Geometry of Locomotion
Chapter 1.2

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Chapter 1 Key Ideas

- Configuration Space
  - Configuration Manifold
  - Configuration Group
- Rigid Body
- Degrees of Freedom

Group

- Motivation: need to add, subtract, multiply and perform other operations on a manifold
- Elements of groups correspond to positions on a manifold and an operation (which can be performed on other group elements)
- An set of elements and an operation
- Examples: $(R, +)$ or (R^+, x)
- Positions are translations from origin and sum of two positions is the sum of the two displacements
- Products of groups inherit properties from their subgroups
- We can identify the identity element of the group with any point on the manifold which is like saying we can chose our origin

Definition of a Group

Groups. A *group* $(G, \circ)$ is the combination of a set G and an operation $\circ$ that satisfies the following properties:

1. **Closure**: The product of any element of G acting on another by the group operation must also be an element of G. More formally, for $g_1, g_2 \in G$,
$$\begin{aligned} (g_1 \circ\ ) &: G \to G \\ g_2 &\mapsto g_1 \circ g_2. \end{aligned} \tag{1.ii}$$
2. **Associativity**: The order in which a sequence of group operations are evaluated must not affect their net product: for all $g_1, g_2, g_3 \in G$,
$$g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3. \tag{1.iii}$$
3. **Identity element**: The set must contain an identity element $\mathbf{e}$ that leaves other elements unchanged when it interacts with them: for $g \in G$, there exists $\mathbf{e} \in G$ such that
$$\mathbf{e} \circ g = g = g \circ \mathbf{e}. \tag{1.iv}$$
4. **Inverse**: The inverse (with respect to the group operation) of each group element must be an element of the group and produce the identity element when operating on or operated on by its respective element: for $g \in G$, there must exist $g^{-1} \in G$ such that
$$g^{-1} \circ g = \mathbf{e} = g \circ g^{-1}. \tag{1.v}$$

When the choice of group action is unambiguous, as in the case of the $SO(n)$ and $SE(n)$ groups discussed below, a group may be referred to simply by its underlying set. As additional shorthand notation, the $\circ$ symbol is often dropped, with $g_1 g_2$ being read as equivalent to $g_1 \circ g_2$.

Other group stuff

- Common Example

Common group actions. Commonly encountered group actions include addition ($+$), for which the identity is a zero element and the inverse operation is subtraction; and multiplication ($\times$), for which the identity has unit value and the inverse is division. Additive group actions are *commutative*, meaning that the left-right order of the group elements in a sequence does not affect the result of their composition,

$$g_1 + g_2 = g_2 + g_1. \tag{1.vi}$$

Multiplicative group actions may be commutative, but are often not, giving rise to the notion of "left" and "right" group actions.

- Left vs. Right

Left and right actions. If the order of operation is important (as in the case of most matrix multiplications), any group action may be conducted as a *left action*

$$L_g = g \circ \tag{1.vii}$$

or a *right action*

$$R_g = \circ\, g \tag{1.viii}$$

with the acting element placed correspondingly at the beginning or end of the execution sequence. For the groups we consider in this book, we use a convention in which the left transformation $L_{g_2} g_1$ is interpreted as "moving the group element g_1 by g_2," whereas the right action $R_{g_2} g_1$ is used to find "the group element at g_2 relative to g_1."

Ring

Rings. A *ring* $(R, +, \cdot)$ is the combination of a set R with two operations: one $(+)$ that acts like addition, and a second $(\cdot)$ that acts like multiplication. These operations satisfy the following properties:

1. **Addition**: A ring forms a commutative group under addition, with closure, associativity, and the existence of identity and inverse elements. The additive identity is referred to as the *zero element* of the ring, or "**0**."
2. **Multiplication**: A ring is closed and associative under multiplication, but does not necessarily contain the multiplicative inverse of each element or a multiplicative identity element. If a multiplicative identity exists, it is referred to as the *unit element* of the ring, or "1."
3. **Distributivity**: The multiplicative operation must distribute over the additive operation, so that the product of a sum is equal to the sum of the individual products: for $a, b, c \in R$,

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c). \tag{1.ix}$$

For example, the real numbers $\mathbb{R}$ form a ring under the standard addition and multiplication operations:

- Elements of $\mathbb{R}$ form the group $(\mathbb{R}, +)$ under addition,
- Multiplication of real numbers produces real numbers and is associative, and
- Multiplication distributes over addition.
- The real numbers do not form a group under multiplication, as the inverse of 0 is not a real number, but this property is not required for ring structure.

