Geometry of Locomotion
Chapter 1.1

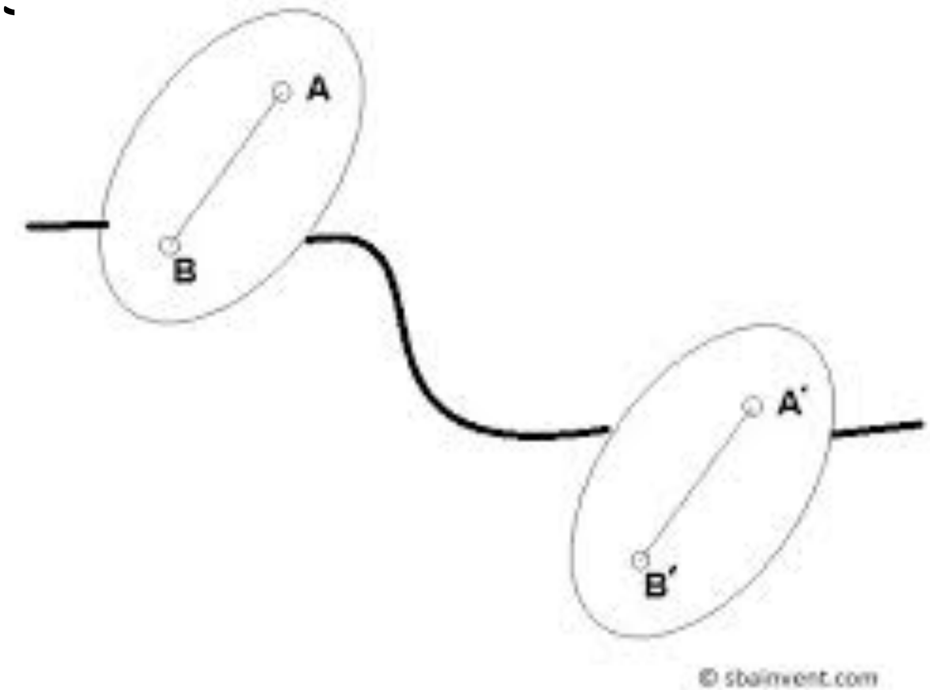
Howie Choset

Chapter 1 Key Ideas

- Configuration Space
  - Configuration Manifold
  - Configuration Group
- Rigid Body
- Degrees of Freedom

Rigid Body

- Infinitely many points
- Distance between points remains fixed
- Orientation remains

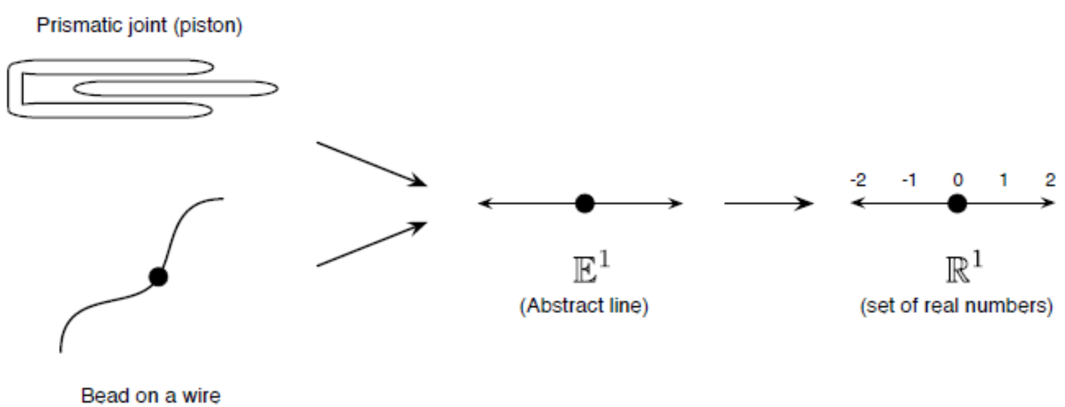


Degrees of Freedom

- Rigid body
- Joint
- Entire System – Configuration Q
- q_i set of numbers that parameterizes space

