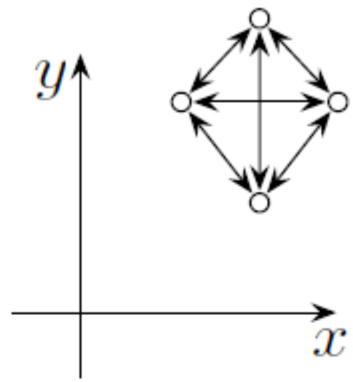
<u>Physical</u>

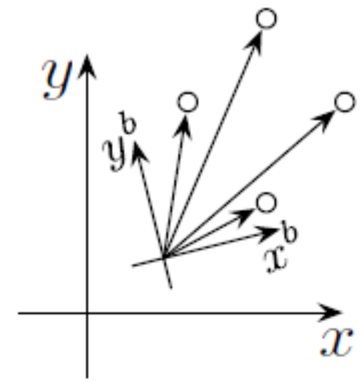
an object that does not deform in response to external forces.

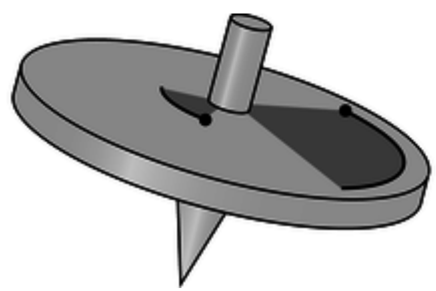
<u>Mathematical</u>

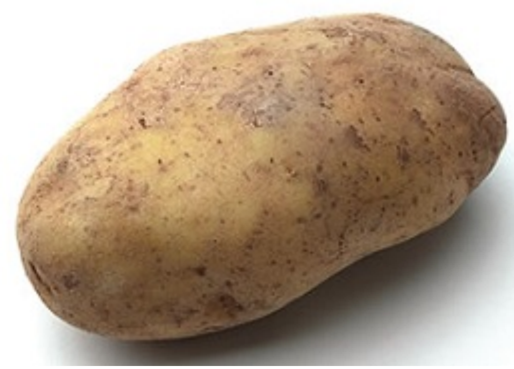
set of points with fixed interpoint distances; a *proper rigid* body also has a fixed handedness, and so can be translated and rotated, but not reflected.

a movable reference frame (the body frame) and a set of points whose positions are fixed with respect to the body frame.









Under this last definition, it becomes clear that the configuration (position and orientation) of the body frame completely defines the location of all points in the body, and therefore that **we can treat the configuration spaces of the body and the frame equivalently.**

Frame Approach

- Choice of frame?
  - Center of mass
  - Not in confines of body
  - Orientation?
- What configuration space and Lie group?
- In plane
  - $(R^2 \times S^1, +)$ - position and orientation
  - *SE(2)* – relative motions, symmetries

SE(2) – Special Euclidean Group

- SE(2) is the set of proper rigid (non-reflecting and nondistorting) transformations in the plane.
  - Includes translation and rotations
  - Excludes shearing, scaling, and mirroring motions.
- An element $g \in SE(2)$ can be parameterized as (x, y, θ)
- Represented by

$$g = (x, y, \theta) = \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix}$$

Semi-Direct Product and Operations on Translation and Rotation

$SO(2) \circledS \mathbb{R}^2$

$g, h \in SE(2)$, with $g = (x, y, \theta)$ and $h = (u, v, \beta)$,

$$gh = \begin{bmatrix} R(\theta) & \begin{pmatrix} x \\ y \end{pmatrix} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} R(\beta) & \begin{pmatrix} u \\ v \end{pmatrix} \\ \mathbf{0} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R(\theta)R(\beta) & R(\theta)\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix} \\ \mathbf{0} & 1 \end{bmatrix}$$

Non commutative

$$hg = \begin{bmatrix} R(\beta) & \begin{pmatrix} u \\ v \end{pmatrix} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} R(\theta) & \begin{pmatrix} x \\ y \end{pmatrix} \\ \mathbf{0} & 1 \end{bmatrix}$$

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Direct and Semi-direct Products of Groups

When we combine groups together to form larger groups, we must consider not only how their underlying sets combine, but also what the overall group action becomes. Often, the combination of two groups A and B into a new group C is taken to mean the creation of a *direct product group*

$$C = A \times B, \tag{1.xix}$$

in which components that started out in A or B only affect other components that started out as elements of the same group, *i.e.*, $c_1c_2 = (a_1a_2, b_1b_2)$. Direct products preserve properties such as being abelian (commutative) – if A or B has this property, then so does the corresponding section of C.

