BME 790

Spring 2017
Weekly Summary

Author: Daniel A Hagen
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Relevant Topics: co-vectors, one-forms, connection vector fields, Noether's Theorem, no-slide and inertial constraints locomotion

Covectors and one-forms

Before we address this week's main topics (connection vector fields, kinematic reconstruction equation, etc.) it is important to discuss covectors and one-forms.

A cotangent space (T_q^*Q) is a dual space to a (tangent) vector space ($\mathrm{T_q}Q$) attached to any point on a manifold. Every element in T_q^*Q (covector) is a linear functional (map) of T_qQ that returns the velocity vector into a scalar quantity.

$$\forall \alpha \in T_q^*Q \quad \alpha : T_qQ \longrightarrow \mathbb{R}$$
$$\dot{q} \longmapsto a$$

More generally, a covector can be thought of as a vector quantity that has the same dimension as a tangent vector that produces a scalar value through the product with a tangent vector.

$$\langle \omega, \dot{q} \rangle = (w^1, \ldots, w^n) \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} = w^\# \cdot \dot{q} = a \in \mathbb{R}$$

Note: Superscripts are used when indexing covectors instead of subscripts, and the musical notion of "sharp" (#) is used to reflect that "lifting" of the indexing.

Whereas tangent vectors possess "velocity"-like terms that describe motion through the space, covectors possess "gradient"-like terms to describe how a value varies across the space.

If ω is a function of the configuration space (q) this represents a collection of covectors, i.e. a vector field. These functions ($\omega(q)$) are known as (differential) one-forms.

$$\text{For } \omega(q) \in T^*Q$$
$$\omega(q) : TQ \longrightarrow \mathbb{R}$$
$$\dot{q} \longmapsto \langle \omega(q), \dot{q} \rangle$$

As demonstrated on the next slide, multiple covector fields may be combined to produce vector-valued one-forms.

Covectors and one-forms

For a vector-valued one-form, covectors become the rows of a matrix that – when multiplied by the vector – produce the desired scalar. This total operation produces a vector of desired scalar values. An obvious example of this is the Jacobian matrix ($J(q)$):

$$J(q)\dot{q} = \begin{bmatrix} J^{1,1}(q) & \cdots & J^{1,n}(q) \\ \vdots & \ddots & \vdots \\ J^{k,1}(q) & \cdots & J^{k,n}(q) \end{bmatrix} \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} = \begin{pmatrix} \langle J^1 q, \dot{q} \rangle \\ \vdots \\ \langle J^k q, \dot{q} \rangle \end{pmatrix}$$

Here, the product of these two dual covectors and vectors produces the scalar value corresponding to the velocity of a particular position variable.

Additionally, covectors can be thought of as local representations of the derivative of some implied function (f) w.r.t. the configuration space. This function is selected depending on the desired scalar produced through the product of vectors/covectors. Suppose,

$$\omega = \vec{d}f = \left(\frac{\partial f}{\partial q_1}, \quad \cdots \quad , \frac{\partial f}{\partial q_n} \right)$$

then the product of ω an $\dot{q}$ represents the directional derivative of f along q.

$$D_v f = \langle \vec{d}f, \dot{q} \rangle = \frac{\partial f}{\partial \vec{q}} q$$

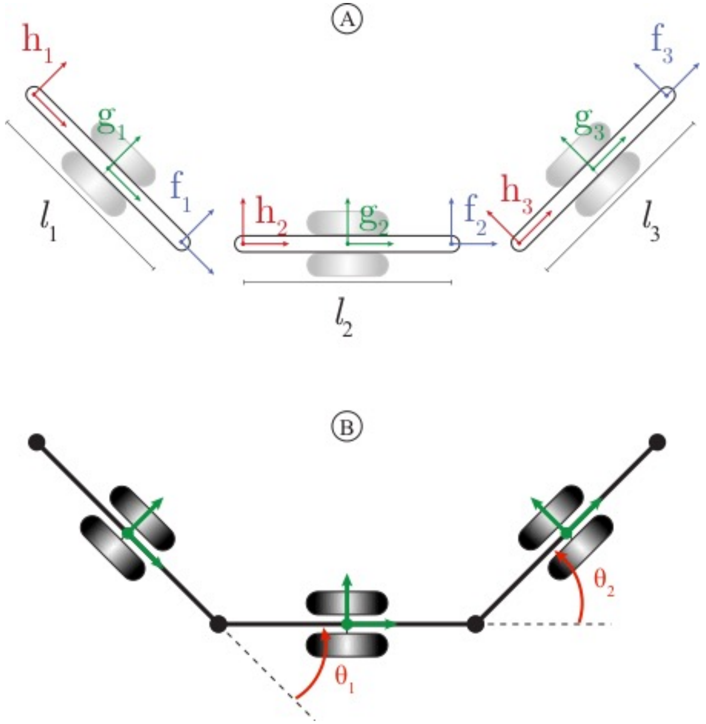
If f is defined over the entire space, then these covectors reflect a gradient vector field (∇f) that will point in the direction in which (a given function) f increase the most quickly over the configuration.