Informal Manifold

- K-dimensional space
- Locally looks like a k-dimensional Euclidean space
- Basic examples: line (prismatic joint), circle (rotary joint), sphere, SO(3)

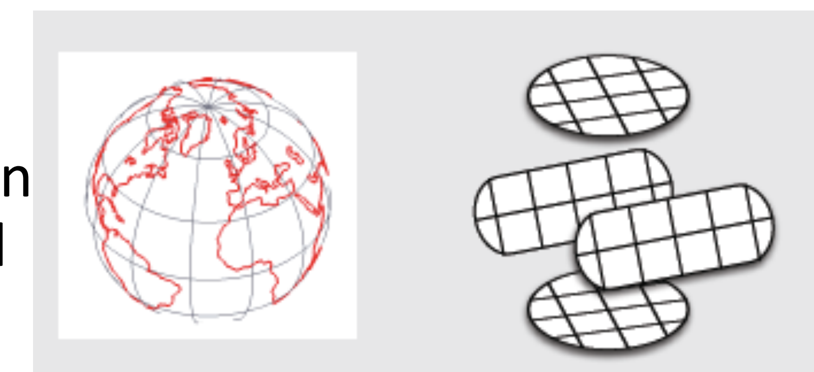


Euclidean Space

- Named after Euclid, 365-300BC?
- Support translation, rotation, and reflection
- Affine – no origin, different of points is vector
- Flat
- Independence
  - operations here, same as there
  - Go up then left is left then up
- E^n vs. R^n(parameterization and has an origin)

Manifold, slightly less informal

- A space locally Euclidean everywhere
- Important properties
  - Curvature: ways at which distance varies at points
  - Topology: in particular, connectivity
- Parameterization
  - Chart: map from a Euclidean space to subset of manifold
  - Atlas: collection of charts



Homeo / Diffeo morphisms

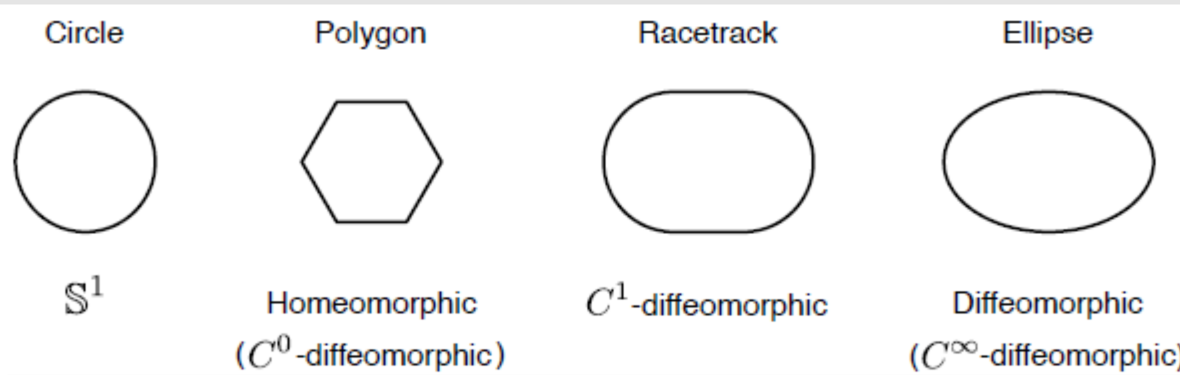
Homeomorphism. A *homeomorphism* is a function f mapping between two spaces that is:

1. **Bijective**, *i.e.*, invertible. Bijective functions are both
   a. **Surjective**, or *onto*, meaning that every point in the range is the function of *at least* one point in the domain, and
   b. **Injective**, or *one-to-one*, meaning that each point in the the domain is mapped to a *unique* point in the range.