Other examples of rings include the complex numbers (under addition and complex multiplication) and square matrices (under elementwise addition and matrix multiplication).

Field

Fields. A *field* is a ring for which the inverse of every non-zero element is also an element. Both the real and complex numbers form fields, but the ring of square matrices does not—it includes singular matrices (containing linearly dependent rows) that do not have a well-defined inverse.

Note that this meaning of "field" is unrelated to terms like "vector field" (an assignment of a vector to each point in an underlying space, as discussed in §2.1.2). The term "scalar field" in particular should be approached with caution, as it has two distinct meanings:

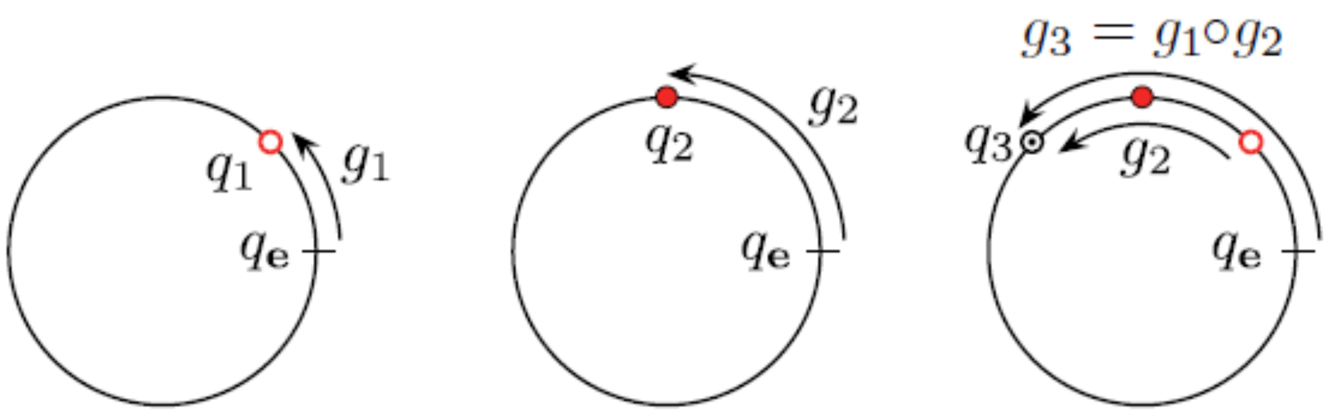
- A field, in the sense here of a special class of group or ring, whose elements can act as *scalars* for a vector space as discussed on page 24.
- By back-formation from "vector field," a function that assigns a single real number to each point in the domain.

In German (the language in which the concept of a field was first defined), fields are called "körper," which translates to "body" or "corpus."

