Geometry of Locomotion
Chapter 3.5 Velocity Kinematics

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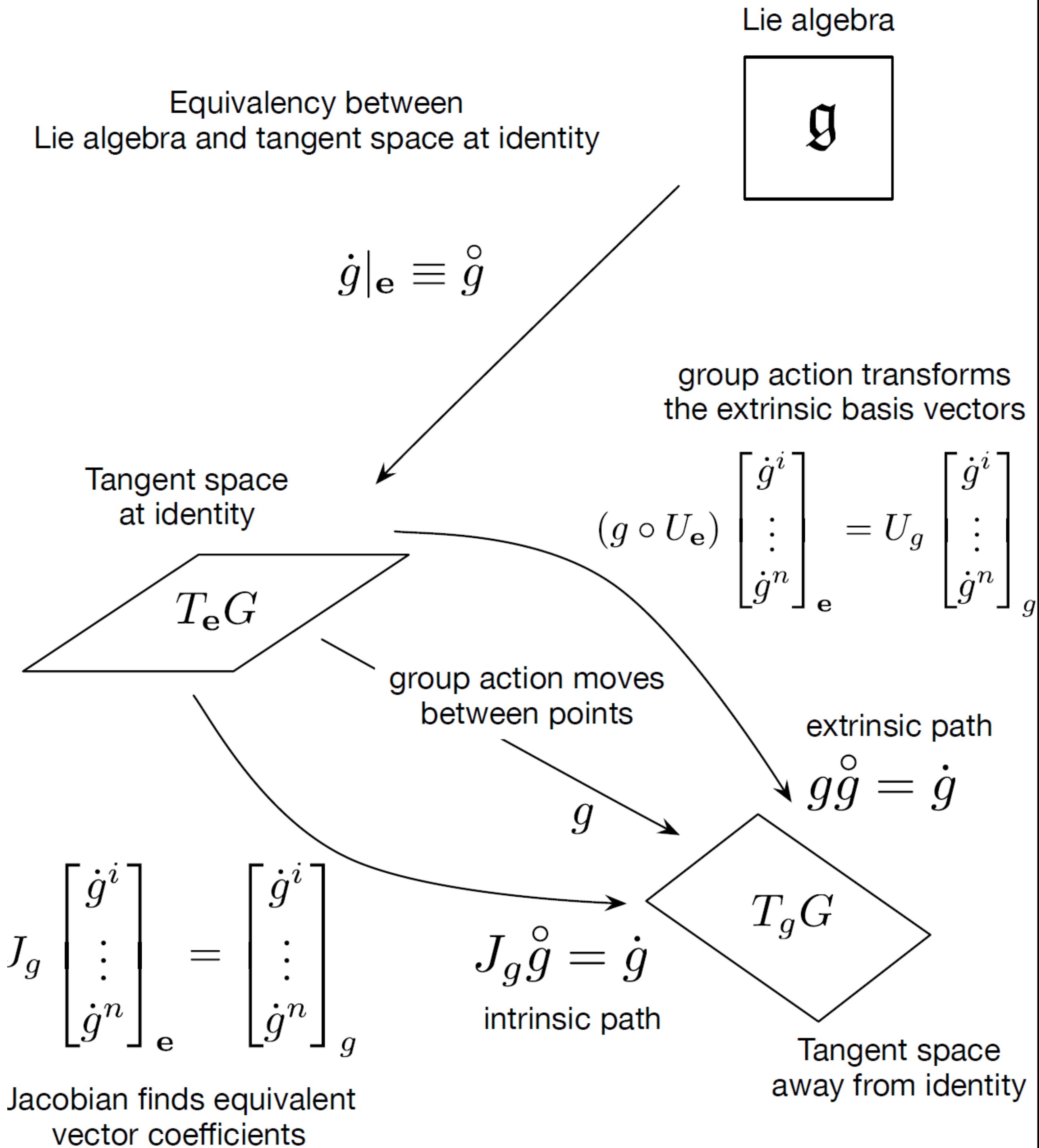