   Combining these properties means that there is a full one-to-one correspondence in both directions between points in the domain and points in the range, and thus that

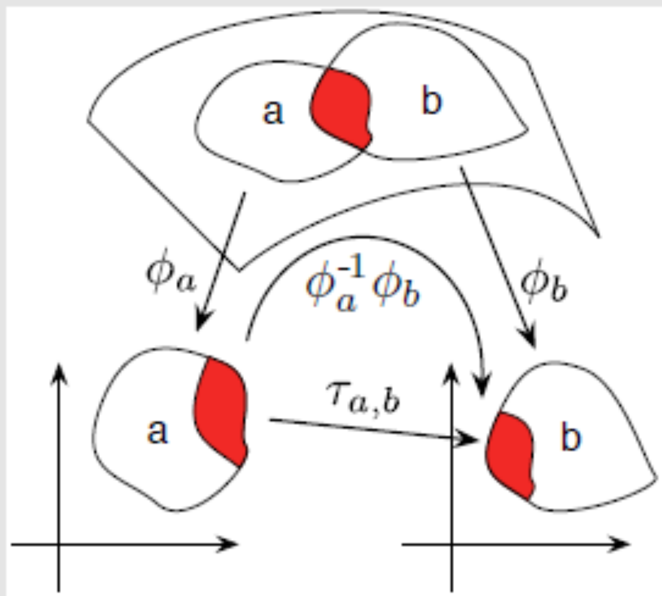
   a. f^{-1} is a true (single-valued) function whose domain is the entire range of f, and
   b. $f \circ f^{-1}$ and $f^{-1} \circ f$ are both identity mappings.
2. **Continuous**, with a continuous inverse.

Diffeomorphism. A *diffeomorphism* is a homeomorphism for which both f and f^{-1} are additionally differentiable, *i.e.*, they are not only continuous, but their derivatives are also continuous. A C^k-diffeomorphism is k-times differentiable, meaning that its first k derivatives are continuous.[a] C^∞-diffeomorphisms, for which



Getting to Formal Definition

Transition map. On regions parameterized by more than one chart (their *overlaps*), coordinates on one chart can be translated to those on a second chart via the *transition map* (also known as the *overlap map*) between the charts. These transition maps encode the idea of a "reparameterization" of the manifold and, as illustrated below, the transition map between charts a and b can be constructed from the parameterization functions ϕ for the two charts as $\tau_{a,b} = \phi_a^{-1}\phi_b$. Note that if there are multiple non-contiguous overlaps between a pair of charts (as in the case of the circle in Figure 1.2), each of these overlaps gets its own transition function, so that the transitions functions are each continuous over their respective domains.