Group Operations

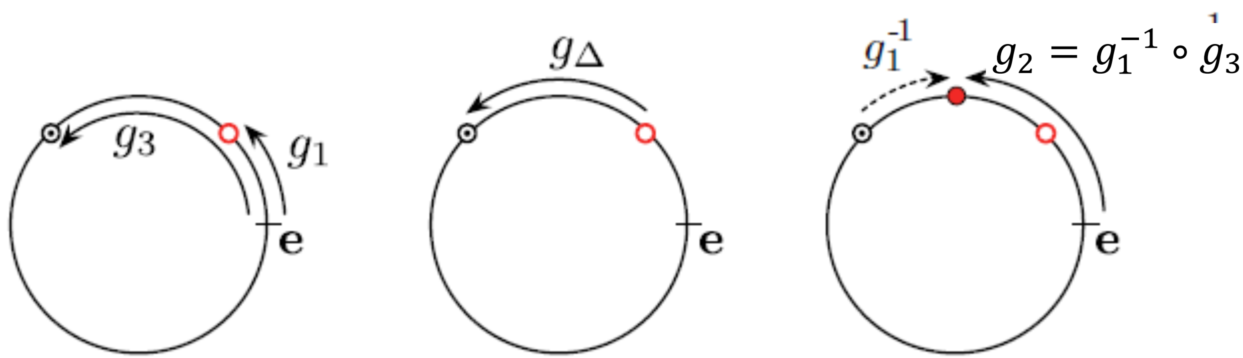
Select a point as q_e as the identity element e and this is the null operation
All other points are displacements g from q_e

$g_3 = g_1 \circ g_2$ g_3 is the same displacement from g_1 and g_2 is from the origin

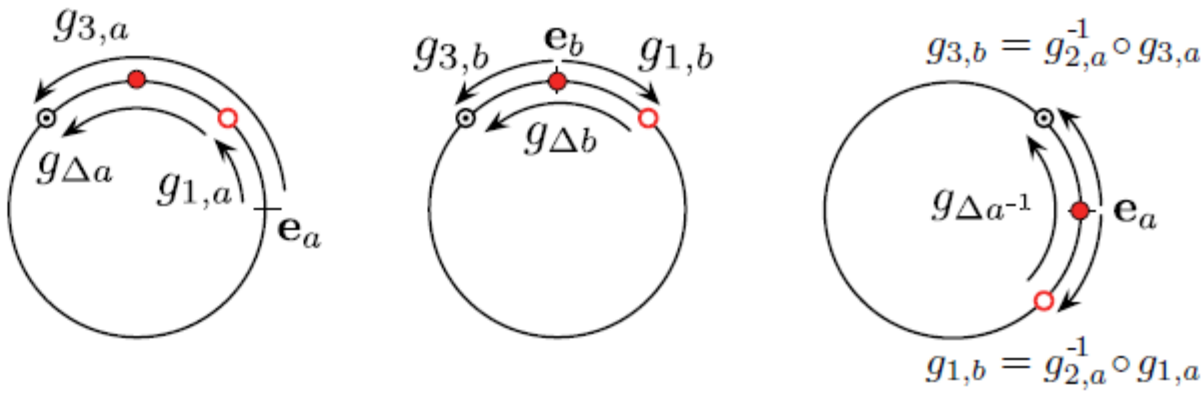


Inverse Operation

- Composition: $g_3 = g_1 \circ g_2$
- Inverse: $g_\Delta = g_1^{-1} \circ g_3$ takes you from 3 to 1
- $g_\Delta$corresponds to a point too and in this case corresponds to q_2 and hence g_2



Change the Origin



Moving the origin in this way changes the group element associated with each point, but does not affect the groupwise displacement between pairs of points.

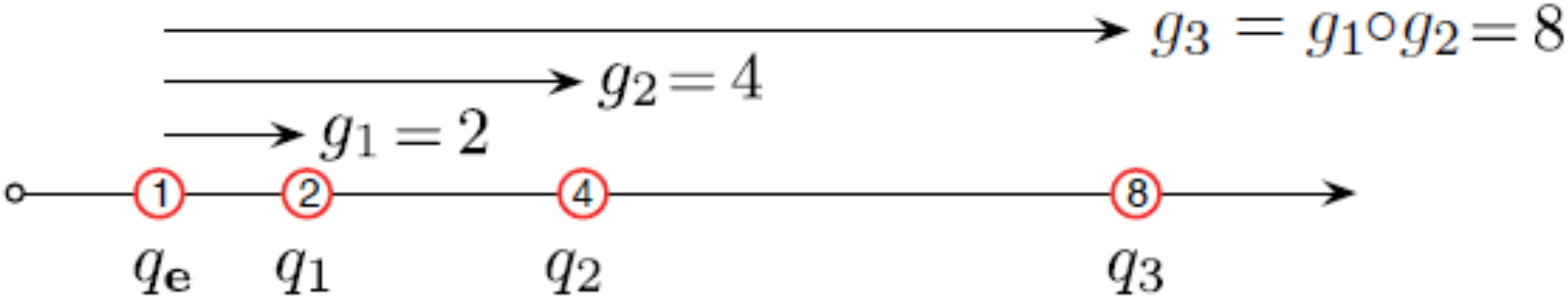
The new origin point is counterclockwise of q_1— changing the sign of $g_{1,b}$ as compared to $g_{1,a}$—and is closer to q_3—making $g_{1,b}$ a smaller displacement than $g_{1,a}$.