In a *semi-direct product group*,

$$D = A \circledS B, \tag{1.xx}$$

which may also be written as

$$D = A \rtimes B \tag{1.xxi}$$

(with a bar connecting the rightmost points of the times symbol), elements of A act not only on each other, but also on elements of B. For example, the structure of $SE(2)$ takes the form

$$d_1d_2 = (b_1(a_1b_2), a_1a_2). \tag{1.xxii}$$

A key aspect of such groups is that even though they do not possess the full orthogonality of a direct product group, the A components do preserve their original properties, and thus results that depend on these properties can be applied to the corresponding elements of D.

Interpretations of SE(2)

1. The position and orientation of a rigid body (or a frame), relative to a choice of origin represented by the identity element of the group.

2. A translation and rotation of a rigid body, relative to the chosen origin.

3. A translation and rotation of a rigid body, relative to its current position and orientation.

Interpretations of SE(2)

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Captures the manifold structure with two non-cyclic and one cyclic dims

2. A translation and rotation of a rigid body, relative to the chosen origin.

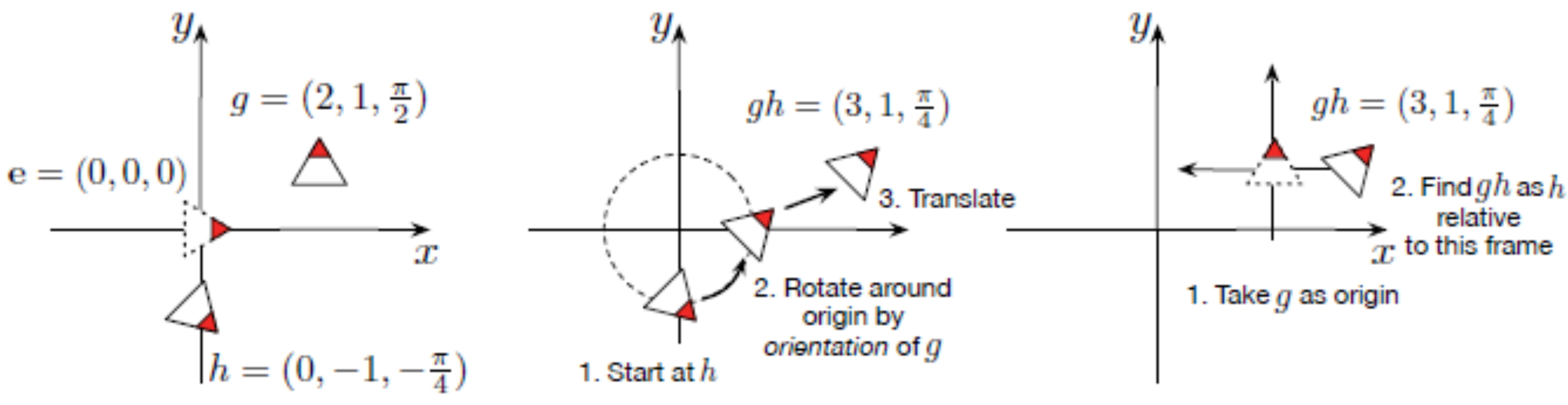
Captures the group structure of SE(2) but with a <u>left operator</u> (fixed/pre-mult)

3. A translation and rotation of a rigid body, relative to its current position and orientation.

Captures the group structure of SE(2) but with a <u>right operator</u> (relative/post mult)

$$g, h \in SE(2), gh \neq hg$$

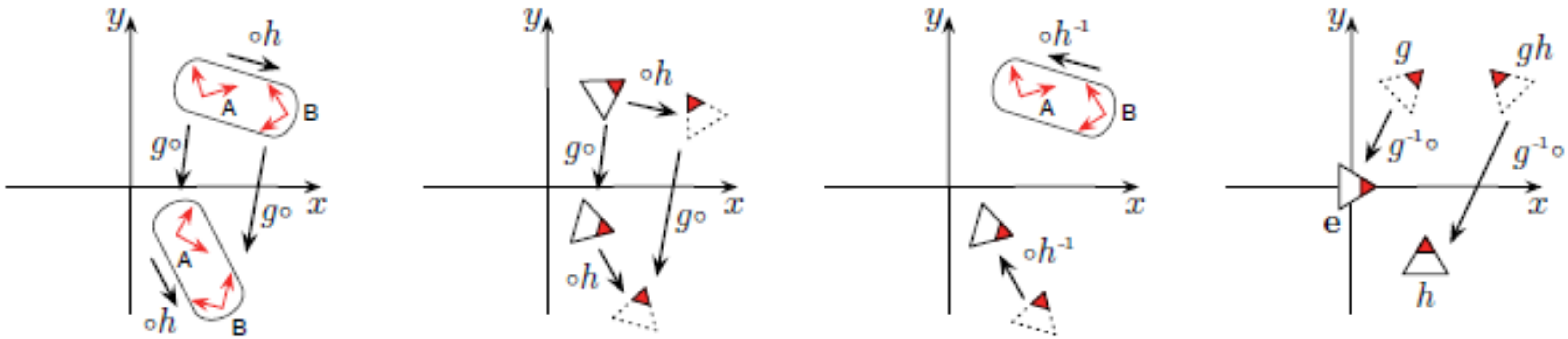
Examples of Left and Right – order matters



