Textbook Supplement 1: Lifted Actions and the Identity

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1 Calculating the Lifted Actions

When calculating lifted actions, it is important to evaluate the derivatives symbolically *before* incorporating any particular frames. For instance:

$$T_e L_g = \left.\frac{\partial(hg)}{\partial g}\right|_{g=e,\ h=g} \neq \frac{\partial(ge)}{\partial e}. \tag{1}$$

The latter expression is ill-defined (what does it mean to take a derivative of a quantity with respect to the identity?), while the correct interpretation allows us to evaluate the lifted action as the variation of hg with respect to the (parameters of) g, and then select g and h to fit the problem.

Note that in this example, we shift g from the first index of the lifted action to the second. This is a slight overloading in notation, but it is good to become comfortable with it, as it allows us to work out general group and lifted operations using the same terms as we use for their instantiations as specific actions.

The other lifted actions relating to the group identity are

$$T_e R_g = \left.\frac{\partial(gh)}{\partial g}\right|_{g=e,\ h=g} \neq \frac{\partial(ge)}{\partial e}, \tag{2}$$

$$T_g L_{g^{-1}} = \left.\frac{\partial(hg)}{\partial g}\right|_{g=g,\ h=g^{-1}} \neq \frac{\partial(g^{-1}g)}{\partial g}, \tag{3}$$

and

$$T_g R_{g^{-1}} = \left.\frac{\partial(gh)}{\partial g}\right|_{g=g,\ h=g^{-1}} \neq \frac{\partial(gg^{-1})}{\partial g}. \tag{4}$$

2 Inverting Group Elements

In the lifted actions involving $h = g^{-1}$, we rely on the property that the inverse of any group element is also an element of that group. When computing these lifted actions, we calculate the inverse of the group elements, then extract the parameter values of the corresponding group element. For example, if we take

$$g^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & -x\cos\theta - y\sin\theta \\ -\sin\theta & \cos\theta & x\sin\theta - y\cos\theta \\ 0 & 0 & 1 \end{bmatrix}, \tag{5}$$

then we can extract the elements of $h = g^{-1}$ as

$$h = (u, v, \beta) = (-x\cos\theta - y\sin\theta, x\sin\theta - y\cos\theta, -\theta). \tag{6}$$