Geometry of Locomotion

Chap 4.2 - Pfaffian Constraints and Local Connection

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Topics

- Pfaffian Constraint on $G \times M$
- **Non-holonomic constraints**
- Symmetry and Pfaffian Form Unite + Diff Drive Example
- Noether's Theorem

Kinematics Reconstruction Equation

$$\xi = -\mathbf{A}(r)\dot{r}$$

Local connection

Pfaffian Constraint

$$\mathbf{0} = \omega(r)\begin{bmatrix}\xi \\ \dot{r}\end{bmatrix} \qquad \mathbf{0}^{m\times 1} = \omega^{m\times(3+n)}\begin{bmatrix}\xi^{3\times 1} \\ \dot{r}^{n\times 1}\end{bmatrix}$$

m individual Pfaffian constraints and n shape variables

Kinematic locomoting systems have Pfaffians composed of at least three independent constraints, one per degree of freedom in the position space

system with fewer constraints gains the ability to drift through the position space without changing shape, and thus is not fully kinematic

The systems we consider below all have three independent constraints, and thus are kinematic without being overconstrained

And onto the local connection

- Given a three-constraint Pfaffian (m = 3),

$$\mathbf{0}^{m\times 1} = \omega^{m\times(3+n)} \begin{bmatrix} \xi^{3\times 1} \\ \dot{r}^{n\times 1} \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \omega_{\xi}^{3\times 3} & \omega_{\dot{r}}^{3\times 3} \end{bmatrix} \begin{bmatrix} \xi \\ \dot{r} \end{bmatrix}$$

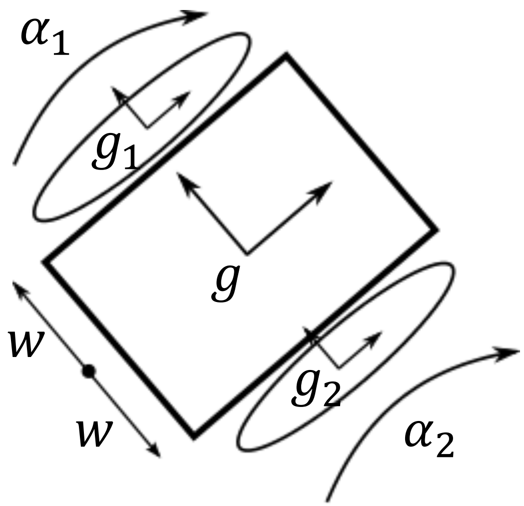
$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \omega_{\xi}\xi + \omega_{\dot{r}}\dot{r}$$

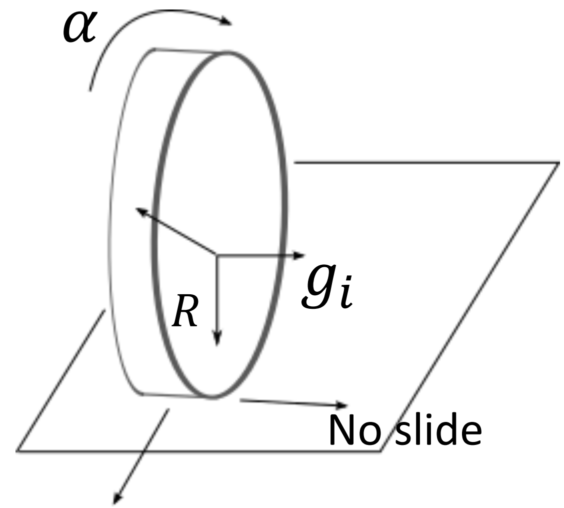
$$\omega_{\xi}\xi = -\omega_{\dot{r}}\dot{r}$$

$$\xi = -\omega_{\xi}^{-1}\omega_{\dot{r}}\dot{r}$$

$$\mathbf{A} = -\omega_{\xi}^{-1}\omega_{\dot{r}}$$

Differential Car





$$\xi_{g_i}^x - R\dot{\alpha}_i = 0 \quad \text{(no slip)}$$

$$\xi_{g_i}^y = 0 \quad \text{(no slide)}$$

2 or 3?

$$\xi_{g_1} = Ad_{g_{1,g}}^{-1}\xi = \begin{bmatrix} \xi^x - w\xi^\theta \\ \xi^y \\ \xi^\theta \end{bmatrix}$$

$$g_{1,g} = (0, w, 0)$$

$$g_{2,g} = (0, -w, 0)$$

$$\xi_{g_2} = Ad_{g_{2,g}}^{-1}\xi = \begin{bmatrix} \xi^x + w\xi^\theta \\ \xi^y \\ \xi^\theta \end{bmatrix}.$$

And onward to the connection

$$\begin{aligned} \xi^x_{g_i} - R\dot{\alpha}_i &= 0 \\ \xi^y_{g_i} &= 0 \end{aligned}$$

$$\begin{aligned} \xi^x - w\xi^\theta - R\dot{\alpha}_1 &= 0 && \text{(no-slip wheel 1)} \\ \xi^x + w\xi^\theta - R\dot{\alpha}_2 &= 0 && \text{(no-slip wheel 2)} \\ \xi^y &= 0 && \text{(no-slide wheels 1 \& 2),} \end{aligned}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \Bigg[\overbrace{\begin{matrix} 1 & 0 & -w \\ 1 & 0 & w \\ 0 & 1 & 0 \end{matrix}}^{\omega_\xi} \; \overbrace{\begin{matrix} -R & 0 \\ 0 & -R \\ 0 & 0 \end{matrix}}^{\omega_{\dot{r}}} \Bigg] \begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \\ \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} \xi \\ \left.\vphantom{\begin{matrix} a \\ a \end{matrix}}\right\} \dot{r} \end{matrix} .$$

Finally there

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix} \overbrace{\begin{matrix}1 & 0 & -w\\ 1 & 0 & w\\ 0 & 1 & 0\end{matrix}}^{\omega_\xi} & \overbrace{\begin{matrix}-R & 0\\ 0 & -R\\ 0 & 0\end{matrix}}^{\omega_{\dot r}} \end{bmatrix} \begin{bmatrix}\xi^x\\ \xi^y\\ \xi^\theta\\ \dot\alpha_1\\ \dot\alpha_2\end{bmatrix} \begin{matrix}\left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\}\xi\\ \left.\vphantom{\begin{matrix}a\\a\end{matrix}}\right\}\dot r\end{matrix}.$$

$$\begin{aligned}
\xi &= -\overbrace{\begin{bmatrix}1 & 0 & -w\\ 1 & 0 & w\\ 0 & 1 & 0\end{bmatrix}}^{\omega_\xi}{}^{-1} \overbrace{\begin{bmatrix}-R & 0\\ 0 & -R\\ 0 & 0\end{bmatrix}}^{\omega_{\dot r}} \dot r\\
&= -\begin{bmatrix}1/2 & 1/2 & 0\\ 0 & 0 & 1\\ -1/(2w) & 1/(2w) & 0\end{bmatrix} \begin{bmatrix}-R & 0\\ 0 & -R\\ 0 & 0\end{bmatrix} \dot r\\
&= -\underbrace{\begin{bmatrix}-R/2 & -R/2\\ 0 & 0\\ R/(2w) & -R/(2w)\end{bmatrix}}_{\mathbf{A}} \dot r,
\end{aligned}$$