Lie algebra
$\mathfrak{g}$
Equivalency between
Lie algebra and tangent space at identity
$\dot{g}\big|_{\mathbf{e}} \equiv \overset{\circ}{g}$
group action transforms
the extrinsic basis vectors
$(g \circ U_{\mathbf{e}}) \begin{bmatrix} \dot{g}^i \\ \vdots \\ \dot{g}^n \end{bmatrix}_{\mathbf{e}} = U_g \begin{bmatrix} \dot{g}^i \\ \vdots \\ \dot{g}^n \end{bmatrix}_g$
Tangent space
at identity
$T_{\mathbf{e}}G$
group action moves
between points
g
extrinsic path
$g\overset{\circ}{g} = \dot{g}$
T_gG
$J_g \begin{bmatrix} \dot{g}^i \\ \vdots \\ \dot{g}^n \end{bmatrix}_{\mathbf{e}} = \begin{bmatrix} \dot{g}^i \\ \vdots \\ \dot{g}^n \end{bmatrix}_g$
$J_g\overset{\circ}{g} = \dot{g}$
intrinsic path
Tangent space
away from identity
Jacobian finds equivalent
vector coefficients

Commutative Diagram

↓ Vectors corresponding to a given **right** action
→ are equivalent under **left** group actions

$$
\begin{tikzcd}
\mathbf{e} \arrow[r, "g\circ"] \arrow[d, "\circ g_\delta \; +dg"'] & g \arrow[d, "\circ g_\delta \; +J_{g\circ}dg"] \\
g_\delta \arrow[r, "g\circ"'] & g g_\delta
\end{tikzcd}
$$

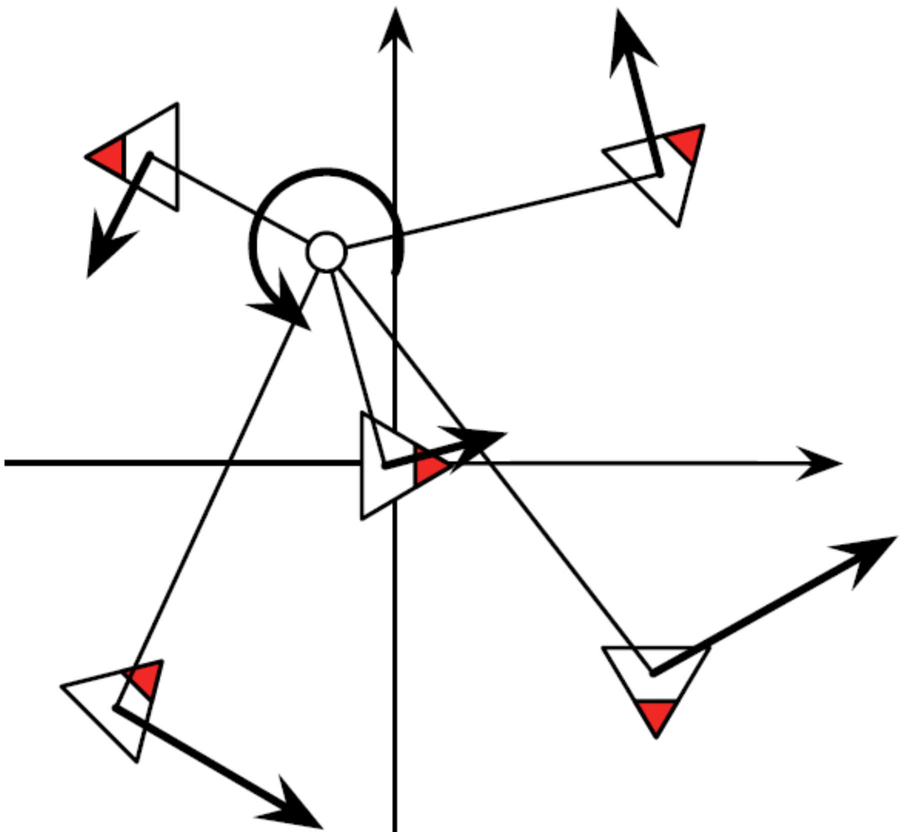
↓ Vectors corresponding to a given **left** action
→ are equivalent under **right** group actions

$$
\begin{tikzcd}
\mathbf{e} \arrow[r, "\circ g"] \arrow[d, "g_\delta\circ \; +dg"'] & g \arrow[d, "g_\delta\circ \; +J_{\circ g}dg"] \\
g_\delta \arrow[r, "\circ g"'] & g_\delta g
\end{tikzcd}
$$