Because q_1 and q_3 have not themselves moved, however, $g_{\Delta b}$ is the same displacement as $g_{\Delta a}$.

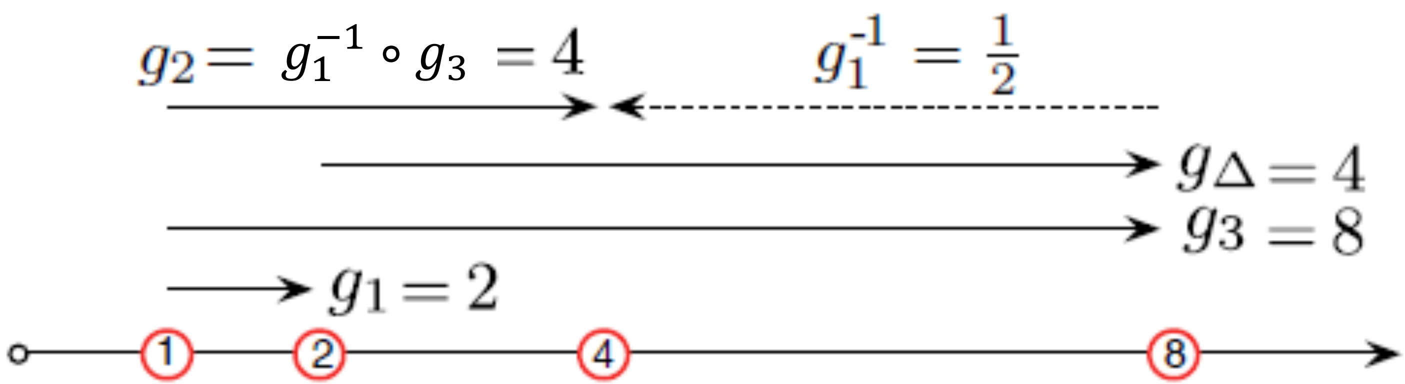
$$
\begin{aligned}
g_{\Delta b} &= g_{1,b}^{-1} \circ g_{3,b} \\
&= (g_{b,a}^{-1} \circ g_{1,a}^{-1}) \circ (g_{b,a}^{-1} \circ g_{3,a}) \\
&= (g_{1,a}^{-1} \circ \overbrace{g_{b,a}) \circ (g_{b,a}^{-1}}^{\mathbf{e}} \circ g_{3,a}) \\
&= g_{1,a}^{-1} \circ g_{3,a} \\
&= g_{\Delta a}.
\end{aligned}
$$

These effects correspond to the principle that displacing the reference point by a given group action $g_{b,a}$ is equivalent to displacing each point on the configuration manifold by the inverse of that action, which can be represented symbolically as