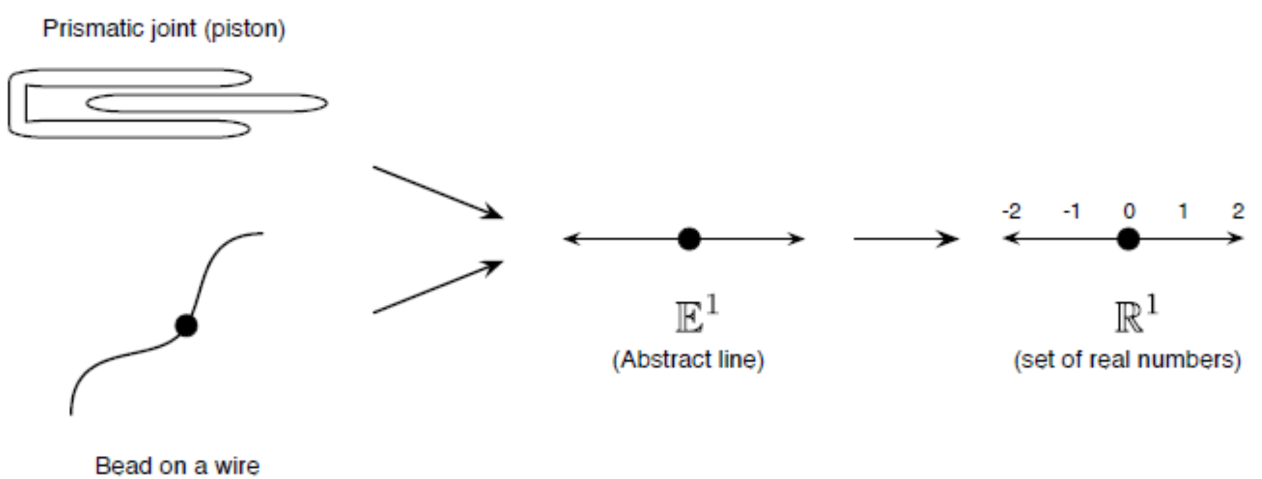


Manifold, The formal Def

- Space where the neighborhood of every point is homeomorphic to R^n and transition map is continuous. Manifold has dimension n
- C^k *differential* manifold is one where nbhd of every point is C^k diffeo to R^n and transision map is C^k. Manifold has dimension n
