Geometry of Locomotion
Chap 4.1 – Kinematic Locomotion

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Topics

- Paths in shape and position space
- Linearity (really briefly)
- **Symmetry**
- Differential Drive and Ackerman Steered Car

Trajectories

As a general rule, any configuration trajectory executed by a locomoting system over a time interval [0, T] can be decomposed into a shape trajectory ψ in which each shape variable is defined as a function of time

$$\begin{aligned} \psi : [0, T] &\to M \\ t &\mapsto r, \end{aligned}$$

an associated *induced a position trajectory* g^{ψ}

$$\begin{aligned} g^{\psi} : [0, T] &\to G \\ t &\mapsto g, \end{aligned}$$

Kinematic Locomotion

- Pacing does not matter

$$\psi(t) = \psi'(\tau(t))$$

$$g^{\psi}(t) = g^{\psi'}(\tau(t))$$

$$\begin{tikzcd}
\psi(t) \arrow[r, "\text{Apply constraints}"] \arrow[d, "\tau = f(t)"', "\text{Repace timing}"] & g^{\psi}(t) \arrow[d, "\tau = f(t)"', "\text{Repace timing}"] \\
\psi'(\tau) \arrow[r, "\text{Apply constraints}"] & g^{\psi'}(t)
\end{tikzcd}$$

Linear Kinematic

- Linear Kinematic

$$\dot{g} = f(q)\dot{r}$$

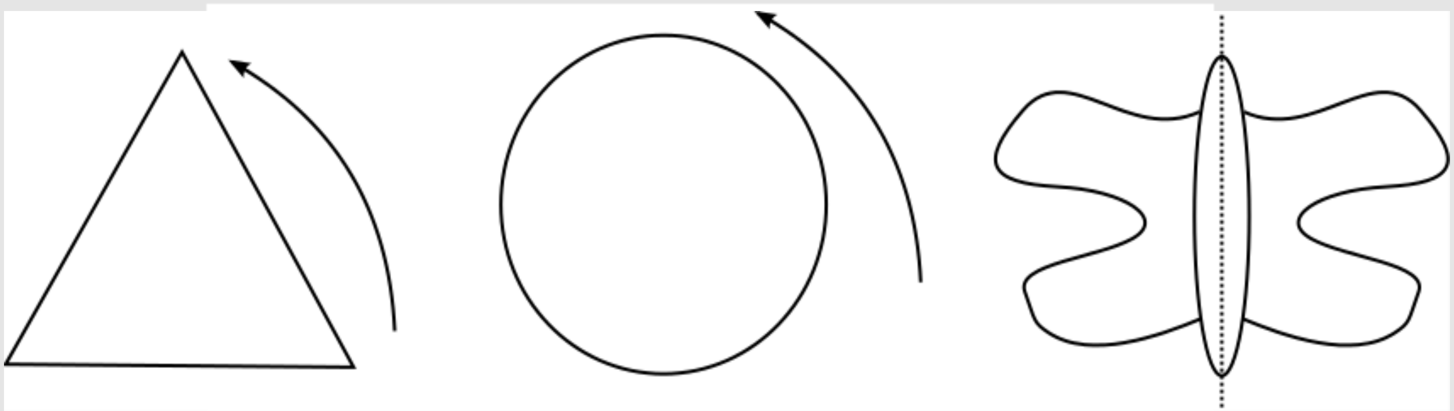
- Symmetric Linear Kinematic

$$\xi = f(r)\dot{r}$$

Dimension reduction
Lie Algebra uses

Symmetry and Groups

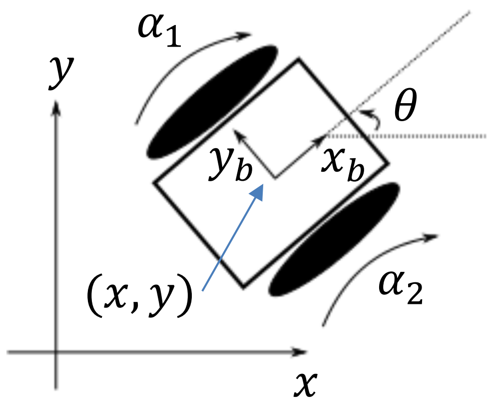
When we say that something has *symmetry*, we mean that there is some action we can perform on it so that its starting and ending configurations are indistinguishable. For example, an equilateral triangle is symmetric with respect to rotations by $120°$, a circle is symmetric with respect to any rotation, and a butterfly is symmetric with respect to reflections across its central axis.



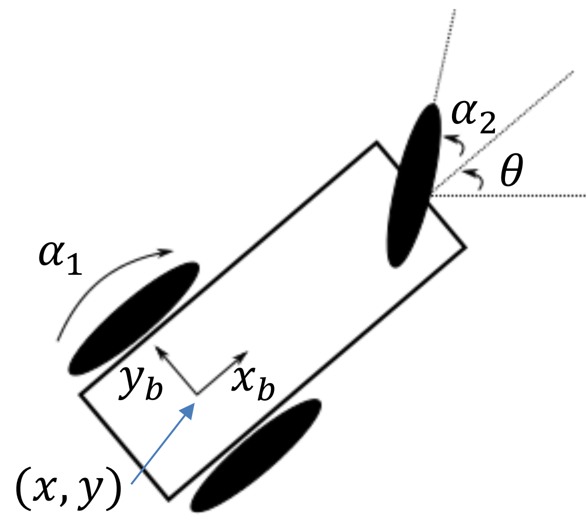
Symmetry is closely linked to mathematical groups structure—the set of actions under which an object is symmetric form a group, satisfying the four basic properties outlined on on page 9:

1. **Closure**: Two symmetry-preserving actions can be concatenated into a third action that is also symmetry-preserving. Example: two rotations by $120°$ form a $240°$ rotation, which is also a symmetric action on the equilateral triangle.
2. **Associativity**: If a series of transformations is conducted, the order of operations does not matter (though left-right order may still be important). Example: rotating the triangle twice by $120°$ is equivalent to concatenating the rotations into a single $240°$ rotation and applying it to the triangle.
3. **Identity**: Objects are trivially symmetric under null actions, which can be incorporated as zero-magnitude members of the symmetry group. Example: the rotational symmetry group for the triangle is rotations by $k \cdot 120°$ for integer values of k, and the identity element (no rotation) is the element for whcih $k = 0$.
4. **Inverse**: Any symmetry-preserving transformation may be "undone" or reversed, and concatenating this reversal with the original action produces a null action. Example: the inverse of a $120°$ rotation is a $-120°$ rotation, and together these actions produce a $0°$ rotation.

Differential and Ackerman



Differential drive



Ackermann steered

$$\begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}$$

$$\begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \tan \alpha_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}$$