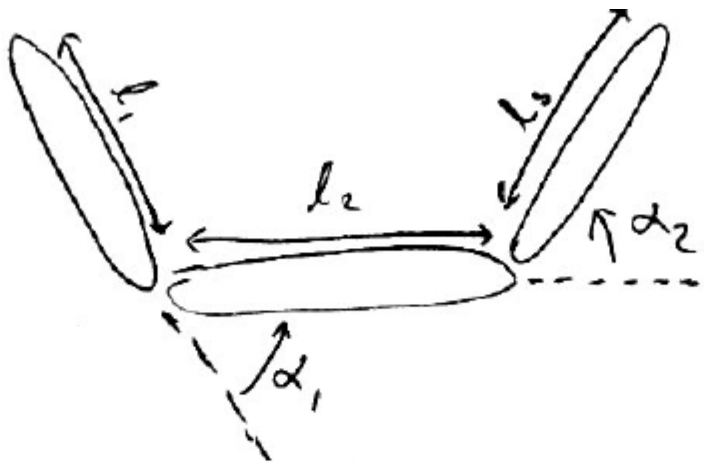
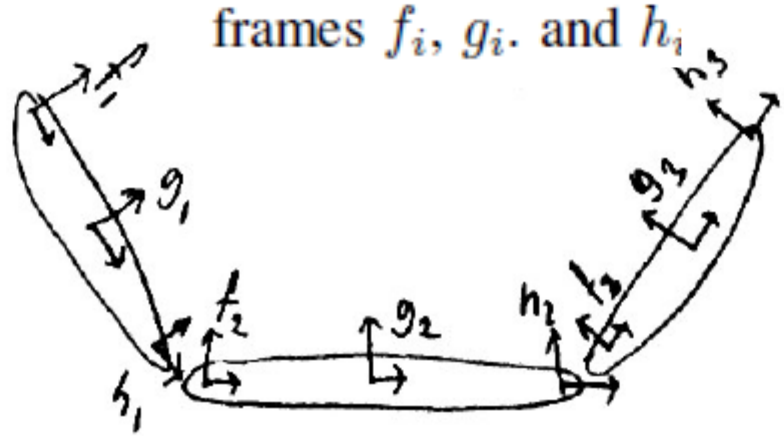
Geometry of Locomotion
Chapter 3.6
Jacobian for Mobile Systems

Howie Choset

Three-Link System





$$\xi_{g_2} = \xi = \begin{bmatrix} I^{3\times 3} & 0^{3\times 2} \end{bmatrix} \begin{bmatrix} \xi \\ \dot{r} \end{bmatrix}$$

Chose middle of middle link to be body frame

Body Velocity of Link 1

body velocity of the proximal end of link 2.

$$\xi_{f_2} = Ad^{-1}_{f_{2,g_2}} \xi_{g_2} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\ell_2/2 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{f_{2,g_2}=(-\ell_2/2,0,0)}} \overbrace{\begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \end{bmatrix}}^{\xi_{g_2}} = \begin{bmatrix} \xi^x \\ \xi^y - (\xi^\theta \ell_2)/2 \\ \xi^\theta \end{bmatrix}$$

calculate the body velocity of the distal end of link 1, which has rotational velocity of $-\dot{\alpha}_1$ with respect to f_2.

$$\xi_{h_1} = Ad^{-1}_{h_{1,h_1'}} \xi_{h_1'} = \overbrace{\begin{bmatrix} \cos\alpha_1 & -\sin\alpha_1 & 0 \\ \sin\alpha_1 & \cos\alpha_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{h_{1,h_1'}=(0,0,-\alpha_1)}} \overbrace{\left(\underbrace{\begin{bmatrix} \xi^x \\ \xi^y - (\xi^\theta \ell_2)/2 \\ \xi^\theta \end{bmatrix}}_{\xi_{f_2}} + \begin{bmatrix} 0 \\ 0 \\ -\dot{\alpha}_1 \end{bmatrix} \right)}^{\xi_{h_1'}}$$

$$= \begin{bmatrix} \xi^x \cos\alpha_1 - (\xi^y - (\xi^\theta \ell_2)/2) \sin\alpha_1 \\ \xi^x \sin\alpha_1 + (\xi^y - (\xi^\theta \ell_2)/2) \cos\alpha_1 \\ \xi^\theta - \dot{\alpha}_1 \end{bmatrix}.$$

