Geometry of Locomotion
Chapter 2.x – Rigid Body Velocities

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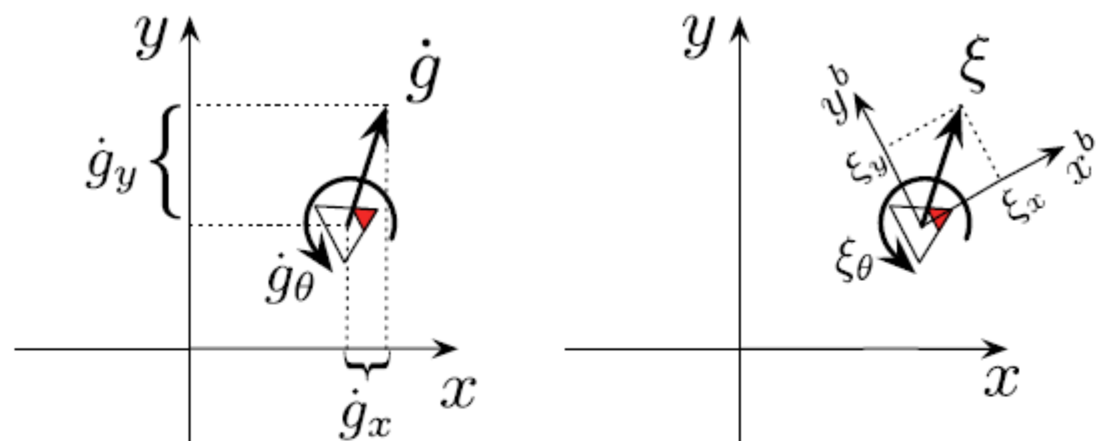
Body Velocity

A system's *body velocity* ξ is its velocity expressed in the instantaneous local coordinate frame,



(a) World velocity. (b) Body velocity.

why

$$\begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot g_x \\ \dot g_y \\ \dot g_\theta \end{bmatrix}$$

rotating the translational component by $-\theta$

(equivalent to rotating the reference frame by θ)

$$\dot g = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \xi$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \xi$$

Left Lifted Action

$$g = (x, y, \theta) \qquad T_g L_h = \frac{\partial(hg)}{\partial g}$$

$$h = (u, v, \beta)$$

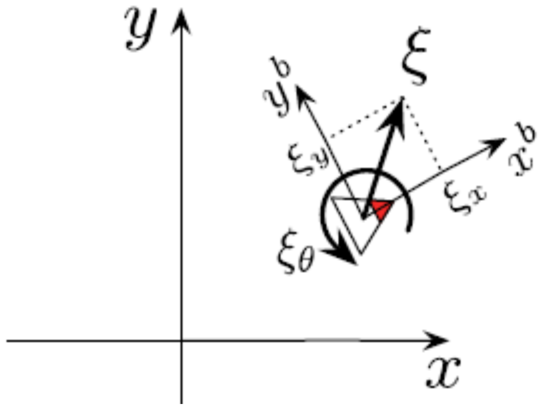
$$= \begin{bmatrix} \frac{\partial(x\cos\beta - y\sin\beta + u)}{\partial x} & \frac{\partial(x\cos\beta - y\sin\beta + u)}{\partial y} & \frac{\partial(x\cos\beta - y\sin\beta + u)}{\partial \theta} \\ \frac{\partial(x\sin\beta + y\cos\beta + v)}{\partial x} & \frac{\partial(x\sin\beta + y\cos\beta + v)}{\partial y} & \frac{\partial(x\sin\beta + y\cos\beta + v)}{\partial \theta} \\ \frac{\partial(\theta+\beta)}{\partial x} & \frac{\partial(\theta+\beta)}{\partial y} & \frac{\partial(\theta+\beta)}{\partial \theta} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$T_g L_h$ preserves body velocities:

For any left action L_h that rotates the body frame by an angle β, the accompanying lifted action $T_q L_h$ rotates the velocity vector by the same amount, leaving its expression in the body frame unchanged.

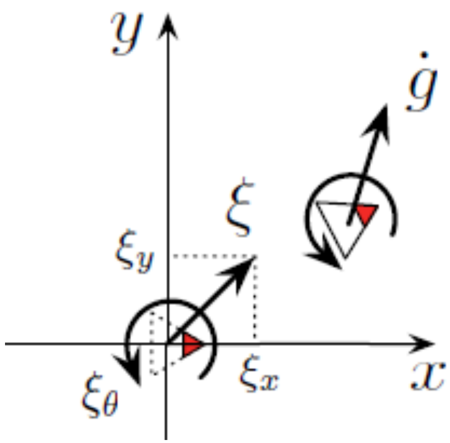
More left lift



h is set equal to g^{-1}

$L_{g^{-1}}$ and $T_g L_{g^{-1}}$ take a rigid body from g

place it at the origin with equivalent body velocity,



$$\xi = T_g L_{g^{-1}} \dot{g} \in T_{\mathbf{e}} G,$$

interchangeably treat the body velocity either in its standard interpretation as the set of forward, lateral, and rotational velocities of the body, or as a vector in the tangent space at the origin

$$\dot{g} = (T_g L_{g^{-1}})^{-1} \xi = T_e L_g \xi.$$

Right Action

$R_h g$ on $SE(2)$ finds the frame at position and orientation h with respect to g.

