Geometry of Locomotion
Chapter 3 Forward Kinematics

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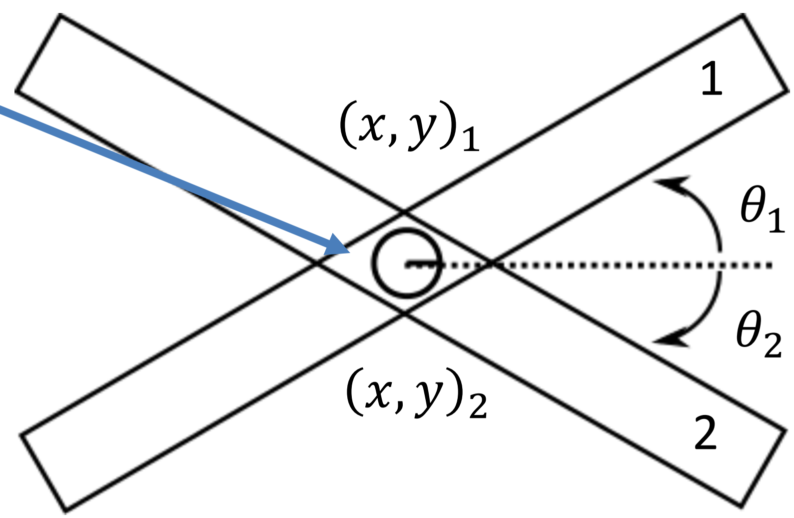
Constraints on a System

- 1 rigid body $SE(2)$
- N rigid bodies $SE(2)\times\cdots\times SE(2) = SE(2)^n$
- 2 bodies pinned at a joint $SE(2)\times S^1$
- Holonomic constraint: $(x, y)_1 = (x, y)_2$



Forward Kinematics Defined

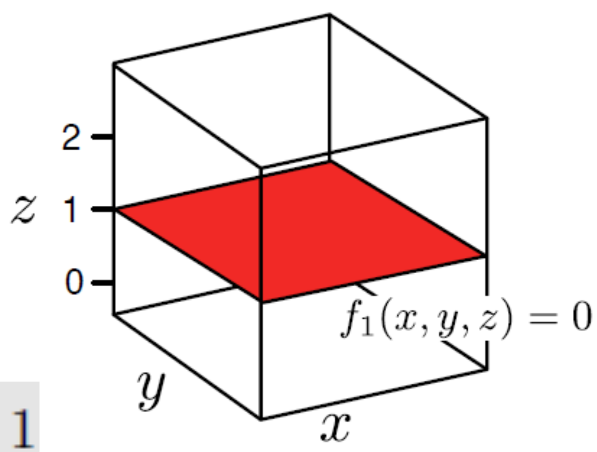
Often, dynamics are not specified in terms of all of the points but rather the positions of the component bodies and/or end-effector position

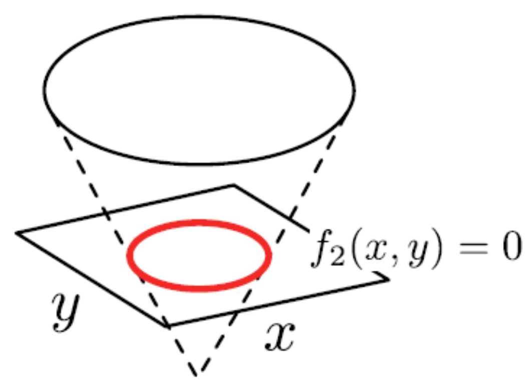
Forward kinematics describe the relationship between the physical positions of all points on the system and the system's generalized coordinates.

Holonomic Constraints

- Remove a dimension of configuration space
- Constraint is $f(q_0, t) = 0$ where $T_{\{q_0\}} f \neq 0$
- Pre-image Theorem
- Multiple constraints act together

Set of points in three dimensions constrained to stay on a circle in plane at z = 1