(a) Individual group elements (b) Left action interpretation of gh (c) Right action interpretation of gh

the sequence order of the group elements is preserved

Applications of SE(2) to Objects



(a) Group actions on and between rigid bodies

(b) Group actions as relative and world-referenced displacements

(c) Inverse right actions

(d) Inverse left actions

B is right h away from A. If we know how A moves subject to left g, then applying the same left g to B will result in a frame that is right h away from the moved A

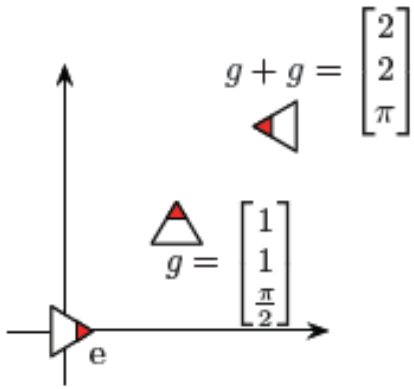
Start a location and undergo a relative motion. Start at another location and undergo the same relative motion. The left action relationship between the two locations remains fixed. Start at two locations that are left g separated. Both locations undergo a right h. Their result is still left g separated.

Note that (a) and (b) are the same because $R\_h(L\_g(\,))$ is the same as $L\_g(R\_h(\,))$ because they are both $g \circ h$ Note, left and right operators commute!

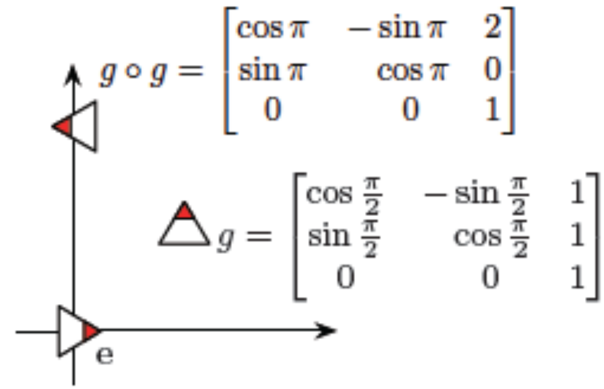
If we have one object at position g and a second object at an (unknown) relative position h, we can call its (known) absolute position gh. If we then apply a left g^{-1} action to both configurations, the second is still h relative to the first. But since the first is now at e, we know that the absolute position of the second is h.

Left SE(2) actions on physical objects move them through the world while preserving their relative displacements but a right action moves relative to its starting point. If I apply a single right action to two different frames, their relative displacement will change. If I want to have a rigid body move through space, I can't apply a single right action to all the frames -- I would need to calculate an individual right action for each of them. I can, however, apply a single left action to all the frames to get the rigid body motion

$R^2 \times S^1$ vs. SE(2)



(a) $\mathbb{R}^2 \times \mathbb{S}^1$



(b) $SE(2)$

Associativity

Associativity. Associativity means that once the placement of elements in an expression has been assigned (e.g., a acting on b acting on c), the order of resolution doesn't matter: For binary operations with infix notation, associativity appears as the ability to arbitrarily group the elements of a multi-operator expression,

$$a \circ (b \circ c) = (a \circ b) \circ c, \tag{1.xxiii}$$

and for function composition (which is always associative), it appears as the statement that precomposing two functions, then applying them to an input is equivalent to applying them to the input in succession,

$$\big(f_2 \circ f_1\big)(a) = f_2(f_1(a)). \tag{1.xxiv}$$

Commutativity. Commutativity means that elements in an expression can exchange positions without affecting the result of the expression. Binary operations are *commutative* if their operands (inputs to the operation) can be swapped for all values of a and b in the domain of the operation,