  - Together, these properties permit the definition of C^k functions on the manifold that maintain their differentiability across regions mapped by different charts.
  - C^0 is a homeomorphism
- Smooth manifold is k = infinity

Linear Joint: 1-D Manifold

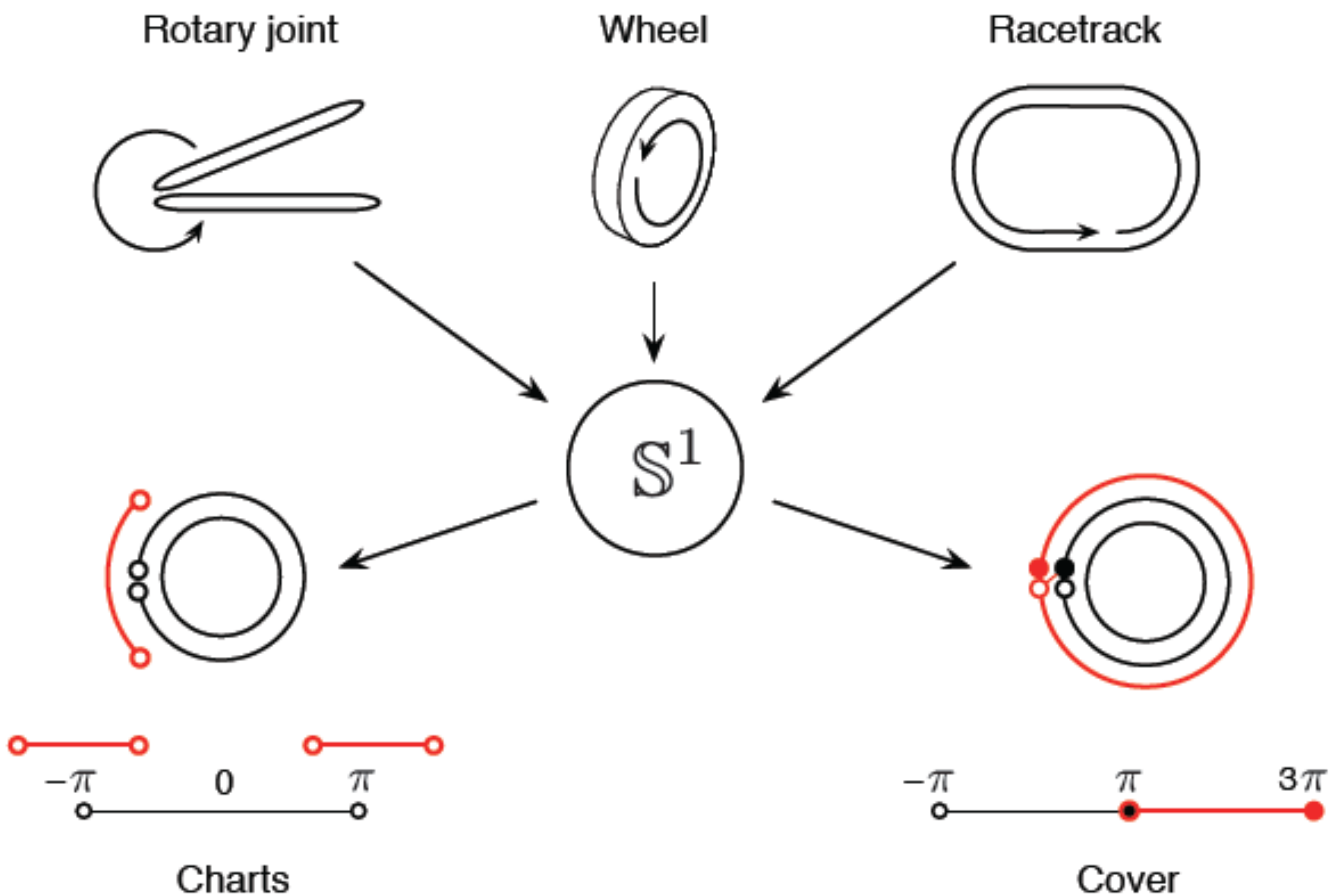
- R^k is a parameterization of E^k
- Configuration space of prismatic joint, bead on a wire



