Where does the Exponential come from?

Howie Choset

September 29, 2015

1 Solution of the first order differential equation

We have two ways to show the solutions to

$$\dot{x} = Ax$$

where the solution is $x(t) = \exp(At)x(0)$.
(1) Convert to the frequency domain

$$\begin{aligned}
sX(s) &= AX(s) \\
(sI - Ax)X(s) &= 0 \\
L[(sI - A)X(s)] &= 0 \\
\exp(-At)x(t) &= x(0) \\
x(t) &= \exp(At)x(0)
\end{aligned}$$

(2) Multiply both sides by $\exp(-At)$

$$\begin{aligned}
\exp(-At)\dot{x} - \exp(-At)Ax &= 0 \\
\int_0^t \frac{d}{ds}[\exp(-As)x]\, ds &= \int_0^t 0\, ds \\
\exp(-At)x(t) - \exp(-A0)x(0) &= 0 \\
x(t) &= \exp(At)x(0)
\end{aligned}$$

There is a third way if x were scalar. What is it?

2 Simple version

Consider the velocity of a point q attached to a rigid body rotating around a fixed axis ω.

$$
\begin{aligned}
\dot{q} &= \omega \times q \\
\dot{q} &= \hat{w}q \text{ (we know the solution here)} \\
q(t) &= \exp(\hat{w}t)q(0)
\end{aligned}
$$

Now, lets do the same thing but for $SE(3)$. Now, we are rotating about an arbitrary axis in space. Let p be a point on that axis and consider the velocity of a point q.

$$
\begin{gathered}
\dot{q} = \omega \times (q - p) \\
\begin{bmatrix} \dot{q} \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \hat{\omega} & -\omega \times p \\ 0 & 0 \end{bmatrix}}_{\hat{\xi}} \begin{bmatrix} q \\ 1 \end{bmatrix}
\end{gathered}
$$

$$
\dot{\bar{q}} = \hat{\xi}\bar{q} \implies \bar{q}(t) = \exp(\hat{\xi}t)q(0)
$$

3 Lie Algebra

A *Lie Algebra* is a vector space over $\mathbb{R}$ that posses a Lie bracket operation that satisfies

- bilinearity
- skew commutativity
- the Jacobi identity

The Lie algebra is denoted by $\mathfrak{g}$. It is important to note that the Lie Algebra is *not* the tangent space at the group at the origin. It can be shown that $\mathfrak{g}$ is isomorphic to T_eG which allows us to act as if they are the same.
Lie groups allow for group velocities to be represented in terms of Lie algebra elements. This is accomplished by using the lifted action, or in this case the left lifted action

$$
\xi = T_g L_{g^{-1}} \dot{g}
$$

Note that the above lifted action is a differential but in this case we are hiding the step that maps us from T_eG to $\mathfrak{g}$.
We want to pull things back to the Lie algebra so we can add vectors from different tangent spaces. For example, $\dot{x} = J(q)\dot{q}$ uses the Jacobian to add all of the joint velocities in the end-effector frame (written with respect to the base frame). If anyone is interested, we can take side bar on that topic.
Alternatively, one can think of creating a left invariant vector field on G by pushing the Lie algebra elements forward

$$X_\xi(g) = T_eL_g\xi$$

Note again, we are mapping from $\mathfrak{g}$ to TG.
We can associate with a left invariant vector field a curve in the group G.

$$\phi_\xi : \mathbb{R} \;\rightarrow\; G \qquad t \mapsto \exp(\hat{\xi}t)$$

Note that ϕ_ξ is the integral curve of X_ξ passing through the group identity at $t = 0$.
The function

$$\exp \;:\; \mathfrak{g} \;\rightarrow\; G \qquad \xi \mapsto \phi_\xi(1)$$

is called the exponential mapping of $\mathfrak{g}$ to G.

4 Fiber Bundles

When we start talking about locomoting systems, we will need additional structure called principle fiber bundles. This is a fancy way of saying the space we are considering becomes

$$Q = \underbrace{G}_{\substack{\text{group}\\\text{(fiber)}}} \times \overbrace{M}^{\substack{\text{shape}\\\text{(base)}}}$$

When we talk about actions on the bundle, the action only occurs on the group portion of the configuration, or the group variable. So consider $q = (g, r)$ then

$$\Phi_hq = (L_hg, r) = (hg, r) = \Phi(h, q)$$

Note that if this can be done globally, then this bundle is called trivial.

5 Infinitesimal Generator

If $\xi \in T_e G$, then

$$\Phi^{\xi} : \mathbb{R} \times Q \to Q \qquad (t, q) \mapsto \Phi(\exp(t\hat{\xi}), q)$$

is a right action on Q.
In other words, Φ^{ξ} is a flow on Q.
The corresponding vector field on Q is

$$\xi_Q(q) = \frac{d}{dt}\Phi(\exp(\hat{\xi}t), q)|_{t=0}$$

where ξ_Q is called the infinitesimal generator.
Let $q = (g, r)$, then

$$\begin{aligned}
\xi_Q((g, r)) &= \frac{d}{dt}\underbrace{\Phi(\exp(\hat{\xi}t), q)}_{(\exp(\hat{\xi}t)g, r)}|_{t=0} \\
&= \frac{d}{dt}(\exp(\hat{\xi}t)g, r)|_{t=0} \\
&= (T_e R_g \hat{\xi} \exp(\hat{\xi}t), 0)|_{t=0} \\
&= T_e R_g \hat{\xi}
\end{aligned}$$