BME 790

Spring 2017
Weekly Summary

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Relevant Topics: mobile articulated systems, Jacobians, generalized body frames

Mobile Articulated Systems

An articulated system is *completely* defined by the location and orientation of its body frame (i.e., its position – g) and the variable(s) that reflect the relative placement of the component rigid bodies w.r.t. the body frame (i.e., its shape – r).

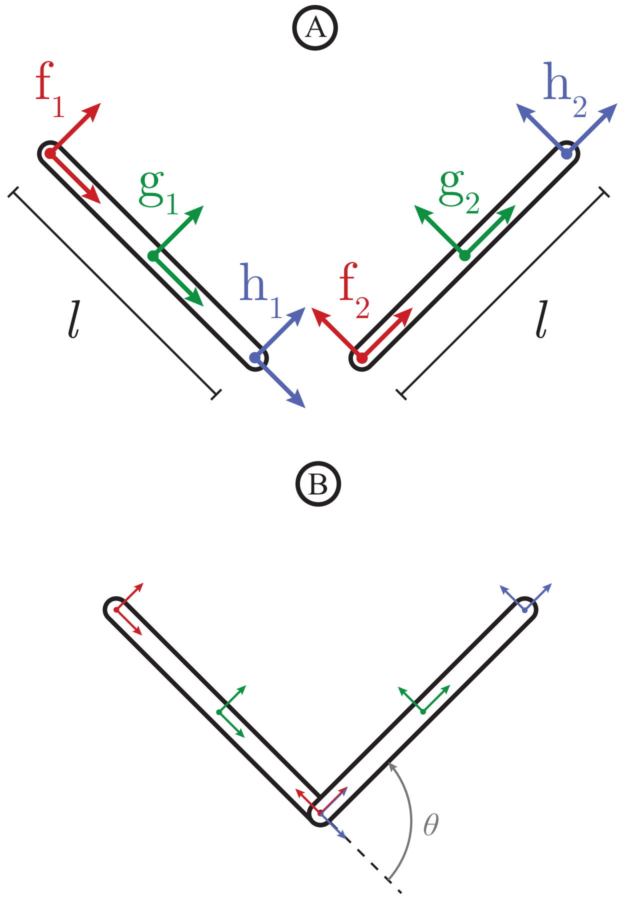
$$g = g_1 \in G \qquad\qquad r = \theta \in M$$

$$q = (g, r) \in G \times M = SE(2) \times \mathbb{S}^1 = Q$$

The remaining frames (and therefore any point on a given rigid body) can then be defined by position and shape through the kinematic model.

Recall that $g_{i,h_j} = h_j^{-1} g_i$ and $h \circ g_h = h \circ h^{-1} \circ g = g$

$$\text{Therefore, } g_2 = \underbrace{g_1 \circ h_{1,g_1}}_{\text{Link 1}} \circ \underbrace{f_{2,h_1} \circ g_{2,f_2}}_{\text{Link 2}} = g_1 \begin{bmatrix} cos\theta & -sin\theta & \frac{l}{2} + \frac{l}{2}cos\theta \\ sin\theta & cos\theta & \frac{l}{2} + \frac{l}{2}sin\theta \\ 0 & 0 & 1 \end{bmatrix}$$

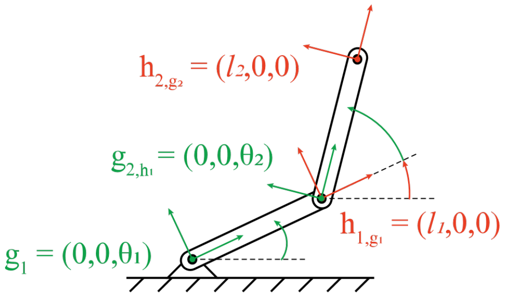


Jacobians

The forward kinematic model, in addition for mapping the configuration of a system to its physical position, can relate the system's physical and configuration space velocities - this is accomplished through the Jacobian. Let $g \in G$ be the *forward kinematic map*. Then,

$$\dot{g}(q,\dot{q}) = J_g \dot{q} = \frac{\partial g}{\partial q}\dot{q}$$

is the Jacobian of the system. Discussed now are two ways in which the Jacobian can be found; differentiation of the kinematic map & by finding the Jacobian of each individual joint.



