An Introduction to Geometric Mechanics and Differential Geometry

Ross L. Hatton and Howie Choset

Draft: 19:54 Saturday 26th September, 2015

Contents

Illustrations

Tables

Contributors

1
Position

The first step in modeling any system is to identify an appropriate mathematical structure to describe it—its *configuration space*. A point in a mechanical system's configuration space uniquely identifies the locations of its constituent bodies, and the structure of this space captures the number and character of its degrees of freedom. Once this structure has been determined, the system's dynamics can be used to find the system's velocity through its configuration space (i.e., how the configuration changes over time).

In this chapter, we outline a mathematical framework for studying configuration spaces that draws on fundamental ideas from differential geometry. In particular, we introduce the notions of structured spaces such as *manifolds* and *groups*, and consider how their constructions affect notions of relative position within the space. An important aspect of this analysis is a rigorous and principled selection of configuration space structure for *rigid bodies*.

Rigid bodies play a key role in the study and application of geometric mechanics. From a theoretical standpoint, mathematical notions of rigid bodies provide intuitive examples of nontrivial group operations, and serve as a foundation for a range of differential geometric concepts such as Lie groups, lifted actions, and exponential maps that we will consider in later chapters. On the applications side, mathematical rigid bodies correspond directly to physical rigid bodies, such as the links of a robot or the components of other mechanical systems. This correspondence provides an avenue for applying deep mathematical results to practical systems. Further, the basic principles of rigid body motion provide the *body frame* paradigm for mobile articulated systems that we explore in subsequent chapters.

1.1 Configuration Manifolds

A mechanical system's *degrees of freedom* are the independent ways in which it can move, such as by translating, rotating, or bending at a joint. Its *configuration*, denoted q, is an arrangement of these degrees of freedom that uniquely determines the location in the world of each point on the system. The system's *configuration space*, Q, is the set of configurations the system can assume; it has as many dimensions as the system has degrees of freedom, and a structure that encodes how these degrees of freedom are coupled or connect back on themselves. It is often convenient (e.g., when performing computations) to represent configurations by their *generalized coordinates*, a set of numbers q_i that identifies configurations in terms of a parameterization of the configuration space.

The configuration spaces of the mechanical systems we consider in this book are differentiable *manifolds*.[1] Three basic examples of manifolds are the line, the circle, and the sphere. Many mechanical elements, such as rotary or prismatic joints, have configuration spaces that can be represented by these manifolds (or, more formally, are *homeomorphic* or *diffeomorphic*[2] to them). Combinations of mechanical elements can in turn be represented by manifolds assembled from the basic manifolds.

Parameterizing the configuration spaces of these systems into generalized coordinates corresponds to assigning one or more *coordinate charts* to the manifold. Note that the configuration-space equivalences we discuss

[1] See the box on the following page.
[2] See the box on page 4.

Euclidean Spaces and Manifolds

Euclidean space. Mathematical spaces are called *Euclidean* if they support the standard operations of Euclidean geometry—translation, rotation, and reflection. Such spaces are *affine* (meaning that the difference between any two points is a vector with magnitude and direction, but that no point is inherently identified as an origin). Euclidean spaces are distinctive for the independence and flatness of their dimensions: moving along one dimension will not lead to at a point reachable by moving along a combination of other dimensions. Additionally, measurements of distance and direction are consistent across the space, so that applying the Euclidean operations to a set of points in the space neither dilates nor distorts them.

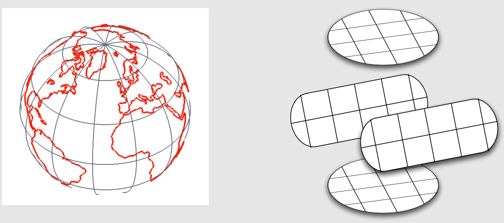
Lin and Burdick for euclidean/cartesian difference

Euclidean spaces with n dimensions, $\mathbb{E}^n$, are closely identified with n-dimensional spaces of real numbers, designated $\mathbb{R}^n$. Strictly speaking, the association of $\mathbb{E}^n$ with $\mathbb{R}^n$ is a smooth, structure-preserving mapping[a] that parameterizes the Euclidean space, allowing individual points to be called out by n-tuples of real numbers. Such parameterizations are so inherent to representation and computation on Euclidean spaces, however, as to be almost transparent in operation. One consequence of this transparency is that Euclidean spaces are often referred to directly as $\mathbb{R}^n$, with the $\mathbb{E}^n$ notation deprecated. We adopt this convention for the rest of the book, except where it is necessary to explicitly contrast the two spaces.

[a] Formally, a diffeomorphism and an isomorphism, as defined in the boxes on pages 4 and 16.

Manifold. A *manifold* is a space that is locally Euclidean (i.e., for which distance and orientation are well defined in the vicinity of each point), but which may have a more complicated global structure. Such structure includes the manifold's *curvature* (the ways in which distance measurements vary at different points, which we consider in Chapter 6) and its *topology* (e.g., the ways in which the manifold connects back on itself). The structure of a given manifold is described by an *atlas* of *charts*. Each chart is a region of Euclidean space that maps to a region of the manifold, and an atlas is a set of overlapping charts that collectively describe the whole manifold; this terminology mirrors nautical terminology, in which a "chart" is a literal "map" of a section of the globe depicted on a flat sheet of paper, and an "atlas" is a collection of maps that together show the whole surface of the earth.

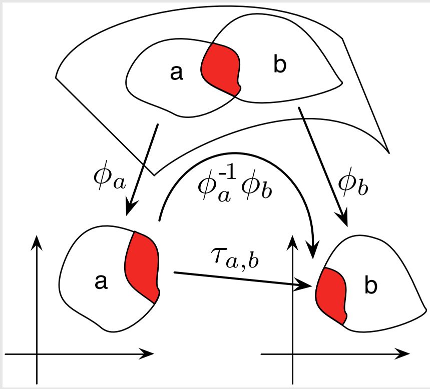
By assigning points in Euclidean space (and thus n-tuples of real numbers) to points in the manifold, charts inherently parameterize the space and so are often referred to as *coordinate charts*. Multi-chart atlases provide a means of "papering over" regions that cannot be adequately captured by a single parameterization. Two canonical examples of the need for multiple charts appear on maps of the globe: If we were to use a single latitude-longitude chart, the poles would correspond to every point on the chart at $\pm 90°$ latitude, and points in the region near $\pm 180°$ longitude would be represented as much farther away from each other than they really are, with a discontinuity that would make it difficult to consider paths crossing the "dateline." Including multiple charts, such as polar projections and latitude-longitude maps centered on different meridians, provides alternate parameterizations of the globe that are better suited for working in these regions.



continued ...

Euclidean Spaces and Manifolds, *continued*

Transition map. On regions parameterized by more than one chart (their *overlaps*), coordinates on one chart can be translated to those on a second chart via the *transition map* (also known as the *overlap map*) between the charts. These transition maps encode the idea of a "reparameterization" of the manifold and, as illustrated below, the transition map between charts a and b can be constructed from the parameterization functions ϕ for the two charts as $\tau_{a,b} = \phi_a^{-1}\phi_b$. Note that if there are multiple non-contiguous overlaps between a pair of charts (as in the case of the circle in Figure 1.2), each of these overlaps gets its own transition function, so that the transitions functions are each continuous over their respective domains.



For many applications, the neighborhoods must share more differentiable structure with $\mathbb{R}^n$ than simple homeomorphism guarantees. Manifolds with this structure are designated as C^k*-(differentiable) manifolds*, and, when paramaterized onto charts, meet two additional requirements:

1. The mappings from the manifolds to the charts must each be k-times differentiable, *i.e.*, they must be C^k-diffeomorphisms.
2. All transition maps between charts in the atlas must be C^k-diffeomorphisms.

Together, these properties permit the definition of C^k functions on the manifold that maintain their differentiability across regions mapped by different charts. C^∞-manifolds, on which functions can be differentiated arbitrarily many times, are also known as *smooth* or *differential* manifolds.

in this chapter are *topologically-based*—i.e., based on the structure and connectedness of the configuration space—and do not take into account any measures of distance or curvature on the manifolds. We will discuss these aspects of the configuration manifolds in Chapter 6.

The first of our basic manifolds, the line, serves as the configuration space of simple one-dimensional systems, some examples of which appear at the left of Figure 1.1. These systems may be physically linear, such as a prismatic joint, or not, as in a bead on a curved wire. The important aspect is that they have a single degree of freedom—e.g., the extension of the joint or the position of the bead along the wire—and so the set of such positions can be mapped continuously (homeo- or diffeomorphically) to points on the abstract line, which is denoted $\mathbb{E}^1$ as the one-dimensional Euclidean space. The abstract line is itself diffeomorphic and isomorphic[3] to the set of ordered real numbers, denoted $\mathbb{R}$ or $\mathbb{R}^1$, providing parameterizations in which each point on the line is identified by a single real number. This final diffeomorphism from $\mathbb{E}^1$ to $\mathbb{R}^1$ is often handled transparently, and for much of the book we adopt the convention that $\mathbb{R}^n$ can refer both to Euclidean space or the set of real numbers, as indicated by context.

Some one-dimensional systems, such as unrestricted rotary joints, wheels, or systems moving on closed tracks, have *cyclic* configuration spaces. A motion far enough through these spaces in a given direction (such

[3] See the box on page 16.

Homeomorphisms and Diffeomorphisms

Homeomorphism. A *homeomorphism* is a function f mapping between two spaces that is:

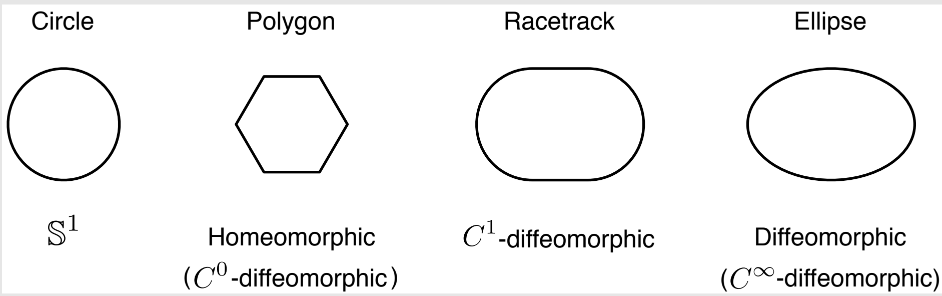
1. **Bijective**, *i.e.*, invertible. Bijective functions are both
   a. **Surjective**, or *onto*, meaning that every point in the range is the function of *at least* one point in the domain, and
   b. **Injective**, or *one-to-one*, meaning that each point in the the domain is mapped to a *unique* point in the range.

   Combining these properties means that there is a full one-to-one correspondence in both directions between points in the domain and points in the range, and thus that
   a. f^{-1} is a true (single-valued) function whose domain is the entire range of f, and
   b. $f \circ f^{-1}$ and $f^{-1} \circ f$ are both identity mappings.
2. **Continuous**, with a continuous inverse.

Diffeomorphism. A *diffeomorphism* is a homeomorphism for which both f and f^{-1} are additionally differentiable, *i.e.*, they are not only continuous, but their derivatives are also continuous. A C^k-diffeomorphism is k-times differentiable, meaning that its first k derivatives are continuous.[a] C^∞-diffeomorphisms, for which all of the derivatives are continuous, are referred to as *smooth* diffeomorphisms. Technically, a homeomorphism that is continuous but not differentiable is a C^0-diffeomorphism, but the adjective "diffeomorphic" is generally reserved for mappings that are at least once-differentiable, and should be taken to indicate a smooth mapping function if not further qualified.

[a] C^k is more generally the set of all k-times differentiable functions.

Example. Comparing the circle $\mathbb{S}^1$ to a sequence of closed-loop shapes provides useful illustration of homeo- and diffeomorphism. Polygons are homeomorphic to $\mathbb{S}^1$, as we can define continuous and invertible mappings from points on the circle to points on the polygon, but not diffeomorphic, because these mapping functions would have discontinuous derivatives at the corners of the polygon. Mappings from the circle to a racetrack composed of straight segments connected by semi-circular arcs can have continuous first derivatives, but their second derivatives are discontinuous where the segments meet, making the racetrack C^1-diffeomorphic to the circle. Ellipses and other shapes for which we can define infinitely-differentiable mappings to and from $\mathbb{S}^1$ are considered to be fully diffeomorphic to the circle.



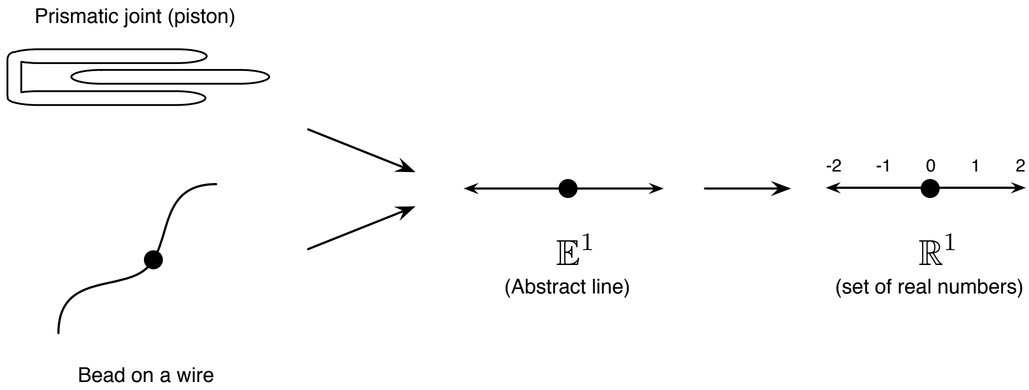


Figure 1.1 The configuration spaces of simple single-degree-of-freedom systems are homeomorphic (and often diffeomorphic) to the abstract line $\mathbb{E}^1$. Because the abstract line is diffeomorphic to the set of real numbers, $\mathbb{R}^1$, we can assign a unique real number to each configuration of these systems and use $\mathbb{R}^1$ for their coordinate charts. A standard convention, which we adopt for much of the book, is to drop the distinction between $\mathbb{E}$ and $\mathbb{R}$, referring to both Euclidean spaces and real numbers as $\mathbb{R}$.

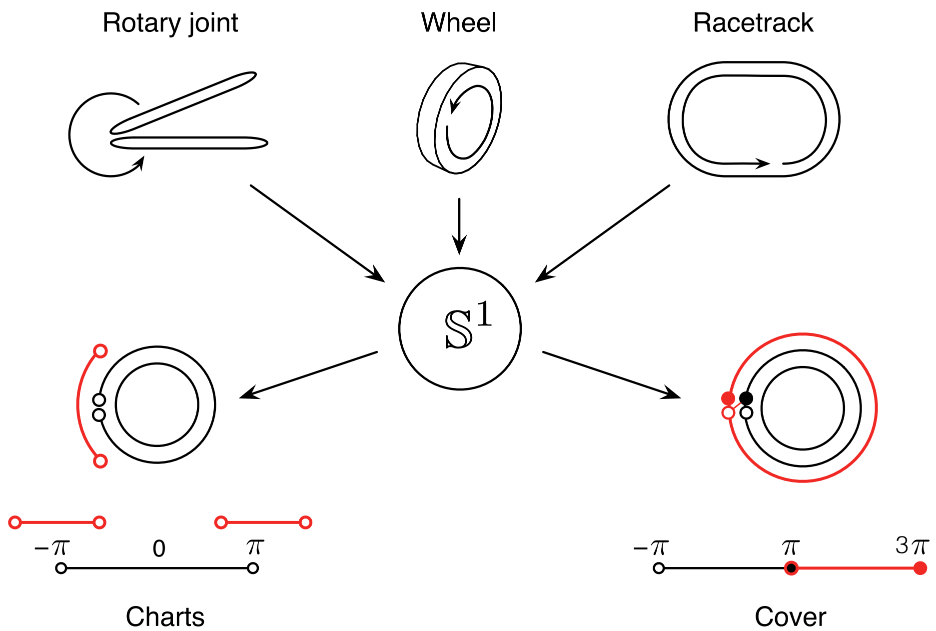


Figure 1.2 The circle $\mathbb{S}^1$ is used to represent the configuration spaces of cyclic systems, such as rotary joints in articulated mechanisms, wheels, and systems moving on closed tracks. A single chart is sufficient to parameterize points on the circle, but becomes discontinuous where $\mathbb{S}^1$ connects back on itself. A continuous parameterization of the circle can be achieved either by using a second chart to patch the discontinuity or by using a covering space that multiply parameterizes points on $\mathbb{S}^1$.

as by rotating through 2π radians) returns to the original configuration. This self-connectedness is not captured by the structure of $\mathbb{R}^1$, so we use a second basic manifold, the circle $\mathbb{S}^1$, to describe such systems as illustrated in Figure 1.2. The circle is the simplest example of a *closed* manifold (one that is finite without boundary), and the notation $\mathbb{S}^1$ is derived from a definition of the circle as the "sphere" with a one-dimensional surface. An important conceptual point is that although we depict the circle as a two-dimensional figure, this is an *embedding*[4] of $\mathbb{S}^1$ into the space of the page, and the circle exists independently of such embeddings as a one-dimensional system that connects back on itself.

A single coordinate chart is almost sufficient to parameterize the circle, but this parameterization would become discontinuous at the point where the manifold connects back to itself. Its atlas (which is required to

[4] See the box on page 15.

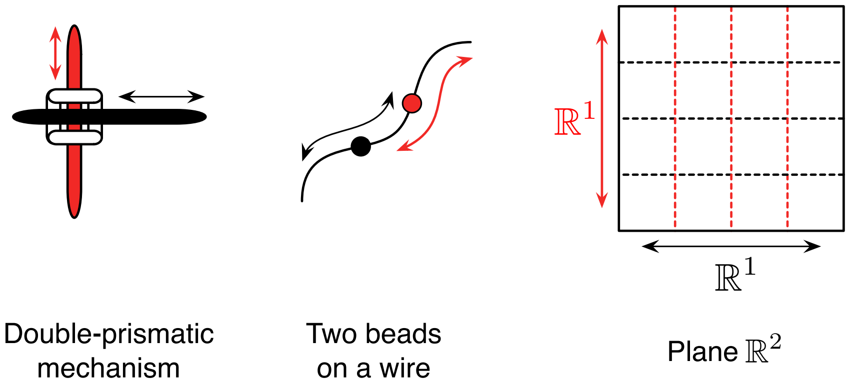


Figure 1.3 Systems composed of two elements with $\mathbb{R}^1$ configuration spaces have configuration spaces homeomorphic to the abstract plane $\mathbb{R}^2$.

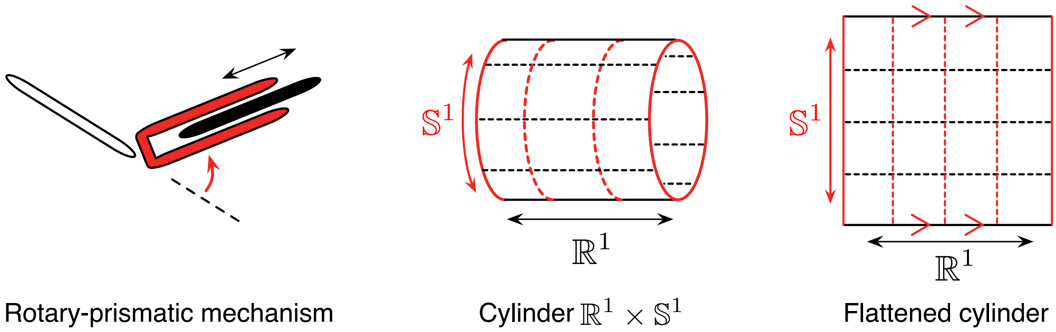


Figure 1.4 Systems one $\mathbb{R}^1$ component and one $\mathbb{S}^1$, such as rotary-prismatic mechanism, have a composite configuration space homeomorphic to the cylinder $\mathbb{R}^1 \times \mathbb{S}^1$. It is often useful to visualize the cylinder in a flattened representation, with the "cut edges" marked as being identified with each other.

provide a continuous parameterization around each point) must therefore include a second chart, overlapping the endpoints of the first. An alternative means of parameterizing the circle is to take the line as a *covering space* of the circle, in which points on the full $\mathbb{R}^1$ space are mapped cyclicaly onto $\mathbb{S}^1$. This approach removes the need to handle discontinuities, but loses the inherent identity between points that are one cycle apart from each other.

Combinations of lines and circles provide configuration spaces for more complex systems. The simplest of these, the plane $\mathbb{R}^2 = \mathbb{R}^1 \times \mathbb{R}^1$, is formed as the *direct product*[5] of two lines. As illustrated in Figure 1.3, the plane is homeomorphic to the configuration space of systems with two non-cyclic degrees of freedom, such as double-prismatic mechanisms, or two beads moving on a non-looped wire.[6] Replacing one of the $\mathbb{R}^1$ components of the plane with a circle generates the cylinder $\mathbb{R}^1 \times \mathbb{S}^1$, shown in Figure 1.4. Following the same pattern as with other configuration spaces, this space may represent a point moving over a literal cylinder or a mechanical system such as a prismatic joint in combination with an unrestricted rotational joint. For ease of representation, the cylinder is often depicted in a flattened view. In this view, the ">" symbols indicate that the two "edges" formed by "cutting" the cylinder are in fact the same line. Two cyclic elements together, such as a pair of rotational joints, form the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ illustrated in Figure 1.5. As with the cylinder, it is often convenient to work with the torus opened out into a planar section with opposite edges identified.

The torus is topologically distinct from the sphere $\mathbb{S}^2$, even though they are both two-dimensional closed spaces. This distinction may be expressed at a variety of technical levels. Most intuitively, embeddings of the torus have "holes" in them, but spheres do not, as illustrated in Figure 1.6. More formally, the lack of holes on the sphere means that it is *simply connected*, i.e., that any loop on it can be continuously contracted to a point

[5] See the box on the facing page.
[6] For the sake of example here, we assume the beads can pass through each other. If they cannot, then their configuration space would be half of $\mathbb{R}^2$, with the partition line running diagonally to the primary axes.

Direct Product

The *direct product*, or *Cartesian product* of two sets or spaces combines them without mixing the elements together. For instance, if we have two systems with configurations $a \in \mathbb{R}^1_a$ and $b \in \mathbb{R}^1_b$, the direct product of the configuration spaces, $\mathbb{R}^2 = \mathbb{R}^1_a \times \mathbb{R}^1_b$, is structured to preserve the independence of the component subspaces,

$$(a, b) \in \mathbb{R}^2 \equiv (a \in \mathbb{R}^1_a, b \in \mathbb{R}^1_b). \tag{1.i}$$

The direct product stands in contrast to the *semi-direct product* discussed on page 18.

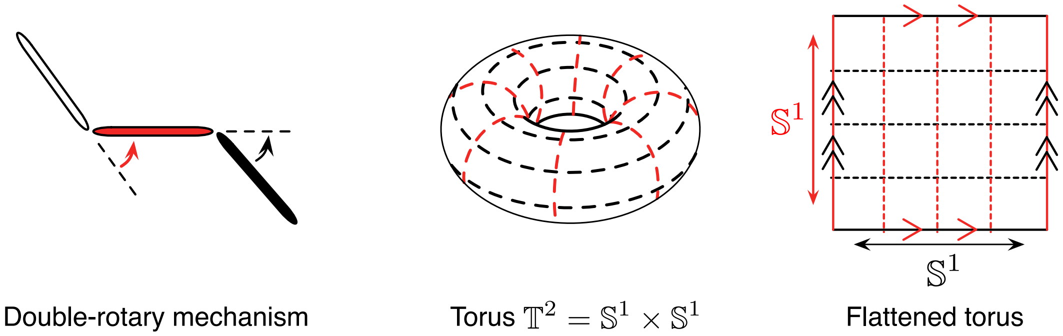


Figure 1.5 The torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, an example system it represents, and its flattened representation.

(and therefore that any loop can be continuously deformed into any other loop). By contrast, loops on the torus that wrap around one of its fundamental $\mathbb{S}^1$ spaces cannot be continuously contracted to a point or deformed into loops that wrap around the other fundamental $\mathbb{S}^1$ space.

Configuration spaces for systems with more than two degrees of freedom may have more complex structure than the examples discussed above, but follow the same general principles. In §1.3, we will examine combining lines and circles to build configuration spaces for rigid bodies moving in the plane, and in Chapter 3, we will extend this approach to articulated systems.

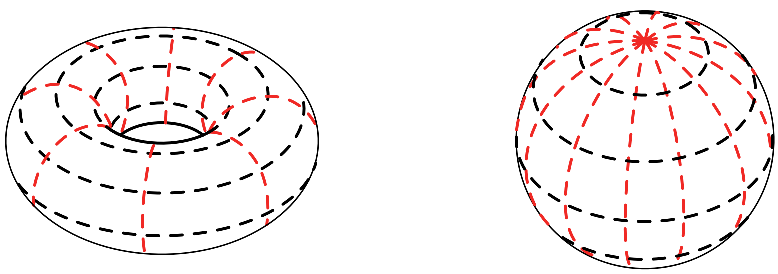


Figure 1.6 The torus and the sphere. The torus has two independent, noncontractable loops, whereas any loop on the sphere can be smoothly deformed into any other loop on the sphere. Note that the illustrated shapes are the embeddings of the torus and sphere into three dimensions; the manifolds themselves are two-dimensional.