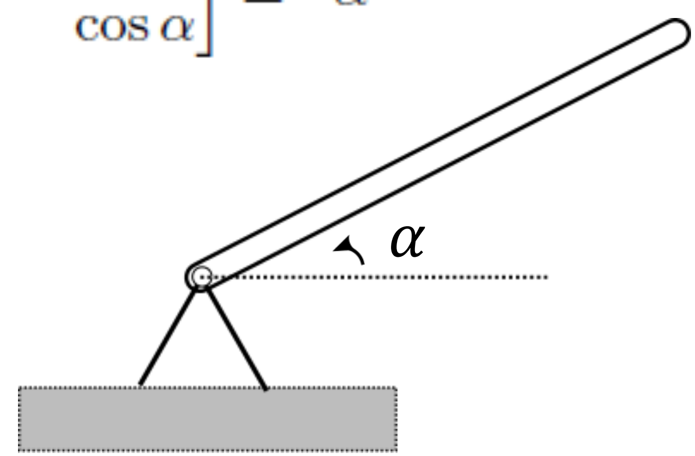
$f_1(p, t) = z - 1$

$f_2(p_1, t) = \sqrt{(x^2 + y^2)} - 1.$

Single DOF system

- Configuration space: $\alpha \in S^1$
- Configuration space: $SE(2)$ with two holonomic constraints $x = 0$ and $y = 0$

$$g = \overbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{SE(2),\ x,y=0} \equiv \overbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}}^{SO(2)} \equiv \overbrace{\alpha}^{\mathbb{S}^1}$$



More frames (for more bodies)

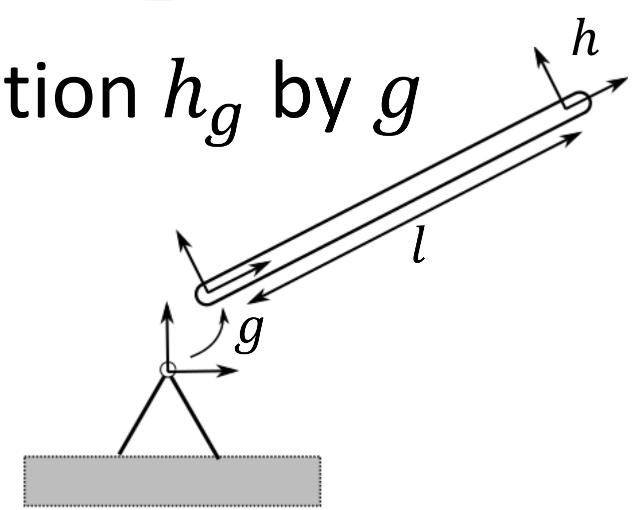
- Want an h frame at other end of body
- Let $h_g = (l, 0{,}0)$ with respect to g

$$h = gh_g = \begin{bmatrix} \cos\alpha & -\sin\alpha & \ell\cos\alpha \\ \sin\alpha & \cos\alpha & \ell\sin\alpha \\ 0 & 0 & 1 \end{bmatrix}$$

$L_g h_g$: transform the frame position h_g by g

$R_{h_g} g$: placing h_g into g

(later is used to help derivation of velocities)



A Note on Notation

frame

base frame

$$g_{1,h}$$

identification number

$$g_{1,h} = h^{-1} g_1$$

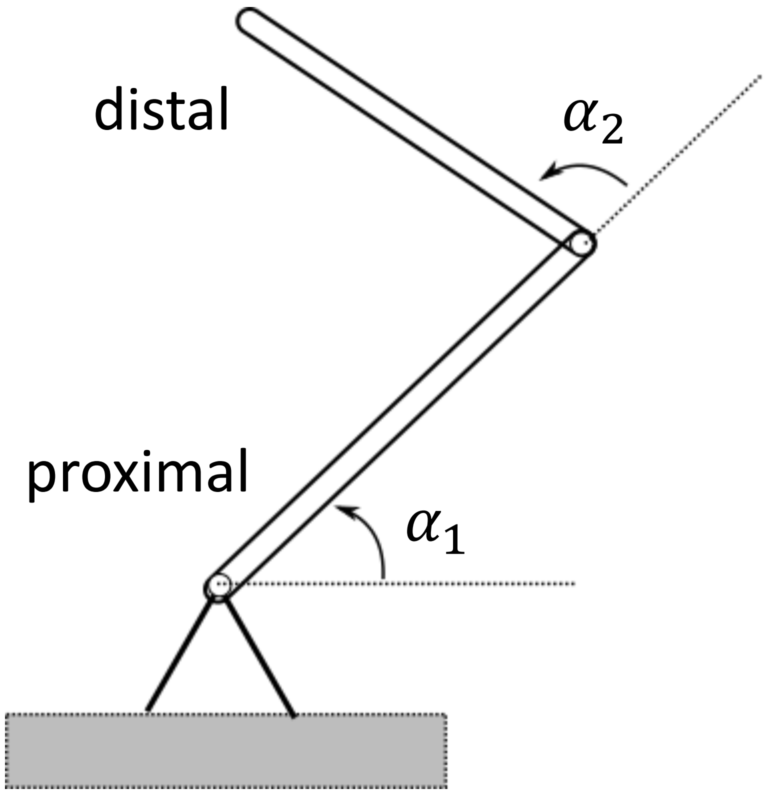
Frames on the left cancel with subscripted frames on the right,