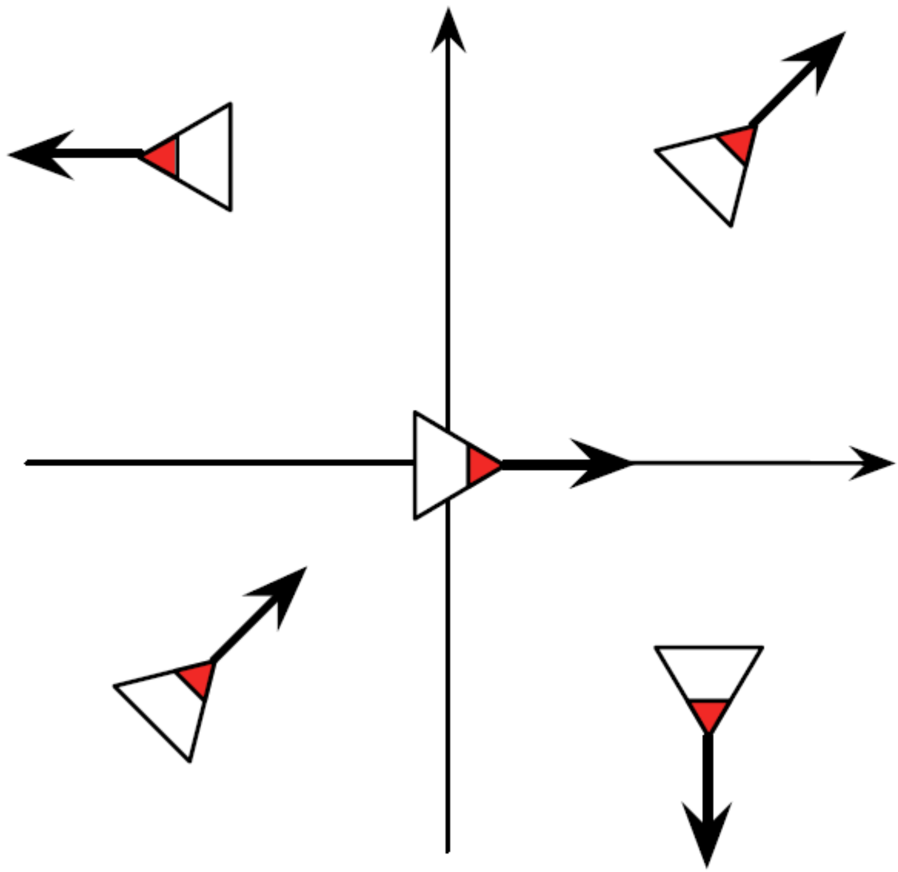
$$\dot{g} = J_g \mathring{g}$$

Left and Right Generated Vector Fields

Objects moving along a left generator



Objects moving along a right generator



Some confusion

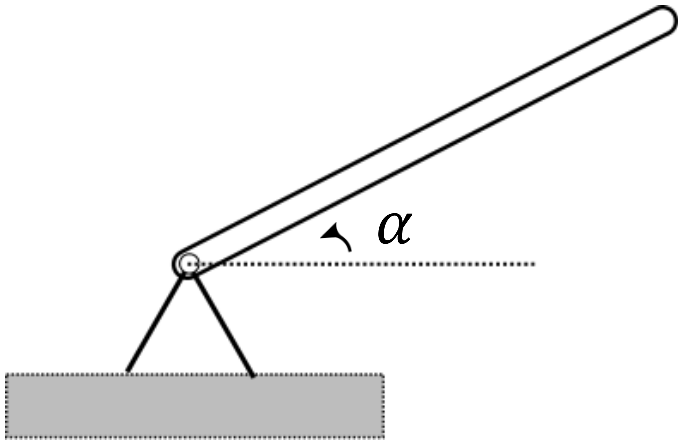
- "generated by a left action" vs "left lifted action“
- Vector field generated by left (right) action has vectors that are invariant with the right (left) lifted action
- Objects with the same spatial velocity are moving along the same left-generator (they are both experiencing the same infinitesimal left action). The instantiations of this ***infinitesimal left action*** at different points in the space are ***right-invariant***, and related by the ***right lifted action***

Exponential Map Again

- Intrinsic form: The exponential map is the solution to the ODE - $\dot{g} = T_e L_g \, g^{circ}$