Repeat with Multiplicative Group

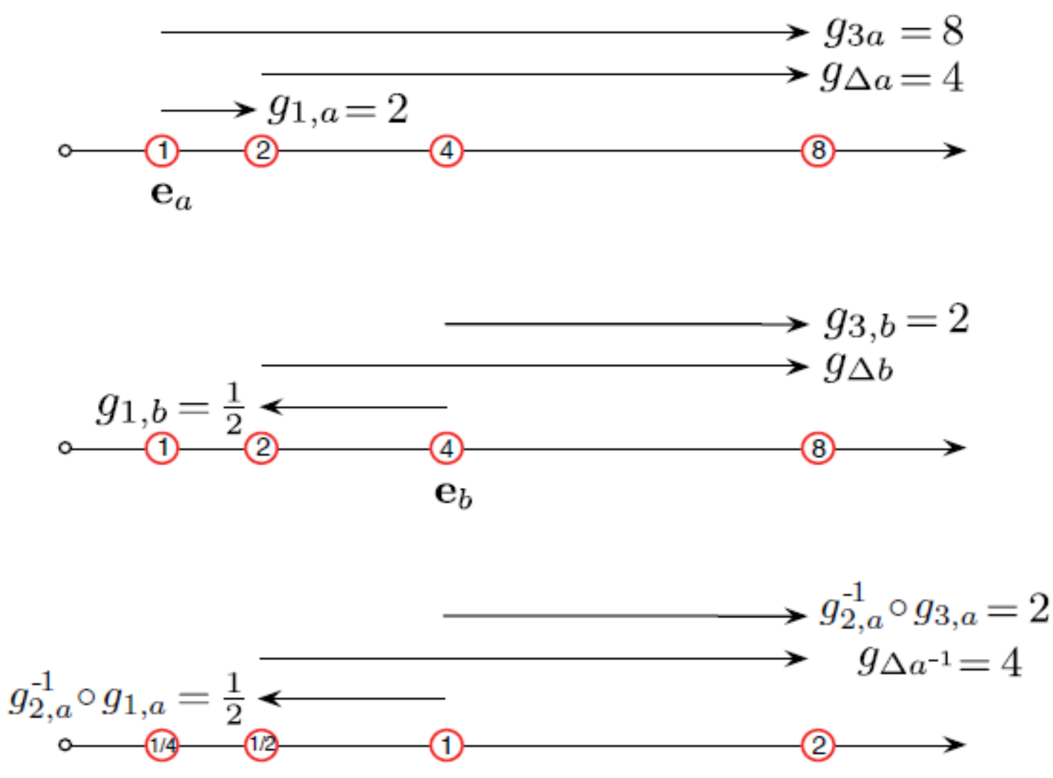


each point q is associated with a group element g that encodes its groupwise displacement from a designated origin point q_e. If we interpret the composition of two displacements as a location on the manifold, the resulting point's displacement from q_1 is the same as that of q_2 from the origin.



Inverse group actions, which correspond to operations like subtraction and division, allow us to compare two group elements. Here, g (the difference between g_1 and g_3) is equal to that encoded by the group element $g_1^{-1} \circ g_3$.

Repeat with Multiplicative Group



Designating a new origin on the manifold changes the displacement associated with each point. This change affects the results of composing two group elements—here, with the new identity $\mathbf{e}_b$ at q_2 (directly between q_1 and q_3), the composition of $g_{1,b}$ and $g_{3,b}$ is the identity. It does not, however, affect the result of taking inverse actions—the points remain separated by the same displacement. The effect of moving the origin by a given transformation is equivalent to transforming all the points on the manifold by the inverse of that transformation.

Group Parameterization

$$\begin{bmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{bmatrix} \times \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{bmatrix}$$

GL(n) is the set of n x n invertible linear matrices

Group Parameterization

$$\rho(g) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{bmatrix} \times \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{bmatrix}$$

GL(n) is the set of n x n invertible linear matrices

GL(n) is a group

Many abstract groups are isomorphic to GL(n) or a subgroup of them
E.g., Planar rotations are isomorphic to SO(2)

This is like, or really is, a parameterization. Taking advantage of the isomorphism
allows us to perform group actions as if they are matrix multiplications