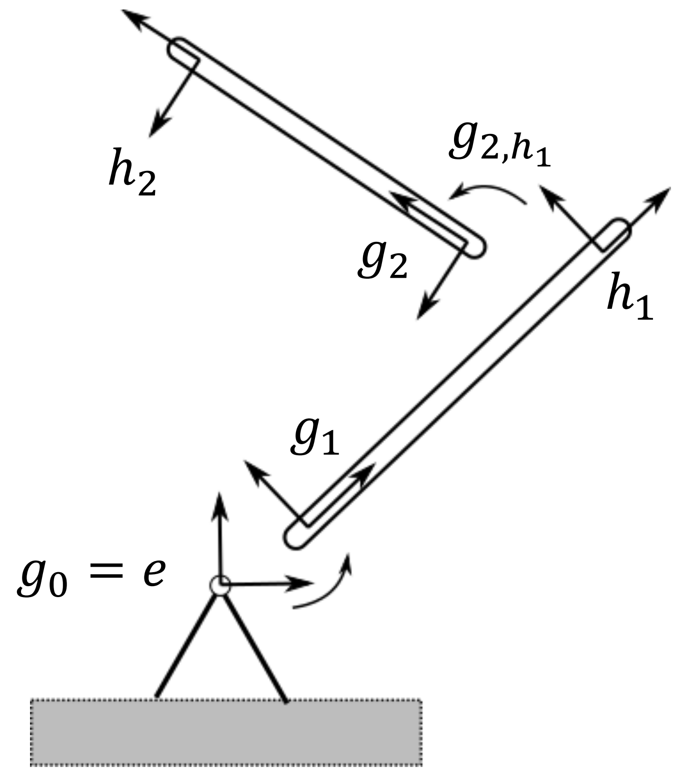
$$g h_g = g g^{-1} h = h$$

During this cancellation, base-frame subscripts on the left are transferred to the right