Recall,

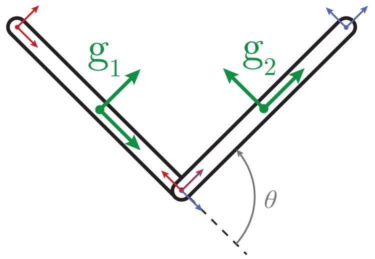
$$\boldsymbol{h_2} = \begin{pmatrix} l_1 cos\theta_1 + l_2 cos(\theta_1+\theta_2) \\ l_1 sin\theta_1 + l_2 sin(\theta_1+\theta_2) \\ \theta_1+\theta_2 \end{pmatrix}$$

through the kinematic model.

Therefore, the Jacobian through differentiation is:

$$\dot{h}_2 = \begin{bmatrix} \frac{\partial h_2^x}{\partial \theta_1} & \frac{\partial h_2^x}{\partial \theta_2} \\ \frac{\partial h_2^y}{\partial \theta_1} & \frac{\partial h_2^y}{\partial \theta_2} \\ \frac{\partial h_2^\theta}{\partial \theta_1} & \frac{\partial h_2^\theta}{\partial \theta_2} \end{bmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} = J_{h_2}\dot{\theta}$$

For the 2-link, 4 *DOF* system (B) previously discussed:



Let $g_1 = g = (x,y,\alpha)$ be the position and $r=\theta$ be the shape. Then g_2 from the geometric model is,

$$g_2 = \begin{pmatrix} \frac{l}{2}(cos\alpha - sin\alpha) + \frac{l}{2}cos(\alpha+\theta) + x \\ \frac{l}{2}(sin\alpha + cos\alpha) + \frac{l}{2}sin(\alpha+\theta) + y \\ \alpha+\theta \end{pmatrix}$$

$$\dot{g}_2 = \begin{bmatrix} \frac{\partial g_2}{\partial g} & \frac{\partial g_2}{\partial r} \end{bmatrix} \begin{pmatrix} \dot{g} \\ \dot{r} \end{pmatrix} = \begin{bmatrix} 1 & 0 & -\frac{l}{2}(sin\alpha + cos\alpha) - \frac{l}{2}sin(\alpha+\theta) & -\frac{l}{2}sin(\alpha+\theta) \\ 0 & 1 & \frac{l}{2}(cos\alpha - sin\alpha) + \frac{l}{2}cos(\alpha+\theta) & \frac{l}{2}cos(\alpha+\theta) \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\alpha} \\ \dot{\theta} \end{pmatrix}$$

Jacobians (cont.)

The second way of finding the Jacobian is to follow an iterative assembly along the links of the system. This allows for a "pre-differentiation" of the relative movements of links as (1) velocities of any two frames on the *same* rigid body are linked by the right lifted action and (2) velocities at the joints are equal, modulo the relative motion allowed by the joint constraint.

Let $g_0 = (0,0,0) = e$ and $\dot{g_0} = (0,0,0)$

Therefore, $\dot{g_1} = \dot{g_0} + (0,0,\dot{\theta}_1) = (0,0,\dot{\theta}_1)$ by (2).

By (1), h_1 has the same spatial velocity as g_1

(i.e., $T_{h_1}R_{h_1}^{-1}\dot{h_1} = T_{g_1}R_{g_1}^{-1}\dot{g_1}$). Therefore,