1.2 Configuration Groups

In many cases, it is useful to perform algebraic operations, such as addition or subtraction, on configurations. We may, for instance, want to know the absolute configuration of an object whose position is specified relative to another (combining the configurations), or to find the relative configurations of two objects whose position is known absolutely (taking their difference). Executing these additions and subtractions requires interpreting configurations not only as points in the configuration space, but also as transformations that can be applied to other configurations. This dual interpretation corresponds to the mathematical concept of a *group*.

A group is the combination of a set (which could be a manifold) and an operation, such that the products of members acting on each other via the operation are themselves within the set.[7] Group structure is often implicit and underlies many basic mathematical concepts, such as the addition of real numbers (the group $(\mathbb{R}, +)$, in which the inverse operation is subtraction), the multiplication of positive reals (the group $(\mathbb{R}_+, \times)$, in which the inverse operation is division), or the addition of integers (the group $(\mathbb{Z}, +)$).

For the basic manifolds we considered above, the commonly encountered group operations are addition on the line and modular addition on the circle, respectively producing the groups $(\mathbb{R}^1, +)$ and $(\mathbb{S}^1, +)$. The group properties match an intuitive understanding of how addition works on these spaces: configurations represent translational offsets through the space, so the natural combination of two configurations is the sum of their offsets from a chosen origin. Combinations of lines and circles created via the direct product naturally inherit the group structures of their component spaces, with (modular) addition acting independently along each degree of freedom. Note, however, that there are other means of constructing groups in which the system degrees of freedom become coupled. Two common examples of such group structures are the group of planar translations and rotations a rigid body may make, which we discuss in §1.3, and the group of three-dimensional rotation, discussed in §7.2.

An important principle when working with groups is that we can associate any point on the underlying set with the group's identity element, and that this freedom corresponds to our freedom in selecting the origin when parameterizing the set. Further, changing the group identity element is equivalent to finding the group action that moves from the old to new choice of origin and applying its inverse to each point on the manifold. Together, these principles make groups a powerful tool for expressing the relative relationships among points in a space *as seen from the perspective of each point*.

1.2.1 Group Operations

When we assign a group structure G to a configuration manifold Q, we are associating a set of *displacements* (the group) with a set of *positions* (the configuration space). A critical step in this process is designating an origin or reference point $q_\mathbf{e}$ on the manifold. Once $q_\mathbf{e}$ has been selected, each other point q can be associated with the group element g that encodes its displacement from $q_\mathbf{e}$. The origin point $q_\mathbf{e}$ itself corresponds to the group's identity element $\mathbf{e}$, which encodes "null" displacements.

If we compose two group elements as

$$g_1 \circ g_2 = g_3, \tag{1.1}$$

the resulting group element g_3 combines their displacements. As illustrated in Figure 1.7, g_3 can be interpreted as identifying a point q_3 whose position relative to q_1 (as measured by the group action) is the same as that of q_2 relative to the origin. We can find the difference between two group elements (and thus the relative displacement between their associated points) by making compositions with inverse group elements. For example, the groupwise difference between two elements g_1 and g_3 is the *inverse composition*

fix left/right discrepancy between text and figures

$$g_\Delta = g_1^{-1} \circ g_3, \tag{1.2}$$

and represents the action required to move between their associated points q_1 and q_3. Because it is a group element in its own right, g_Δ also corresponds to the position of a point on the manifold; if we incorporate the relationship in (1.1), then $g_\Delta = g_2$ in this instance, and corresponds to q_2. These relationships are shown for circular addition and the multiplication of real numbers in Figure 1.8.

[7] Group structure also requires several other properties, as described in the box on the next page.

Groups

Groups. A *group* $(G, \circ)$ is the combination of a set G and an operation $\circ$ that satisfies the following properties:

1. **Closure**: The product of any element of G acting on another by the group operation must also be an element of G. More formally, for $g_1, g_2 \in G$,
$$\begin{aligned}(g_1 \circ\ ) &: G \to G \\ g_2 &\mapsto g_1 \circ g_2.\end{aligned} \tag{1.ii}$$
2. **Associativity**: The order in which a sequence of group operations are evaluated must not affect their net product: for all $g_1, g_2, g_3 \in G$,
$$g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3. \tag{1.iii}$$
3. **Identity element**: The set must contain an identity element $\mathbf{e}$ that leaves other elements unchanged when it interacts with them: for $g \in G$, there exists $\mathbf{e} \in G$ such that
$$\mathbf{e} \circ g = g = g \circ \mathbf{e}. \tag{1.iv}$$
4. **Inverse**: The inverse (with respect to the group operation) of each group element must be an element of the group and produce the identity element when operating on or operated on by its respective element: for $g \in G$, there must exist $g^{-1} \in G$ such that
$$g^{-1} \circ g = \mathbf{e} = g \circ g^{-1}. \tag{1.v}$$

When the choice of group action is unambiguous, as in the case of the $SO(n)$ and $SE(n)$ groups discussed below, a group may be referred to simply by its underlying set. As additional shorthand notation, the $\circ$ symbol is often dropped, with $g_1 g_2$ being read as equivalent to $g_1 \circ g_2$.

Common group actions. Commonly encountered group actions include addition $(+)$, for which the identity is a zero element and the inverse operation is subtraction; and multiplication $(\times)$, for which the identity has unit value and the inverse is division. Additive group actions are *commutative*, meaning that the left-right order of the group elements in a sequence does not affect the result of their composition,
$$g_1 + g_2 = g_2 + g_1. \tag{1.vi}$$
Multiplicative group actions may be commutative, but are often not, giving rise to the notion of "left" and "right" group actions.

Left and right actions. If the order of operation is important (as in the case of most matrix multiplications), any group action may be conducted as a *left action*,
$$L_g : h \mapsto g \circ h \tag{1.vii}$$
or a *right action*,
$$R_g : h \mapsto h \circ g \tag{1.viii}$$
with the acting element placed correspondingly at the beginning or end of the execution sequence. For the groups we consider in this book, we use a convention in which the left transformation $L_{g_2}(g_1)$ is interpreted as "moving the group element g_1 by g_2," whereas the right action $R_{g_2}(g_1)$ is used to find "the group element at g_2 relative to g_1."

continued . . .

Groups, *continued*

The basic concept of a group—a set with an operation for combining its elements—extends to several other mathematical structures, including *rings* and *fields*.

Rings. A *ring* $(R, +, \cdot)$ is the combination of a set R with two operations: one $(+)$ that acts like addition, and a second $(\cdot)$ that acts like multiplication. These operations satisfy the following properties:

1. **Addition**: A ring forms a commutative group under addition, with closure, associativity, and the existence of identity and inverse elements. The additive identity is referred to as the *zero element* of the ring, or "**0**."
2. **Multiplication**: A ring is closed and associative under multiplication, but does not necessarily contain the multiplicative inverse of each element or a multiplicative identity element. If a multiplicative identity exists, it is referred to as the *unit element* of the ring, or "**1**."
3. **Distributivity**: The multiplicative operation must distribute over the additive operation, so that the product of a sum is equal to the sum of the individual products: for $a, b, c \in R$,

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c). \tag{1.ix}$$

For example, the real numbers $\mathbb{R}$ form a ring under the standard addition and multiplication operations:

- Elements of $\mathbb{R}$ form the group $(\mathbb{R}, +)$ under addition,
- Multiplication of real numbers produces real numbers and is associative, and
- Multiplication distributes over addition.
- The real numbers do not form a group under multiplication, as the inverse of 0 is not a real number, but this property is not required for ring structure.

Other examples of rings include the complex numbers (under addition and complex multiplication) and square matrices (under elementwise addition and matrix multiplication).

Fields. A *field* is a ring for which the inverse of every non-zero element is also an element. Both the real and complex numbers form fields, but the ring of square matrices does not—it includes singular matrices (containing linearly dependent rows) that do not have a well-defined inverse.

Note that this meaning of "field" is unrelated to terms like "vector field" (an assignment of a vector to each point in an underlying space, as discussed in §2.1.2). The term "scalar field" in particular should be approached with caution, as it has two distinct meanings:

- A field, in the sense here of a special class of group or ring, whose elements can act as *scalars* for a vector space as discussed on page 27.
- By back-formation from "vector field," a function that assigns a single real number to each point in the domain.

In German (the language in which the concept of a field was first defined), fields are called "körper," which translates to "body" or "corpus."

A common convention when modeling systems via groups is to assume a choice of origin, drop the distinction between positions and displacements, and directly refer to the system's configuration space as G. Even within such a convention, it is often useful to select a new reference point on the configuration manifold, and express the location of points with respect to the new origin. Moving the origin in this way changes the group element associated with each point, but does not affect the groupwise displacement between pairs of points. These properties are illustrated in Figure 1.9, where we have designated our original placement of the identity element as $\mathbf{e}_a$, and selected a new origin/identity placement $\mathbf{e}_b$ at q_2. The new origin point is counterclockwise of q_1—changing the sign of $g_{1,b}$ as compared to $g_{1,a}$—and is closer to q_3—making $g_{3,b}$ a smaller displacement than $g_{3,a}$. Because q_1 and q_3 have not themselves moved, however, $g_{\Delta b}$ is the same displacement as $g_{\Delta a}$.

These effects correspond to the principle that (absent an external frame of reference) displacing the reference point by a given group action $g_{b,a}$ is equivalent to displacing each point on the configuration manifold by the inverse of that action, which can be represented symbolically as

$$g_{i,b} = g_{b,a}^{-1} \circ g_{i,a}. \tag{1.3}$$

Modifying the group actions in this way propagates into their composition,

$$g_{1,b} \circ g_{3,b} = (g_{b,a}^{-1} \circ g_{1,a}) \circ (g_{b,a}^{-1} \circ g_{3,a}), \tag{1.4}$$

which, if the group is commutative, becomes

$$g_{1,b} \circ g_{3,b} = (g_{b,a}^{-1})^2 (g_{1,a} \circ g_{3,a}). \tag{1.5}$$

In an inverse composition (regardless of whether or not the group is commutatitve), the change-of-identity transformation self-cancels, leaving the group difference unchanged:

$$\begin{aligned} g_{\Delta b} &= g_{1,b}^{-1} \circ g_{3,b} && (1.6)\\ &= (g_{b,a}^{-1} \circ g_{1,a}^{-1}) \circ (g_{b,a}^{-1} \circ g_{3,a}) && (1.7)\\ &= (g_{1,a}^{-1} \circ \overbrace{g_{b,a}) \circ (g_{b,a}^{-1}}^{\mathbf{e}} \circ g_{3,a}) && (1.8)\\ &= g_{1,a}^{-1} \circ g_{3,a} && (1.9)\\ &= g_{\Delta a}. && (1.10) \end{aligned}$$

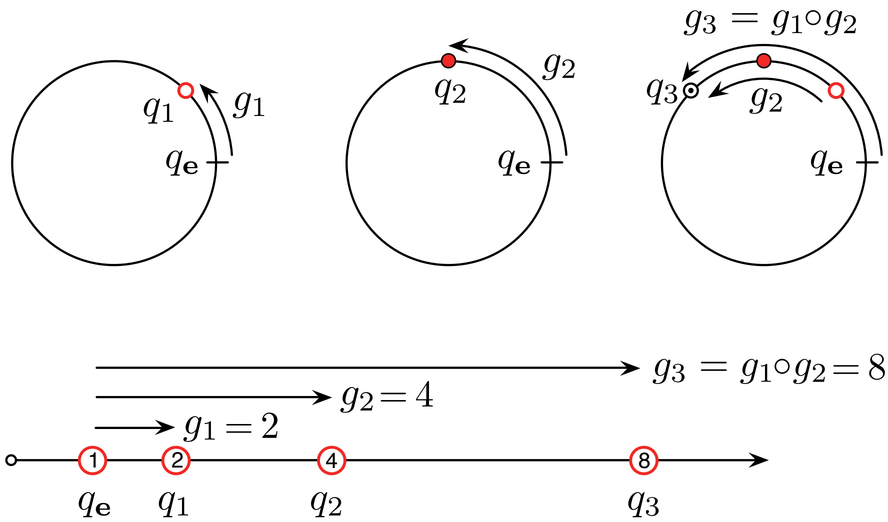


Figure 1.7 Composition of group elements on $(\mathbb{S}^1, +)$ and $(\mathbb{R}_+, \times)$. Augmenting a manifold with group structure allows points on a manifold to be combined by treating them as composable displacements from a chosen origin. Here, each point q is associated with a group element g that encodes its groupwise displacement from a designated origin point $q_\mathbf{e}$. If we interpret the composition of two displacements as a location on the manifold, the resulting point's displacement from q_1 is the same as that of q_2 from the origin.

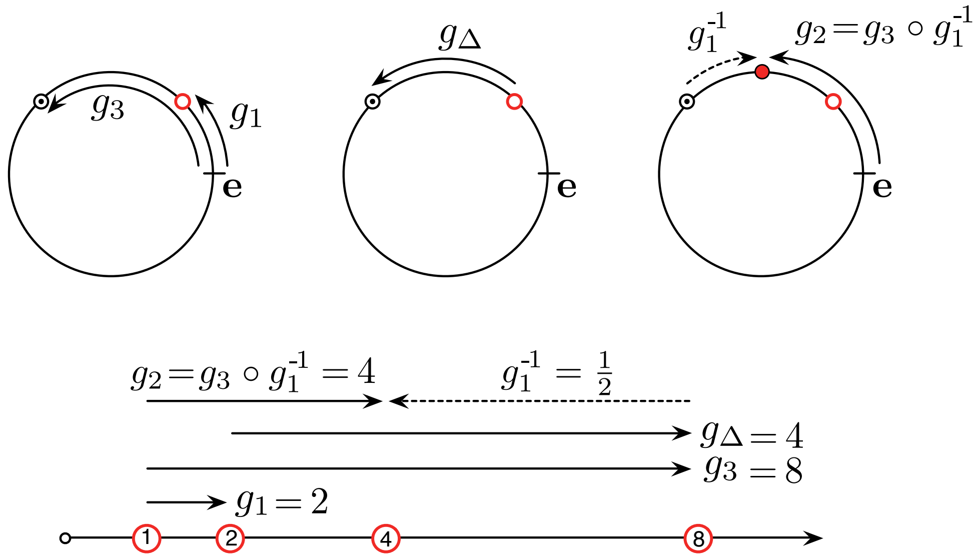


Figure 1.8 Inverse composition of group elements on $(\mathbb{S}^1, +)$ and $(\mathbb{R}_+, \times)$. Inverse group actions, which correspond to operations like subtraction and division, allow us to compare two group elements. Here, g_Δ (the difference between g_1 and g_3) is equal to that encoded by the group element $(g_3 \circ g_1^{-1})$. Because we have used the same group elements as in Figure 1.7, this difference is also equal to g_2.

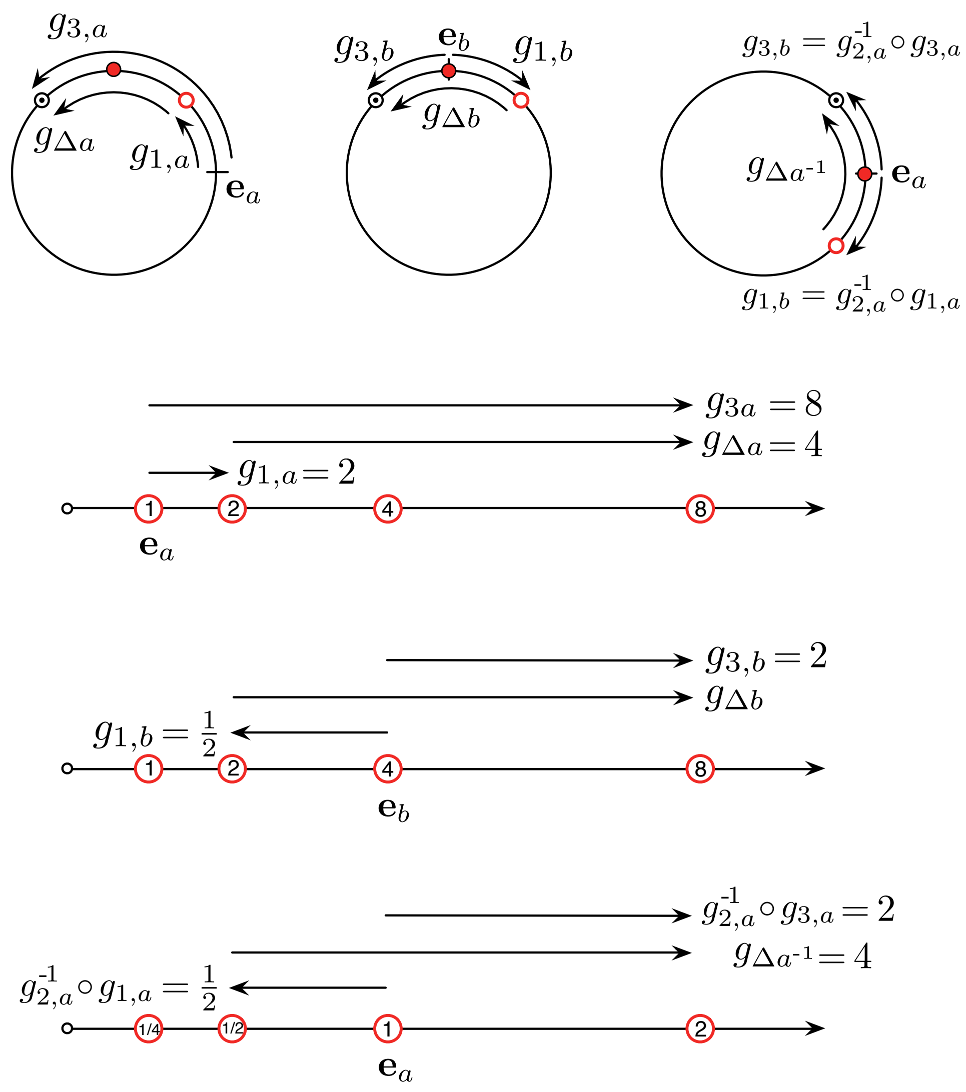
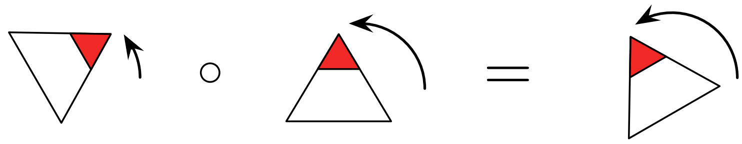


Figure 1.9 Designating a new origin on the manifold changes the displacement associated with each point. This change affects the results of composing two group elements—here, with the new identity $\mathbf{e}_b$ at q_2 (directly between q_1 and q_3), the composition of $g_{1,b}$ and $g_{3,b}$ is the identity. It does not, however, affect the result of taking inverse actions—the points remain separated by the same displacement. The effect of moving the origin by a given transformation is equivalent to transforming all the points on the manifold by the inverse of that transformation. In the illustrated examples, this inverse precomposition rotates the points a quarter turn around the circle and moves each point on the real number line to the point at a quarter of its original value.



$$\begin{bmatrix}\cos\frac{\pi}{6} & -\sin\frac{\pi}{6}\\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6}\end{bmatrix} \times \begin{bmatrix}\cos\frac{\pi}{2} & -\sin\frac{\pi}{2}\\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2}\end{bmatrix} = \begin{bmatrix}\cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3}\\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3}\end{bmatrix}$$

Figure 1.10 The abstract group of planar rotations is isomorphic to $SO(2)$, the group of 2×2 matrices with determinant 1. Representing the rotations via the matrix group allows us to compute the result of composing multiple rotations.

1.2.2 Group Representation

The *general linear group* of $n \times n$ invertible matrices, $GL(n)$, plays an important role in studying groups. These matrices form a group under matrix multiplication, and are particularly significant because many abstract groups are *isomorphic*[8] to $GL(n)$ or a subgroup within it. This isomorphism allows us to numerically *represent* elements of the abstract group as matrices of real numbers by *embedding*[9] the group in an $\mathbb{R}^{(n^2)}$ space, and then compute the effect of group actions via matrix multiplication. For example, the group of planar rotations is isomorphic to $SO(2)$, the group of 2×2 matrices with a positive unit determinant. As illustrated in Figure 1.10, we can use this isomorphism to compute the result of composing two rotations together: multiplying their respective matrix representations produces the matrix representation of the composed rotations.

Identifying a representation of a group is analogous to, and an extension of, parameterizing the set or manifold that underlies the group. If we take parameterization as associating real numbers with points on an abstract manifold, then identifying an isomorphic matrix group provides *a computable structure relating those numbers under the group action*. In the following text, we will not make a strong distinction between elements of a group and their matrix representations, in the same way that we do not make a strong distinction between points on a manifold and the parameter values that identify them. Where such a distinction is necessary, we will identify representation maps from abstract groups to their matrices as ρ, following the pattern of designating parameterization maps as ϕ and transition map as τ.

Canonically, group representation functions are selected so that the origin of the parameter space corresponds to the identity element of the group,

$$\rho(\mathbf{e}) = \rho(\{g_i = 0\}) = I \tag{1.11}$$

for a set of group parameters g_i. This choice of representation is natural for many groups—in the rotations of Figure 1.10, for example, having

$$\rho(g) = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} \tag{1.12}$$

with $\rho(\mathbf{e})$ a 2×2 identity matrix makes $\mathbf{e}$ equivalent to $\theta = 0$. Similarly, the translation-rotation matrices we consider in §1.3 naturally become identity matrices at the origin of their parameter spaces.

For other groups, however, this construction may be less intuitive. A good example of this case is the multiplicative group $(\mathbb{R}_+, \times)$. It would be straightforward to represent this group as

$$\rho(g) = [g] \tag{1.13}$$

(where $[g]$ indicates that the right-hand side is a 1×1 matrix, not a scalar), but this representation is the identity when $g = 1$, which doesn't satisfy the condition in (1.11). To generate a canonical representation of this group in which the identity element is at the origin (zero-value) of the parameter space, we can introduce a

[8] See the box on page 16.
[9] See the box on the facing page.

Embedding

Embedding. *Embedding* is an operation that allows one mathematical structure to be treated as a subset of another. As an noun, "an embedding" most strictly describes the function that maps the first structure into the second, but is also used to refer to the function's *image*—the set of points in the second structure corresponding to the whole of the first structure. Making an embedding allows mathematical tools defined on the enclosing structure to be applied to the embedded structure. For example, embedding the circle into $\mathbb{R}^2$ or the sphere into $\mathbb{R}^3$ allows the use of projective geometry to draw these manifolds on the page, and more generally enables us to perform computations about points on the manifold using real numbers.

By linking abstract spaces to the real numbers, embeddings of manifolds into $\mathbb{R}^n$ serve a similar purpose to coordinate charts, but with a key difference: Whereas a manifold's charts are of the same dimension as the manifold itself but multiple charts may be necessary to capture its structure, an embedding of the manifold captures the whole structure at once but may require that the enclosing space have more dimensions than the manifold in order to do so.

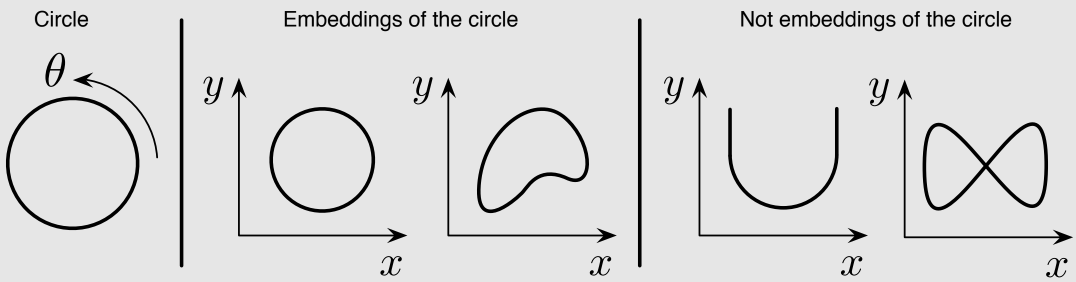
For instance, the circle $\mathbb{S}^1$—a one-dimensional manifold—cannot be embedded in the dimension-matched $\mathbb{R}^1$ space, but can be embedded in $\mathbb{R}^2$, where the extra dimension provides a means of encoding the way in which $\mathbb{S}^1$ connects back on itself. Similarly, the sphere $\mathbb{S}^2$ and torus $\mathbb{T}^2$ cannot be embedded in $\mathbb{R}^2$, but can be embedded in $\mathbb{R}^3$. The number of extra dimensions needed for an embedding depends on the manifold structure, but (by the *Whitney embedding theorem*), no more than $2m$ dimensions are ever required to enclose an m-dimensional manifold.

Formal definition. To qualify as an embedding of a structure A into space B, a function $E : A \to B$ must be:

1. **Injective.** Every point in A must be associated with a unique point in B, and
2. **Structure-preserving.** Structural properties of A, such as connectivity, must be carried over into its embedding $E(A)$.