$$g_{1,g_0} h_{g_1} = g_0^{-1}(g_1 g_1^{-1}) h = g_0^{-1} h = h_{g_0}$$

Second Link

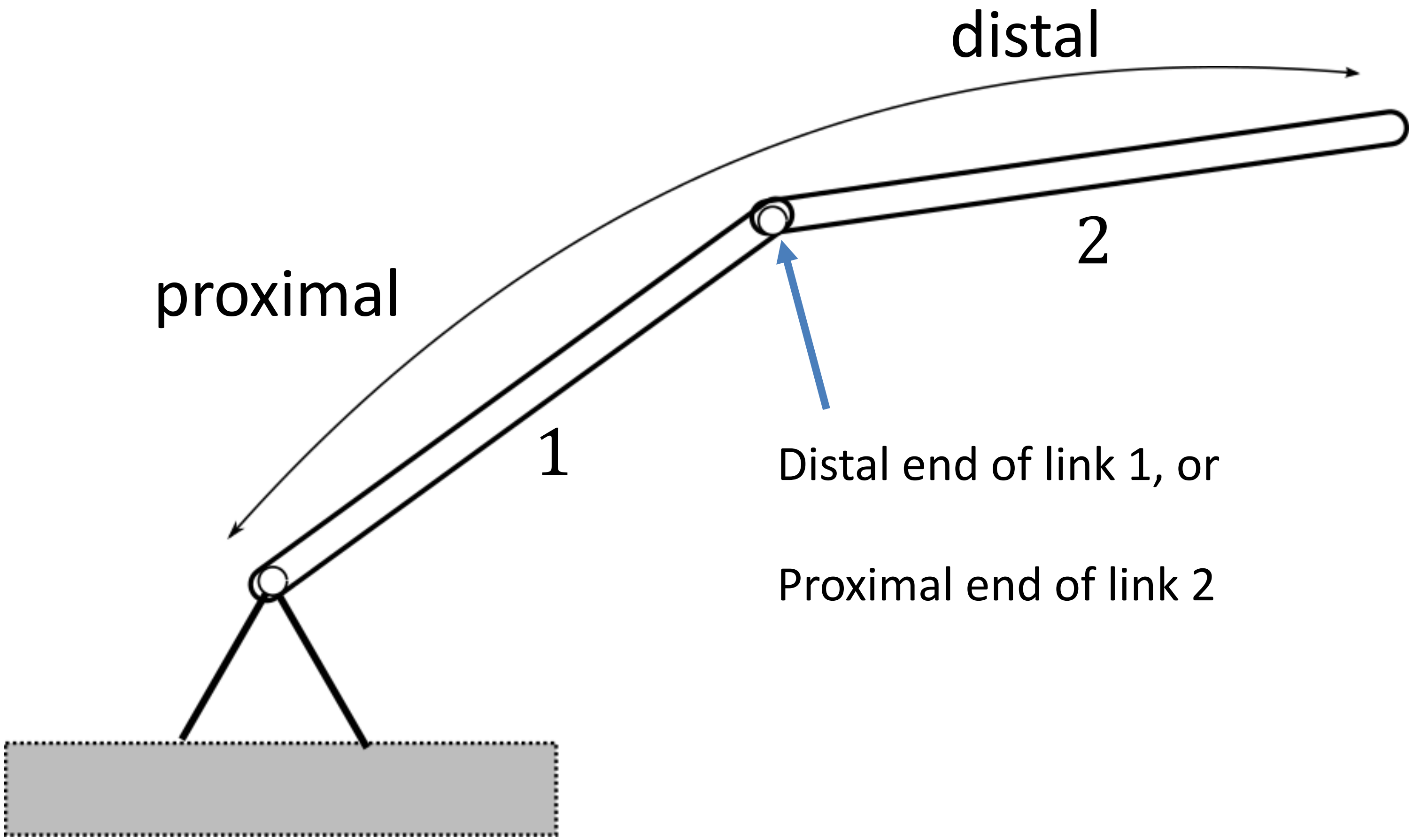




$$g_{2,h_1} = (x_{2,h_1}, y_{2,h_1}, \theta_{2,h_1})$$

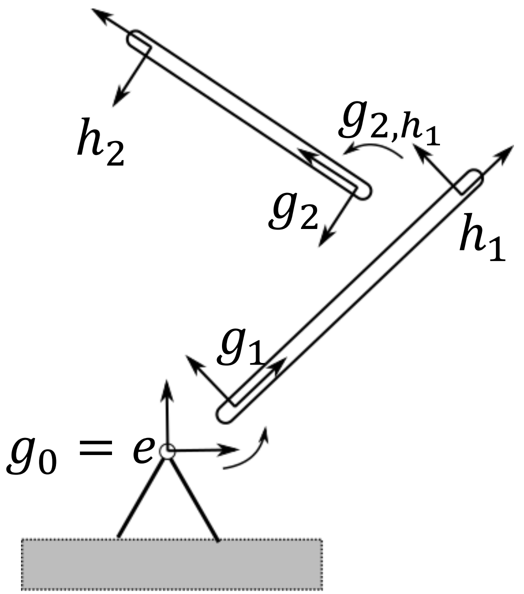
$$\theta_{2,h_1} = \alpha_2$$

Proximal, medial, distal



("link 1 is proximal to link 2") ("link 2 is the distal link") ("the distal end of link 1")