Finish Link 1

Finally, we arrive at the body velocity of g_1 by moving along the link by $\ell_1/2$,

$$\xi_{g_1} = Ad^{-1}_{g_1,h_1}\xi_{h_1} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\ell_1/2 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{g_1,h_1=(0,-\ell_1/2,0)}} \overbrace{\begin{bmatrix} \xi^x \cos\alpha_1 - (\xi^y - (\xi^\theta \ell_2)/2)\sin\alpha_1 \\ \xi^x \sin\alpha_1 + (\xi^y - (\xi^\theta \ell_2)/2)\cos\alpha_1 \\ \xi^\theta - \dot{\alpha}_1 \end{bmatrix}}^{\xi_{h_1}}$$

$$= \begin{bmatrix} \xi^x \cos\alpha_1 - (\xi^y + (\xi^\theta \ell_2)/2)\sin\alpha_1 \\ \xi^x \sin\alpha_1 + (\xi^y - (\xi^\theta \ell_2)/2)\cos\alpha_1 - (\ell_1/2)(\xi^\theta - \dot{\alpha}_1) \\ \xi^\theta - \dot{\alpha}_1 \end{bmatrix}$$

Link 3

we calculate the body velocity of link 3 by first taking the body velocity of Distal end of link 2

$$\xi_{h_2} = Ad^{-1}_{h_2,g_2}\xi_{g_2} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \ell_2/2 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{f_2,g_2=(\ell_2/2,0,0)}} \overbrace{\begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \end{bmatrix}}^{\xi_{g_2}} = \begin{bmatrix} \xi^x \\ \xi^y + (\xi^\theta \ell_2)/2 \\ \xi^\theta \end{bmatrix}$$

converting it into the proximal velocity of link 3

$$\xi_{f_3} = Ad^{-1}_{f_3,f_3'}\xi_{f_3'} = \overbrace{\begin{bmatrix} \cos\alpha_2 & \sin\alpha_2 & 0 \\ -\sin\alpha_2 & \cos\alpha_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{f_3,f_3'=(0,0,\alpha_2)}} \overbrace{\left(\underbrace{\begin{bmatrix} \xi^x \\ \xi^y + (\xi^\theta \ell_2)/2 \\ \xi^\theta \end{bmatrix}}_{\xi_{h_2}} + \begin{bmatrix} 0 \\ 0 \\ \dot{\alpha}_2 \end{bmatrix} \right)}^{\xi_{f_3'}}$$

$$= \begin{bmatrix} \xi^x \cos\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\sin\alpha_2 \\ -\xi^x \sin\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\cos\alpha_2 \\ \xi^\theta + \dot{\alpha}_2 \end{bmatrix},$$

Link 3

- Move to middle

$$\xi_{g_3} = Ad^{-1}_{g_3,f_3}\xi_{f_3} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \ell_3/2 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{g_3,f_3=(0,\ell_3/2,0)}} \overbrace{\begin{bmatrix} \xi^x \cos\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\sin\alpha_2 \\ -\xi^x \sin\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\cos\alpha_2 \\ \xi^\theta + \dot{\alpha}_2 \end{bmatrix}}^{\xi_{f_3}}$$

$$= \begin{bmatrix} \xi^x \cos\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\sin\alpha_2 \\ -\xi^x \sin\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\cos\alpha_2 + (\ell_3/2)(\xi^\theta + \dot{\alpha}_2) \\ \xi^\theta + \dot{\alpha}_2 \end{bmatrix}$$