- The exponential map is the solution to the ODE - $\dot{g} = g \, g^{circ}$

Direct Differentiation



$$g = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

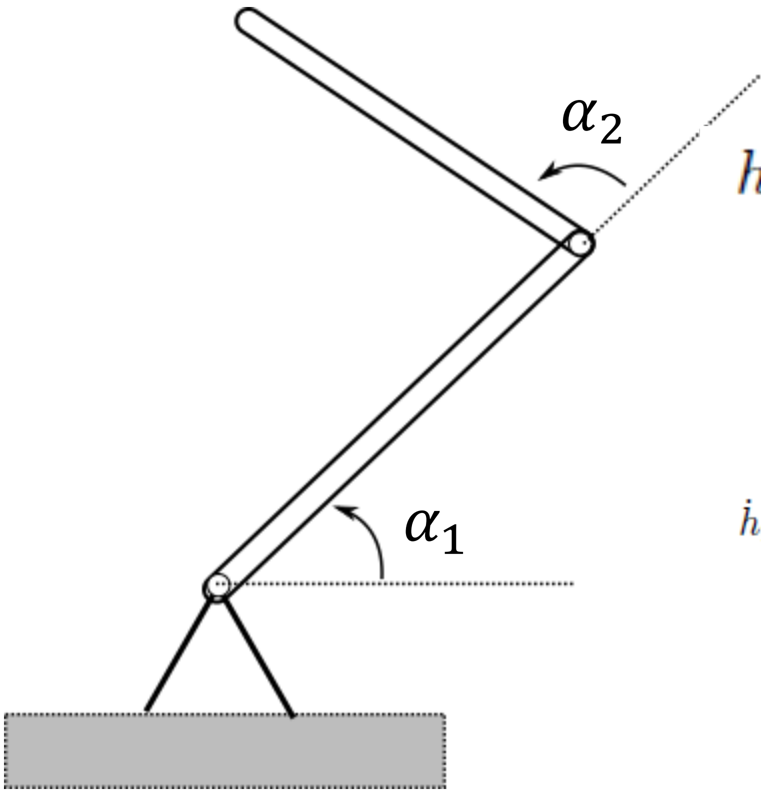
$$h = gh_g = \begin{bmatrix} \cos\alpha & -\sin\alpha & \ell\cos\alpha \\ \sin\alpha & \cos\alpha & \ell\sin\alpha \\ 0 & 0 & 1 \end{bmatrix}$$

$$h = (\ell\cos\alpha, \ell\sin\alpha, \alpha)$$

$$\dot{g}(q,\dot{q}) = J_g\dot{q} = \frac{\partial g}{\partial q}\dot{q}$$

$$\dot{h} = J_h\dot{\alpha} = \frac{\partial h}{\partial \alpha}\dot{\alpha} = \begin{bmatrix} -\ell\sin\alpha \\ \ell\cos\alpha \\ 1 \end{bmatrix}\dot{\alpha}$$

Direct Differentiation



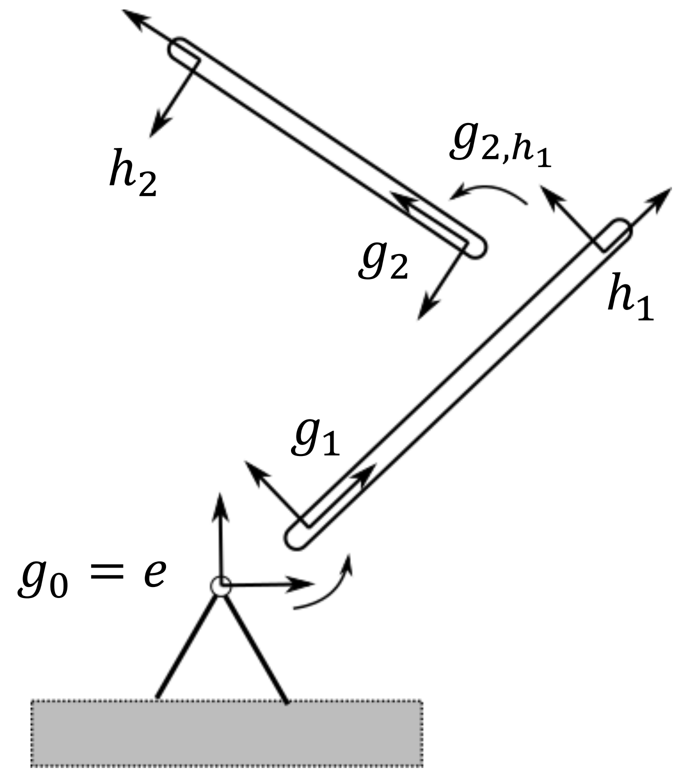
$$h_2(q) = \Big((\ell_1 \cos\alpha_1 + \ell_2 \cos(\alpha_1 + \alpha_2)),\\ (\ell_1 \sin\alpha_1 + \ell_2 \sin(\alpha_1+\alpha_2)), (\alpha_1 + \alpha_2) \Big)$$