Having these properties requires that A and $E(A)$ are at least homeomorphic[a] to each other; in many useful embeddings the two structures are additionally diffeomorphic or isomorphic[b] to each other.

Example. The circle $\mathbb{S}^1$ is a one-dimensional closed manifold. It can be embedded into a two-dimensional space as any closed, non-selfintersecting curve. Neither an open curve nor a figure-eight may be taken as an embedding of the circle: The map from the circle to the open curve is discontinuous, and the crossing point on the figure-eight is associated with two points on the circle, so neither is homeomorphic to the circle. Note that although the figure-eight is not an embedding of the circle, it does qualify as an *immersion*[c] of the circle as long as the two lobes are non-parallel at their crossing point.



[a] See the box on page 4.
[b] See the box on the following page.
[c] See the box on page ??.

Isomorphism

An *isomorphism* is a structure-preserving relationship between two mathematical objects. Two groups A and B are considered isomorphic if there exists a bijective function

$$f : A \to B \tag{1.x}$$
$$a \mapsto b \tag{1.xi}$$

such that

$$f(a_1 \circ a_2) = f(a_1) \circ f(a_2), \tag{1.xii}$$

and (because of the bijectivity requirement),

$$f^{-1}(b_1 \circ b_2) = f^{-1}(b_1) \circ f^{-1}(b_2). \tag{1.xiii}$$

Two simple examples of group isomorphisms are

1. The relationship between the multiplicative group of positive real numbers, $(\mathbb{R}_+, \times)$, and the additive group of real numbers, $(\mathbb{R}, +)$. The natural logarithm and exponential functions isomorphically map between these two groups: for $x, y \in (\mathbb{R}_+, \times)$,

$$\log(xy) = \log x + \log y \tag{1.xiv}$$

   and for $a, b \in (\mathbb{R}, +)$

$$\exp(a + b) = (\exp a)(\exp b) \tag{1.xv}$$

2. The relationship between the modular additive group of numbers on the circle, $(\mathbb{S}^1, + \bmod 2\pi)$, and the group of two-dimensional rotation matrices, $SO(2)$. Under the matrix construction function for elements of $SO(2)$,

$$R : \mathbb{S}^1 \to SO(2) \tag{1.xvi}$$
$$\theta \mapsto \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{1.xvii}$$

   addition of $\mathbb{S}^1$ elements is equivalent to multiplications of $SO(2)$ elements,

$$R(\theta_1 + \theta_2) = R(\theta_1)R(\theta_2). \tag{1.xviii}$$

parameterization $\gamma = g - 1$ for the group, such that the representation

$$\rho(g) = [1 + \gamma], \tag{1.14}$$

has its identity element at $\gamma = 0$.

1.3 Rigid Body Configurations

With basic definitions of configuration spaces and groups in place, we can now turn our attention to a major theme of this book: the motion of rigid bodies. First, what is a rigid body? Physically, it is an object that does not deform in response to external forces. Mathematically, it is a (possibly continuous) set of points with fixed interpoint distances; a *proper* rigid body also has a fixed handedness, and so can be translated and rotated, but not reflected. A more useful mathematical definition, illustrated in Figure 1.11, is that a rigid body is composed of a movable reference frame (the *body frame*) and a set of points whose positions are fixed with respect to the body frame. Under this last definition, it becomes clear that the configuration (position and orientation) of the body frame completely defines the location of all points in the body, and therefore that we can treat the configuration spaces of the body and the frame equivalently.

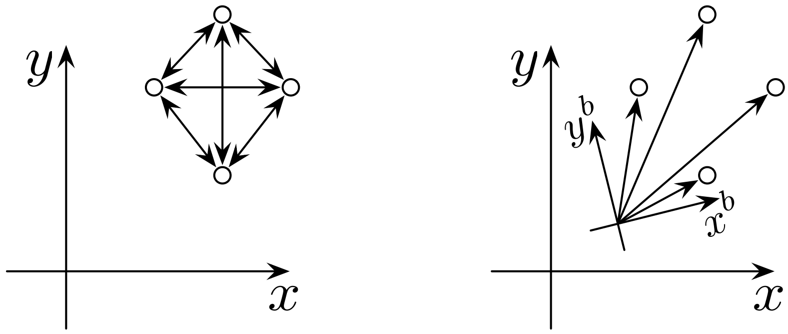


Figure 1.11 A rigid body can be defined either as a set of points at fixed distances from each other (left) or more usefully as a set of points at fixed locations in a movable body frame (right).

When modeling a rigid body, we have some freedom in our selection of the body frame, corresponding to our freedom to select different parameterizations for the system. For instance, we may place the body frame at the center of mass of an object, at one end, or even, as illustrated at the right of Figure 1.11, at a point that is not actually included within the physical bounds of the body, so long as the positions of all points in the body are constant with respect to the chosen frame.[10]

Once a body frame has been selected, the question arises of what configuration space and Lie group structure best describes its configuration. In the case of bodies moving in the plane, for which the frame is defined by its two-dimensional position and a single orientation component, it might be natural to take these components independently and use an $(\mathbb{R}^2 \times \mathbb{S}^1, +)$ configuration group. There are several advantages, however, to instead describing the frame's configuration via the *special Euclidian group* on 2-dimensional space, $(SE(2), \times)$. Most relevantly to our current discussion, $SE(2)$ better captures the *relative* motion of rigid bodies than does $(\mathbb{R}^2 \times \mathbb{S}^1, +)$, giving rise to *symmetries* that we exploit in future chapters.[11] Before discussing this point further, we will examine the details of working on $SE(2)$.

1.3.1 The Special Euclidean Group $SE(2)$

The special Euclidean group in two dimensions, $SE(2)$ is the set of *proper rigid* (non-reflecting and non-distorting) transformations in the plane. It includes translations and rotations, while excluding shearing, scaling, and mirroring motions. An element $g \in SE(2)$ can be parameterized as (x, y, θ), with two components for translation and one for rotation, and is typically represented as a *homogeneous matrix*,

$$g = (x, y, \theta) = \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix}, \tag{1.15}$$

with the group action is taken as matrix multiplication. This structure, containing an $SO(2)$ element for orientation and an $\mathbb{R}^2$ element for position, represents the *semi-direct product*[12] $SO(2) \circledS \mathbb{R}^2$, in which rotations act on each other and on the translational components, but the translational components only act on each other: for group elements $g, h \in SE(2)$, with $g = (x, y, \theta)$ and $h = (u, v, \beta)$, the group composition is

$$gh = \begin{bmatrix} R(\theta) & \begin{pmatrix} x \\ y \end{pmatrix} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} R(\beta) & \begin{pmatrix} u \\ v \end{pmatrix} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} R(\theta)R(\beta) & R(\theta)\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix} \\ \mathbf{0} & 1 \end{bmatrix}. \tag{1.16}$$

[10] Some problems do become easier to solve under certain choices of body frame, as discussed further in §5.6.

[11] Additionally, although we will not consider this until Chapter 7, $SE(n)$ generalizes much better to three dimensional motion than does $(\mathbb{R}^2 \times \mathbb{S}^1, +)$.

[12] See the box on the following page.

Direct and Semi-direct Products of Groups

When we combine groups together to form larger groups, we must consider not only how their underlying sets combine, but also what the overall group action becomes. Often, the combination of two groups A and B into a new group C is taken to mean the creation of a *direct product group*

$$C = A \times B, \tag{1.xix}$$

in which components that started out in A or B only affect other components that started out as elements of the same group, *i.e.*, $c_1c_2 = (a_1a_2, b_1b_2)$. Direct products preserve properties such as being abelian (commutative) – if A or B has this property, then so does the corresponding section of C.

In a *semi-direct product group*,

$$D = A \circledS B, \tag{1.xx}$$

which may also be written as

$$D = A \rtimes B \tag{1.xxi}$$

(with a bar connecting the rightmost points of the times symbol), elements of A act not only on each other, but also on elements of B. For example, the structure of $SE(2)$ takes the form

$$d_1d_2 = \big(b_1(a_1b_2), a_1a_2\big). \tag{1.xxii}$$

A key aspect of such groups is that even though they do not possess the full orthogonality of a direct product group, the A components do preserve their original properties, and thus results that depend on these properties can be applied to the corresponding elements of D.

As we discuss further below, an important aspect of $SE(2)$ is that it is *noncommutative*—changing the order in which we compose the group elements from gh to hg changes their resulting product:

$$hg = \begin{bmatrix} R(\beta) & \begin{pmatrix} u \\ v \end{pmatrix} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} R(\theta) & \begin{pmatrix} x \\ y \end{pmatrix} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} R(\beta)R(\theta) & R(\beta)\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix} \\ \mathbf{0} & 1 \end{bmatrix}. \tag{1.17}$$

1.3.2 Interpreting Elements of $SE(2)$

Elements of $SE(2)$ have three common (and related) interpretations for planar systems:

1. The position and orientation of a rigid body (or a frame), relative to a choice of origin represented by the identity element of the group.
2. A translation and rotation of a rigid body, relative to the chosen origin.
3. A translation and rotation of a rigid body, relative to its current position and orientation.

The first interpretation corresponds to the inherent nature of $SE(2)$'s underlying set as the space of possible positions and orientations forming a smooth mathematical space with two non-cyclic and one cyclic component. Interpretations two and three capture the group nature of $SE(2)$, in which elements act on one another to produce new elements of the group. The presence of two interpretations related to group actions highlights a feature of groups that did not appear in §**??**: groups have both a *left action* and a *right action*. Additive groups and scalar multiplication, such as we considered in §1.2, are abelian (commutative), making the left/right distinction disappear, but sequences of relative translation and rotation do not commute, so the two actions take on distinct meanings on $SE(2)$.

The left action of a group element $g \in SE(2)$ transforms other group elements by first rotating them around the origin, and then by translating them along the absolute x and y axes. These transformations, which are

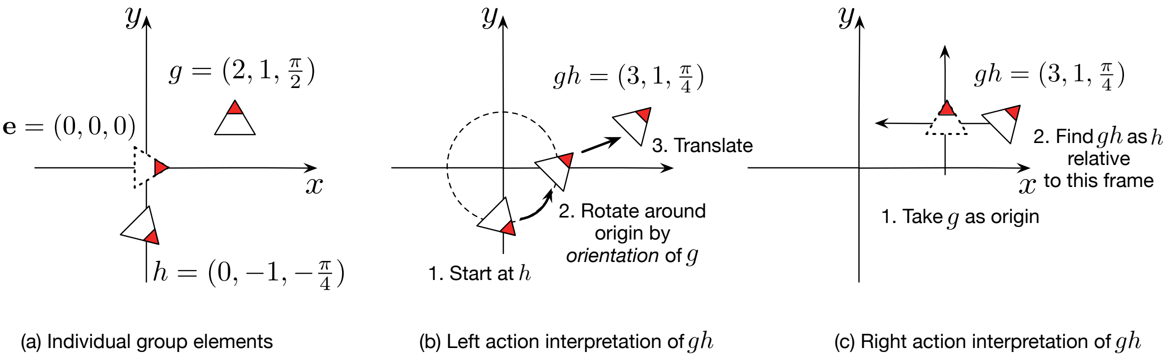


Figure 1.12 Group actions on $SE(2)$.

illustrated in Figure 1.12(b), can be seen in (1.16)'s expansion of the group composition: $R(\theta)$ is applied to both $R(\beta)$ and $\binom{u}{v}$, and $\binom{x}{y}$ is incorporated directly into the position component of gh.

The right action of a group element h on a second element g finds the position of gh by taking g as the starting point and finding the group element whose position relative to g is h. This action, illustrated in Figure 1.12(c), can also be seen in (1.16): In the orientation component, $R(\beta)$ is a rotation appended onto $R(\theta)$, and the $R(\theta)$ applied to $\binom{u}{v}$ in the position component means that u and v are taken as displacements along a rotated set of axes.

In these examples, we have worked with the left action of g and the right action of h to emphasize that the left and right group actions have different meanings on non-commutative groups like $SE(2)$ and that *as long as the sequence order of the group elements is preserved* the same result is reached for treating gh as the result of a left or right action. We leave it as an exercise to the reader to show that hg can be similarly calculated as a left or right action and that—outside of trivial cases—this quantity is different from gh. Note that when $h = g^{-1}$, both the left and right action interpretations return the identity element of $SE(2)$, *i.e.* the origin of the space, as

$$g^{-1}g = gg^{-1} = \mathbf{e} \equiv (0, 0, 0) \equiv I. \tag{1.18}$$

1.3.3 Applications of $SE(2)$ to Physical Objects

In the context of physical objects moving through space, right actions on $SE(2)$ represent rigid connections between two frames or displacements taken relative to an object's current configuration: two frames with a fixed relative displacement—e.g., because they are attached to the same rigid body as in Figure 1.13(a)—are always separated by the same right action, independent of their absolute location. Similarly, a motion like "move forward, right, and spin counterclockwise" is always represented by the same right action, regardless of the starting position, as illustrated in Figure 1.13(b). The start and end points of these actions can be exchanged by taking their inverses: where h describes the configuration of B relative to A, its inverse h^{-1} gives the configuration of A relative to B, as illustrated in Figure 1.13(c). Likewise, the right action of h^{-1} encodes the motion that "undoes" a motion encoded by h.

Applying a left $SE(2)$ action to multiple frames moves them through the world while preserving their relative displacements. This means that if we know how frame A moves in Figure 1.13(a), we can apply the same left action g to find the new position of frame B that maintains the rigid connection (right action) encoded by h. This same principle carries over to locally-defined motions like those in Figure 1.13(b): when two different starting points for a given motion are separated by a left action g, their ending points are also separated by the left action of g.

Left inverse actions on $SE(2)$ are useful for extracting relative displacements between two objects from their known absolute positions. As illustrated in Figure 1.13(d), if we have one object known to be at position g and a second object at an (unknown) relative position h, we can call its (known) absolute position gh. If we then apply a left g^{-1} action to both configurations, the first object is moved to the origin and the g component is stripped out of the position of the second object, leaving only the (previously-unknown) relative displacement

Associativity and Commutativity

Associativity. Associativity means that once the placement of elements in an expression has been assigned (e.g., a acting on b acting on c), the order of resolution doesn't matter: For binary operations with infix notation, associativity appears as the ability to arbitrarily group the elements of a multi-operator expression,

$$a \circ (b \circ c) = (a \circ b) \circ c, \tag{1.xxiii}$$

and for function composition (which is always associative), it appears as the statement that precomposing two functions, then applying them to an input is equivalent to applying them to the input in succession,

$$\big(f_2 \circ f_1\big)(a) = f_2(f_1(a)). \tag{1.xxiv}$$

Commutativity. Commutativity means that elements in an expression can exchange positions without affecting the result of the expression. Binary operations are *commutative* if their operands (inputs to the operation) can be swapped for all values of a and b in the domain of the operation,

$$a \circ b = b \circ a. \tag{1.xxv}$$

For operations that are not commutative, specific elements of the domain may still *commute* with each other if they satisfy (1.xxv). For example, any pair of pure-translation elements of $SE(2)$ commute with each other, as

$$(x, y, 0) \circ (u, v, 0) = (x + u, y + v, 0) = (u, v, 0) \circ (x, y, 0), \tag{1.xxvi}$$

and a similar rule holds for pure-rotation elements.[a] Pairs of functions commute with each other if they are commutative with respect to the composition operation,

$$f_1 \circ f_2 = f_2 \circ f_1, \tag{1.xxvii}$$

or, in functional notation,

$$f_2(f_1(a)) = f_1(f_2(a)) \tag{1.xxviii}$$

Commutativity of Left and Right actions. Combining the associativity of group operations with the functional forms of the left and right group actions[b] leads to an interesting and powerful property of group actions: left actions applied to a group element commute with right actions applied to that element,

$$\begin{aligned} L_g(R_h(\mathbf{e})) &= g \circ (\mathbf{e} \circ h) && \text{(1.xxix)} \\ &= (g \circ \mathbf{e}) \circ h && \text{(1.xxx)} \\ &= R_h(L_g(\mathbf{e})) && \text{(1.xxxi)} \end{aligned}$$

(where $\mathbf{e}$ is the group's identity element, and could be replaced by any other element without affecting the equality). This property, which is illustrated in Figure 1.12 and Figure 1.13(a–b), plays an important role in understanding velocities on groups, which we consider in §2.2.

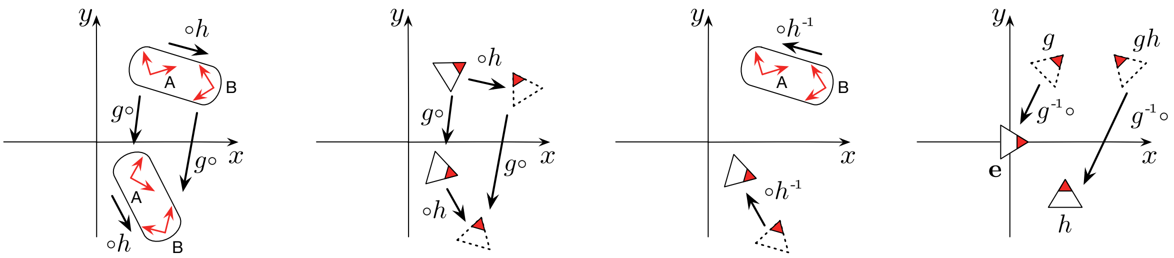
continued . . .

[a] This relates to $SE(2)$'s structure as the semi-direct product of two commutative groups, as discussed on page 18
[b] See the box on page 9.

Associativity and Commutativity, *continued*

Commutative Diagrams. Commutative diagrams are a useful tool for identifying transformations that commute, without getting caught up in the differences between infix and functional notation. The core idea of a *diagram* (which originates in a branch of mathematics called *category theory*) is that transformations can be represented by arrows, and a sequence of operations can then be represented by chaining arrows together. The resulting diagram commutes (or "is commutative") if all directed paths with the same start and end point produce the same result; pairs of transformations commute with each other if we can construct parallel paths in which the order of their arrows is exchanged while preserving the commutativity of the diagram. The commutative diagram for left and right group operations is shown below.

$$
\begin{tikzcd}
\mathbf{e} \arrow[r, "g\circ"] \arrow[d, "\circ h"'] & g \arrow[d, "\circ h"] \\
h \arrow[r, "g\circ"'] & gh
\end{tikzcd}
$$



(a) Group actions on and between rigid bodies

(b) Group actions as relative and world-referenced displacements

(c) Inverse right actions

(d) Inverse left actions

Figure 1.13 Frames that are fixed to each other are offset by a constant right action h; applying a left action g to both frames preserves this offset. The same principle holds when considering body-relative displacements. Inverse right actions swap the origin and destination of the relative motion. Inverse right actions extract the relative components of quantities whose global values are known.

Adjoint Actions

Given a group element g, each group element h_1 acting on g from the left can be paired with a second element h_2 whose right action on g produces the same result, such that

$$h_1 \circ g = g \circ h_2. \tag{1.xxxii}$$

Pairs of h_1 and h_2 elements that satisfy (1.xxxii) are *adjoint* to each other, and are related by the group's *adjoint action*.[a]

Adjoint actions combine left and right actions

note that we're using left and right actions of the starting point to transform between left and right representations of the motion

Adjoint relationship is directional – use adjoint inverse, not repeated adjointing to go other direction

$$AD_g : G \to G \tag{1.xxxiii}$$

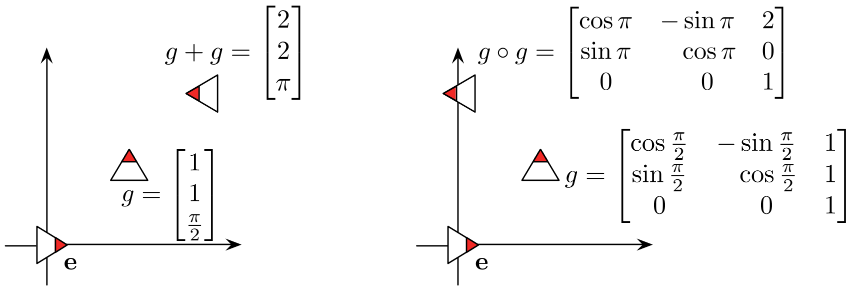
$$h \mapsto ghg^{-1} \tag{1.xxxiv}$$

$$h_1 = g \circ h_2 \circ g^{-1} = AD_g(h_2) \tag{1.xxxv}$$

$$h_2 = g^{-1} \circ h_1 \circ g = AD_g^{-1}(h_1) \tag{1.xxxvi}$$

[a] Specifically, this is the *adjoint action of the group on itself.* Groups also have adjoint actions that apply to velocity vectors through the group; these actions are closely related to the adjoint action discussed here, and are themselves discussed on page ??.

h. Another way of thinking about this operation is that applying g^{-1} to the objects is equivalent to applying g to the reference frame, so that the position of each object is now measured relative to a frame at g, as in Figure 1.12(c).



(a) $\mathbb{R}^2 \times \mathbb{S}^1$ (b) $SE(2)$

Figure 1.14 In $(\mathbb{R}^2 \times \mathbb{S}^1, +)$, displacements are specified with respect to the global reference frame, whereas in $SE(2)$ they are with respect to the system's reference frame at the start of the motion (as right actions), or take into account its starting position (as left actions).

1.3.4 $SE(2)$ and Points

Each point p in a rigid body is characterized by its coordinates p_x^b and p_y^b. For a body configuration g, the world positions (p_x, p_y), of these points can be calculated as a special case of the general $SE(2)$ action,

$$\begin{bmatrix} \varnothing & & p_x \\ & & p_y \\ 0 & 0 & 1 \end{bmatrix} = \overbrace{\begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix}}^{g} \begin{bmatrix} \varnothing & & p_x^b \\ & & p_y^b \\ 0 & 0 & 1 \end{bmatrix} \tag{1.19}$$

$$\begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x^b \\ p_y^b \\ 1 \end{bmatrix}, \tag{1.20}$$

in which, as points have only positions, the orientation component of p is undefined. The process of mapping from g to the world position of p is an aspect of the *forward kinematics* process we discuss in more depth in §3.1.

1.3.5 $SE(2)$ vs. $(\mathbb{R}^2 \times \mathbb{S}^1, +)$

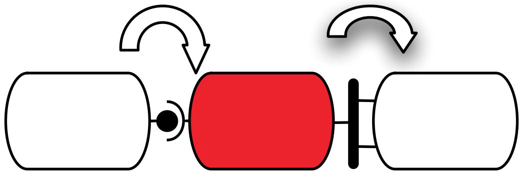
In the choice of group structures to use for rigid body configurations, $(\mathbb{R}^2 \times \mathbb{S}^1, +)$ initially appears to have an advantage from a complexity standpoint—simply adding translations and changes in orientation together is easier than working with matrix multiplications that mix translation and rotation. The advantage of $SE(2)$ lies in the way the group action corresponds to relative positioning. For example, Figure 1.14 shows a rigid body at positions $\mathbf{e} = (0, 0, 0)$ (the group identity element, and the origin of the space), $g = (1, 1, \pi/2)$, and at $g \circ g$ as calculated with each group action.

In $\mathbb{R}^2 \times \mathbb{S}^1$, the move from $\mathbf{e}$ to g is the same as that from g to $g \circ g$ in *world* terms: the system experiences the same displacements in x, y, and θ during each transformation. From a *local* perspective, however, the displacements appear very different. In the body frame, looking down the x axis, the translation from $\mathbf{e}$ to g is a motion "forward and to the left," but the translation from g to $g \circ g$ is "forward and to the right." With $SE(2)$, the group action takes into account the orientation of the frame at g, ensuring that the second transformation is, like the first, a motion "forward and to the left."

This automatic inclusion of locality is the strength of $SE(2)$. It provides a solid framework for describing relative motion that supports describing the motions of systems in terms of body velocities (§**??**) and intuitive tools for working with kinematic chains of linked bodies. Ultimately, the locality underlies the *symmetries* in system dynamics that enable the locomotion analysis starting in Chapter 4.

Exercises

1.1 What are the configuration spaces of (a) a universal joint and (b) a similar joint where the rotational axes do not intersect, as depicted below?



1.2 How many charts do we need to construct an atlas for

a. A cylinder?
b. A torus?

Explain why.

1.3 Does the set of integers form a group under multiplication? (i.e., is $(\mathbb{Z}, \times)$ a group?) Justify your answer.

1.4 Generate a canonical matrix-multiplication representation for the additive group $(\mathbb{R}, +)$.

1.5 Something about rotation/scaling matrix group

1.6 Carry out Figure 1.12 examples with hg instead of gh

1.7 $SE(2)$ is a noncommutative group, but Figure 1.13(a–b) are commutative diagrams showing that $g\circ$ commutes with $\circ h$. Explain why this is allowable.