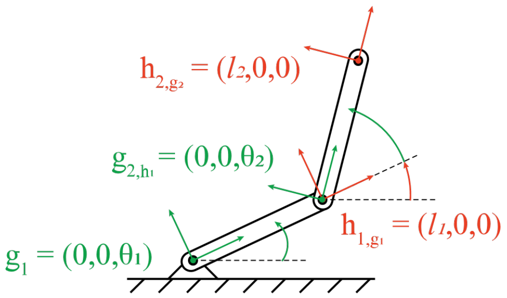
$\dot{h_1} = (T_eR_{h_1})(T_{g_1}R_{g_1}^{-1})\dot{g_1} = T_{g_1}R_{h_{1,g_1}}\dot{g_1}$

$$\dot{h_1}\begin{bmatrix}1 & 0 & -l_1 sin\theta_1\\ 0 & 1 & l_1 cos\theta_1\\ 0 & 0 & 1\end{bmatrix}\begin{pmatrix}0\\0\\\dot{\theta_1}\end{pmatrix} = \begin{bmatrix}-l sin\theta_1\\ l cos\theta_1\\ 1\end{bmatrix}\dot{\theta}_1 = J_{h_1}\dot{\theta}_1$$

Similarly, $\dot{g_2} = \dot{h_1} + (0,0,\dot{\theta}_2)$ from (2) and $\dot{h_2} = T_{g_2}R_{h_{2,g_2}}\dot{g_2}$ from (1).

$$\dot{h_2} = \begin{bmatrix}-(l_1 sin\theta_1 + l_2 sin(\theta_1+\theta_2)) & -l_2 sin(\theta_1+\theta_2)\\ l_1 cos\theta_1 + l_2 cos(\theta_1+\theta_2) & l_2 cos(\theta_1+\theta_2)\\ 1 & 1\end{bmatrix}\begin{pmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{pmatrix}$$

It can be seen that the result from the this technique provides the same as the differentiation.



From this approach it can be seen that, in general,

$$\dot{h}_i = \underbrace{(T_eR_{h_i})\overbrace{(T_{g_i}R_{g_i}^{-1})\underbrace{(\dot{h}_{i-1}+v_i)}_{\dot{g}_i}}^{\textit{Spatial Velocity for } \dot{g}_i}}_{\textit{Returning to } \dot{h}_i \textit{ from Spatial Velocity}}$$

where v_i is the velocity of body i w.r.t. body $i-1$ at joint i.

Sometimes it is useful to deal with body velocities (ξ) rather than absolute velocities. This is done by:

$$\overbrace{(T_{h_i}L_{h_i}^{-1})}^{\textit{New}}\dot{h}_i = \overbrace{(T_{h_i}L_{h_i}^{-1})}^{\textit{New}}(T_eR_{h_i})(T_{g_i}R_{g_i}^{-1})\overbrace{(T_eL_{g_i})(T_{g_i}L_{g_i}^{-1})}^{\textit{Identity}}\dot{g}_i$$

$$\xi_{h_i} = \mathrm{Ad}_{h_i}^{-1}\mathrm{Ad}_{g_i}\xi_{g_i}$$

Jacobians (cont.)

$$\underbrace{\overbrace{(T_{h_i}L_{h_i}^{-1})}^{New}\dot{h}_i}_{\xi_{h_i}} = \underbrace{\overbrace{(T_{h_i}L_{h_i}^{-1})}^{New}(T_eR_{h_i})}_{\mathrm{Ad}_{h_i}^{-1}}\underbrace{(T_{g_i}R_{g_i}^{-1})(T_eL_{g_i})}_{\mathrm{Ad}_{g_i}}\underbrace{(T_{g_i}L_{g_i}^{-1})\dot{g}_i}_{\xi_{g_i}}$$

with $\overbrace{(T_eL_{g_i})(T_{g_i}L_{g_i}^{-1})}^{Identity}$

$$\xi_{h_i} = \mathrm{Ad}_{h_i}^{-1}\mathrm{Ad}_{g_i}\xi_{g_i}$$

Note that this equation is only useful for body velocities of frames on the *same* rigid body. In order to calculate the generalized form of body velocities, it is important to consider three frames g_i, h_{i-1}, and g_i'. This last frame (g_i') is the frame on body i that is instantaneously placed in frame h_{i-1}. Its body velocity can be found by adding the body velocity of h_{i-1} with the body velocity of g_i' relative to h_{i-1} (i.e., v_i):

$$\xi_{g_i'} = \xi_{h_{i-1}} + v_i$$

Because g_i and g_i' are on the same rigid body, we can solve for ξ_{g_i} from $\xi_{g_i'}$.

$$\xi_{g_i} = \mathrm{Ad}_{g_i}^{-1}\mathrm{Ad}_{g_i'}\xi_{g_i'}$$

(Note that if the x, y components of g_i and g_i' are equal, then this conversion is simply rotational.)