$$\dot{h}_2 = \begin{bmatrix} -(\ell_1 \sin\alpha_1 + \ell_2 \sin(\alpha_1+\alpha_2)) & -\ell_2 \sin(\alpha_1+\alpha_2) \\ (\ell_1 \cos\alpha_1 + \ell_2 \cos(\alpha_1+\alpha_2)) & \ell_2 \cos(\alpha_1+\alpha_2) \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha_1} \\ \dot{\alpha_2} \end{bmatrix}$$

Mobile systems

$$\dot{g}_2 = \begin{bmatrix} \frac{\partial g_2}{\partial g} & \frac{\partial g_2}{\partial r} \end{bmatrix} (\dot{g}, \dot{r})$$

Iterative Buildup



the fixed pivot point (at $g_0 = e$) is grounded

$$\dot{g}_0 = (0,0,0)$$

The proximal frame of link 1 has equal translational velocity to the pivot and a relative velocity of $\dot{\alpha}_0$

$$\dot{g}_1 = \dot{g}_0 + (0,0,\dot{\alpha}_1) = (0,0,\dot{\alpha}_1)$$

At the distal end of the first link, h_1 has the same spatial velocity as g_1,

$$T_{h_1} R_{h_1^{-1}} \dot{h}_1 = T_{g_1} R_{g_1^{-1}} \dot{g}_1$$

$$\dot{h}_1 = \overbrace{(T_{\mathbf{e}} R_{h_1})(T_{g_1} R_{g_1^{-1}})}^{T_{g_1} R_{h_1,g_1}} \dot{g}_1$$

$$\dot{h}_1 = \overbrace{\begin{bmatrix} 1 & 0 & -\ell_1 \sin\alpha_1 \\ 0 & 1 & \ell_1 \cos\alpha_1 \\ 0 & 0 & 1 \end{bmatrix}}^{T_{\mathbf{e}} R_{h_1}} \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{T_{g_1} R_{g_1^{-1}}} \overbrace{\begin{bmatrix} 0 \\ 0 \\ \dot{\alpha}_1 \end{bmatrix}}^{\dot{g}_1} = \overbrace{\begin{bmatrix} -\ell_1 \sin\alpha_1 \\ \ell_1 \cos\alpha_1 \\ 1 \end{bmatrix}}^{J_{h_1}} \dot{\alpha}_1$$

The next link

relative velocity of the proximal end of link 2 and the distal end of link 1 is given by the second joint's rotation, making its net velocity

$$\dot{g}_2 = \dot{h}_1 + (0, 0, \dot{\alpha}_2)$$
$$= ((-\ell_1 \sin\alpha_1)\dot{\alpha}_1, (\ell_1 \cos\alpha_1)\dot{\alpha}_1, (\dot{\alpha}_1 + \dot{\alpha}_2))$$

$$\dot{g}_2 = \begin{bmatrix} -\ell_1 \sin\alpha_1 & 0 \\ \ell_1 \cos\alpha_1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}$$

$$\dot{h}_2 = \overbrace{\begin{bmatrix} 1 & 0 & -(\ell_1 \sin\alpha_1 + \ell_2 \sin(\alpha_1 + \alpha_2)) \\ 0 & 1 & (\ell_1 \cos\alpha_1 + \ell_2 \cos(\alpha_1 + \alpha_2)) \\ 0 & 0 & 1 \end{bmatrix}}^{T_{\mathbf{e}} R_{h_2}} \overbrace{\begin{bmatrix} 1 & 0 & \ell_1 \sin\alpha_1 \\ 0 & 1 & -\ell_1 \cos\alpha_1 \\ 0 & 0 & 1 \end{bmatrix}}^{T_{g_2} R_{g_2^{-1}}} \overbrace{\begin{bmatrix} -\ell_1 \sin\alpha_1 & 0 \\ \ell_1 \cos\alpha_1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}}^{\dot{g}_2}$$

$$\dot{h}_2 = \overbrace{\begin{bmatrix} -(\ell_1 \sin\alpha_1 + \ell_2 \sin(\alpha_1 + \alpha_2)) & -\ell_2 \sin(\alpha_1 + \alpha_2) \\ (\ell_1 \cos\alpha_1 + \ell_2 \cos(\alpha_1 + \alpha_2)) & \ell_2 \cos(\alpha_1 + \alpha_2) \\ 1 & 1 \end{bmatrix}}^{J_{h_2}} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}$$