2

Velocity

After selecting an appropriate configuration space for a system, the next step in modeling its behavior is to consider its *velocity*—the speed and direction with which it is moving. As a general principle, velocities can be interpreted geometrically as *tangent vectors* to the system's configuration manifold, corresponding to infinitesimal changes in the configuration over time. In this chapter, we examine several means of representing these tangent vectors. Paralleling the progression of ideas in Chapter 1, we begin with tangent vectors to generic manifolds, then incorporate the extra structural information available for manifolds that are also Lie groups. We finish the chapter by applying these velocity representations to the motion of rigid bodies, for which the Lie group structure provides a well-defined paradigm for discussing an object's velocity relative to its current position or that of other objects.

2.1 Tangent Spaces

If we take a system's configuration q as a point on a manifold Q, then its velocity $\dot{q}$ is the direction and speed with which this point is moving through the manifold—*i.e.* a vector that we can think of as being "attached" to the manifold at the current configuration. The velocity vectors that the system can have at a given configuration q are elements of a *vector space*[1] called the *tangent space* to Q at that configuration, T_qQ, that more generally contains all possible differential changes in the configuration from point q. The tangent spaces to an n-dimensional manifold are each $\mathbb{R}^n$ vector spaces, and can be thought of as "linearizations" of the manifold, or the vector spaces that most closely approximate the manifold at each point.

The name "tangent space" reflects the idea that small changes in the configuration are always in a direction contained by the manifold, and are thus "tangent" to it, as illustrated for the circle shown in Figure 2.1. This notion of tangency extends naturally to points moving on a line (for which the tangent spaces are overlaid with the line itself), and to higher-dimension manifolds, such as the sphere, for which the tangent space can be illustrated as a two-dimensional plane touching the sphere. The collection of all the tangent spaces to a manifold form the manifold's *tangent bundle*, TQ. Like manifolds, tangent spaces and the vectors they contain exist as

[1] See the box on page 27.

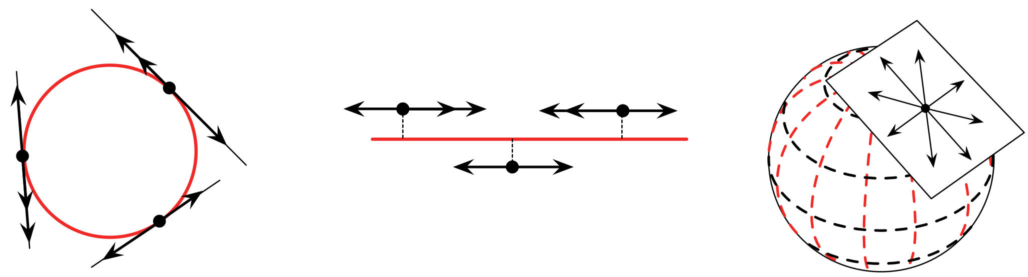


Figure 2.1 Vectors in the tangent spaces to the circle, line, and sphere. The tangent vectors to the line are offset for visual clarity.

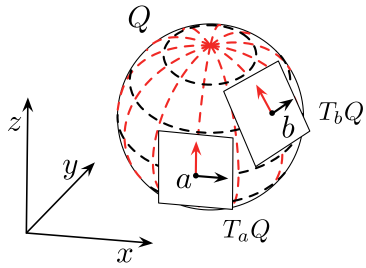


Figure 2.2 Tangent vectors on a manifold can be regarded both intrinsically as elements of the abstract tangent space, and extrinsically as small steps through the manifold. In the extrinsic interpretation, it is important to remember that even the longest vectors in the tangent space are still only of infinitesimal magnitude with respect to the manifold.

geometric objects independent of any coordinates used to describe them. To perform computations on these vectors, however, it is useful to assign bases[2] to the tangent spaces, so that each vector is parameterized by a set of real numbers.

On a parameterized manifold, the tangent spaces inherit a natural (but not unique) set of bases from the parameterization. In these bases, each basis vector u_i represents the infinitesimal change in configuration corresponding to an infinitesimal change in the parameter q_i,

$$u_i = \left.\frac{\partial q}{\partial q_i}\right|_q . \tag{2.1}$$

Any differential change in the system configuration can then be represented as the sum of differential changes in the parameter values multiplied by their respective basis vectors,

$$dq = \sum_i u_i \, dq^i, \tag{2.2}$$

or, if we take the set of basis vectors as being implicit,

$$dq = (dq^1, dq^2, \ldots, dq^n). \tag{2.3}$$

Velocities are tangent vectors in which the magnitudes represent differential displacements over time, rather than simple differential displacements. The velocities use the same set of differential bases, as

$$\dot{q} = \frac{dq}{dt} = \sum_i \frac{\partial q}{\partial q_i}\frac{dq^i}{dt} = \sum_i u_i \, \dot{q}^i, \tag{2.4}$$

which, with implicit basis vectors, can be represented as

$$\dot{q} = (\dot{q}^1, \dot{q}^2, \ldots, \dot{q}^n). \tag{2.5}$$

2.1.1 Extrinsic and Intrinsic Bases

A manifold's tangent vectors can be regarded both *extrinsically*, as small steps through the manifold, and *intrinsically*, as elements of the individual tangent spaces. In an extrinsic treatment, the vectors' magnitudes and orientations are interpreted with respect to the manifold's structure; in an intrinsic treatment, magnitude and orientation are defined within the tangent space. The tangent vectors to a sphere embedded in $\mathbb{R}^3$ provide a useful example of the difference between these two interpretations, as illustrated in Figure 2.2:

Each tangent space T_qQ to the sphere Q can be identified by a pair of basis vectors. If we have chosen a longitude-latitude parameterization (θ, ϕ) for the sphere, a natural choice for these basis vectors is

$$u_{1,q} = \left.\frac{\partial q}{\partial \theta}\right|_{\theta,\phi} \quad \text{and} \quad u_{2,q} = \left.\frac{\partial q}{\partial \phi}\right|_{\theta,\phi} . \tag{2.6}$$

[2] See the box on page 28.

Vectors and Vector Spaces

Vectors. The term “vector” carries several meanings in physics, mathematics, and engineering, including:

1. A quantity with magnitude and direction,
2. An element of a *vector space*, and
3. An n-tuple (or list) of values.

These meanings are distinct from each other, but related. Vectors in the physical sense of directed quantities can be treated as elements of a vector space, which provides rules for adding them together and multiplying them by *scalar* terms that change their magnitude but not their direction. Within a given vector space, vectors can be characterized as weighted sums of a set of *basis vectors*. The values of these weights (the vector’s *components*) uniquely identify each vector with respect to the chosen basis, leading to the interpretation of a vector as a list of numbers.

Representing vectors as n-tuples provides a convenient means of performing computations with them. For example, linear transformations of a vector can be encoded as matrices that are multiplied by the column of component values. This convention, however, can obscure important differences between operations that act *within* a vector space and operations that act *on* the vectors themselves. Where such distinctions are important, they can be made clear by turning to more formal treatments of vector spaces and bases.

Vector spaces. A *vector space* is a set V of objects called *vectors*, associated with a set S of objects called *scalars* and four operations:

1. *Vector addition*, in which two vectors are combined to generate a third,
2. *Scalar addition*, in which two scalars are summed to generate another scalar,
3. *Scalar multiplication*, in which the product of two scalars is another scalar,
4. *Scalar multiplication of vectors*, in which a scalar is combined with a vector to produce another vector.

These sets and operations must satisfy the following conditions:

1. The set V must form a commutative group[a] under vector addition.
2. The set S must form a commutative field[b] under the scalar addition and multiplication operations.
3. Scalar multiplication of vectors must have the following properties for $v, w \in V$ and $a, b \in S$:
   a. **Closure**: the product of a scalar with a vector is always also a vector: $av \in V$
   b. **Distributivity I**: Multiplying a scalar by the sum of two vectors is equivalent to multiplying the scalar by each of the two vectors and summing the result: $a(v + w) = av + aw$
   c. **Distributivity II**: Summing two scalars and multiplying them by a vector is equivalent to multiplying each of them by the vector and summing the result: $(a + b)v = av + bv$
   d. **Associativity**: Multiplying two scalars together and then by a vector is equivalent to sequentially multiplying the scalars by the vector: $(ab)v = a(bv)$
   e. **Identity**: The multiplicative identity element of S is also the identity for scalar multiplication: $\mathbf{1}_S v = v$

The term “scalar” is commonly used to refer to quantities described by a single value. This usage is derived from the fact that many commonly-encountered vector spaces have the real numbers $\mathbb{R}$ as their set of scalars. More generally, the scalars for a vector space may be drawn from any (potentially multidimensional) commutative field, such as the space of complex numbers.

continued . . .

[a] See the box on page 9.
[b] See the box on page 10. Note that this use of the word “field” is unrelated to the idea of a “vector field” discussed in §2.1.2.

Vectors and Vector Spaces, *continued*

Bases. A *basis* U on a vector space V is a set of vectors $u_i \in V$ that *minimally span* the space. Spanning V means that any vector v in the space can be expressed as a weighted sum of the basis vectors,

$$v = \sum_i u_i \, v^i, \tag{2.i}$$

in which the *components* of the vector, the coefficients v^i that are multiplied by the corresponding basis elements u_i, are scalars from the space's associated field. Minimally spanning V means that each vector is identified by a *unique* sum of basis vectors, and thus by a unique set of components. If the scalar field associated with V is the set of real numbers $\mathbb{R}$, then each vector is uniquely identified with a point in $\mathbb{R}^n$, giving rise to the notion of a vector as a "list of numbers."

Bases can also be defined as sets of vectors with the following properties, which are equivalent to the minimal spanning condition:

1. **Number of basis vectors**: There are exactly as many basis vectors in B as there are dimensions in V.
2. **Linear independence**: No basis vector in B may be expressible as a weighted sum of the other basis vectors.

Matrix notation. Using matrix notation, the summing operation in (2.i) can be expressed as the product of the basis vectors arranged in a row and the coefficients arranged in a column,[a]

$$v = \overbrace{\begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix}}^{U} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}. \tag{2.ii}$$

Many linear transformations $w = Av$ between vectors may then be encoded as matrices that multiply columns of vector coefficients,

$$\begin{bmatrix} w^1 \\ \vdots \\ w^m \end{bmatrix} = A^{m \times n} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}. \tag{2.iii}$$

Note however, that this use of a "matrix multiplying by a vector" maps between intrinsic representations of vectors. Linear transformations that act on the extrinsic representation of the vector transform the basis, rather than the coefficients, as $U_w = BU_v$. Under these circumstances, the full operation becomes

$$U_1 \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} = BU_2 \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}, \tag{2.iv}$$

with the dimensionality of B corresponding to the dimensionality of the elements of U_1 and U_2. This second sense plays a central role in the behavior of vectors on Lie groups, which is explored in §2.2.

[a] The choice of which elements form the row and which form the column is of course arbitrary, but the convention is that vector components are given as columns, and that the components of *covectors* (a category which includes vector bases) are placed into rows. Covectors are discussed on page 79.

Extrinsically, these basis vectors correspond to small steps through the manifold, with a sense of length and orientation derived from the manifold's structure. For an embedding that takes θ as the longitude and ϕ as the latitude on the sphere,

$$q = (x, y, z) = (\cos\theta\cos\phi,\ \sin\theta\cos\phi,\ \sin\phi), \tag{2.7}$$

the basis vectors evaluate to

$$u_{1,q} = \begin{bmatrix} -\cos(\phi)\sin(\theta) \\ \cos(\phi)\cos(\theta) \\ 0 \end{bmatrix} \text{ and } u_{2,q} = \begin{bmatrix} -\cos(\theta)\sin(\phi) \\ -\sin(\phi)\sin(\theta) \\ \cos(\phi) \end{bmatrix}. \tag{2.8}$$

By contrast, an intrinsic treatment of these vectors takes them as the bases of the $\mathbb{R}^2$ tangent spaces at their respective locations on the sphere, which exist independently of the manifold or its embedding. Under this interpretation, vectors in each tangent space are defined in terms of u_1 and u_2 and the basis vectors themselves thus take on the role of principle axes in their respective tangent spaces,

$$u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{2.9}$$

Intrinsic and extrinsic vector representations both serve useful roles in dynamic analysis of systems. Intrinsic representations have trivial bases, which allows linear mappings between vectors to be encoded as matrices multiplied by lists of coefficients, as in (2.iii). Extrinsic representations relate the velocity vectors back to the motion on the manifold and allow for additional operations, such as applying group actions directly to vectors tangent to the group, as we discuss in §2.2.

2.1.2 Vector Fields

A *vector field* is a (possibly time-varying) assignment of a single vector to each point in a subset of a manifold. Formally, a vector field $\mathbf{X}$ on $Q_1 \subset Q$ is a mapping

$$\mathbf{X} : Q_1 \times \mathbb{R} \to T_qQ \tag{2.10}$$

$$(q, t) \mapsto v, \tag{2.11}$$

where v is a vector in T_qQ.

When the manifold Q is a configuration space, vector fields on the manifold can be used to represent *velocity fields* describing how the system's configuration evolves under the influence of a first-order differential equation

$$\dot{q} = \mathbf{X}(q, t). \tag{2.12}$$

Solutions to this differential equation (given an initial configuration $q_0 = q(0)$) are the *integral curves*, or *flows*, of $\mathbf{X}$. These solutions take the forms of trajectories γ through the configuration space,

$$\gamma : [0, T] \to Q \tag{2.13}$$

$$t \mapsto q, \tag{2.14}$$

whose tangent vector (*i.e.* time derivative) at each point is equal to the tangent vector at that point in the underlying vector field,

$$\dot{\gamma}(t) = \mathbf{X}(\gamma(t), t). \tag{2.15}$$

Several examples of integral curves on vector fields are shown in Figure 2.3(a).

On a static (time-invariant) field, no two integral curves may cross each other, and no integral curve may cross itself; either occurrence would require that $\mathbf{X}$ contain two different vectors at the intersection point. Similarly, two integral curves will never merge with each other, and single integral curve that is not a closed cycle will never merge back into itself (though it may become asymptotically close to an earlier section), as a complete merger would require that the corresponding flow on $-\mathbf{X}$ split into two solutions, and thus that $\mathbf{X}$ have two vectors at the split/merge point.[3]

[3] Together, these properties give rise to the theorem of uniqueness of the solutions of differential equations.

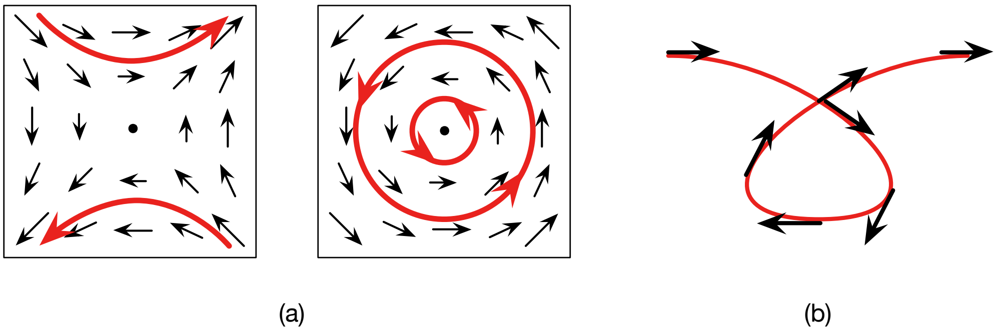


Figure 2.3 A vector field's integral curves flow along the field. (a) Sample integral curves on vector fields. (b) Vector fields tangent to a path are defined over the path—not the embedding space—and so may take on multiple values at a given point in the embedding space (any crossing points).

It is often useful to work with vector fields along paths through a space—for example, the sets of vectors tangent to or normal to a curve. Because such curves may cross themselves, there at first appears to be an incompatibility between these fields and the definition that each tangent space has a single vector: where the curve crosses itself, there are two tangent vectors, as illustrated in Figure 2.3(b). This conflict can be resolved by defining vector fields along paths as being along the path-as-one-dimensional-manifold, and not over the image of the path embedded in the higher-dimensional space. Treating the parameterization of the path as a "time" variable then allows the embedded vector field along the image to be regarded as time-varying, and thus not subject to the uniqueness requirement.

2.1.3 "Constant" Vector Fields

In some contexts, it is common to encounter the notion of "constant" vector fields in which the vectors at each point are "the same." Typically this equivalence is expressed with a statement like "each vector in the field has the same x and y components," so that the vector field looks something like that in Figure 2.4(a). Constant vector fields in this sense are an intuitive idea, easy to communicate, and usefully correspond to many physical phenomena in Euclidean spaces.

From a more rigorous perspective—in which tangent spaces at different points in the manifold are fundamentally distinct—saying that two vectors at different points are "the same" requires that we declare an equivalence between specific sets of bases in the corresponding tangent spaces, and then that the two vectors have identical relationships to their respective bases. Equivalence between vectors is commonly based on some shared extrinsic property. Under the standard definition of a constant vector field, the tangent spaces each inherit a set of $\frac{\partial}{\partial x}$

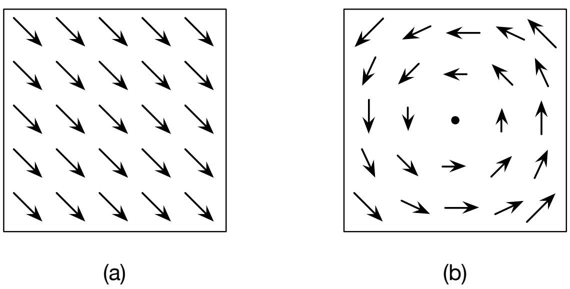


Figure 2.4 "Constant" vector fields made up of the "same" vector in each tangent space are typically those with the same x and y components as in (a). This idea of "sameness" is context-specific, however: the vectors in (b) all share the same r and θ components (outside of the singularity at the center). Note that the lengths of the vectors in (b) increase proportionally with distance from the origin—the distance moved in the plane for a given $\Delta\theta$ increases with radius, and this scaling propagates into the unit vectors formed by taking the derivative of the parameterization as in (2.1).

and $\frac{\partial}{\partial y}$ basis vectors as in (2.1), and we declare equivalence between these bases on the grounds of symmetry: for points moving through the plane, a change in x or y has a useful physical meaning that does not depend on the current value of x and y.

We point out this distinction because choices of parameterization and symmetry are not unique, and different choices lead to different notions of which vectors are "the same." As a simple example, the vectors in Figure 2.4(b) are all $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ vectors under the $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ bases inherited from a polar parameterization, and hence are just as equivalent to each other as are the $\left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]$ vectors in Figure 2.4(a). More importantly, in many cases we want to separate the idea of vector equivalence from our choice of coordinates (and thus generate *coordinate-invariant* models of system dynamics), and so more rigorous methods must be employed to define vectors that are "the same" at different points.

Under the more rigorous approaches, equivalence between bases at different points is derived from the transformations that are used to move between the points. Two examples of these approaches are *parallel transport along manifolds* and *equivalency under the differential of a function.* Parallel transport relates to the curvature of the manifold, so we will save it until Chapter 6. Equivalence with respect to functions, however, can be expressed in terms of mathematical tools already at our disposal, and lays the groundwork for considering velocities on groups.

2.1.4 Vectors that are Equivalent with respect to Functions

When we say that two vectors are equivalent with respect to a function, the central concept is that they have *corresponding effects on the input and output of the function*. Given a function $f : q \mapsto p$, two vectors $dq \in T_qQ$ and $dp \in T_pP$ are equivalent with respect to f if stepping along dq, then applying f reaches the same point in P as is reached by first applying f, then stepping along dp:

$$dq \equiv_f dp \tag{2.16}$$

if dq and dp satisfy the relationship

$$f(q + dq) = \overbrace{f(q)}^{p} + dp. \tag{2.17}$$

Similarly, two velocity vectors $\dot{q} \in T_qQ$ and $\dot{p} \in T_pP$ are equivalent with respect to f if $\dot{p}$ is the time derivative of $f(q)$ when the configuration is changing at $\dot{q}$:

$$\dot{q} \equiv_f \dot{p} \tag{2.18}$$

if $\dot{q}$ and $\dot{p}$ satisfy the relationship

$$\left.\frac{d}{dt} f(q)\right|_{\dot{q}} = \dot{p}. \tag{2.19}$$

Vectors that are equivalent to each other under this definition are related to each other by the Jacobian[4] of f,

$$dp = J_f\, dq \quad \text{and} \quad \dot{p} = J_f\, \dot{q}. \tag{2.20}$$

The idea underlying this relationship is that the Jacobian is the derivative of every degree of freedom in the output of f with respect to every degree of freedom in its input. When we multiply the Jacobian by a dq or $\dot{q}$ vector, it collapses down from describing all possible changes in the output to describing the specific change corresponding to that dq or $\dot{q}$, and hence the equivalent dp or $\dot{p}$.

- As per comment in greybox (to be added) that the columns of the Jacobian are the equivalent vectors in the output space to the basis vectors in the input space, the columns of the Jacobian of an embedding function are the extrinsic form of the parameter-induced basis
- three examples: Embed $\mathbb{S}^1$

$$f_1 : \theta \mapsto \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} \implies J_1 = \begin{bmatrix}-\sin\theta \\ \cos\theta\end{bmatrix}. \tag{2.21}$$

[4] See the box on the next page.

Differential Mappings and Jacobians

A function mapping between manifolds A and B,

$$\begin{aligned} f : A &\to B \\ a &\mapsto b, \end{aligned} \tag{2.v}$$

has a set of associated *differential mappings* (also called *tangent mappings*) between the tangent spaces of A and B, defined so that the output velocity $\dot{b}$ is the rate at which the output is changing, given a known rate at which the input is changing,

$$\begin{aligned} f' : T_a A &\to T_b B \\ \dot{a} &\mapsto \dot{b} = \tfrac{d}{dt} f(a)\big|_{a,\dot{a}}. \end{aligned} \tag{2.vi}$$

By the chain rule, this derivative decomposes into the product of the derivative of f over the input space and the rate at which the input is changing,

$$f'(\dot{a}) = \left.\frac{\partial f(a)}{\partial a}\right|_a \dot{a}. \tag{2.vii}$$

The derivative term in (2.vii) is the *Jacobian* of f, denoted J. If A or B are multi-dimensional spaces, J includes the derivative of each degree of freedom in the range of f with respect to each component of its domain. In coordinates, these derivatives can be grouped into matrix form as

$$\begin{bmatrix} \dot{b}_1 \\ \vdots \\ \dot{b}_n \end{bmatrix} = \overbrace{\begin{bmatrix} \partial f_1/\partial a_1 & \cdots & \partial f_1/\partial a_m \\ \vdots & \ddots & \vdots \\ \partial f_n/\partial a_1 & \cdots & \partial f_n/\partial a_m \end{bmatrix}}^{J} \begin{bmatrix} \dot{a}_1 \\ \vdots \\ \dot{a}_m \end{bmatrix}. \tag{2.viii}$$

in which each row is the derivative of one component of the output and each column is the derivative with respect to one component of the input.

$$J = \begin{bmatrix} \partial f_1/\partial a_1 & \cdots & \partial f_1/\partial a_m \\ \vdots & \ddots & \vdots \\ \partial f_n/\partial a_1 & \cdots & \partial f_n/\partial a_m \end{bmatrix} \tag{2.ix}$$

expands into a matrix in which each row is the derivative of one component of the output and each column is the derivative with respect to one component of the input.

As discussed in §2.1.4, Jacobians are very useful in identifying velocities that are in some sense equivalent to each other, without resorting to coordinate-dependent definitions of "sameness." The core principle here is that if we take a snapshot of a trajectory $a(t)$ and its image $b(t)$ under the function f, we arrive the same velocity $\dot{b}(t)$ of the output trajectory whether we directly differentiate the output of the function or we apply the tangent mapping to the input velocity,

$$\dot{b}(t) = \frac{d}{dt} f(a(t)) = f'(\dot{a}(t)) = J\dot{\alpha}(t), \tag{2.x}$$

which can be demonstrated via the chain rule for derivatives. This equivalence is described by the *commutative diagram* below, which illustrates that both mapping-first and derivative first pathways reach the same $\dot{b}$.

$$\begin{tikzcd} a \arrow[r, "d/dt"] \arrow[d, "f"'] & \dot{a} \arrow[d, "f'"] \\ b \arrow[r, "d/dt"'] & \dot{b} \end{tikzcd}$$

continued ...

Differential Mappings and Jacobians, *continued*

Four common circumstances in which vectors equivalent under Jacobians appear are:

1. **Mapping between the intrinsic and extrinsic forms of a vector.** Given a parameterization function[a] ϕ, its inverse maps from the parameters to the manifold,
$$\phi^{-1} : \{q_i\} \mapsto q. \tag{2.xi}$$
Each column of the Jacobian of this function matches the corresponding basis vector from (2.1),
$$J_{\phi^{-1}} = \begin{bmatrix} \frac{\partial \phi^{-1}}{\partial q_1} & \cdots & \frac{\partial \phi^{-1}}{\partial q_n} \end{bmatrix}\Big|_{\{q_i\}} = \begin{bmatrix} \frac{\partial q}{\partial q_1} & \cdots & \frac{\partial q}{\partial q_n} \end{bmatrix} = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix}, \tag{2.xii}$$
and thus provides the mapping from the intrinsic representation (the set of dq_i or $\dot{q}_i$ coefficients) to the full extrinsic form of the vector.
2. **Change-of-basis operations linked to changes in parameterization.** Given a transition map[b] $\tau_{a,b} = \phi_a^{-1}\phi_b$ that reparameterizes a space as
$$\tau_{a,b} : \{q_{i,a}\} \mapsto \{q_{i,b}\}, \tag{2.xiii}$$
the Jacobian of this function maps vectors from their intrinsic representation in the first parameter-induced basis to their intrinsic representation in the second parameter-induced basis,
$$J\tau_{a,b} : \{dq_{i,a}\} \mapsto \{dq_{i,b}\}. \tag{2.xiv}$$
3. **Identifying vectors represent the "same" motion.** Given a function $f : Q \to Q$ mapping between points in a single space, two vectors related by J_f correspond to a *single* action that commutes with the function.
4. **Finding motion through one space that is induced by motion in a second space**.