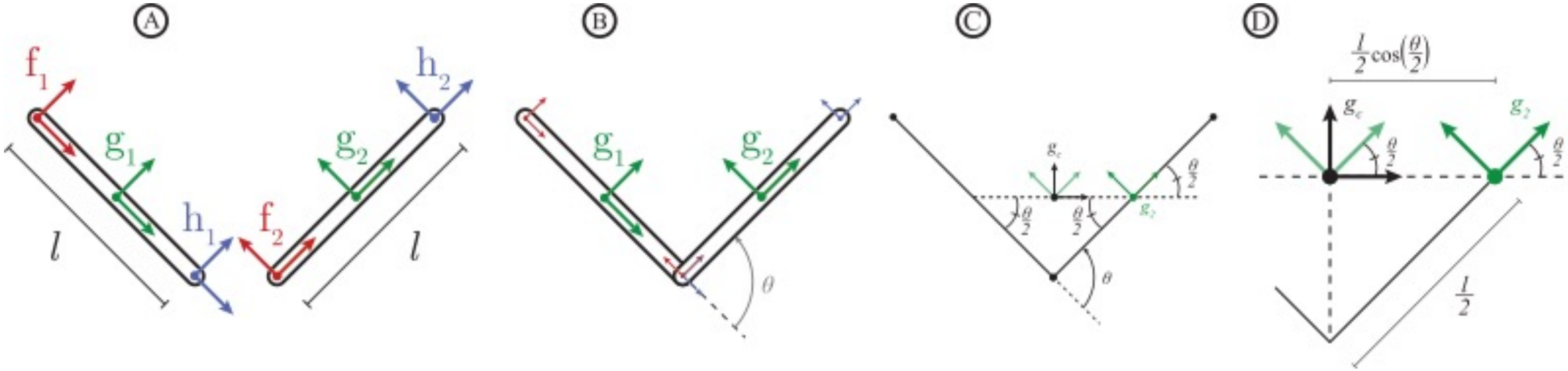
These values can then be substituted back into the original equation to reveal the general form for the conversion of one distal frame body velocity to the distal frame body velocity of the adjoining link.

$$\xi_{h_i} = \mathrm{Ad}_{h_i}^{-1}\mathrm{Ad}_{g_i'}(\xi_{h_{i-1}} + v_i)$$

Additionally, if the frames are reduced to *strictly relative* frames, then the equation reduces even further to,

$$\xi_{h_i} = \mathrm{Ad}_{h_{i,g_i}}^{-1}\mathrm{Ad}_{g_{i,g_i}}\xi_{g_i} = \mathrm{Ad}_{h_{i,g_i}}^{-1}\mathrm{Ad}_e\xi_{g_i} = \mathrm{Ad}_{h_{i,g_i}}^{-1}\xi_{g_i}$$