A pattern is beginning to emerge

$$\dot{h}_i = (T_{\mathbf{e}} R_{h_i})(T_{g_i} R_{g_i^{-1}}) \overbrace{(\dot{h}_{i-1} + v_i)}^{\dot{g}_i},$$

where v_i is the velocity of body i with respect to body $i-1$ at joint i.

Body Velocity Formulation

$$\underbrace{\overbrace{(T_{h_i}L_{h_i^{-1}})}^{\text{new}}\dot{h}_i}_{\xi_{h_i}} = \underbrace{\overbrace{(T_{h_i}L_{h_i^{-1}})}^{\text{new}}(T_{\mathbf{e}}R_{h_i})}_{Ad_{h_i}^{-1}}\underbrace{(T_{g_i}R_{g_i^{-1}})\overbrace{(T_{\mathbf{e}}L_{g_i})}^{\text{Identity}}}_{Ad_{g_i}}\underbrace{(T_{g_i}L_{g_i^{-1}})\dot{g}_i}_{\xi_{g_i}}$$

Three frames at play

1. h_{i-1} Distal i-1
2. ${g_i}'$ Instantaneously aligned with h_{i-1}
3. g_i proximal i

$$\xi_{g_i'} = \xi_{h_{i-1}} + v_i$$

$$v_i = (\dot{\delta}_{x,i}, \dot{\delta}_{y,i}, \dot{\alpha}_i)$$

$$\xi_{g_i'} = \xi_{h_{i-1}} + v_i$$

$$\xi_{g_i} = Ad_{g_i}^{-1} Ad_{g_i'} \xi_{g_i'}.$$ g_i' and g_i are attached to the same rigid body

Note that when the x and y components of g_i' and g_i are equal, this conversion reduces to rotation by $-\alpha_i$

$$\xi_{h_i} = (Ad_{h_i}^{-1})(Ad_{g_i'})(\xi_{h_{i-1}} + v_i),$$

Even simpler

- Place the origin at g_i'
- Transforms everything by $(g_i')^{-1}$

g_i' and h_i respectively become e and $h_{i,g_i'}$.

adjoint action at the origin is an identity matrix

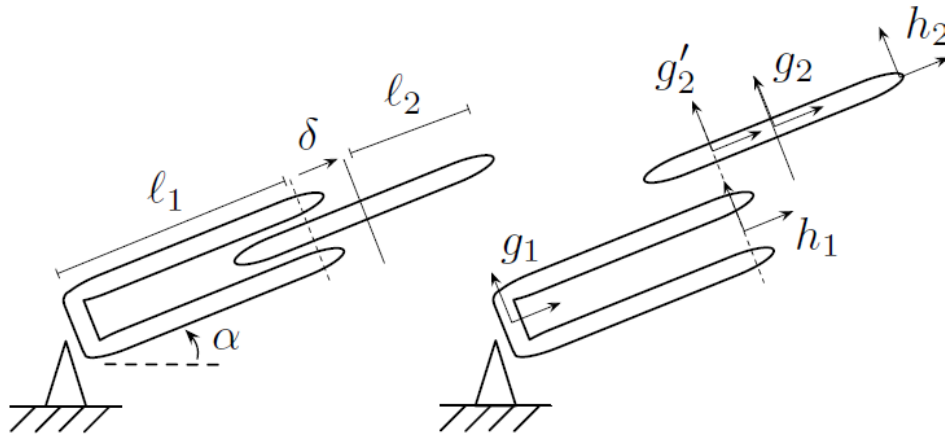
$$\xi_{h_i} = (Ad^{-1}_{h_i,g_i'})(\xi_{h_{i-1}} + v_i).$$

inverse adjoint action of that frame relative to g_i'.