The fourth point is a core subject of Chapter 3, so we will hold off discussing it until then. Points 1 and 2, however, provide some useful examples that are worth investigating before returning to consideration of "constant" vector fields in Point 3.

[a] See the box on page 3.
[b] See the box on page 3.

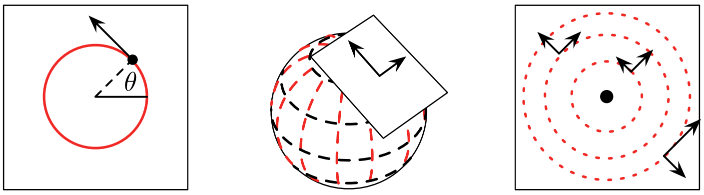


Figure 2.5 Jacobians of embedding functions are extrinsic basis for vectors.

Embed $\mathbb{S}^2$

$$f_2 : \theta \mapsto \begin{bmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ \sin\phi \end{bmatrix} \implies J_2 = \begin{bmatrix} -\cos(\phi)\sin(\theta) & -\cos(\theta)\sin(\phi) \\ \cos(\phi)\cos(\theta) & -\sin(\phi)\sin(\theta) \\ 0 & \cos(\phi) \end{bmatrix}. \tag{2.22}$$

Embed $\mathbb{R}^1 \times \mathbb{S}^1$

$$f_3 : (r,\ \theta) \mapsto \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} r \implies J_3 = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}. \tag{2.23}$$

Returning to our discussion of "constant" vector fields, we can use equivalence with respect to transformations to shore up our intuition that the vector fields in Figure 2.4 are built up from multiple copies of the "same" vector, and that doing so provides several simple examples of this principle in action: In Figure 2.4a, the vectors are invariant with respect to translation. Once we make a Euclidean parameterization the plane as (x, y), any translation in the plane by (a, b) can be expressed as the function

$$f_1 : \mathbb{R}^2 \to \mathbb{R}^2 \tag{2.24}$$
$$(x, y) \mapsto (x + a, y + b). \tag{2.25}$$

The Jacobian of this function evaluates to an identity matrix,

$$J_1 = \begin{bmatrix} \frac{\partial(x+a)}{\partial x} & \frac{\partial(x+a)}{\partial y} \\ \frac{\partial(y+b)}{\partial x} & \frac{\partial(y+b)}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{2.26}$$

which means that all vectors that share the same x and y components are invariant with respect to translation, as fits with intuition.

Similarly, in a polar parameterization of the space in Figure 2.4b, radial motions and rotations are accomplished by adding a radial distance and a change in angle to the starting position,

$$f_2 : \mathbb{R} \times \mathbb{S} \to \mathbb{R} \times \mathbb{S} \tag{2.27}$$
$$(r, \theta) \mapsto (r + \rho, \theta + \phi). \tag{2.28}$$

Once again, the Jacobian of the function that maps between points in the space evaluates to an identity matrix,

$$J_2 = \begin{bmatrix} \frac{\partial(r+\rho)}{\partial r} & \frac{\partial(r+\rho)}{\partial \theta} \\ \frac{\partial(\theta+\phi)}{\partial r} & \frac{\partial(\theta+\phi)}{\partial \theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{2.29}$$

supporting our intuitive definition that vectors that have the same expression in parameter-induced bases are equivalent.

In both of the above examples, the action of the mapping function (Euclidean or polar translation) corresponded to the parameterization of the space, with the result that the Jacobians evaluated to simple identity matrices. As we noted above, however, it is not always possible to generate parameterizations that naturally generate useful equivalences. Using the Jacobian framework allows us to declare notions of equivalence that are independent of the coordinates used to express the problem.

For example, we can use Cartesian coordinates to encode the polar equivalence of the vectors in Figure 2.4b by starting with the function that applies a polar translation to a set of Cartesian coordinates by extending the original (x, y) vector by a length ρ, then rotating the new vector by an angle ϕ,

$$f_3 : \mathbb{R}^2 \to \mathbb{R}^2 \tag{2.30}$$

$$(x, y) \mapsto R(\phi) \begin{bmatrix} x + \frac{x}{\sqrt{(x^2+y^2)}}\rho \\ y + \frac{y}{\sqrt{(x^2+y^2)}}\rho \end{bmatrix}, \tag{2.31}$$

Taking the Jacobian of this function results in the expression

$$J_3 = R(\phi) \begin{bmatrix} \frac{\partial(1+\rho(x^2+y^2)^{-\frac{1}{2}})x}{\partial x} & \frac{\partial(1+\rho(x^2+y^2)^{-\frac{1}{2}})x}{\partial y} \\ \frac{\partial(1+\rho(x^2+y^2)^{-\frac{1}{2}})y}{\partial x} & \frac{\partial(1+\rho(x^2+y^2)^{-\frac{1}{2}})y}{\partial y} \end{bmatrix} \tag{2.32}$$

$$= R(\phi) \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{\rho}{(x^2+y^2)^{\frac{3}{2}}} \begin{bmatrix} y^2 & -xy \\ -xy & x^2 \end{bmatrix} \right). \tag{2.33}$$

Although this expression looks complicated, breaking it down into components highlights the ways in which f_3' transforms vectors into their equivalents in different tangent spaces:

add figure here, showing each of the transformations in the list

1. If the distance from the origin does not change ($\rho = 0$), then the Jacobian is a simple rotation of the vector by the same angle as was used to move to the new point.
2. For initial positions along a principal axis of the parameterization (e.g., along the x axis with $y = 0$), the Jacobian scales the transverse component of the velocity (here, the y component) proportionally to the new distance of the point from the origin before applying the rotation,

$$J_3|_{y=0} = R(\phi) \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 + \frac{\rho}{x} \end{bmatrix} \right), \tag{2.34}$$

   capturing the idea that a larger translational velocity is required to produce a given rotational velocity when further away from the center of rotation.
3. For initial (x, y) positions not on a principal axis, the terms multiplied by ρ in (2.33) generalize the transverse-scaling principle in (2.34) to arbitrary angles.

The key principle to take away from this example is that the Jacobian provides a structured way to lift information from mappings on a manifold into equivalence relationships between vectors in different tangent spaces, without relying on any particular choice of coordinates.

2.2 Velocities on Groups

Our treatment of velocities in §2.1 was built on the idea that a system's motion through its configuration space can be broken down into a (continuous) sequence of incremental steps (vectors) that are *added* together into a trajectory. In coordinates, the natural bases for these vectors are the derivatives of the parameterization functions, so that the system velocity is characterized by the rate at which the parameters are changing.

If the configuration space has group structure, we can also think of a system's velocity as the rate at which group operations are being applied, and thus that the incremental steps are composed together according to the group action instead of addition. In this view, the tangent spaces on the configuration space inherit a second natural set of bases, corresponding to the set of infinitesimal group actions. Considering velocities from this perspective provides a powerful tool for describing a system's motion relative to its current configuration, or in a way that preserves groupwise displacements between pieces of the system.

For example, in rigid body motion, it is often useful to talk about an object's velocity in a body-aligned frame (e.g., "forward" or "sideways" motion), or to say that two frames rigidly attached to each other have the "same" velocity, even if their relative positions and rotational motion mean that they are moving in different absolute directions. Both of these ideas are cumbersome to express in terms of the $\dot{x}$, $\dot{y}$, and $\dot{\theta}$ components of the objects' velocities: the forward and lateral velocities are

$$\dot{x}^b = \quad \dot{x}\cos\theta + \dot{y}\sin\theta \tag{2.35}$$

$$\dot{y}^b = -\dot{x}\sin\theta + \dot{y}\cos\theta, \tag{2.36}$$

and there is no inherently-obvious combination of $\dot{x}$, $\dot{y}$, and $\dot{\theta}$ components that can be assigned to the pair of linked frames.

Drawing on our treatment of rigid body motions in §1.3.2, however, we can state these ideas much more cleanly: Motions relative to the system's current frame are right actions on $SE(2)$, so an object moving with a given body velocity is experiencing a corresponding infinitesimal right action that is independent of its current configuration. Similarly, two objects moving through the world while rigidly attached to each other experience the same sequence of left actions, and their shared velocity at any instant can be characterized by an infinitesimal left action.

In this section, we provide an introduction to Lie[5] group theory, the body of work which supplies a framework for taking groupwise derivatives and integrals. This introduction includes basic definitions of calculus on groups, two choices of notation that illuminate different aspects of groupwise velocity, a mapping from groupwise derivatives back to parameter-based notions of velocity, and a specific treatment of how these concepts are instantiated in rigid body motion.

2.2.1 Groupwise Velocity Definitions and Notation

The core principles underlying groupwise treatments of velocity are:

1. Groupwise velocities are calculated on *Lie groups*,[6] whose underlying manifolds are (C^∞-)differentiable.
2. Displacements on groups are fundamentally characterized by the group actions separating configurations,

$$g_\Delta = g_1^{-1} \circ g_2, \tag{2.37}$$

which, as illustrated in Figure 1.13d, should be read as "g_Δ is the configuration of g_2 relative to g_1" or, equivalently, "g_Δ is the step that takes the system from g_1 to g_2,"

$$g_1 \circ g_\Delta = g_2. \tag{2.38}$$

3. Velocities on Lie groups are fundamentally characterized by the rate at which infinitesimal group actions are being applied to the configuration. For infinitesimal time δt and groupwise displacement

$$g_\delta = g^{-1}(t) \circ g(t + \delta t), \tag{2.39}$$

the groupwise velocity $\mathring{g}$ is calculated as the difference between g_δ and the identity element,[7] divided by the time over which the displacement occurs:

$$\mathring{g} = \lim_{\delta t \to 0} \frac{g_\delta - \mathbf{e}}{\delta t}. \tag{2.40}$$

4. If there is a distinction between the group's left and right actions (because the group is noncommutative), then the formula for g_δ in (2.39) is specifically the *right* displacement from $g(t)$ to $g(t + \delta t)$, and (2.40) gives the *right groupwise velocity*, $\overrightarrow{\mathring{g}}$. Substituting a left displacement

$$g_\delta = g(t + \delta t) \circ g^{-1}(t) \tag{2.41}$$

into (2.40) produces the *left groupwise velocity*, $\overleftarrow{\mathring{g}}$.

[5] "Lie" is pronounced "Lee." Sophus Lie originated the study of continuous groups.
[6] See the box on page ??.
[7] Small groupwise displacements are close in value to the group's identity element. For groups whose action behaves like multiplication, the identity is a generalization of "**1**" (rather than "**0**" for addition), and so must be subtracted from g_δ to produce a "small quantity" that can be divided by δt. For further discussion of this and other aspects of *multiplicative calculus*, see the box on page 39.

5. Groupwise velocities are elements of a vector space $\mathfrak{g}$, called the group's *Lie algebra*.[8] This vector space is a linearization of the group around its identity, much in the same way that a tangent space is the linearization of a manifold around a given point. A natural basis for $\mathfrak{g}$ can be taken from the first-order terms in the series expansion of g, which for group elements representing infinitesimally small displacements is

$$g_\delta = \mathbf{e} + \sum_i \underbrace{\left.\frac{\partial g}{\partial g_i}\right|_{\mathbf{e}}}_{\text{basis for } \mathfrak{g}} g_\delta^i \,. \tag{2.42}$$

That this is also the basis for groupwise velocities $\mathring{g}$ can be seen by substituting (2.42) into (2.40), which strips off the identity element and divides the remainder by δt, giving

$$\mathring{g} = \sum_i \left.\frac{\partial g}{\partial g_i}\right|_{\mathbf{e}} \left(\lim_{\delta t \to 0} \frac{g_\delta^i}{\delta t}\right) \tag{2.43}$$

$$= \sum_i \left.\frac{\partial g}{\partial g_i}\right|_{\mathbf{e}} \mathring{g}^i \tag{2.44}$$

6. Because the $\mathfrak{g}$ is a linearization of the group around its identity element, the Lie algebra is isomorphic to the tangent space of the underlying manifold at that point,

$$\mathfrak{g} \simeq T_{\mathbf{e}} G. \tag{2.45}$$

This isomorphism means that each infinitesimal group action and groupwise velocity has a corresponding vector in the tangent space at the origin that represents the same change in configuration,

$$g_\delta \equiv dg|_{g=\mathbf{e}} \quad \text{and} \quad \mathring{g} \equiv \dot{g}|_{g=\mathbf{e}}, \tag{2.46}$$

and that because the basis for $\mathfrak{g}$ in (2.42) is constructed in the same fashion as the general basis for tangent vectors in (2.1), the coefficients of g_δ and dg will be the same at the origin for any given parameterization, as will the coefficients of $\mathring{g}$ and $\dot{g}$.

7. Any point on the manifold can be chosen as the origin, which means that the Lie algebra is isomorphic to *every* tangent space on the group. Each $\dot{g}$ vector in the tangent bundle can thus be associated with a groupwise velocity $\mathring{g}$ that represents the same infinitesimal change in configuration. The set of vectors corresponding to a given $\mathring{g}$ (one in each tangent space) form a vector field $\mathring{g}_G$ over the group, called a *fundamental vector field* or an *infinitesimal generator*.

8. Away from the origin,

Because the result of applying a given group action depends on the group element to which it is applied, the coefficients of vectors in a given $\mathring{g}_G$ at points away from the identity element are in general not the same as those of $\mathring{g}$. Fundamental vector fields are, however, *group-invariant*. This means that all the vectors in a given $\mathring{g}_G$ are equivalent under the group action (in the sense of equivalency discussed in §2.1.4) such that the $\dot{g}$ vector corresponding to a given $\mathring{g}$ at a point g away from the identity can be found

Maybe bring in the $g^{-1}\dot{g} = \mathring{g}$ notation to emphasize the meaning of invariant

$$\dot{g} = \mathring{g}_G|_g = J_g \mathring{g}, \tag{2.47}$$

$\dot{g}$ vector corresponding to a given $\mathring{g}$ at a point g away from the identity can be found either extrinsically by multiplying $\mathring{g}$ by g,

$$\dot{g} = \mathring{g}_G|_g = g\mathring{g}, \tag{2.48}$$

or intrinsically by multiplying the coefficients of $\mathring{g}$ by the Jacobian of the group action,

$$\dot{g} = J_g \mathring{g}. \tag{2.49}$$

For noncommutative groups, the left fundamental field is right-invariant and the right fundamental field is left-invariant,.

Deciding which set of notations for the lifted action to describe here. Certainly $T_e L_g$ style, but $L_{g\star}$ has some appeal

$$\dot{g} = \overleftarrow{\mathring{g}} g \quad \text{and} \quad \dot{g} = g \overrightarrow{\mathring{g}} \tag{2.50}$$

[8] See the box on page 45.

↓ Vectors corresponding to a given **right** action
→ are equivalent under **left** group actions

$$\begin{tikzcd}[column sep=large, row sep=large] \mathbf{e} \arrow[r, "g\circ"] \arrow[d, "{\circ g_\delta \atop +dg}"'] & g \arrow[d, "{\circ g_\delta \atop +J_{g\circ}dg}"] \\ g_\delta \arrow[r, "g\circ"'] & g g_\delta \end{tikzcd}$$

↓ Vectors corresponding to a given **left** action
→ are equivalent under **right** group actions

$$\begin{tikzcd}[column sep=large, row sep=large] \mathbf{e} \arrow[r, "\circ g"] \arrow[d, "{g_\delta\circ \atop +dg}"'] & g \arrow[d, "{g_\delta\circ \atop +J_{\circ g}dg}"] \\ g_\delta \arrow[r, "\circ g"'] & g_\delta g \end{tikzcd}$$

Figure 2.6 Left and right equivalence of vectors (will have more text here).

2.2.2 Groupwise Velocity Examples

Many of the salient characteristics of groupwise velocity are exhibited by three basic groups or group types—additive groups, the group of scalar multiplications, and the group of planar rotations. Understanding how the structures outlined in §§2.3.2–2.3.3 apply to these groups sets the stage for defining a group's lifted actions in §2.3.4 and for considering velocities on Lie groups with more complicated structure, such as the $SE(2)$ group of rigid body motions.

On additive groups, the group identity is zero and the inverse operation is subraction. Under these rules, the group velocity defined in (2.78) becomes the familiar definition of a time-derivative,

$$\overset{\circ}{g} = \lim_{\delta t \to 0} \frac{g(t+\delta t) - g(t)}{\delta t} = \dot{g}. \tag{2.51}$$

This should not be surprising, as standard calculus is rooted in the idea of adding and subtracting differential quantities, and the availability of these actions depends on $\mathbb{R}^n$ forming a group under addition.

A more interesting example of groupwise velocity is provided by the multiplicative group of real numbers, $(\mathbb{R}_+, \times)$, for which the identity element is 1 and the inverse operation is division. The groupwise velocity on this group is thus

$$\overset{\circ}{g} = \lim_{\delta t \to 0} \frac{g(t+\delta t)/g(t) - 1}{\delta t}, \tag{2.52}$$

corresponding to the *ratio* of the group values at two closely separated times, rather than their absolute difference.[9] The structure of this equation provides some concrete illustrations of the group velocity properties introduced in §2.3.2 and §2.3.3.

First, the differential group action in the numerator of (2.52),

$$g_\delta = \lim_{\delta t \to 0} g(t+\delta t)/g(t) \tag{2.53}$$

is within an infinitesimal neighborhood of 1; values of g_δ that are greater than 1 indicate that g is growing over time, g_δ less than 1 indicates that g is shrinking, and g_δ equal to 1 (the identity) indicates no change in g over time. Simply dividing g_δ by δt to find the rate of change would not provide a meaningful result, as

$$\lim_{\substack{a \to 1 \\ b \to 0}} \frac{a}{b} = \infty, \tag{2.54}$$

regardless of the manner in which a and b approach their limits. Subtracting 1 (the identity element) from dg before dividing by the timestep re-centers the numerator to be in the neighborhood of 0, allowing the limit in (2.52) to approach a meaningful finite value, in which the sign of $\overset{\circ}{g}$ indicates whether g is growing or shrinking over time.

Second, because g and its derivatives are real numbers, the relationship

$$\overset{\circ}{g} = g^{-1}\dot{g} \tag{2.55}$$

[9] This velocity is the *quotient-derivative* of g with respect to time. General definitions for the quotient-derivative and other aspects of *multiplicative calculus* are presented in the grey box on on the next page.

Multiplicative Calculus

Standard calculus in the Newton-Leibniz tradition is fundamentally based on the addition and subtraction of infinitesimal quantities. This property is well-suited to working on additive Lie groups, but does not adapt naturally to problems on multiplicative Lie groups, in which products and quotients replace sums and differences. On these groups, it is much more natural to use the *multiplicative calculus* paradigm introduced by Volterra. The two chief tools of multiplicative calculus are the *quotient-derivative*, which describes the rate of change of a function in terms of the ratio of its values at closely separated points, and the *product-integral*, which generalizes integration into a cumulative product.

Derivative. In standard calculus, the *(difference-)derivative*, u, represents a differential quantity that is continuously being added to the total value and which can be calculated as the limit of the difference between the function values at adjacent points as the step-size between the points goes to zero,

$$u(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}. \tag{2.xv}$$

In multiplicative calculus, the *quotient-derivative*, v, represents a differential quantity that is being continuously multiplied into the total, and is the limit of the *quotient* between consecutive function values, rather than their difference:

$$v(x) = \lim_{\delta x \to 0} \frac{\big(f(x + \delta x)/f(x)\big) - I}{\delta x}. \tag{2.xvi}$$

The inclusion of the identity element in the definition of the product-integral reflects this multiplicative nature—if v is a small change to the total, it must be incorporated as a small variation on the identity element.

Integral. The *product-integral* is the continuous counterpart to the product of a sequence, much in the same way that the standard integral is the continuous counterpart to summation:

	Continuous	Discrete
Addition	$\int$ Integral	$\sum$ Sum
Multiplication	$\prod\!\!\!\!\!\!\!\!\;\int$ Product-integral	$\prod$ Product

Just as the standard integral is defined via the limit

$$f(b) - f(a) = \int_a^b u(x)\, dx = \lim_{n \to \infty} \sum_{k=1}^{n} x(t_k)\, \delta x, \tag{2.xvii}$$

with $t_k = k\delta x$ and $\delta x = \frac{b-a}{n}$, the product integral is

$$\frac{f(b)}{f(a)} = \prod\!\!\!\!\!\!\!\!\;\int_a^b (I + v(x)\, dx) = \lim_{n \to \infty} \prod_{k=1}^{n} (I + v(x_k)\, \delta x), \tag{2.xviii}$$

with I the identity matrix (or equivalent identity element), and x_k and δx the same as for the standard integral.

continued ...

Multiplicative Calculus, *continued*

Left and right variations. If the order in which the multiplications are executed matters, (2.xviii) is specifically referred to as the *left product integral*,

$$\prod_a^b (I + v_L(x)\,dx) = \lim_{n\to\infty} (I + v_L(x_n)\,\delta x) \times (I + v_L(x_{n-1})\,\delta x) \times \ldots \times (I + v_L(x_0)\,\delta x), \tag{2.xix}$$

in which each element added to the sequence left-multiplies the previous total, and $v_L(t)$ is calculated as

$$v_L(x) = \lim_{\delta x\to 0} \frac{\big(f(x+\delta x)\,f^{-1}(x)\big) - I}{\delta x}. \tag{2.xx}$$

For many problems, it is more appropriate to use the *right product integral*

$$(I + v_R(x)\,dx) \prod_a^b = \lim_{n\to\infty} (I + v_R(x_k)\,\delta x) \prod_{k=1}^{n} \tag{2.xxi}$$

$$= \lim_{n\to\infty} (I + v_R(x_0)\,\delta x) \times (I + v_R(x_1)\,\delta x) \times \ldots \times (I + v_R(x_n)\,\delta x), \tag{2.xxii}$$

which places new elements at the right of the sequence, thus representing small changes *from* what came before, rather than *applied to* the existing total. In this case, the quotient-derivative should be calculated as

$$v_R(x) = \lim_{\delta x\to 0} \frac{\big(f^{-1}(x)\,f(x+\delta x)\big) - I}{\delta x}. \tag{2.xxiii}$$

Converting between multiplicative and traditional derivatives. For practical calculations, it is useful to convert between multiplicative and additive representations of velocity. To this end, we can convert the quotient-derivative expressions in (2.xx) and (2.xxiii) into difference-derivatives by extracting an $f^{-1}(x)$ term from the limit contents,

$$v_L(x) = \left(\lim_{\delta x\to 0} \frac{f(x+\delta x) - f(x)}{\delta x}\right) f^{-1}(x) = u(x)\,f^{-1}(x) \tag{2.xxiv}$$

and

$$v_R(x) = f^{-1}(x) \left(\lim_{\delta x\to 0} \frac{f(x+\delta x) - f(x)}{\delta x}\right) = f^{-1}(x)\,u(x). \tag{2.xxv}$$

Note that when these operations are applied to multiplicative groups of square matrices, v, u, and f^{-1} will be matrices of the same size as f.

has a concrete interpretation: for a given groupwise velocity $\overset{\circ}{g}$, the absolute velocity of a point g moving through the manifold $\mathbb{R}_+$ is *scaled proportionally to the value of its current position*, as illustrated in Figure 2.7. This interpretation is the differential form of the principle that incrementing a value by a given amount results in a greater proportional change when the starting value is small—an addition of 1 will double a starting value of 1, but only add ten percent to a starting value of 10. Applying the equality in (2.55) allows for the conversion between absolute and relative velocities through the group, for example finding the rate at which money should be added to a bank account (an absolute change), for a given interest rate (a relative velocity) and amount of money in the account (the current group value).