$$a \circ b = b \circ a. \tag{1.xxv}$$

For operations that are not commutative, specific elements of the domain may still *commute* with each other if they satisfy (1.xxv). For example, any pair of pure-translation elements of $SE(2)$ commute with each other, as

$$(x, y, 0) \circ (u, v, 0) = (x + u, y + v, 0) = (u, v, 0) \circ (x, y, 0), \tag{1.xxvi}$$

and a similar rule holds for pure-rotation elements.[a] Pairs of functions commute with each other if they are commutative with respect to the composition operation,

$$f_1 \circ f_2 = f_2 \circ f_1, \tag{1.xxvii}$$

or, in functional notation,

$$f_2(f_1(a)) = f_1(f_2(a)) \tag{1.xxviii}$$

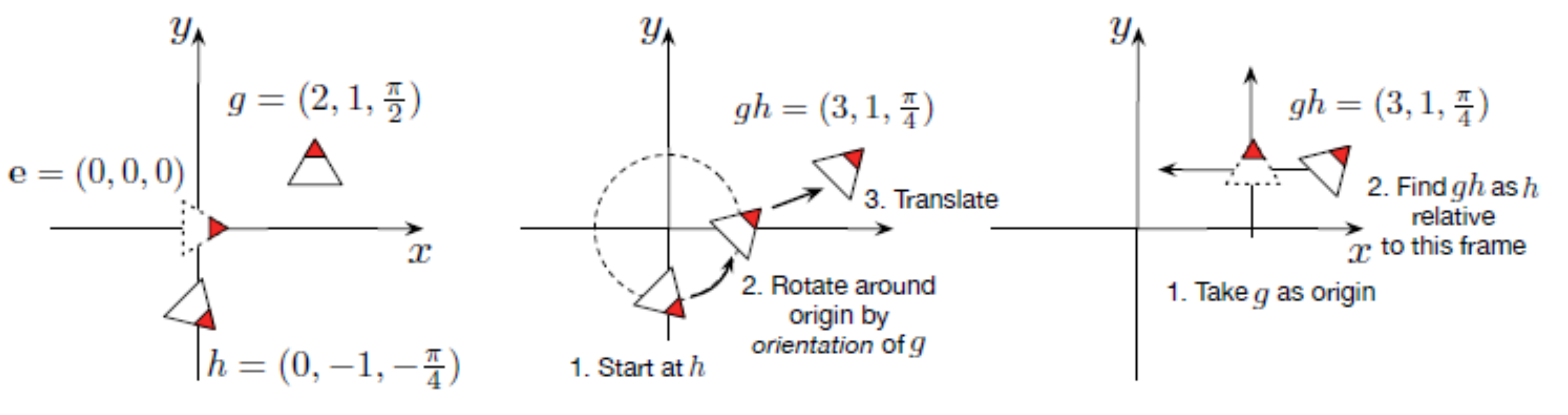
Commutativity of actions

Commutativity of Left and Right actions. Combining the associativity of group operations with the functional forms of the left and right group actions[b] leads to an interesting and powerful property of group actions: left actions applied to a group element commute with right actions applied to that element,

$$L_g(R_h(\mathbf{e})) = g \circ (\mathbf{e} \circ h) \quad (1.xxix)$$

$$= (g \circ \mathbf{e}) \circ h \quad (1.xxx)$$

$$= R_h(L_g(\mathbf{e})) \quad (1.xxxi)$$



(a) Individual group elements (b) Left action interpretation of gh (c) Right action interpretation of gh

Commutative Diagram

Associativity and Commutativity, *continued*

Commutative Diagrams. Commutative diagrams are a useful tool for identifying transformations that commute, without getting caught up in the differences between infix and functional notation. The core idea of a *diagram* (which originates in a branch of mathematics called *category theory*) is that transformations can be represented by arrows, and a sequence of operations can then be represented by chaining arrows together. The resulting diagram commutes (or "is commutative") if all directed paths with the same start and end point produce the same result; pairs of transformations commute with each other if we can construct parallel paths in which the order of their arrows is exchanged while preserving the commutativity of the diagram. The commutative diagram for left and right group operations is shown below.

$$
\begin{tikzcd}
\mathbf{e} \arrow[r, "g\circ"] \arrow[d, "\circ h"'] & g \arrow[d, "\circ h"] \\
h \arrow[r, "g\circ"'] & gh
\end{tikzcd}
$$