$$(\dot{gh}) = \begin{bmatrix} 1 & 0 & -(u\sin\theta + v\cos\theta) \\ 0 & 1 & u\cos\theta - v\sin\theta \\ 0 & 0 & 1 \end{bmatrix} \dot{g},$$

$$\begin{aligned} T_g R_h &= \frac{\partial(gh)}{\partial g} \\ &= \begin{bmatrix} \frac{\partial(x+u\cos\theta - v\sin\theta)}{\partial x} & \frac{\partial(x+u\cos\theta - v\sin\theta)}{\partial y} & \frac{\partial(x+u\cos\theta - v\sin\theta)}{\partial \theta} \\ \frac{\partial(y+u\sin\theta + v\cos\theta)}{\partial x} & \frac{\partial(y+u\sin\theta + v\cos\theta)}{\partial y} & \frac{\partial(y+u\sin\theta + v\cos\theta)}{\partial \theta} \\ \frac{\partial(\theta+\beta)}{\partial x} & \frac{\partial(\theta+\beta)}{\partial y} & \frac{\partial(\theta+\beta)}{\partial \theta} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -(u\sin\theta + v\cos\theta) \\ 0 & 1 & u\cos\theta - v\sin\theta \\ 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

Right Action

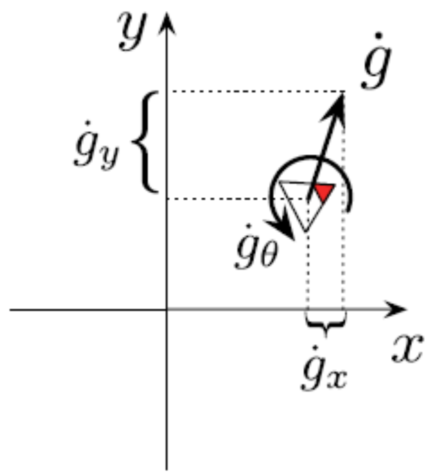
- an action R_h maps between rigidly attached frames
- the "equivalent" vector selected by the accompanying lifted action $T_g R_h$ ensures that the new frame's velocity is compatible with the rigid attachment.
- right lifted action preserves their *spatial velocity*
- velocity of the (possibly imaginary) point on that body that is instantaneously over the origin, calculated as

$$\xi^s = T_g R_{g^{-1}} \dot{g} = \begin{bmatrix} 1 & 0 & y \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{bmatrix} \dot{g}$$

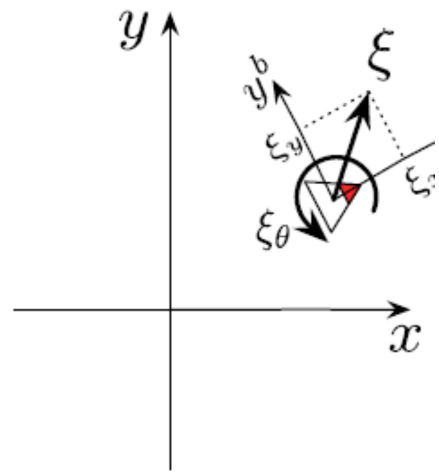
$$\dot{g} = (T_g R_{g^{-1}})^{-1} \xi^s = T_{\mathbf{e}} R_g \xi^s = \begin{bmatrix} 1 & 0 & -y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} \xi^s$$

Lie albebra

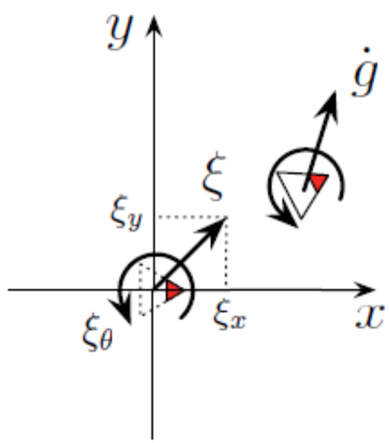
Velocities Summarized



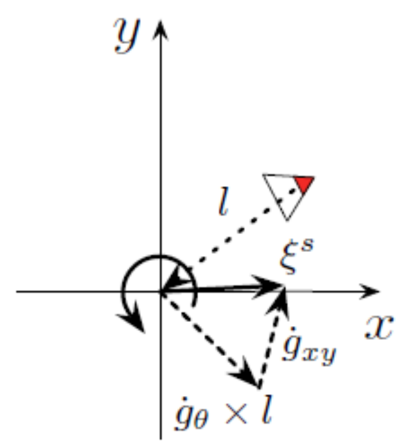
(a) World velocity.



(b) Body velocity.



(c) Alternate interpretation of body velocity



(d) Spatial velocity.

Adjoint Operation

$$\xi^s = \overbrace{(T_g R_{g^{-1}})(T_{\mathbf{e}} L_g)}^{Ad_g} \xi,$$

$$\xi = \overbrace{(T_g L_{g^{-1}})(T_{\mathbf{e}} R_g)}^{Ad_g^{-1}} \xi^s$$

adjoint inverse action $Ad_g^{-1} = Ad_{g^{-1}}$

Adjoint on SE(2)

$$\xi^s = \overbrace{\begin{bmatrix} 1 & 0 & y \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{bmatrix}}^{T_g R_{g^{-1}}} \underbrace{\overbrace{\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{T_{\mathbf{e}} L_g} \xi}_{\dot{g}}$$

$$= \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta & y \\ \sin\theta & \cos\theta & -x \\ 0 & 0 & 1 \end{bmatrix}}_{Ad_g} \xi.$$

$$\xi = \overbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{T_g L_{g^{-1}}} \underbrace{\overbrace{\begin{bmatrix} 1 & 0 & -y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}}^{T_{\mathbf{e}} R_g} \xi^s}_{\dot{g}}$$

$$= \underbrace{\begin{bmatrix} \cos\theta & \sin\theta & x\sin\theta - y\cos\theta \\ -\sin\theta & \cos\theta & x\cos\theta + y\sin\theta \\ 0 & 0 & 1 \end{bmatrix}}_{Ad_g^{-1}} \xi^s.$$