A second illustrative example of groupwise velocity is provided by the group of planar rotations, $SO(2)$. This group can be parameterized by a single value θ and represented as the group of 2×2 matrices with determinant 1,

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{2.56}$$

Velocities on this group are likewise described by single parameters, $\dot\theta$ for the manifold velocity and $\overset{\circ}{g}_\theta$ for the

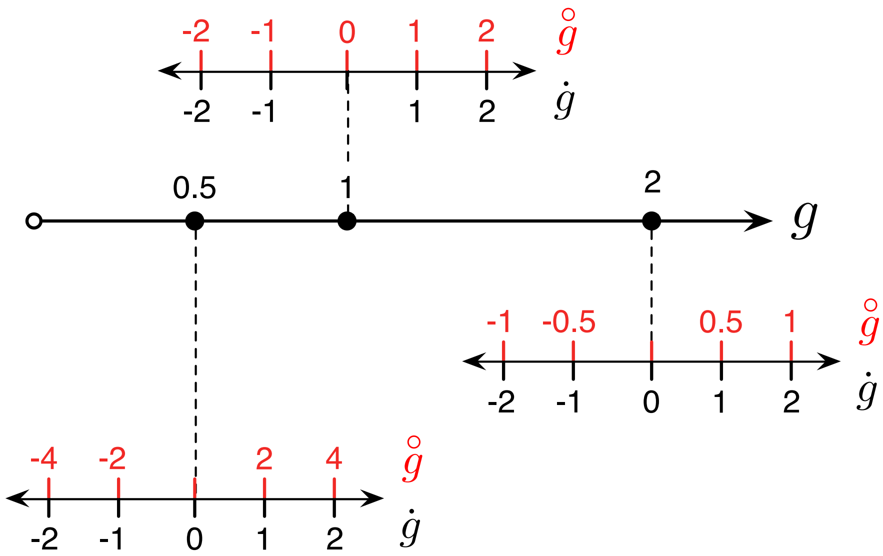


Figure 2.7 The group $G = (\mathbb{R}_+, \times)$ and three of its tangent spaces, $T_{0.5}G$, T_1G, and T_2G. If we assign one set of bases to these tangent spaces that measures $\dot{g}$ and a second basis set that measures $\overset{\circ}{g}$, these bases agree at the group identity, $g = 1$. Away from the identity, the $\overset{\circ}{g}$ bases are scaled relative to the $\dot{g}$ bases by the value of the group element—a given groupwise (proportional) velocity corresponds to greater or smaller parameter (absolute) velocities when the group value is respectively larger or smaller.

groupwise velocity. Applying the relationships in (2.82) and (2.83) to convert between these two representations of velocity highlights an important aspect of such transformations: g and g^{-1} are not applied directly to the set of $\overset{\circ}{g}_i$ or $\dot{g}_i$ coefficients, but instead *to their basis vectors*, as indicated in (2.91).

As a demonstration of this principle, consider the relationship $\dot{g} = g\,\overset{\circ}{g}$ applied to $SO(2)$. It clearly does *not* state that the two velocity parameters are related by a rotation matrix,

$$\dot{\theta} = R(\theta)\,\overset{\circ}{g}_\theta, \tag{2.57}$$

as this would be equating a scalar value on the left with a matrix on the right.[10] Rather, the relationship is between the full-basis forms of the manifold velocity, given in (2.1), and the groupwise velocity, given in (2.87). On the representation of $SO(2)$, these velocities respectively evaluate to

$$\dot{g}|_g = \overbrace{\left.\frac{\partial R(\theta)}{\partial \theta}\right|_\theta}^{\text{basis vector}} \dot{\theta} = \overbrace{\begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}}^{\text{basis vector}} \dot{\theta} \tag{2.58}$$

and

$$\overset{\circ}{g} = \overbrace{\left.\frac{\partial R(\theta)}{\partial \theta}\right|_{\theta=0}}^{\text{basis vector}} \overset{\circ}{g}_\theta = \overbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}^{\text{basis vector}} \overset{\circ}{g}_\theta, \tag{2.59}$$

in which the basis vectors are *matrix-valued*. If we insert the groupwise velocity expression from (2.59) into the

[10] There are of course deeper reasons why (2.57) is wrong, but the fact that it doesn't even parse at a numerical level indicates an incorrect formulation.

relationship in (2.83), we can then calculate the manifold velocity as

$$\dot{g}|_g = g\,\overset{\circ}{g} \tag{2.60}$$

$$= R(\theta)\left.\frac{\partial R(\theta)}{\partial \theta}\right|_{\theta=0} \overset{\circ}{g}_\theta \tag{2.61}$$

$$= \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}0 & -1 \\ 1 & 0\end{bmatrix}\overset{\circ}{g}_\theta \tag{2.62}$$

$$= \begin{bmatrix}-\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta\end{bmatrix}\overset{\circ}{g}_\theta. \tag{2.63}$$

By comparing this result with our initial expression for $\dot{g}$ in (2.58),

$$\begin{bmatrix}-\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta\end{bmatrix}\dot{\theta} = \begin{bmatrix}-\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta\end{bmatrix}\overset{\circ}{g}_\theta, \tag{2.64}$$

we see that the two expressions are compatible—both sides of (2.64) are a scalar value multiplied by matching matrices.

put extrinsic bases and scale/rotation matrix here

$$\begin{bmatrix}\dfrac{\partial kR}{\partial k} & \dfrac{\partial kR}{\partial \theta}\end{bmatrix}\begin{bmatrix}\dot{k} \\ \dot{\theta}\end{bmatrix} = kR\begin{bmatrix}\left.\dfrac{\partial kR}{\partial k}\right|_{\substack{\theta=0\\k=1}} & \left.\dfrac{\partial kR}{\partial \theta}\right|_{\substack{\theta=0\\k=1}}\end{bmatrix}\begin{bmatrix}\overset{\circ}{g}_k \\ \overset{\circ}{g}_\theta\end{bmatrix} \tag{2.65}$$

$$\tag{2.66}$$

$$\left(\overbrace{\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}}^{\text{first basis vector at }(k,\theta)}\quad\overbrace{\begin{bmatrix}-k\sin\theta & -k\cos\theta \\ k\cos\theta & -k\sin\theta\end{bmatrix}}^{\text{second basis vector at }(k,\theta)}\right)\begin{bmatrix}\dot{k} \\ \dot{\theta}\end{bmatrix} = kR\left(\overbrace{\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}}^{\text{first basis vector at }\mathbf{e}}\quad\overbrace{\begin{bmatrix}0 & -1 \\ 1 & 0\end{bmatrix}}^{\text{second basis vector at }\mathbf{e}}\right)\begin{bmatrix}\overset{\circ}{g}_k \\ \overset{\circ}{g}_\theta\end{bmatrix} \tag{2.67}$$

2.3 END OF CURRENT TEXT

The first principle behind groupwise definitions of velocity is that displacements on a groups are fundamentally characterized by group actions,

$$g_\Delta = g_1^{-1} \circ g_2, \tag{2.68}$$

and that if the group's underlying manifold is continuously differentiable, (i.e., the group is a *Lie group*[11]), then g_1 and g_2 can be infinitesimally close to each other, and the group action separating two configurations an infinitesimal timestep apart is

$$g_\delta = g^{-1}(t) \circ g(t+\delta t). \tag{2.69}$$

Given this axiom, it then follows that velocities on the group are fundamentally characterized by the rate at which group actions are being applied to the configuration,

$$\overset{\circ}{g} = \lim_{\delta t \to 0} \frac{g_\delta - \mathbf{e}}{\delta t} = \lim_{\delta t \to 0} \frac{\big(g^{-1}(t) \circ g(t+\delta t)\big) - \mathbf{e}}{\delta t}, \tag{2.70}$$

where the group's identity element $\mathbf{e}$ is subtracted[12] from the infinitesimal group element prior to dividing by the change in time so that the limit converges.

This subtraction leaves a remainder that is the difference between g_δ and the identity element. Such differences (and consequently, group velocities $\overset{\circ}{g}$) are in the tangent space to G around its identity element, $T_{\mathbf{e}}G$, which corresponds to the group's *Lie algebra*, $\mathfrak{g}$.[13]

[11] See the box on page ??.

[12] Note that the "subtraction" operation that removes the identity element in (2.78) and the "addition" operations in the series expansions of (2.85) and (2.42), are not themselves related to the group operation. Instead, they are operations that allow us to decompose or construct group elements.

[13] See the box on page 45.

Much in the same way that the motion of a system with a specified configuration velocity $\dot{q}$ can be integrated as

$$q(t + dt) = q(t) + \dot{q}\, dt, \tag{2.71}$$

the evolution of a system's configuration on a Lie group with a specified group velocity $\overset{\circ}{g}$ can be integrated as

$$g(t + dt) = g(t) \circ (\mathbf{e} + \overset{\circ}{g}\, dt). \tag{2.72}$$

Many analytical and numerical integration techniques, however, are much more suited to working with equations of the form in (2.79) than those in (2.80). To this end, it is often useful to define a problem in terms of $\overset{\circ}{g}$ vectors—which capture the essential behavior of the system—and then convert them into $\dot{g}$ vectors that represent the same differential change in configuration over time, but can be integrated via addition rather than the group operation.

The first step in making this conversion is to note that because a group's Lie algebra is the tangent space at its identity element, vectors in $\mathfrak{g}$ have a natural interpretation as infinitesimal displacements from, or velocities through, the origin/identity point on the configuration manifold. As noted in §1.2.1, however, we can select any point on the manifold to correspond to the identity element. Here, this principle means that we can associate $\mathfrak{g}$ with the tangent space at any or all points on the manifold and use it to propagate a set of groupwise-equivalent bases across the tangent bundle. With these bases in place, any given $\overset{\circ}{g}$ in the Lie algebra has a well-defined equivalent $\dot{g}$ vector in each tangent space.

To construct a mapping between group velocities and their corresponding velocity vectors in the configuration manifold's tangent spaces, we can use the principle that $\mathbf{e} = g^{-1}g$ to factor out a $g^{-1}(t)$ term from the group velocity formulation in (2.78),[14]

$$\overset{\circ}{g} = g^{-1}(t) \left(\lim_{\delta t \to 0} \frac{g(t + \delta t) - g(t)}{\delta t} \right), \tag{2.73}$$

to find

$$\overset{\circ}{g} = g^{-1}\, \dot{g} \tag{2.74}$$

and

$$\dot{g} = g\, \overset{\circ}{g}. \tag{2.75}$$

These operations can be respectively interpreted as finding the groupwise velocity corresponding to a given velocity in the tangent space at g, and sending out a groupwise velocity from $T_{\mathbf{e}}G$ to find its equivalent velocity in T_gG.

These notions of relative velocity can be captured by *Lie groups* and their associated structures, *Lie algebras*.[15] A Lie[16] group is one in which the underlying set forms a smooth manifold, and thus has a tangent bundle. This category includes most of the example groups from Chapter 1, including $(\mathbb{R}, +)$, $(\mathbb{R}_+, \times)$, $SO(2)$, and $SE(2)$, but excludes discrete groups such as $(\mathbb{Z}, +)$. The group's Lie algebra is the set of all infinitesimal actions on the group. finish this paragraph

The core concept in using Lie groups and algebras to study system velocities is that each tangent vector dq on the manifold corresponds to an infinitesimal group action g_δ, and thus that each velocity vector $\dot{q} = dq/dt$ correspond to the rate at which infinitesimal group actions are being applied to the configuration, $\overset{\circ}{g} = g_\delta/dt$. Because the infinitesimal group element conform to the group structure, they carry the same sense of locality and relative motion as does the full group: $\overset{\circ}{g}$ is the velocity of the system relative to its current configuration, and a system moving at a constant $\overset{\circ}{g}$ experiences the same group displacement g over each unit time,[17] as illustrated in Figure 2.8.

The relationship between Lie algebra elements and tangent vectors is most directly seen at the group identity,

[14] This factorization makes use of the property that as differentiable manifolds, Lie groups are locally diffeomorphic to $\mathbb{R}^n$; this means that subtraction between infinitesimally-separated elements is a supported operation, even if the group action itself is not additive. It also assumes that the group operation $\circ$ is distributive as in multiplication, so that $a \circ (b - c) = (a \circ b) - (a \circ c)$. Group velocities on additive groups cannot be factored in this way, but as we discuss in §2.2.2, they simplify directly to $\overset{\circ}{g} = \dot{g}$.

[15] See the box on page 45.

[16] "Lie" is pronounced "Lee." Sophus Lie originated the study of continuous groups.

[17] The relationship between $\overset{\circ}{g}$ and g is given by the generalized exponential and logarithmic functions, as we discuss in §**??**.

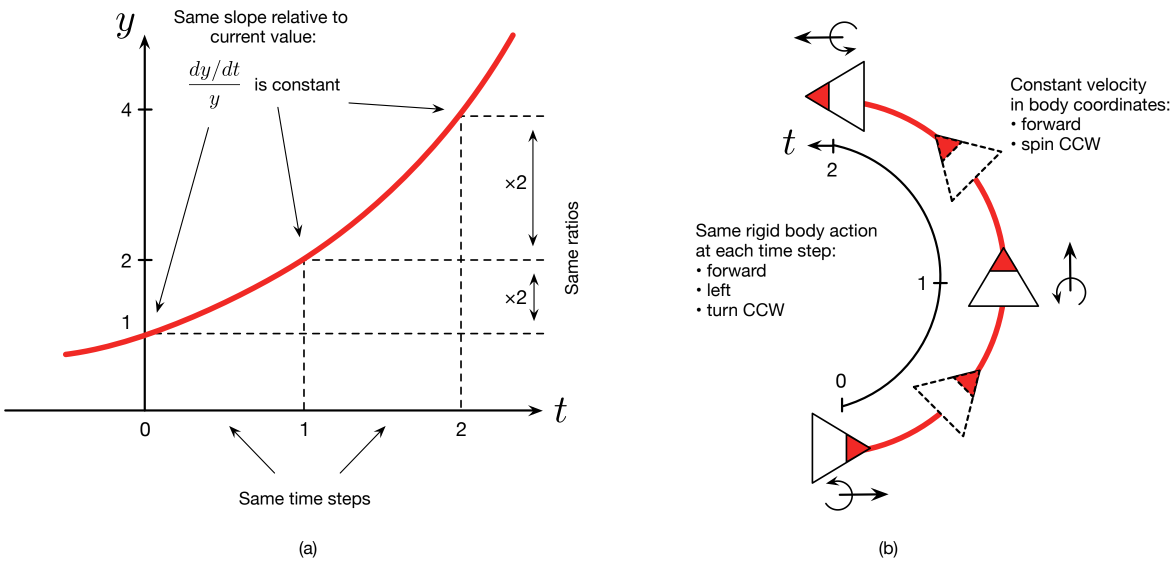


Figure 2.8 Groupwise velocity.

as $\mathfrak{g}$ is a linearization of G around $\mathbf{e}$, and thus has an inherent equivalence with $T_{\mathbf{e}}G$. One of the principles of using groups as configuration spaces, however, is that we can choose any group element as the identity—consequently, $\mathfrak{g}$ is identifiable with *every* tangent space T_gG on the manifold. The way in which the Lie algebra propagates out along the tangent bundle is itself closely tied to the group structure, and may be handled in two ways: intrinsically, where all velocities that are equivalent under the Jacobian of the group action (as per §2.1.4) correspond to the same element of $\mathfrak{g}$; and extrinsically, where transforming the basis of $\mathfrak{g}$ by g gives its representation at T_gG.

In §**??**, we introduced group structure as a way of composing and comparing configurations more meaningfully than by simply adding and subtracting parameter values. On Lie groups (which are by definition continuous and differentiable), we can lift this concept into the tangent bundle, using group structure to interpret system velocities in the context of the current configuration.

For example, actions on the multiplicative group $(\mathbb{R}_+, \times)$ scale elements proportionally to their current value, and the groupwise velocity on this space measures the instantaneous rate at which the scaling is taking place. This "rate of scaling" corresponds to the way in which interest on a bank account is typically specified: not in terms of the *absolute* rate at which money is added to the account (which changes as the account grows), but as a *proportional* rate at which money is added, relative to the current value. Similarly, group velocities on $SE(2)$ identify the velocity of a rigid body *relative to its current position*, rather than with respect to *absolute directions* in a fixed frame.

Groupwise velocities are elements of a vector space called the group's *Lie algebra*. The Lie algebra is a linearization of the group around its identity element, and has a natural equivalence and shared basis with the tangent space at the origin of the group's underlying manifold. Because Lie groups are self-similar, the Lie algebra can also be associated with any other tangent space on the manifold, using information from the group structure to align their respective bases. The group's *lifted actions* (derivatives of the group actions) then relate velocities (expressed in the manifold's parameter bases) that represent the same groupwise velocities across different locations in the manifold.

mention exponential map and logarithm here

Update this to reflect new ordering with lifted actions definition first

The concepts and notation for groupwise velocity are similar to those encountered in more traditional treatments of calculus and differential equations, but include some subtle differences. In the following subsections, we introduce this notation in stages, starting with basic, coordinate-free definitions in §2.3.2. We follow these definitions with expressions for group velocity in terms of coordinates in §2.3.3 and examples on simple groups in §2.2.2 and §2.3.4.

Lie Algebras

The *Lie algebra* of a Lie group G is the vector space of infinitesimal group actions, and is closely related to the tangent space around the identity element of the group, $T_{\mathbf{e}}G$. Lie algebras are equipped with operators called *Lie brackets* that capture structural information about their associated groups, such as whether or not they are commutative. Vectors in the Lie algebra can also be *exponentiated* to generate elements of the group. Bracketing and exponentiation play important roles in Lie group theory and in the analysis of systems modeled as Lie groups. We discuss exponentiation in §**??**, and Lie brackets are a dominant theme in Chapter 5.

Notation. Lie algebras are customarily denoted by a lowercase Fraktur ("German-style") symbol with the same spelling as the group. For example, the Lie algebra of G is $\mathfrak{g}$ and the Lie algebra of $SE(2)$ is $\mathfrak{se}(2)$.

The calligraphic style of Fraktur is difficult to achieve in handwriting on paper or a blackboard, so various conventions are used in these situations. In some cases, such as the group G, Sütterlin script (the handwriting style contemporary with Fraktur) provides a symbol $\mathfrak{g}$ that is close enough to the standard English alphabet to be recognizable as a variation on the group name, but distinct enough not to cause confusion with other symbols already in use. Many other Sütterlin letters, however, are not simple cognates to their English equivalents, and may even be confusingly close to other letters. If it will not lead to ambiguity, a simple lowercase rendition of the groupname may be sufficient when handwriting, as in rendering $\mathfrak{se}(2)$ as $se(2)$.

Representation. Lie algebra elements have matrix representations corresponding to the representations of their parent groups. The extrinsic bases for the Lie algebra representation are the derivatives of the group representation around the identity,

$$u_i = \left.\frac{\partial \rho(g)}{\partial g_i}\right|_{g=\mathbf{e}}. \tag{2.xxvi}$$

In the same manner as the notation $\rho(g)$ explicitly identifies the representation of a group element, the "hat" operator constructs the representation of a Lie algebra vector from its component values and extrinsic bases: for $\xi \in \mathfrak{g}$,

$$\xi \equiv \begin{bmatrix} \xi^1 \\ \xi^2 \\ \vdots \\ \xi^n \end{bmatrix} \quad \text{and} \quad \hat{\xi} \equiv \sum \left.\frac{\partial \rho(g)}{\partial g_i}\right|_{g=\mathbf{e}} \xi^i. \tag{2.xxvii}$$

2.3.1 Groupwise Velocity via Lifted Actions

2.3.2 Groupwise Velocity Definitions and Notation

The core idea behind groupwise definitions of velocity is that displacements on a Lie group are fundamentally characterized by group actions, bring first part of this into section introduction

$$g_\Delta = g_1^{-1} \circ g_2, \tag{2.76}$$

and thus that the group action separating two configurations an infinitesimal timestep apart is

$$g_\delta = g^{-1}(t) \circ g(t + \delta t). \tag{2.77}$$

Given this axiom, it then follows that velocities on the group are fundamentally characterized by the rate at which group actions are being applied to the configuration,

$$\overset{\circ}{g} = \lim_{\delta t \to 0} \frac{g_\delta - \mathbf{e}}{\delta t} = \lim_{\delta t \to 0} \frac{g^{-1}(t) \circ g(t + \delta t) - \mathbf{e}}{\delta t}, \tag{2.78}$$

where the group's identity element $\mathbf{e}$ is subtracted[18] from the differential group action prior to dividing by the

[18] Note that the "subtraction" operation that removes the identity element in (2.78) and the "addition" operations in the series expansions

↓ Vectors corresponding to a given **right** action
→ are equivalent under **left** group actions

$$\begin{tikzcd}
\mathbf{e} \arrow[r, "g\circ"] \arrow[d, "\circ g_\delta", "+dg"'] & g \arrow[d, "\circ g_\delta", "+J_{g\circ} dg"'] \\
g_\delta \arrow[r, "g\circ"'] & g g_\delta
\end{tikzcd}$$

↓ Vectors corresponding to a given **left** action
→ are equivalent under **right** group actions

$$\begin{tikzcd}
\mathbf{e} \arrow[r, "\circ g"] \arrow[d, "g_\delta\circ", "+dg"'] & g \arrow[d, "g_\delta\circ", "+J_{\circ g} dg"'] \\
g_\delta \arrow[r, "\circ g"'] & g_\delta g
\end{tikzcd}$$

Figure 2.9 Left and right equivalence of vectors (will have more text here).

change in time. This subtraction leaves a remainder that is the difference between g_δ and the identity element. Such differences (and consequently, group velocities $\mathring{g}$) are in the tangent space to G around its identity element,
break for new text $T_{\mathbf{e}}G$, which corresponds to the group's *Lie algebra*, $\mathfrak{g}$.[19]

Much in the same way that the motion of a system with a specified configuration velocity $\dot{q}$ can be integrated as

$$q(t + dt) = q(t) + \dot{q}\, dt, \tag{2.79}$$

the evolution of a system's configuration on a Lie group with a specified group velocity $\mathring{g}$ can be integrated as

$$g(t + dt) = g(t) \circ (\mathbf{e} + \mathring{g}\, dt). \tag{2.80}$$

Many analytical and numerical integration techniques, however, are much more suited to working with equations of the form in (2.79) than those in (2.80). To this end, it is often useful to define a problem in terms of $\mathring{g}$ vectors—which capture the essential behavior of the system—and then convert them into $\dot{g}$ vectors that represent the same differential change in configuration over time, but can be integrated via addition rather than the group operation.

The first step in making this conversion is to note that because a group's Lie algebra is the tangent space at its identity element, vectors in $\mathfrak{g}$ have a natural interpretation as infinitesimal displacements from, or velocities through, the origin/identity point on the configuration manifold. As noted in §1.2.1, however, we can select any point on the manifold to correspond to the identity element. Here, this principle means that we can associate $\mathfrak{g}$ with the tangent space at any or all points on the manifold and use it to propagate a set of groupwise-equivalent bases across the tangent bundle. With these bases in place, any given $\mathring{g}$ in the Lie algebra has a well-defined equivalent $\dot{g}$ vector in each tangent space.

To construct a mapping between group velocities and their corresponding velocity vectors in the configuration manifold's tangent spaces, we can use the principle that $\mathbf{e} = g^{-1}g$ to factor out a $g^{-1}(t)$ term from the group velocity formulation in (2.78),[20]

$$\mathring{g} = g^{-1}(t)\left(\lim_{\delta t \to 0} \frac{g(t+\delta t) - g(t)}{\delta t}\right), \tag{2.81}$$

to find

$$\mathring{g} = g^{-1}\, \dot{g} \tag{2.82}$$

and

$$\dot{g} = g\, \mathring{g}. \tag{2.83}$$

of (2.85) and (2.42), are not themselves related to the group operation. Instead, they are operations that allow us to decompose or construct group elements.

19 See the box on the previous page.

20 This factorization makes use of the property that as differentiable manifolds, Lie groups are locally diffeomorphic to $\mathbb{R}^n$; this means that subtraction between infinitesimally-separated elements is a supported operation, even if the group action itself is not additive. It also assumes that the group operation $\circ$ is distributive as in multiplication, so that $a \circ (b - c) = (a \circ b) - (a \circ c)$. Group velocities on additive groups cannot be factored in this way, but as we discuss in §2.2.2, they simplify directly to $\mathring{g} = \dot{g}$.

These operations can be respectively interpreted as finding the groupwise velocity corresponding to a given velocity in the tangent space at g, and sending out a groupwise velocity from $T_{\mathbf{e}}G$ to find its equivalent velocity in T_gG.