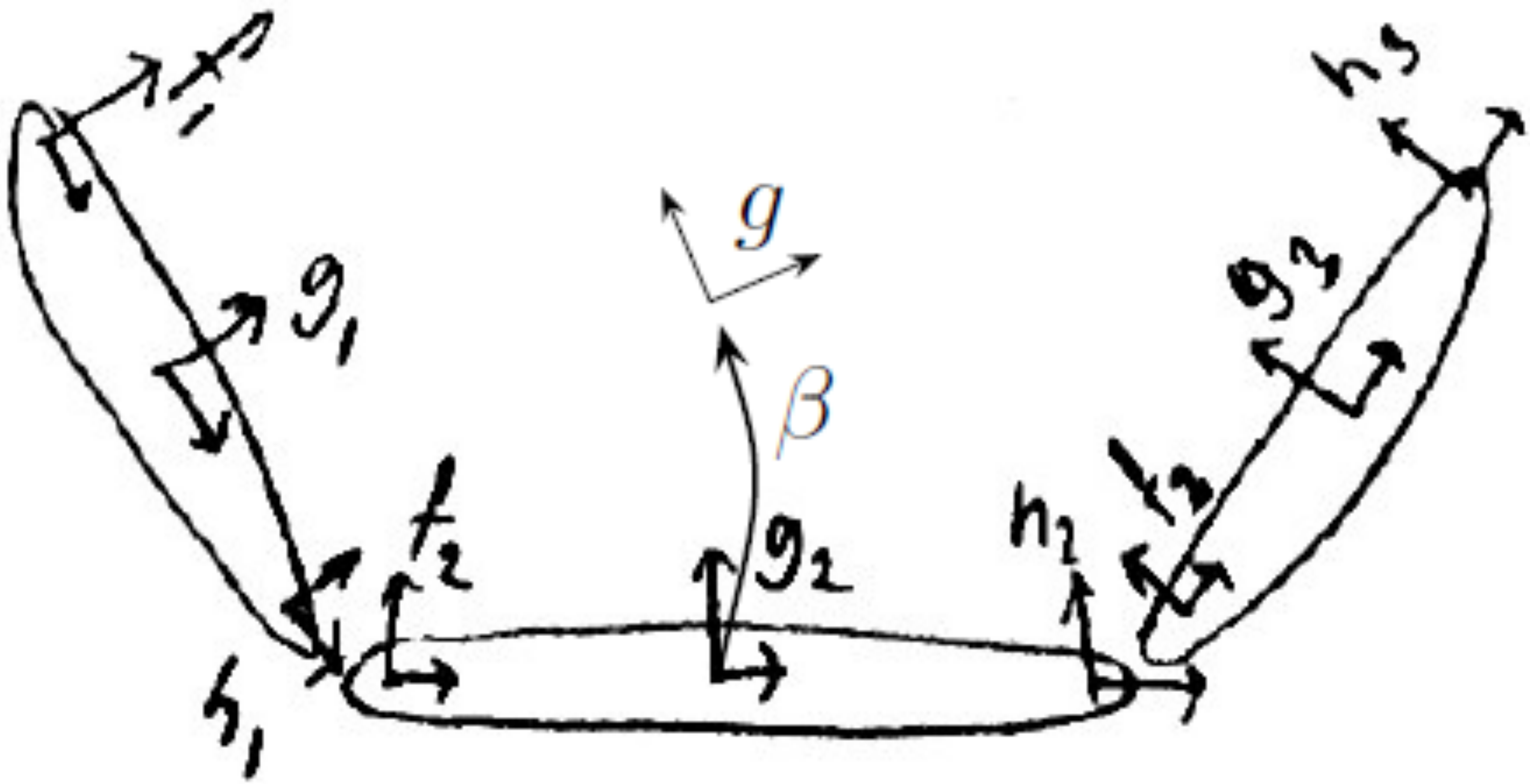
Jacobians

$$\xi_{g_1} = \overbrace{\begin{bmatrix} \cos\alpha_1 & -\sin\alpha_1 & (\ell_2 \sin\alpha_1)/2 & 0 & 0 \\ \sin\alpha_1 & \cos\alpha_1 & -(\ell_2\cos\alpha_1 + \ell_1)/2 & \ell_1/2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}}^{J^b_{g_1}} \begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \\ \dot\alpha_1 \\ \dot\alpha_2 \end{bmatrix}$$

$$\xi_{g_2} = \overbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}}^{J^b_{g_2}} \begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \\ \dot\alpha_1 \\ \dot\alpha_2 \end{bmatrix}$$

$$\xi_{g_3} = \overbrace{\begin{bmatrix} \cos\alpha_2 & \sin\alpha_2 & (\ell_2 \sin\alpha_2)/2 & 0 & 0 \\ -\sin\alpha_2 & \cos\alpha_2 & (\ell_2\cos\alpha_2 + \ell_3)/2 & 0 & \ell_3/2 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}}^{J^b_{g_3}} \begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \\ \dot\alpha_1 \\ \dot\alpha_2 \end{bmatrix}.$$