Second Link



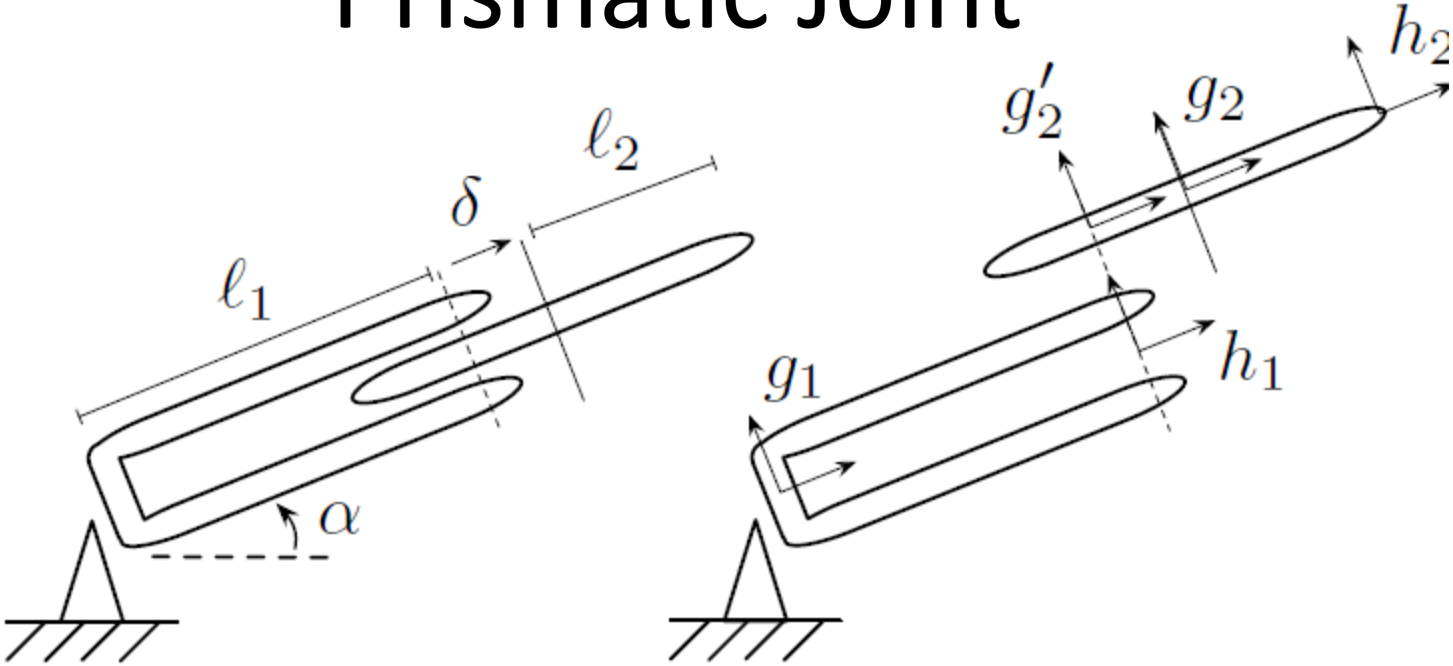
$$g_{2,h_1} = (x_{2,h_1}, y_{2,h_1}, \theta_{2,h_1})$$

$$\theta_{2,h_1} = \alpha_2$$

$$g_2 = \overbrace{(g_1 h_{1,g_1})}^{h_1} g_{2,h_1} = \begin{bmatrix} \cos(\alpha_1+\alpha_2) & -\sin(\alpha_1+\alpha_2) & \ell_1 \cos\alpha_1 \\ \sin(\alpha_1+\alpha_2) & \cos(\alpha_1+\alpha_2) & \ell_1 \sin\alpha_1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} h_2 &= g_2 h_{2,g_2} \\ &= \begin{bmatrix} \cos(\alpha_1+\alpha_2) & -\sin(\alpha_1+\alpha_2) & \ell_1 \cos\alpha_1 + \ell_2 \cos(\alpha_1+\alpha_2) \\ \sin(\alpha_1+\alpha_2) & \cos(\alpha_1+\alpha_2) & \ell_1 \sin\alpha_1 + \ell_2 \sin(\alpha_1+\alpha_2) \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Prismatic Joint



$$g = \overbrace{\begin{bmatrix} 1 & 0 & \delta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{SE(2),\ y,\theta=0} \equiv \overbrace{\begin{bmatrix} \delta \\ 0 \end{bmatrix}}^{\mathbb{R}^2,\ y=0} \equiv \overbrace{\delta}^{\mathbb{R}^1}.$$

$$g_2 = \begin{bmatrix} \cos\alpha & -\sin\alpha & (\ell_1+\delta)\cos\alpha \\ \sin\alpha & \cos\alpha & (\ell_1+\delta)\sin\alpha \\ 0 & 0 & 1 \end{bmatrix}$$

$$h_2 = \begin{bmatrix} \cos\alpha & -\sin\alpha & (\ell_1+\delta+\ell_2)\cos\alpha \\ \sin\alpha & \cos\alpha & (\ell_1+\delta+\ell_2)\sin\alpha \\ 0 & 0 & 1 \end{bmatrix}$$