2.3.3 Groupwise Velocities in Coordinates

If we have assigned a set of parameters g_i to the group G, its Lie algebra $\mathfrak{g}$ acquires a natural set of basis vectors

$$u_{i,\mathbf{e}} = \left.\frac{\partial g}{\partial g_i}\right|_{\mathbf{e}}, \tag{2.84}$$

corresponding to infinitesimal changes in the group element under differential change in the parameters, evaluated at the identity element. They can also be thought of as the first-order component of the series expansion of the group elements around the identity in terms of the parameters,[21]

$$g = \mathbf{e} + \sum_i \left.\frac{\partial g}{\partial g_i}\right|_{\mathbf{e}} g_i + \frac{1}{2}\left(\sum_i \left.\frac{\partial^2 g}{\partial g_i^2}\right|_{\mathbf{e}} g_i^2 + \sum_{i\neq j} \left.\frac{\partial^2 g}{\partial g_i \partial g_j}\right|_{\mathbf{e}} g_i g_j\right) + \cdots. \tag{2.85}$$

The second and higher-order terms in this series become negligible in the neighborhood of the identity element, leaving infinitesimal group elements as

$$g_\delta = \lim_{g\to\mathbf{e}} g \;=\; \mathbf{e} \;+\; \sum_i \left.\frac{\partial g}{\partial g_i}\right|_{\mathbf{e}} g_\delta^i \tag{2.86}$$

and group velocities as

$$\overset{\circ}{g} = \sum_i \left.\frac{\partial g}{\partial g_i}\right|_{\mathbf{e}} \overset{\circ}{g}{}^i, \tag{2.87}$$

where $\overset{\circ}{g}{}^i$ is g_δ^i divided by dt.[22] If we take the basis vectors as implicit, we can identify differential group elements and group velocities by their coefficients as we did in §2.1,

$$g_\delta = (g_\delta^1, g_\delta^2, \ldots, g_\delta^n) \tag{2.88}$$

and

$$\overset{\circ}{g} = (\overset{\circ}{g}{}^1, \overset{\circ}{g}{}^2, \ldots, \overset{\circ}{g}{}^n). \tag{2.89}$$

When using this convention, however, it is important to note that applying a linear transformation (such as a matrix multiplication) to $\overset{\circ}{g}$ is in general *not* equivalent to applying it to the vector of $\overset{\circ}{g}$ coefficients, i.e.,

idea here is to emphasize that groupderiv uses extrinsic vectors. will come back and update

$$A\,\overset{\circ}{g} \neq A \begin{bmatrix} \overset{\circ}{g}{}^1 \\ \vdots \\ \overset{\circ}{g}{}^n \end{bmatrix}. \tag{2.90}$$

This difference is because linear transformations are most correctly thought of as applying to the bases of a vector,

$$A\,\overset{\circ}{g} = \sum_i \left(A \left.\frac{\partial g}{\partial g_i}\right|_{\mathbf{e}} \right) \overset{\circ}{g}{}^i, \tag{2.91}$$

which is only equivalent to multiplying the transformation by the list of coefficients if the basis is a set of orthogonal unit vectors, and the basis vectors defined in (2.84) have a more "interesting" structure. Recognizing this distinction is especially important when mapping between $\overset{\circ}{g}$ and $\dot{g}$ as in (2.82) and (2.83). We will examine the details of such transformations, and the associated notion of *lifted actions* that find transformations of the $\overset{\circ}{g}_i$ coefficients that are consistent with transformations of $\overset{\circ}{g}$, after considering some examples of group velocity.

[21] In (2.85) and (2.42), we assume that the group parameters are zero at the identity element. If the identity is associated with a different point in the parameter space, then the g_i coefficients should instead be $g_i - g_i|_e$.

[22] Note that this quantity is distinct from the rate of change of the ith coordinate of g.

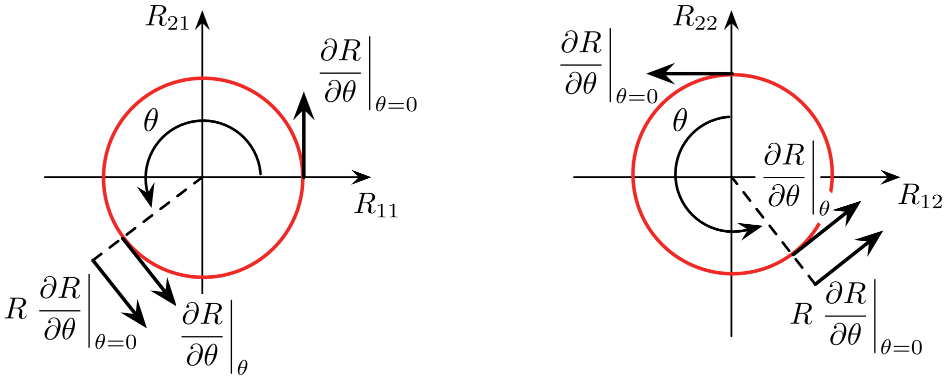


Figure 2.10 The representation $R(\theta) = \left[\begin{smallmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{smallmatrix}\right]$ for $SO(2)$ corresponds to a circle embedded in a 4-dimensional space (the space of all 2×2 matrices), which we can visualize by splitting it into components representing the first and second columns. The basis vector for the tangent space at θ, $\left[\begin{smallmatrix}-\sin\theta & -\cos\theta\\ \cos\theta & -\sin\theta\end{smallmatrix}\right]$, is equal to $R(\theta)$ multiplied by the basis vector at $\theta = 0$, $\left[\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right]$. This means that the bases for $\dot{g}$ and $\mathring{g}$ are identical, so that $\dot{\theta} = \mathring{g}_\theta$ for all θ.

2.3.4 Lifted Actions

It is often convenient to map directly between the coefficients of a system's groupwise and manifold velocities, bypassing the basis operations detailed in (2.60)–(2.63). For example, the key information in (2.55) is that the coefficients of $\mathring{g}$ and $\dot{g}$ are scaled proportionally to each other by g, and the key information in (2.64) is that the coefficients on the two velocities are equal,

$$\dot{\theta} = \mathring{g}_\theta, \tag{2.92}$$

independent of the current value of θ. Such mappings correspond to a group's *lifted actions*.

$$\sum_i u_{i,g}\, \dot{g}_i = \sum_i (g\, u_{i,\mathbf{e}})\, \mathring{g}_i = \dot{g} \tag{2.93}$$

extrinsic: basis vectors get magnitude/direction meaning from manifold intrinsic: basis vectors define magnitudes/directions

text building here about how basis vectors for tangent spaces have length and orientation in 3-space, and how this idea holds even if we don't make an explicit embedding

Extrinsic bases

$f, g, h \in G$

$$f = (f_1, \ldots, f_n) \tag{2.94}$$
$$g = (g_1, \ldots, g_n) \tag{2.95}$$
$$h = (h_1, \ldots, g_n) \tag{2.96}$$

$$f = hg \tag{2.97}$$

$$T_g L_h = \begin{bmatrix} \partial f_1/\partial g_1 & \cdots & \partial f_1/\partial g_n \\ \vdots & \ddots & \vdots \\ \partial f_n/\partial g_1 & \cdots & \partial f_n/\partial g_n \end{bmatrix} \tag{2.98}$$

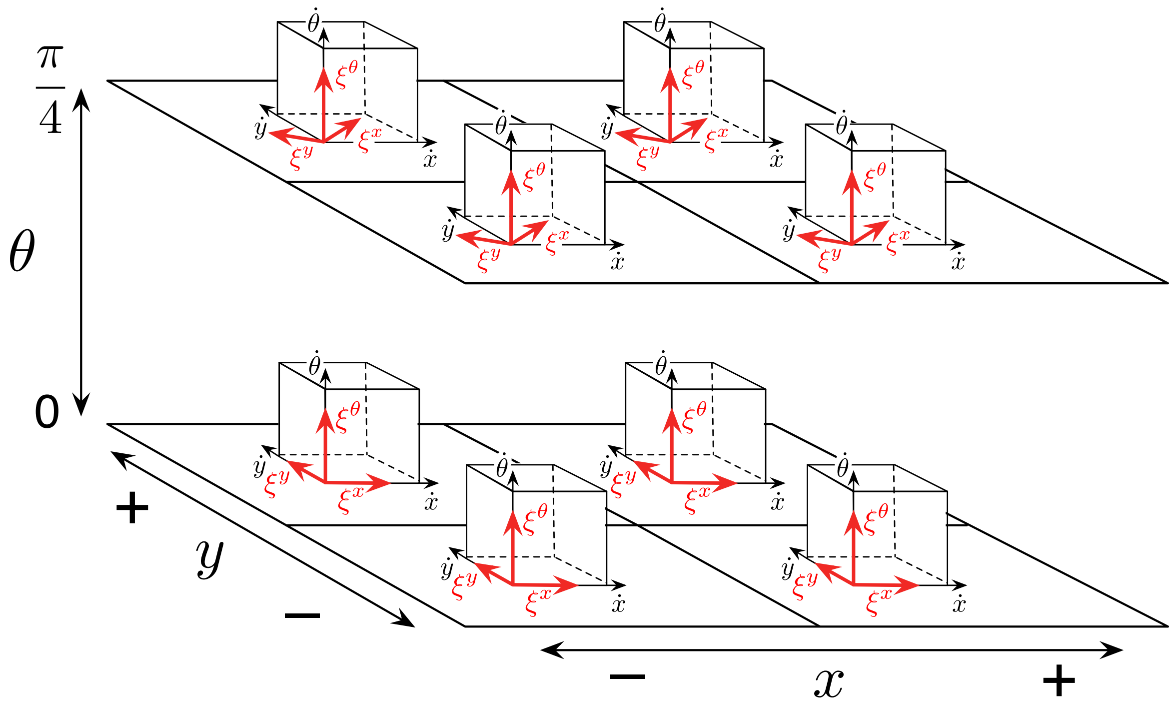


Figure 2.11 Bases for the body velocity, shown relative to the bases for coordinate velocity. The transformation between these bases is independent of the x and y position, but the ξ^x and ξ^y components are rotated relative to $\dot{x}$ and $\dot{y}$ by the θ orientation.

$$\dot{g} = g\mathring{g} = gU|_{\mathbf{e}} \begin{bmatrix} \mathring{g}^1 \\ \vdots \\ \mathring{g}^n \end{bmatrix} = U|_g T_{\mathbf{e}} L_g \begin{bmatrix} \mathring{g}^1 \\ \vdots \\ \mathring{g}^n \end{bmatrix} = U|_g \begin{bmatrix} \dot{g}^1 \\ \vdots \\ \dot{g}^n \end{bmatrix} \tag{2.99}$$

2.4 Rigid Body Velocities

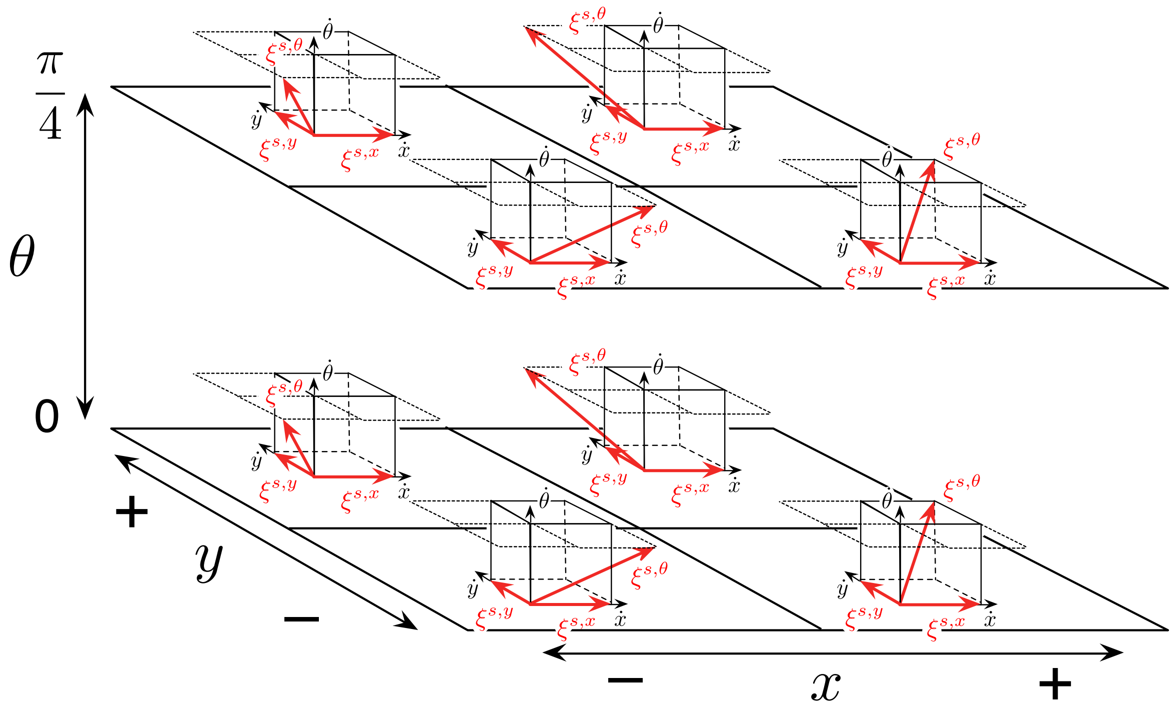


Figure 2.12 Bases for the spatial velocity, shown relative to the bases for coordinate velocity. The transformation between these bases is independent of the θ orientation, and the $\xi^{s,x}$ and $\xi^{s,y}$ components are always aligned with $\dot{x}$ and $\dot{y}$. The $\xi^{s,\theta}$ components, however, also make a contribution in the $\dot{x}$ and $\dot{y}$ directions that depends on the x and y positions: Rotation at the origin sweeps points away from the origin through a circle.

3
Forward Kinematics

In the first two chapters, we presented a framework for describing the configuration and velocity of points and rigid bodies. In this chapter, we extend the basic principles of the framework to articulated systems composed of multiple rigid bodies, and then to flexible systems with continuous modes of deformation. A key aspect of this extension is the notion of *forward kinematics* relating the physical motion of the system to changes in its configuration. As we explore this subject, we introduce connections to fundamental mathematical concepts such as *holonomic constraints* and *Jacobians*.

3.1 Forward Kinematics

The full configuration space of a set of n planar rigid bodies is the direct product of their individual configuration spaces,

$$SE(2) \times SE(2) \times \ldots \times SE(2) = SE(2)^n. \tag{3.1}$$

If the bodies are linked, however, not all of these configurations are accessible. For example, the two bodies in Figure 3.1 are pinned together at the origins of their body frames, so the system as a whole can only achieve configurations in which $(x, y)_1 = (x, y)_2$.

One approach to working with such constraints is to incorporate them into the system's equations of motion. A more elegant approach is to recognize that the pin imposes a pair of *holonomic constraints*[1] on the system, reducing the effective configuration space from six dimensions ($SE(2)^2$) to four. The reduced space, the *accessible manifold*, is the configuration space of the constrained system (in this case $SE(2) \times \mathbb{S}^1$ for the full position of one body and the relative orientation of the second); if generalized coordinates on this space (such as joint angles) are used to represent the system's configuration, the holonomic constraints are automatically respected.

Despite the usefulness of this reduction, one thing that it does not do is to remove the dependence of the

[1] See the box on the following page.

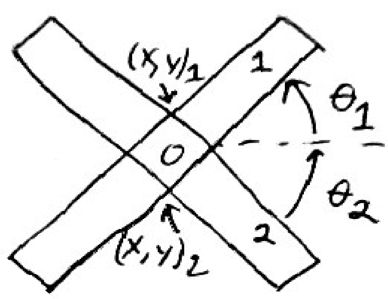


Figure 3.1 Articulated system are composed of multiple rigid bodies with constraints on their relative positions.

Holonomic Constraints

Holonomic constraints remove degrees of freedom from a system, reducing the dimensionality of its configuration space. Formally, a holonomic constraint is defined as a (possibly time-varying) constraint function f on the system's configuration space Q. The *zero set* of the function (set of points $Q_0 \subset Q$ satisfying $f(q_0, t) = 0$ for all $q_0 \in Q_0$) forms the *accessible manifold* of the constrained system, the set of configurations satisfying the constraint.

When a holonomic constraint is applied to an n-dimensional manifold, the resulting accessible manifold is $(n - 1)$-dimensional. As it contains all of the admissible configurations of the system, this accessible manifold is itself the configuration space of the constrained system, and in many cases can be used in place of the full configuration space. Multiple holonomic constraints act in concert: the accessible manifold for a multi-constrained system is the intersection of the individual accessible manifolds, and each independent constraint reduces the manifold dimension by one.

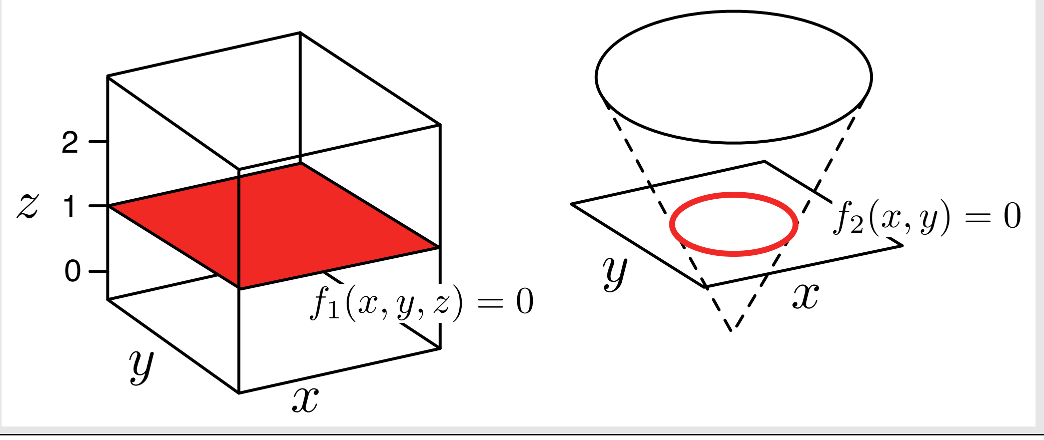
As an example, consider the problem of restricting a point $p = (x, y, z) \in \mathbb{R}^3$ to move only on the boundary of a unit circle on the plane $z = 1$. The planar condition corresponds to the holonomic constraint function

$$f_1(p, t) = z - 1, \tag{3.i}$$

which has a zero set (and accessible manifold) at the $z = 1$ plane. Once this constraint is made, the configuration space of the system is reduced by one dimension and effectively becomes $p_1 = (x, y) \in \mathbb{R}^2$. Restricting the point to the unit circle is accomplished by the constraint function

$$f_2(p_1, t) = \sqrt{(x^2 + y^2)} - 1. \tag{3.ii}$$

with the final accessible manifold formed by the intersection of a cone representing f_2 with the xy plane. Note that points on the f_2 cone are *not* elements of the original $\mathbb{R}^3$ space.



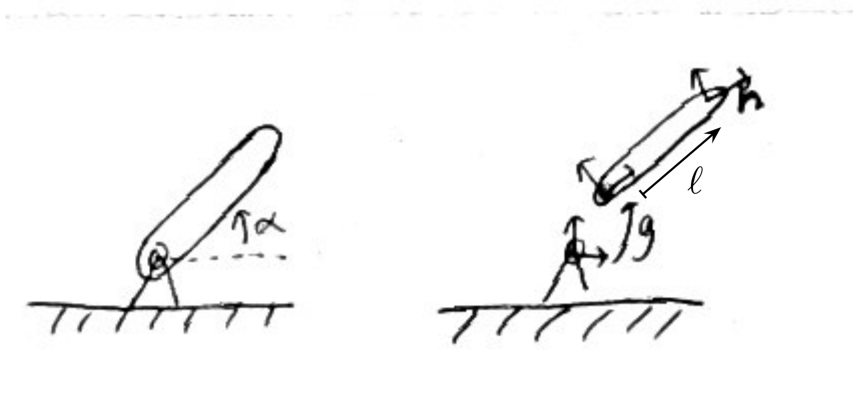


Figure 3.2 A pivoted link (left) can be viewed as a rigid body at position g with respect to the pivot and restrictions $x = 0$ and $y = 0$ on the possible values of g.

system dynamics on the actual physical positions of the component bodies; these positions must be taken into account when constructing a system's equations of motion. Similarly, tasks such as robotic manipulation are generally specified not in terms of the generalized coordinates, but rather with respect to the location and orientation of an end effector. *Forward kinematics* describe the relationship between these physical positions and the system's generalized coordinates.

3.2 Fixed-base Arms

A simple yet informative example of an articulated system is a single link attached to a fixed pivot, as illustrated at the left of Figure 3.2. Clearly, its configuration can be described by as a single generalized coordinate $\alpha \in \mathbb{S}^1$ specifying the angle the link makes with respect to a reference line, as discussed in §1.1. From a broader perspective, however, we can view the link as being a rigid body at position and orientation $g \in SE(2)$ with respect to the pivot, as shown at the right of Figure 3.2. The pin joint imposes the two holonomic constraints

$$x = 0 \qquad \text{and} \qquad y = 0 \tag{3.2}$$

on the link, restricting the free variables in g to only the orientation component of $SE(2)$. As noted in **??**, this orientation component is isomorphic to the circle,

$$g = \overbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{SE(2),\ x,y=0} \equiv \overbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}}^{SO(2)} \equiv \overbrace{\alpha}^{\mathbb{S}^1} \tag{3.3}$$

bringing the full rigid-body interpretation into agreement with the simple interpretation of the system.

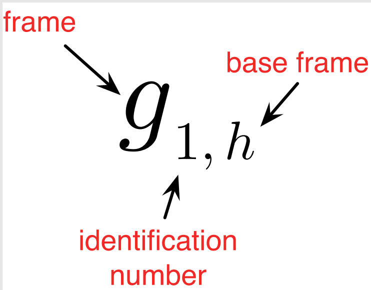
Using the full rigid body interpretation systematizes the process of determining the positions and orientations of other frames attached to the rigid body, and from there the positions and orientations of other linked bodies. Continuing the single-link example, the world position of a second body-attached frame at the other end of the link and aligned with the first frame (*i.e.* at $h_g = (\ell, 0, 0)$ with respect to g as in Figure 3.2) is

$$h = gh_g = \begin{bmatrix} \cos\alpha & -\sin\alpha & \ell\cos\alpha \\ \sin\alpha & \cos\alpha & \ell\sin\alpha \\ 0 & 0 & 1 \end{bmatrix} \tag{3.4}$$

This position can be interpreted as either the left action $L_g h_g$ transforming the frame position h_g by g, or as the right action $R_{h_g} g$ placing h_g into g. In §3.5, we use the right action interpretation to facilitate determination of systems' velocity kinematics.

A Note on Notation

When working with a kinematic chain or other collection of rigid bodies, it is convenient to use notation that concisely describes both the absolute positions of frames on the bodies and their positions relative to each other. In this book, we use the notation illustrated below:



In this notation, a frame and its position are designated by a letter with two optional subscripts. The first subscript is an identification number used to distinguish between frames that share the same letter, such as two frames at corresponding positions on different bodies. This subscript may be omitted if there is no such ambiguity (as in the single link arm at the beginning of §3.2) or to indicate a privileged frame (such as the system body frames in §3.3).

The second subscript indicates the base frame that the frame's coordinates are defined with respect to; for example, $g_{1,h}$ denotes the position of frame g_1 with respect to frame h,

$$g_{1,h} = h^{-1} g_1. \tag{3.iii}$$

If this subscript is absent, the frame position is with respect to the origin, $g_1 = g_{1,\mathrm{e}}$; for such frames, we will only use the latter notation if a single subscript would introduce ambiguity as to its meaning.

This definition gives rise to a pair of simple rule for concatenating transforms together:

1. Frames on the left cancel with subscripted frames on the right,

$$g h_g = g g^{-1} h = h \tag{3.iv}$$

2. During this cancellation, base-frame subscripts on the left are transferred to the right:

$$g_{1,g_0} h_{g_1} = g_0^{-1} (g_1 g_1^{-1}) h = g_0^{-1} h = h_{g_0} \tag{3.v}$$

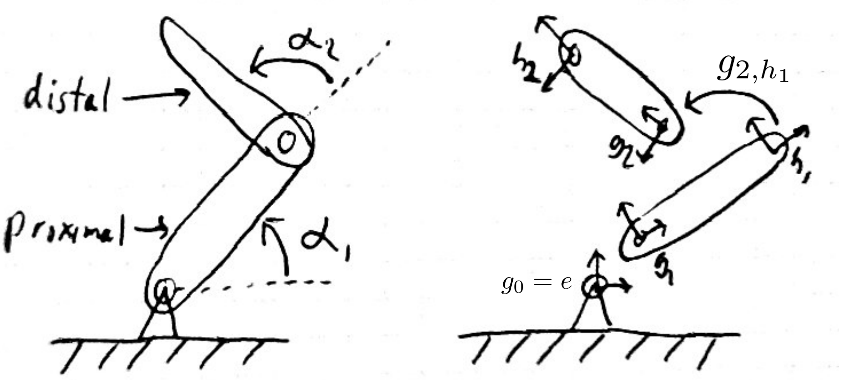


Figure 3.3 A two-link arm, with frames, transforms, and proximal/distal terminology illustrated.