$$\nabla f = \omega(q)$$
$$(\text{where } f \text{ is defined over all of } Q)$$

These distinctions are important because they emphasize the existence of a function that relates configuration space variables to relevant scalar values. It also restates that vectors/matrices are additionally tensors with associated actions.

No-Slide 3-Link Locomotor

Below is a 3-link kinematic locomotor with wheels at the link body frames that each have a ”no slide” condition.



The body frame velocities can be calculated by considering the middle link body frame to be the overall body frame.

$$\xi_{g_1} = (\mathrm{Ad}^{-1}_{g_{1,h_1}})(\mathrm{Ad}^{-1}_{h_{1,h_1'}})[(\mathrm{Ad}^{-1}_{f_{2,g_2}}\xi_{g_2} + \xi_{h_{1,f_2}}]$$
$$= \begin{pmatrix} \xi^x\cos\theta_1 - (\xi^y - \frac{\xi^\theta l_2}{2})\sin\theta_1 \\ \xi^x\sin\theta_1 + (\xi^y - \frac{\xi^\theta l_2}{2})\cos\theta_1 - \frac{l_1}{2}(\xi^\theta - \dot\theta_1) \\ \xi^\theta - \dot\theta_1 \end{pmatrix}$$

$$\xi_{g_2} = \begin{pmatrix} \xi^x \\ \xi^y \\ \xi^\theta \end{pmatrix}$$

$$\xi_{g_3} = (\mathrm{Ad}^{-1}_{g_{3,f_3}})(\mathrm{Ad}^{-1}_{f_{3,f_3'}})[(\mathrm{Ad}^{-1}_{h_{2,g_2}}\xi_{g_2} + \xi_{f_{3,h_2}}]$$
$$= \begin{pmatrix} \xi^x\cos\theta_2 + (\xi^y + \frac{\xi^\theta l_2}{2})\sin\theta_2 \\ -\xi^x\sin\theta_2 + (\xi^y + \frac{\xi^\theta l_2}{2})\cos\theta_2 + \frac{l_3}{2}(\xi^\theta + \dot\theta_2) \\ \xi^\theta + \dot\theta_2 \end{pmatrix}$$

The **second term of each velocity vector** must be **zero** to ensure that the 3-link ”snake” **does not move ”laterally”** at the wheels (i.e., “no slide”). Therefore, the Pfaffian constraints of the system are:

$$\begin{pmatrix} \xi^y_{g_1} \\ \xi^y_{g_2} \\ \xi^y_{g_3} \end{pmatrix} = \begin{bmatrix} \sin\theta_1 & \cos\theta_1 & -\frac{l_2\cos\theta_1 + l_1}{2} & l_1/2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -\sin\theta_2 & \cos\theta_2 & \frac{l_2\cos\theta_2 + l_3}{2} & 0 & l_3/2 \end{bmatrix} \begin{pmatrix} \xi^x \\ \xi^y \\ \xi^\theta \\ \dot\theta_1 \\ \dot\theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} \sin\theta_1 & \cos\theta_1 & -\frac{l_2\cos\theta_1 + l_1}{2} \\ 0 & 1 & 0 \\ -\sin\theta_2 & \cos\theta_2 & \frac{l_2\cos\theta_2 + l_3}{2} \end{bmatrix} \begin{pmatrix} \xi^x \\ \xi^y \\ \xi^\theta \end{pmatrix} + \begin{bmatrix} l_1/2 & 0 \\ 0 & 0 \\ 0 & l_3/2 \end{bmatrix} \begin{pmatrix} \dot\theta_1 \\ \dot\theta_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} \xi^x \\ \xi^y \\ \xi^\theta \end{pmatrix} = -\begin{bmatrix} \sin\theta_1 & \cos\theta_1 & -\frac{l_2\cos\theta_1 + l_1}{2} \\ 0 & 1 & 0 \\ -\sin\theta_2 & \cos\theta_2 & \frac{l_2\cos\theta_2 + l_3}{2} \end{bmatrix}^{-1} \begin{bmatrix} l_1/2 & 0 \\ 0 & 0 \\ 0 & l_3/2 \end{bmatrix} \begin{pmatrix} \dot\theta_1 \\ \dot\theta_2 \end{pmatrix} = -\mathrm{A}(r)\dot r$$