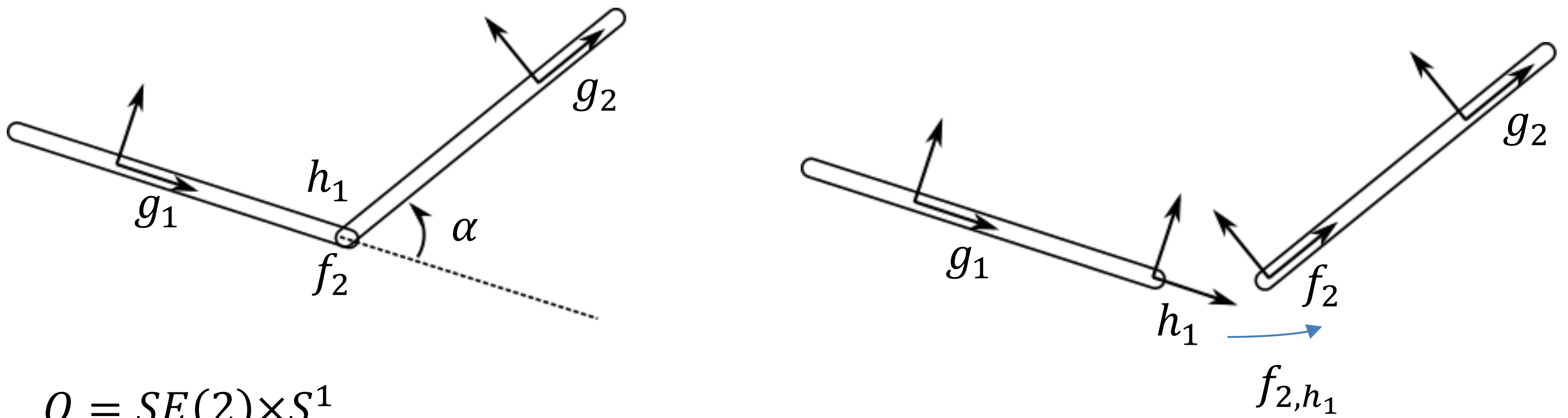
Body Frame

- Position of a system is the position and orientation of its body frame
- All points on system are with respect to body frame
- Placements of rigid bodies are described by shape variables

$$Q = G \times M$$

$$q \in Q, \qquad g \in G, \qquad r \in M$$

Frame Assignments



$Q = SE(2) \times S^1$

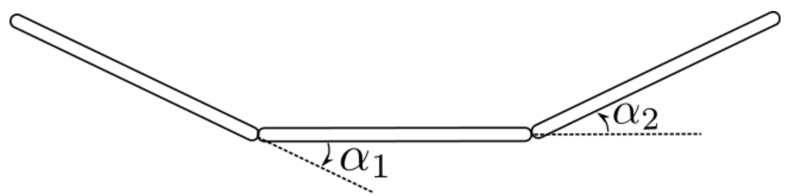
Choose a base link, assign the base frame and then go from there
$g \in S2(2)$ be the position and orientation of the base link

$$g_2 = \overbrace{g_1 \circ h_{1,g_1}}^{\text{link 1}} \circ \overbrace{f_{2,h_1} \circ g_{2,f_2}}^{\text{link 2}}$$

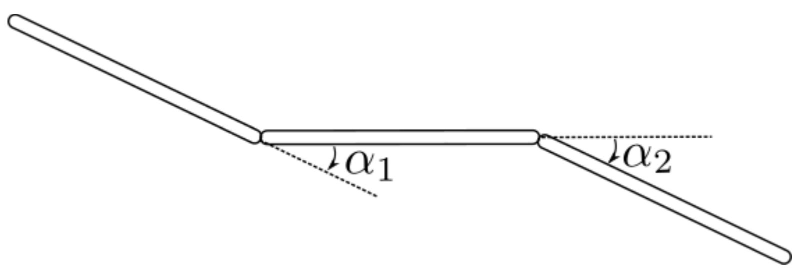
$$= g_1 \begin{bmatrix} \cos\alpha & -\sin\alpha & (\ell_1/2) + (\ell_2/2)\cos\alpha \\ \sin\alpha & \cos\alpha & (\ell_1/2) + (\ell_2/2)\sin\alpha \\ 0 & 0 & 1 \end{bmatrix}$$