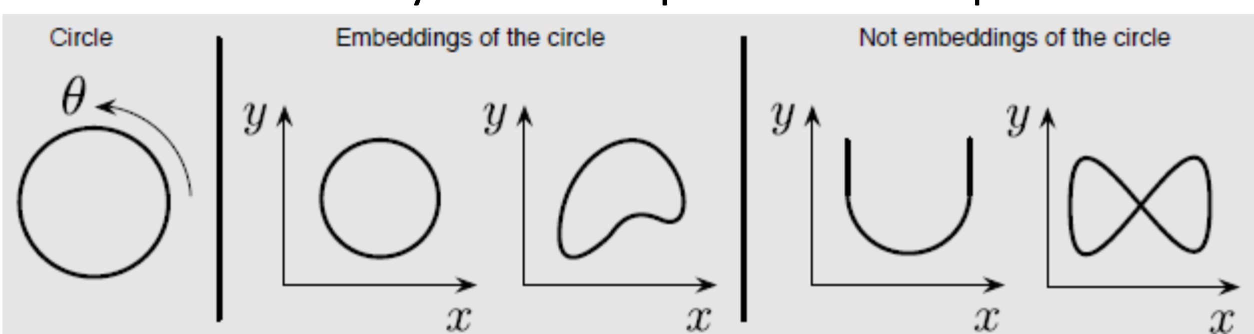
We will casually refer to the matrices as group elements but when necessary, use the $\rho(g) = [g]$

Embedding

- Informal
  - Embedding is an operation that allows one mathematical structure to be treated as a subset of another.
  - Making an embedding allows mathematical tools defined on the enclosing structure to be applied to the embedded structure, e.g., a circle in the plane
  - Similar to parameterizations because of ability to perform operations but different in dimension count

- Formal: To qualify as an embedding of a structure A into space B, a function E : A -> B must be:
  - 1. Injective. Every point in A must be associated with a unique point in B, and
  - 2. Structure-preserving. Structural properties of A, such as connectivity, must be carried over into its embedding E(A).

- Having these properties requires that A and E(A) are at least homeomorphic to each other; in many useful embeddings the two structures are additionally diffeomorphic or isomorphic to each other.



Isomorphism

An *isomorphism* is a structure-preserving relationship between two mathematical objects. Two groups A and B are considered isomorphic if there exists a bijective function

$$f : A \to B \tag{1.x}$$

$$a \mapsto b \tag{1.xi}$$

such that

$$f(a_1 \circ a_2) = f(a_1) \circ f(a_2), \tag{1.xii}$$

and (because of the bijectivity requirement),

$$f^{-1}(b_1 \circ b_2) = f^{-1}(b_1) \circ f^{-1}(b_2). \tag{1.xiii}$$

Two simple examples of group isomorphisms are

1. The relationship between the multiplicative group of positive real numbers, $(\mathbb{R}_+, \times)$, and the additive group of real numbers, $(\mathbb{R}, +)$. The natural logarithm and exponential functions isomorphically map between these two groups: for $x, y \in (\mathbb{R}_+, \times)$,

$$\log (xy) = \log x + \log y \tag{1.xiv}$$

   and for $a, b \in (\mathbb{R}, +)$

$$\exp (a + b) = (\exp a)(\exp b) \tag{1.xv}$$

2. The relationship between the modular additive group of numbers on the circle, $(\mathbb{S}^1, + \bmod 2\pi)$, and the group of two-dimensional rotation matrices, $SO(2)$. Under the matrix construction function for elements of $SO(2)$,

$$R : \mathbb{S}^1 \to SO(2) \tag{1.xvi}$$

$$\theta \mapsto \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{1.xvii}$$

   addition of $\mathbb{S}^1$ elements is equivalent to multiplications of $SO(2)$ elements,

$$R(\theta_1 + \theta_2) = R(\theta_1)R(\theta_2). \tag{1.xviii}$$

Associativity

Associativity. Associativity means that once the placement of elements in an expression has been assigned (e.g., a acting on b acting on c), the order of resolution doesn't matter: For binary operations with infix notation, associativity appears as the ability to arbitrarily group the elements of a multi-operator expression,