$$\xi = \frac{1}{D}\begin{bmatrix} -\frac{l_1(l_3 + l_2\cos\theta_2)}{2} & -\frac{l_3(l_1 + l_2\cos\theta_1)}{2} \\ 0 & 0 \\ -l_1\sin\theta_2 & -l_3\sin\theta_1 \end{bmatrix}\dot r$$

(Where $D = l_2\sin(\theta_1 - \theta_2) - l_1\sin\theta_2 + l_3\sin\theta_1$)

Noether's Theorem

In short, Noether's Theorem states that if a system is symmetric with respect to a given transformation, then some value is conserved in that direction.

A simple example is a system that is symmetrical to horizontal translations (i.e., horizontal translation does not affect the Lagrangian of the system). Because of this symmetry, the linear momentum will be preserved in this direction.

Let δq_k be any deviation in the k-direction. Then, if q' is configuration that has moved by δq_k in the k-direction (i.e., $q'_k = q_k + \delta q_k$). Then,

$$\mathcal{L}(q, \dot{q}) = \mathcal{L}(q', \dot{q}')$$

This theorem allows for the calculation of conserved quantities (invariants) from observable symmetries and vice versa. Therefore, it is possible to take a system with an observable symmetry and find a value that is conserved, or a system can be imagined to have a given conserved value and the resulting symmetry can be obtained (thus allowing for a way to test models).

This becomes particularly important when attempting to reduce the necessary equations of motion by "ignoring" a given coordinate over which the symmetry is observed.

Inertial Constraints (3-Link Locomotor)

An illustration of this concept can be seen in systems with inertial constraints (and cleverly chosen coordinate frames).

For the 3-link locomotor, if we select the center of mass as the x,y location of the body frame, then the Lagrangian will not change with translations in either x or y (i.e. translational velocity does not depend on the Cartesian coordinates of the body frame). Therefore, the local connection can be more easily obtained as:

$$\xi = -\frac{1}{D}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ a_{3,1} & a_{3,2} \end{bmatrix} \dot{r}$$

A formal calculation for the values of $a_{3,1}$, $a_{3,2}$, and D can be found in Shammas et al. (2007)

However, it is possible to find the local connection from the energy of the system. For this "floating snake" there is no potential energy, indicating that:

$$\begin{aligned} \text{Total Energy} &= KE = \frac{1}{2}\begin{bmatrix}\xi & \dot{r}\end{bmatrix} \mathbb{M} \begin{bmatrix}\xi \\ \dot{r}\end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix}\xi & \dot{r}\end{bmatrix} \begin{bmatrix} \mathbb{I}(r) & \mathbb{I}(r)\mathbb{A}(r) \\ (\mathbb{I}(r)\mathbb{A}(r))^T & m(r) \end{bmatrix} \begin{bmatrix}\xi \\ \dot{r}\end{bmatrix} \end{aligned}$$

where it is possible to isolate the connection $\mathbb{A}(r)$. Note that $\mathbb{I}(r)$ is the inertia tensor and $m(r)$ is the mass matrix (which only depends on the shape variables).

These techniques allow for a formulation of the local connection, which allows for insight into the behavior of the system to changes in the shape variables.

Conclusions/Impressions

We are working towards a better understanding of how to incorporate these principles in trajectory (gait) formulation. Consequences to shape variable trajectories on the position of a system are not only elucidated by the connection of a system, but are also utilized to find movement trajectories that satisfy given criteria.

Formulating the system properly, by either the use of a clever coordinate frame or the selection of systems with natural symmetries, allows for simpler formulation of the equations of motion through the use of Noether's theorem. Additionally, this allows for a simpler construction of the Pfaffian constraints and local connections.

References:

- Choset, H. and Hatton, RL, 2015. *An Introduction to Geometric Mechanics and Differential Geometry*
- Shammas, E.A., Choset, H. and Rizzi, A.A., 2007. Geometric motion planning analysis for two classes of underactuated mechanical systems. *The International Journal of Robotics Research*, *26*(10), pp.1043-1073.