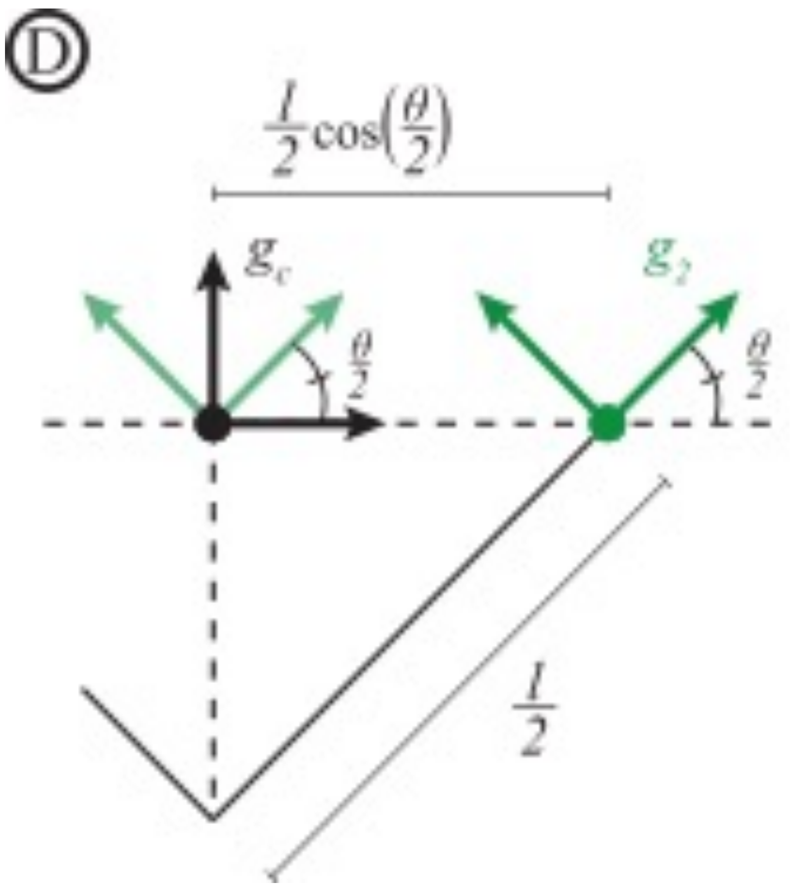
Generalized Body Frames



The body frame for either formalation can be chosen to be any frame (attached or floating) as long as each attached frame defined w.r.t. the chosen body frame is a function of the shape parameters. Note that the simplest choice for body frame is an attached body frame as it automatically satisfies this criterion. For this two-link, 4 *DOF* system, the choice of body frame is adequate because the attached frames are completely defined by transforms that are functions of the shape parameter (θ).

$$g_{2,g_c} = \left(\tfrac{l}{2}cos\left(\tfrac{\theta}{2}\right), \quad 0, \quad -\tfrac{\theta}{2}\right)^T \in SE(2)$$

Generalized Body Frames (cont.)



As the transform to the overall body frame is a function of the shape parameter (defined as the position and orientation of any frame w.r.t. the overall frame), it is possible to convert from the (more easily obtained) body velocity for any attached frame to the overall body velocity, which is generally more difficult to obtain analytically.

$$\begin{aligned}\xi_{g_c} &= \mathrm{Ad}^{-1}_{g_{2,g_c}}(\xi_{g_2} + v_{g_{2,g_c}}) \\ &= \mathrm{Ad}^{-1}_{\beta}(\xi_{g_2} + v_{\beta}) \qquad (\text{Where } \beta = g_{2,g_c})\end{aligned}$$

Here v_β is calculated as the spatial velocity of $\dot{\beta}$ as we are interested in the velocity of the frame *rigidly attached* to g_c and *coincident* with g_2. This way, the inverse adjoint transform of β will return this *relative* spatial velocity, v_β, to the overall body frame.

$$v_\beta = T_\beta R_\beta^{-1}\dot{\beta} = T_\beta R_\beta^{-1}\frac{\partial \beta}{\partial \theta}\dot{\theta} \quad \text{and} \quad \xi_{g_c} = \mathrm{Ad}_\beta^{-1}\left(\xi_{g_2} + T_\beta R_\beta^{-1}\frac{\partial \beta}{\partial \theta}\dot{\theta}\right)$$

Conclusions/Impressions

The calculation of Jacobian operators allows for insight into the relationship between configuration velocities and the physical velocities of the system. But perhaps more interesting, there are individual approaches to deriving this operator that help to reveal key features about the system (e.g., the body frame velocities of each link and physical velocities at each joint through the joint-by-joint Jacobian formulation).

It is also important to have a transform operator that allows for the conversion of the potentially easier-to-derive equations of motion of attached frames to the physical velocity of any overall body frame (so long as this transform is a function of the system's shape). This will be revisited later when determining optimal overall frames.

Biomechanically speaking, this could be potentially useful for evaluating equations of motion from an arbitrary frame (such as the center of gravity) by first evaluating the equations of motion for the links of the system. Additionally, it could also be useful to know the physical velocities of each joint (as opposed to only knowing the endpoint velocity found through the differentiation approach for deriving the Jacobian).