Rotary Joint – 1-D manifold

- Rotary joint without joint limits, car on a track, wheels – cyclic 1-d configuration space
- Self connectedness not captured in R^1
- Compact manifold – closed and bounded
- Circle S^1
- Usually visualized as an embedding in R^2
- No single chart
  - Atlas with two charts (handles discontinuity)
  - Covering – line wraps around forever (lose cyclicness)

Charts vs. Cover



Direct Products

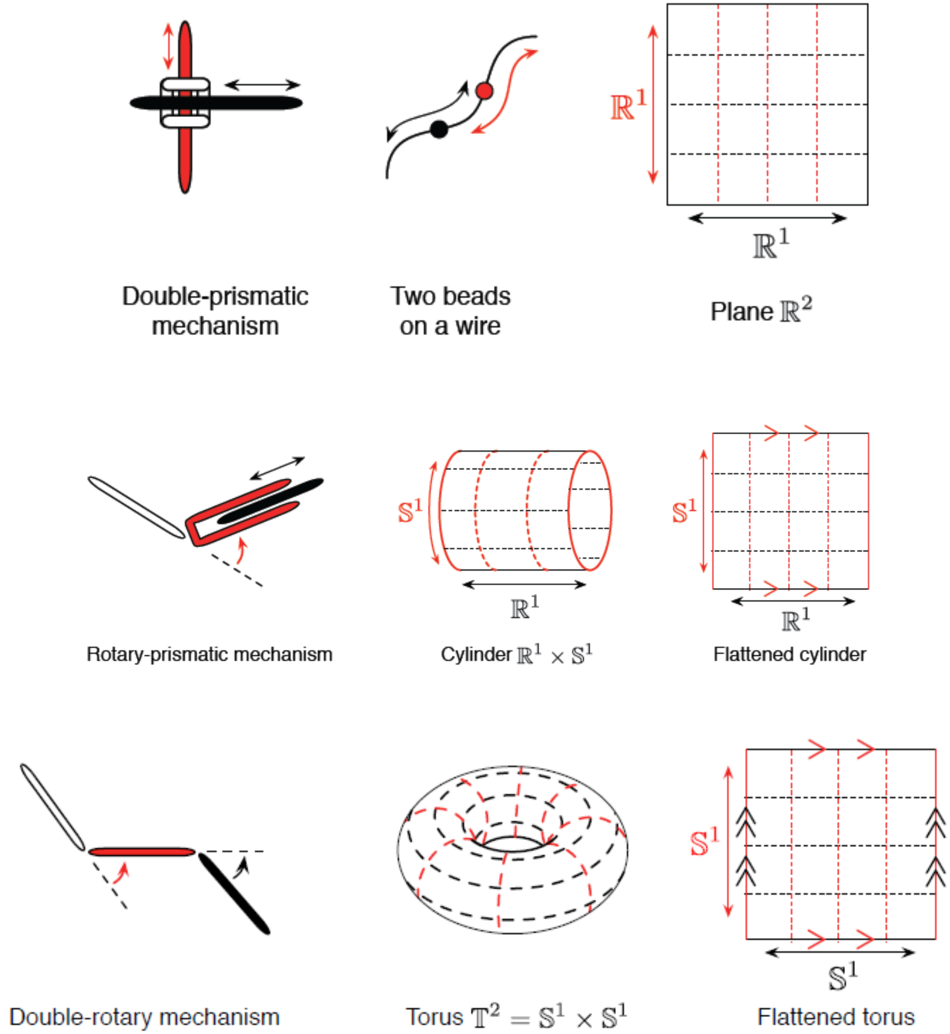
Direct Product

The *direct product*, or *Cartesian product* of two sets or spaces combines them without mixing the elements together. For instance, if we have two systems with configurations $a \in \mathbb{R}^1_a$ and $b \in \mathbb{R}^1_b$, the direct product of the configuration spaces, $\mathbb{R}^2 = \mathbb{R}^1_a \times \mathbb{R}^1_b$, is structured to preserve the independence of the component subspaces,

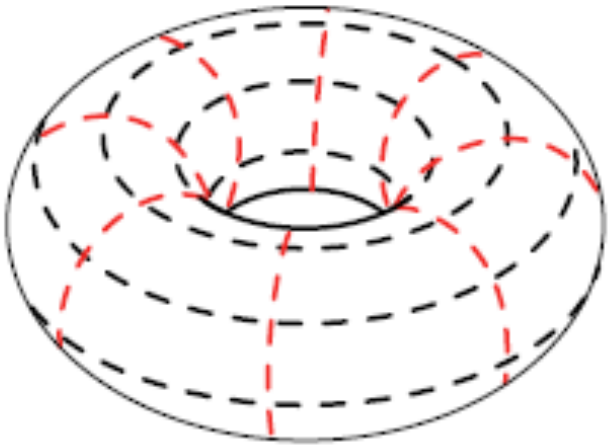
$$(a, b) \in \mathbb{R}^2 \equiv (a \in \mathbb{R}^1_a, b \in \mathbb{R}^1_b). \tag{1.i}$$

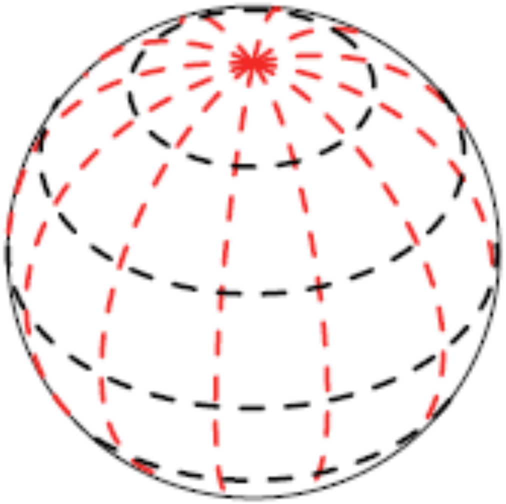
More Complicated Manifolds Assembled from Smaller Ones

- R^2 = R^1 x R^1
  - Cspace with two non-cyclic DOFs
  - Double prismatic joint
- Cylinder = R^1 x S^1
  - A point on a cylinder
  - Prismatic and rotational DOF
- Torus = T^2 = S^1 x S^1



Sphere vs torus





simply connected, i.e., that any loop on it can
be continuously contracted to a point
(and therefore that any loop can be
continuously deformed into any other loop