Other Body Frames



1. If we keep the middle link as the system body frame, what is the body velocity of frame g

2. If we take frame g as the system body frame, what are the body velocities of the links?

Question 1

Taking our body-frame iterative Jacobian formula from $\xi_{h_i} = (Ad^{-1}_{h_i,g_i'})(\xi_{h_{i-1}} + v_i).$

$$\xi_g = Ad_\beta^{-1}(\xi_{g_2} + v_\beta).$$

Want the velocity with respect to g_2 of the frame rigidly attached to g and coincident with g_2

$$v_\beta = T_\beta R_{\beta^{-1}} \dot{\beta} = T_\beta R_{\beta^{-1}} \frac{\partial \beta}{\partial \alpha} \dot{\alpha}.$$

$$\xi_g = Ad_\beta^{-1}(\xi_{g_2} + T_\beta R_{\beta^{-1}} \frac{\partial \beta}{\partial \alpha} \dot{\alpha})$$

Question 1

$$\xi_g = \begin{bmatrix} \cos\beta^\theta & \sin\beta^\theta & \beta^x \sin\beta^\theta - \beta^y \cos\beta^\theta \\ -\sin\beta^\theta & \cos\beta^\theta & \beta^x \cos\beta^\theta + \beta^y \sin\beta^\theta \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} \xi_{g_2}^x \\ \xi_{g_2}^y \\ \xi_{g_2}^\theta \end{bmatrix} + \begin{bmatrix} 1 & 0 & \beta^y \\ 0 & 1 & -\beta^x \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial\beta^x}{\partial\alpha_1} & \frac{\partial\beta^x}{\partial\alpha_2} \\ \frac{\partial\beta^y}{\partial\alpha_1} & \frac{\partial\beta^y}{\partial\alpha_2} \\ \frac{\partial\beta^\theta}{\partial\alpha_1} & \frac{\partial\beta^\theta}{\partial\alpha_2} \end{bmatrix} \begin{bmatrix} \dot\alpha_1 \\ \dot\alpha_2 \end{bmatrix} \right)$$

Reform

Alternatively, we can reform (3.62) to simplify the matrix algebra by separating the adjoint a
components,

$$\xi_g = \overbrace{(T_\beta L_{\beta^{-1}})(T_\mathbf{e} R_\beta)}^{Ad_\beta^{-1}}(\xi_{g_2} + T_\beta R_{\beta^{-1}} \tfrac{\partial \beta}{\partial \alpha}\dot{\alpha}).$$

and distributing the right lifted action over the velocity terms,

$$\xi_g = T_\beta L_{\beta^{-1}}(T_\mathbf{e} R_\beta \xi_{g_2} + \overbrace{(T_\mathbf{e} R_\beta)(T_\beta R_{\beta^{-1}})}^{I} \tfrac{\partial \beta}{\partial \alpha}\dot{\alpha}).$$

This form of the equation expands as

$$\xi_g = \begin{bmatrix} \cos\beta^\theta & \sin\beta^\theta & 0 \\ -\sin\beta^\theta & \cos\beta^\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & -\beta^y \\ 0 & 1 & \beta^x \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi^x_{g_2} \\ \xi^y_{g_2} \\ \xi^\theta_{g_2} \end{bmatrix} + \begin{bmatrix} \frac{\partial \beta^x}{\partial \alpha_1} & \frac{\partial \beta^x}{\partial \alpha_2} \\ \frac{\partial \beta^y}{\partial \alpha_1} & \frac{\partial \beta^y}{\partial \alpha_2} \\ \frac{\partial \beta^\theta}{\partial \alpha_1} & \frac{\partial \beta^\theta}{\partial \alpha_2} \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix} \right),$$