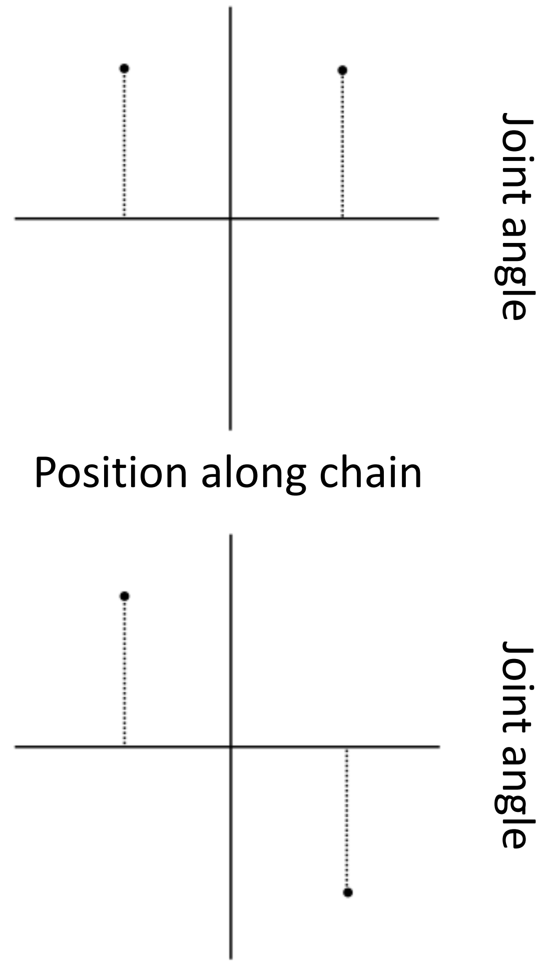
Even and Odd

Even
$\alpha_1 > 0$
$\alpha_2 > 0$



Odd
$\alpha_1 > 0$
$\alpha_2 < 0$





Put that frame anywhere

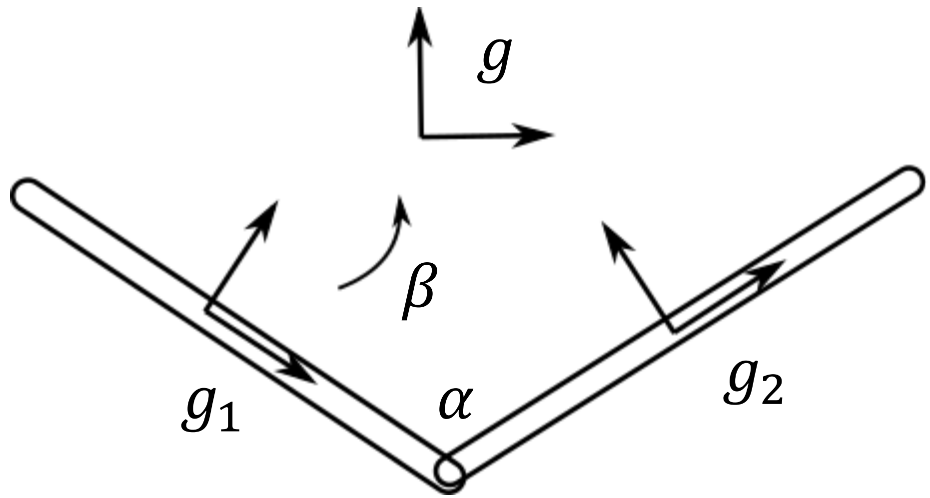
Consider a frame at

$g \in SE(2)$

Which is

$\beta \in SE(2)$

Away from the base link



$$g_{1,g} = \beta^{-1}$$

$$g_{2,g} = \beta^{-1} g_{2,g_1}(\alpha)$$

If (and only if) we can express β as a function of α, then $g_{1,g}$ and $g_{2,g}$ are both functions of the system shape α, meeting the necessary and sufficient conditions for g to serve as the system body frame.