$$a \circ (b \circ c) = (a \circ b) \circ c, \tag{1.xxiii}$$

and for function composition (which is always associative), it appears as the statement that precomposing two functions, then applying them to an input is equivalent to applying them to the input in succession,

$$\big(f_2 \circ f_1\big)(a) = f_2(f_1(a)). \tag{1.xxiv}$$

Commutativity. Commutativity means that elements in an expression can exchange positions without affecting the result of the expression. Binary operations are *commutative* if their operands (inputs to the operation) can be swapped for all values of a and b in the domain of the operation,

$$a \circ b = b \circ a. \tag{1.xxv}$$

For operations that are not commutative, specific elements of the domain may still *commute* with each other if they satisfy (1.xxv). For example, any pair of pure-translation elements of $SE(2)$ commute with each other, as

$$(x, y, 0) \circ (u, v, 0) = (x + u, y + v, 0) = (u, v, 0) \circ (x, y, 0), \tag{1.xxvi}$$

and a similar rule holds for pure-rotation elements.[a] Pairs of functions commute with each other if they are commutative with respect to the composition operation,

$$f_1 \circ f_2 = f_2 \circ f_1, \tag{1.xxvii}$$

or, in functional notation,

$$f_2(f_1(a)) = f_1(f_2(a)) \tag{1.xxviii}$$

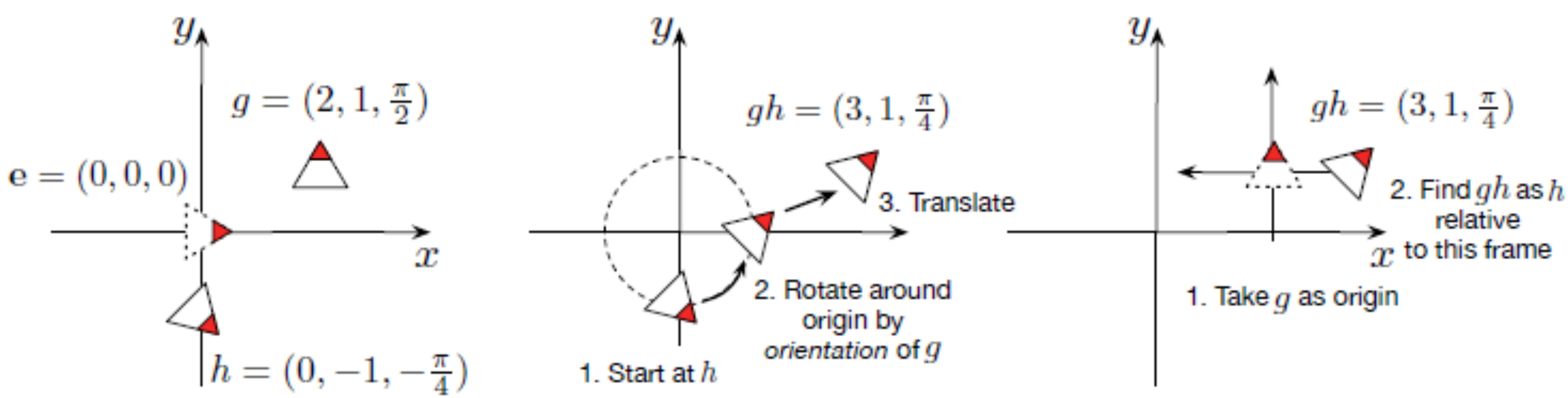
Commutativity of actions

Commutativity of Left and Right actions. Combining the associativity of group operations with the functional forms of the left and right group actions[b] leads to an interesting and powerful property of group actions: left actions applied to a group element commute with right actions applied to that element,

$$L_g(R_h(\mathbf{e})) = g \circ (\mathbf{e} \circ h) \tag{1.xxix}$$

$$= (g \circ \mathbf{e}) \circ h \tag{1.xxx}$$

$$= R_h(L_g(\mathbf{e})) \tag{1.xxxi}$$



(a) Individual group elements

(b) Left action interpretation of gh

(c) Right action interpretation of gh