Rotary Prismatic Arm

$$\xi_{g_0} = (0,0,0)$$

$$\xi_{g_1'} = \xi_{g_0} + \overbrace{(0,0,\dot{\alpha})}^{v_1} = (0,0,\dot{\alpha})$$



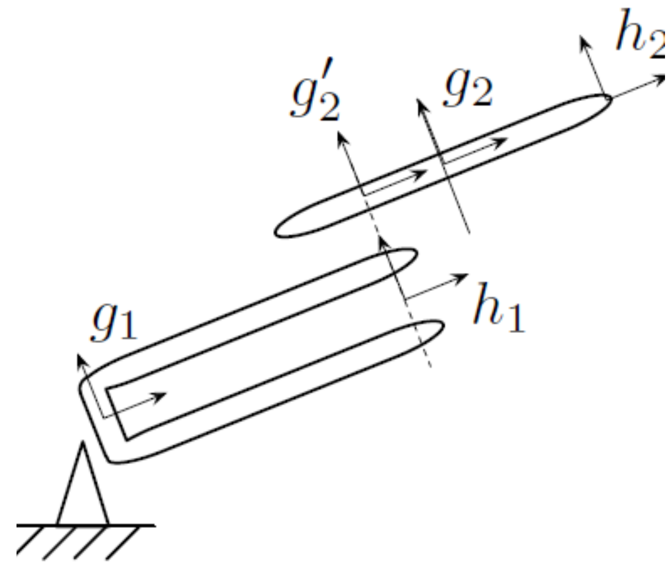
$$\xi_{g_1} = Ad^{-1}_{g_{1,g_1'}} \xi_{g_1'} = \overbrace{\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{g_{1,g_1'}=(0,0,\alpha)}} \overbrace{\begin{bmatrix} 0 \\ 0 \\ \dot{\alpha} \end{bmatrix}}^{\xi_{g_1'}} = \begin{bmatrix} 0 \\ 0 \\ \dot{\alpha} \end{bmatrix} = \overbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}^{J^b_{g_1}} \dot{\alpha}.$$

Jacobian for the proximal end of the link, expressed in its own body frame

The Next Frame (Distal Link 1)

$$\xi_{h_1} = Ad^{-1}_{h_1,g_1}\xi_{g_1} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \ell_1 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{h_1,g_1=(\ell_1,0,0)}} \overbrace{\begin{bmatrix} 0 \\ 0 \\ \dot{\alpha} \end{bmatrix}}^{\xi_{g_1}} = \begin{bmatrix} 0 \\ \ell_1\dot{\alpha} \\ \dot{\alpha} \end{bmatrix} = \overbrace{\begin{bmatrix} 0 \\ \ell_1 \\ 1 \end{bmatrix}}^{J^b_{h_1}} \dot{\alpha},$$

Next Link



$$v_2 = (\dot{\delta}, 0, 0)$$

$$\xi_{g_2'} = \xi_{h_1} + v_2 = (\dot{\delta}, \ell_1\dot{\alpha}, \dot{\alpha})$$

body-frame Jacobians for the midpoint

$$\xi_{g_2} = Ad^{-1}_{g_2,g_2'}\xi_{g_2'} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \delta \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{g_2,g_2'=(\delta,0,0)}} \overbrace{\begin{bmatrix} \dot{\delta} \\ \ell_1\dot{\alpha} \\ \dot{\alpha} \end{bmatrix}}^{\xi_{g_2'}} = \begin{bmatrix} \dot{\delta} \\ (\ell_1+\delta)\dot{\alpha} \\ \dot{\alpha} \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 \\ \ell_1+\delta & 0 \\ 1 & 0 \end{bmatrix}}^{J^b_{g_2}} \begin{bmatrix} \dot{\alpha} \\ \dot{\delta} \end{bmatrix}$$

$$\xi_{h_2} = Ad^{-1}_{h_2,g_2'}\xi_{g_2'} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \delta+\ell_2 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{h_2,g_2'=(\delta+\ell_2,0,0)}} \overbrace{\begin{bmatrix} \dot{\delta} \\ \ell_1\dot{\alpha} \\ \dot{\alpha} \end{bmatrix}}^{\xi_{g_2'}} = \begin{bmatrix} \dot{\delta} \\ (\ell_1+\delta+\ell_2)\dot{\alpha} \\ \dot{\alpha} \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 \\ \ell_1+\delta+\ell_2 & 0 \\ 1 & 0 \end{bmatrix}}^{J^b_{h_2}} \begin{bmatrix} \dot{\alpha} \\ \dot{\delta} \end{bmatrix}$$

body-frame Jacobians for the distal end