Adding a second link to the system (creating the *kinematic chain* with *proximal* and *distal* members in Figure 3.3) is also a well-defined operation in $SE(2)$. Taking a second link with end frames g_2 and h_2 as in Figure 3.3, and relabeling the first link's frames g_1 and h_1, we can define the second body's base position with respect to the end of the first link as $g_{2,h_1} = (x_{2,h_1}, y_{2,h_1}, \theta_{2,h_1})$. Applying the pin constraint in (3.3) to this position attaches the new link to the first, and restricts the degrees of freedom of g_{2,h_1} to the joint angle $\theta_{2,h_1} = \alpha_2$ in the manner of (3.3). The world positions of the frames g_2 and h_2 then follow straightforwardly via $SE(2)$ actions,

$$g_2 = \overbrace{(g_1 h_{1,g_1})}^{h_1} g_{2,h_1} = \begin{bmatrix} \cos(\alpha_1+\alpha_2) & -\sin(\alpha_1+\alpha_2) & \ell_1 \cos\alpha_1 \\ \sin(\alpha_1+\alpha_2) & \cos(\alpha_1+\alpha_2) & \ell_1 \sin\alpha_1 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.5}$$

and

$$h_2 = g_2 h_{2,g_2} \tag{3.6}$$

$$= \begin{bmatrix} \cos(\alpha_1+\alpha_2) & -\sin(\alpha_1+\alpha_2) & \ell_1 \cos\alpha_1 + \ell_2 \cos(\alpha_1+\alpha_2) \\ \sin(\alpha_1+\alpha_2) & \cos(\alpha_1+\alpha_2) & \ell_1 \sin\alpha_1 + \ell_2 \sin(\alpha_1+\alpha_2) \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.7}$$

If the links were instead connected by a prismatic joint with displacement δ, as illustrated in Figure 3.4, the constraint functions in (3.2) would instead take the form

$$y = 0 \qquad \text{and} \qquad \theta = 0, \tag{3.8}$$

replacing the equivalency relationship in (3.3) with

$$g = \overbrace{\begin{bmatrix} 1 & 0 & \delta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{SE(2),\ y,\theta=0} \equiv \overbrace{\begin{bmatrix} \delta \\ 0 \end{bmatrix}}^{\mathbb{R}^2,\ y=0} \equiv \overbrace{\delta}^{\mathbb{R}^1}. \tag{3.9}$$

The world positions of the second link frames would then be

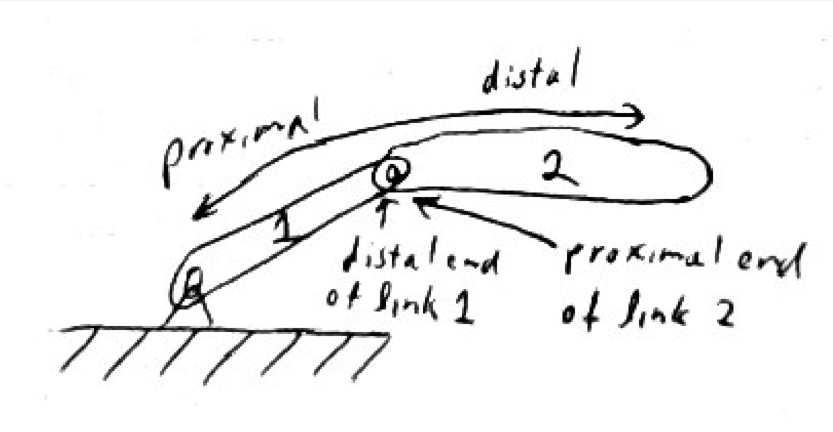
$$g_2 = \begin{bmatrix} \cos\alpha & -\sin\alpha & (\ell_1+\delta)\cos\alpha \\ \sin\alpha & \cos\alpha & (\ell_1+\delta)\sin\alpha \\ 0 & 0 & 1 \end{bmatrix} \tag{3.10}$$

and

$$h_2 = \begin{bmatrix} \cos\alpha & -\sin\alpha & (\ell_1+\delta+\ell_2)\cos\alpha \\ \sin\alpha & \cos\alpha & (\ell_1+\delta+\ell_2)\sin\alpha \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.11}$$

Proximal, Medial and Distal

Elements of a kinematic chain are often described in terms of the local directions *proximal* (closer to the base of the chain), *medial* (at the center), and *distal* (further from the base). For individual links, these terms may describe a relationship between two links ("link 1 is proximal to link 2") or an absolute position in the chain ("link 2 is the distal link"). Other uses include identifying positions on a link ("the distal end of link 1") or more general regions along the chain ("the proximal portion of a chain" to refer to a set of links near the base).



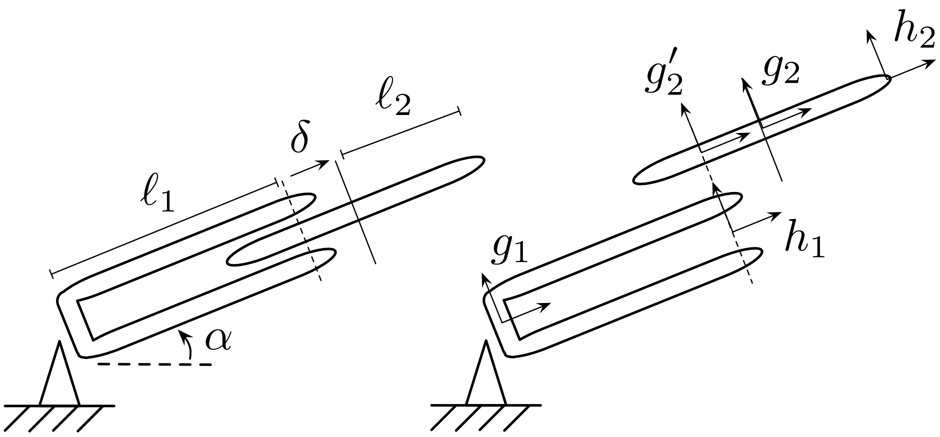


Figure 3.4 A two-link arm with one rotary and one prismatic joint. Note that g_2 is placed at the middle of the link to accommodate positive values of δ, and that we have introduced a new frame g_2' on the second body that is instantaneously aligned with h_1; this frame will be explicitly used in §3.5.2.

Chains of more than two links can naturally be constructed along the same pattern, with constrained elements of $SE(2)$ representing the relative positions of linked joints and fixed elements connecting different frames on the links. The expressions of the frame positions naturally increases in complexity with each link added, but, as they will almost certainly be handled by a computer, this is not a major concern; what matters is that the configuration of any given link can be easily represented as a sequence of the relative link positions leading up to it.

Frame Naming

In our examples using fixed-base systems with rotary joints, we use a naming convention of g for the proximal ends of links and h for their distal ends. This naming convention reflects the primary importance of these two frames in describing the motion of fixed-base chains. When working with mobile systems and those with prismatic joints, we alter this convention to place g at the center of the link. In the case of prismatic joints, this allows us to use a natural joint parameterization in which positive values of δ extend the joint. For mobile systems, the locomotion equations in Chapter 4 are conveniently described in terms of the motion of the center of each link, and so we directly incorporate these positions into our kinematic development and introduce f to denote frames at the proximal end of the link.

3.3 Mobile Articulated Systems

Forward kinematics for articulated systems that move freely in the plane combine aspects of both rigid-body and fixed-base analysis. As with a rigid bodies discussed in §1.3, the *position* of an articulated system is defined by the location and orientation of its body frame. Rather than each point on a system having a fixed location with respect to this body frame, however, the locations of these points depend on the placement of the component rigid bodies relative to the system body frame. These placements are specified by the system's *shape* variables, which correspond to the the configuration variables of fixed-base systems.

Together, the system's position and shape fully define its configuration. We denote the position as $g \in G$ (highlighting the group nature of the position space) and the shape as $r \in M$ (which, for historical reasons, does not follow the capital/lowercase convention we use elsewhere). For the overall configuration, we then have $q = (g, r)$, with a configuration space structured as $Q = G \times M$.

The simplest means of combining the shape and position terms into a kinematic structure is to choose a "base link" for the chain to define its body frame, then iteratively attach links to this first link following the procedure in §3.2. For example, the two-link system in Figure 3.5 has a configuration space $Q = SE(2) \times \mathbb{S}^1$, elements of which we can characterize as g, the position and orientation of link 1, and the shape $r = \alpha$, the angle between the links. Given these parameters, we can apply our general forward kinematics approach to find the position of any other frame on the system, such as the center of link 2,

$$g_2 = \overbrace{g_1 \circ h_{1,g_1}}^{\text{link 1}} \circ \overbrace{f_{2,h_1} \circ g_{2,f_2}}^{\text{link 2}} \tag{3.12}$$

$$= g_1 \begin{bmatrix} \cos\alpha & -\sin\alpha & (\ell_1/2) + (\ell_2/2)\cos\alpha \\ \sin\alpha & \cos\alpha & (\ell_1/2) + (\ell_2/2)\sin\alpha \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.13}$$

with the product of the final multiplication being too long to display here.

Subsequent links can then be added either to the distal end of link 2 or to the proximal end of link 1. As a practical matter, it is often useful (but by no means necessary) to build out evenly from a center link, rather than extending all the links in the same direction. The even approach minimizes the number of matrix multiplications required to find the link positions, simplifying their expression and improving the numerical stability of algorithms incorporating them, such as the coordinate optimizations in Chapter **??**.

During this construction process, the orientation convention for the joint angles should be chosen to meaningfully reflect the nature of the system. In the case of serial chains such as the three-link system examined in the next chapter, it is preferable to define the joint angles as based on the overall proximal-distal relationship of the links (so that positive values of α_1 correspond to positive rotations of link 2 with respect to link 1) as depicted at the left of Figure 3.6, even when using link 2 as the base for the system. First, this convention provides an unambiguous scheme for defining the joint angles that is independent of the base link. Second, it means that configurations in which the joint angles have "even" and "odd" symmetry as functions of distance along the chain correspond to configurations with even (bilateral) and odd (rotational) physical symmetries, as shown at the right of Figure 3.6. In Chapter **??**, we will re-examine this angle convention as it relates to the body's *curvature*.

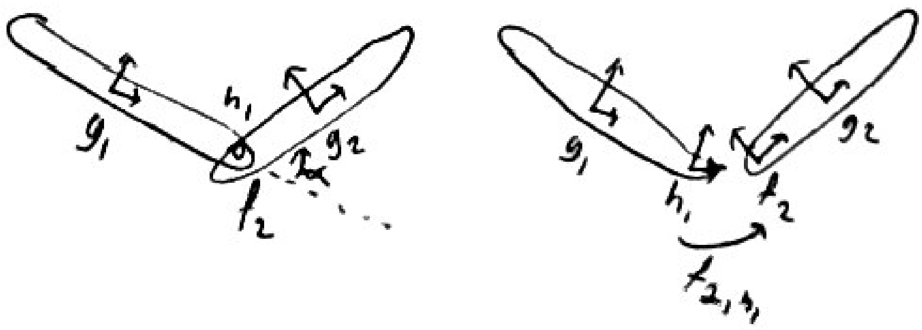


Figure 3.5 Kinematics of a mobile system with two links.

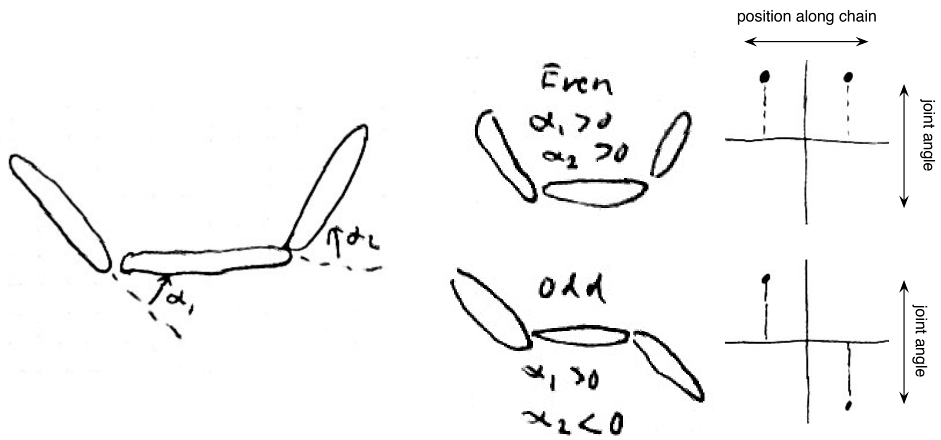


Figure 3.6 The sign convention for mobile articulated systems (left) and the correspondence between odd and even signs on the joint angles and physically odd and even shapes (right).

3.4 Generalized Body Frames

There is a great deal of freedom in selecting the body frame for a given system. As was implicitly noted in our discussion of the choice of base links in §3.3, the position of any rigid body in the system may be used as the position of the system as a whole (and therefore any body frame rigidly attached to that link may be taken as the system's body frame). More generally, however, we may also select *any frame whose position with respect to the base link is a function of the shape variables*, as illustrated in Figure 3.7.

To see that this is the case, consider our definition of a body frame for an articulated system: *a frame in which the position of every component body (and, by extension, any point on those bodies) is a function of the shape r.* Clearly, frames on the base link meet this definition, as do frames on the other links (both by the "choose any base link" argument, and because $SE(2)$ elements are invertible). Now consider a frame g at $\beta \in SE(2)$ with respect to the base link. The positions of links 1 and 2 with respect to g are respectively

$$g_{1,g} = \beta^{-1} \tag{3.14}$$

and

$$g_{2,g} = \beta^{-1} g_{2,g_1}(\alpha). \tag{3.15}$$

If (and only if) we can express β as a function of α, then $g_{1,g}$ and $g_{2,g}$ are both functions of the system shape, meeting the necessary and sufficient conditions for g to serve as the system body frame.

Defining the valid body frames in this manner provides several tools for working with articulated systems

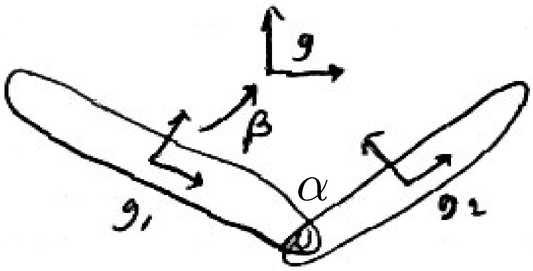


Figure 3.7 Any frame that is a shape-dependent transform away from a link of an articulated system is a valid body frame for that system.

that we make use of in the following chapters. Most importantly, our previous definition for a valid body frame describes a test for validity (for any given frame, we can evaluate whether the link positions are all functions of the shape), but does not provide a *space* of valid body frames. In contrast, the new definition identifies each valid body frame with a function $\beta(r)$, allowing us to, for example, choose an optimal β for locomotion analysis, as in Chapter **??**. Second, defining the valid frames with respect to an obviously valid frame (the base link) simplifies the process of evaluating the system kinematics with respect to a more abstract frame (such as at the center of mass). Rather than directly determining the position of each link relative to an abstract body frame, we can initially evaluate the system kinematics with respect to a base link, and then simply add a β^{-1} transform to change into the new frame.

3.5 Velocity Kinematics

Along with mapping the configuration of a system to its physical position, forward kinematics also relates the system's physical and configuration velocities. At each frame on the system, with position and orientation $g(q)$, the latter relationship is given by the *Jacobian* of the forward kinematic map,

$$\dot{g}(q,\dot{q}) = J_g \dot{q} = \frac{\partial g}{\partial q}\dot{q}. \tag{3.16}$$

Several methods for finding this Jacobian are available. In this section, we examine two: direct differentiation of the forward kinematic map and assembly of the Jacobian from individual joint kinematics.

3.5.1 Jacobians from Differentiation

The most straightforward option for finding this Jacobian is to explicitly evaluate the differential in (3.16). For example, the distal frame on the single-link arm in Figure 3.2 has a position $h = (\ell\cos\alpha, \ell\sin\alpha, \alpha)$. We can calculate its velocity by differentiating this relationship with respect to the configuration variable α,

$$\dot{h} = J_h\dot{\alpha} = \frac{\partial h}{\partial \alpha}\dot{\alpha} = \begin{bmatrix} -\ell\sin\alpha \\ \ell\cos\alpha \\ 1 \end{bmatrix}\dot{\alpha}. \tag{3.17}$$

Systems with more than one configuration variable have a corresponding number of columns in their Jacobians. The distal frame on the two-link chain in Figure 3.3,

$$h_2(q) = \Big((\ell_1\cos\alpha_1 + \ell_2\cos(\alpha_1+\alpha_2)), (\ell_1\sin\alpha_1 + \ell_2\sin(\alpha_1+\alpha_2)), (\alpha_1+\alpha_2)\Big), \tag{3.18}$$

has a two-column Jacobian,

$$\dot{h}_2 = \begin{bmatrix} -(\ell_1 \sin\alpha_1 + \ell_2 \sin(\alpha_1+\alpha_2)) & -\ell_2 \sin(\alpha_1+\alpha_2) \\ (\ell_1 \cos\alpha_1 + \ell_2 \cos(\alpha_1+\alpha_2)) & \ell_2 \cos(\alpha_1+\alpha_2) \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}, \tag{3.19}$$

whereas the mobile three-link system from Figure 3.7 has five columns in its Jacobian, three corresponding to the velocity of the body frame and two to the shape velocity:

$$\dot{g}_2 = \begin{bmatrix} \frac{\partial g_2}{\partial g} & \frac{\partial g_2}{\partial r} \end{bmatrix} (\dot{g}, \dot{r}). \tag{3.20}$$

3.5.2 Iterative Jacobian Assembly

An alternative approach to generating the Jacobian of an articulated system is to build it progressively along with the forward kinematic map. Methods based on this approach take advantage of the iterated $SE(2)$ structure of the system to "pre-differentiate" the relative motions of the links at the joints before placing them into the system structure.

Two basic principles underly this pre-differentiation. First, as we saw in §**??**, the velocities of any two frames on a rigid body are linked by the right lifted action. Second, at a joint the velocities of the interacting links are matched, modulo the relative motion allowed by the joint constraints.

Taking the two-link system in Figure 3.3 as an example, the fixed pivot point (at $g_0 = \mathbf{e}$) is grounded, and so has velocity

$$\dot{g}_0 = (0, 0, 0). \tag{3.21}$$

The proximal frame of link 1 has equal translational velocity to the pivot and a relative velocity of $\dot{\alpha}_1$, and so is

$$\dot{g}_1 = \dot{g}_0 + (0, 0, \dot{\alpha}_1) = (0, 0, \dot{\alpha}_1) \tag{3.22}$$

At the distal end of the first link, h_1 has the same spatial velocity as g_1,

$$T_{h_1} R_{h_1^{-1}} \dot{h}_1 = T_{g_1} R_{g_1^{-1}} \dot{g}_1, \tag{3.23}$$

or, taking advantage of the equality in (**??**),

$$\dot{h}_1 = \overbrace{(T_{\mathbf{e}} R_{h_1})(T_{g_1} R_{g_1^{-1}})}^{T_{g_1} R_{h_1, g_1}} \dot{g}_1. \tag{3.24}$$

Combining this relationship with $\dot{g}_1$ from (3.22) and the kinematic map for h_1 in (3.4) gives the velocity of the distal frame and its Jacobian with respect to the joint angle,

$$\dot{h}_1 = \overbrace{\begin{bmatrix} 1 & 0 & -\ell_1 \sin\alpha_1 \\ 0 & 1 & \ell_1 \cos\alpha_1 \\ 0 & 0 & 1 \end{bmatrix}}^{T_{\mathbf{e}} R_{h_1}} \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{T_{g_1} R_{g_1^{-1}}} \overbrace{\begin{bmatrix} 0 \\ 0 \\ \dot{\alpha}_1 \end{bmatrix}}^{\dot{g}_1} = \overbrace{\begin{bmatrix} -\ell_1 \sin\alpha_1 \\ \ell_1 \cos\alpha_1 \\ 1 \end{bmatrix}}^{J_{h_1}} \dot{\alpha}_1, \tag{3.25}$$

which matches the results of our direct differentiation in (3.17).

For the second link, the relative velocity of the proximal end of link 2 and the distal end of link 1 is given by the second joint's rotation, making its net velocity

$$\dot{g}_2 = \dot{h}_1 + (0, 0, \dot{\alpha}_2) \tag{3.26}$$

$$= ((-\ell_1 \sin\alpha_1)\dot{\alpha}_1, (\ell_1 \cos\alpha_1)\dot{\alpha}_1, (\dot{\alpha}_1 + \dot{\alpha}_2)), \tag{3.27}$$

which can be written as a Jacobian product as

$$\dot{g}_2 = \begin{bmatrix} -\ell_1 \sin\alpha_1 & 0 \\ \ell_1 \cos\alpha_1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix} \tag{3.28}$$

The velocity of the distal end of link 2 is then

$$\dot{h}_2 = \overbrace{\begin{bmatrix} 1 & 0 & -(\ell_1 \sin\alpha_1 + \ell_2 \sin(\alpha_1+\alpha_2)) \\ 0 & 1 & (\ell_1 \cos\alpha_1 + \ell_2 \cos(\alpha_1+\alpha_2)) \\ 0 & 0 & 1 \end{bmatrix}}^{T_{\mathbf{e}}R_{h_2}} \overbrace{\begin{bmatrix} 1 & 0 & \ell_1 \sin\alpha_1 \\ 0 & 1 & -\ell_1 \cos\alpha_1 \\ 0 & 0 & 1 \end{bmatrix}}^{T_{g_2}R_{g_2^{-1}}} \overbrace{\begin{bmatrix} -\ell_1 \sin\alpha_1 & 0 \\ \ell_1 \cos\alpha_1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}}^{\dot{g}_2}, \tag{3.29}$$

which, when the multiplications are carried out, resolves to

$$\dot{h}_2 = \overbrace{\begin{bmatrix} -(\ell_1 \sin\alpha_1 + \ell_2 \sin(\alpha_1+\alpha_2)) & -\ell_2 \sin(\alpha_1+\alpha_2) \\ (\ell_1 \cos\alpha_1 + \ell_2 \cos(\alpha_1+\alpha_2)) & \ell_2 \cos(\alpha_1+\alpha_2) \\ 1 & 1 \end{bmatrix}}^{J_{h_2}} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}, \tag{3.30}$$

once again equaling the results of direct differentiation (this time in (3.19)).

The overall pattern in this example is that the distal velocity of each link in a chain is defined recursively as

$$\dot{h}_i = (T_{\mathbf{e}}R_{h_i})(T_{g_i}R_{g_i^{-1}})\overbrace{(\dot{h}_{i-1} + v_i)}^{\dot{g}_i}, \tag{3.31}$$

where v_i is the velocity of body i with respect to body $i-1$ at joint i.

3.5.3 Body Velocity Formulation of Iterative Jacobian

Depending on how the velocities and relative positions of the links are represented, it may be preferable to modify (3.31) to better accommodate this representation. For instance, in the locomotion problems we examine in Chapter 4, it is more useful to work with the link body velocities than their absolute velocities. A related concern appears in the case of prismatic joints (and, outside the scope of the present material, in the case of joints in three-dimensional systems) where the relative motion at the joint is more easily defined in the frame of the attached link than with respect to the world.

In this case, we can first insert two balanced sets of left lifted actions into (3.31) and regroup the terms into body velocities (employing (**??**)) and *adjoint actions*[2] to produce a map between the body velocities of any two frames on a rigid body:

$$\underbrace{\overbrace{(T_{h_i}L_{h_i^{-1}})}^{\text{new}}\dot{h}_i}_{\xi_{h_i}} = \underbrace{\overbrace{(T_{h_i}L_{h_i^{-1}})}^{\text{new}}(T_{\mathbf{e}}R_{h_i})}_{Ad_{h_i}^{-1}}\underbrace{(T_{g_i}R_{g_i^{-1}})\overbrace{(T_{\mathbf{e}}L_{g_i})}^{}}_{Ad_{g_i}}\overbrace{\underbrace{(T_{g_i}L_{g_i^{-1}})\dot{g}_i}_{\xi_{g_i}}}^{}. \tag{3.32}$$

(3.33) [the line of underbrace labels ξ_{h_i}, $Ad_{h_i}^{-1}$, Ad_{g_i}, ξ_{g_i} is numbered (3.33); the overbrace "Identity" spans $(T_{\mathbf{e}}L_{g_i})(T_{g_i}L_{g_i^{-1}})$]

At the joints, the relative velocity between the links is most easily considered by recognizing that there are in fact three frames involved: h_{i-1}, the distal frame on link $i-1$; g_i, the proximal frame on link i; and g_i', the frame on link i that is instantaneously aligned with h_{i-1}. For a joint with relative motion defined as $v_i = (\dot{\delta}_{x,i}, \dot{\delta}_{y,i}, \dot{\alpha}_i)$ relative to the proximal link, the body velocity of frame g_i' is

$$\xi_{g_i'} = \xi_{h_{i-1}} + v_i. \tag{3.34}$$

As frames g_i' and g_i are attached to the same rigid body, we can convert between their body velocities using the same approach as in (3.33),

$$\xi_{g_i} = Ad_{g_i}^{-1} Ad_{g_i'} \xi_{g_i'}. \tag{3.35}$$

Note that when the x and y components of g_i' and g_i are equal, this conversion reduces to rotation by $-\alpha_i$, as discussed in the box on page 63.

Together, (3.34), (3.33), and (3.35) provide a formulation equation for the body velocities of the distal ends of each link,

$$\xi_{h_i} = (Ad_{h_i}^{-1})(Ad_{g_i'})(\xi_{h_{i-1}} + v_i), \tag{3.36}$$

[2] See the box on the next page.

Adjoint actions

more to come, this is just core information

Adjoint actions convert between right and left velocities.
Left velocity relative to right velocity is

$$\overset{\leftarrow\circ}{g} = g\,\overset{\circ\rightarrow}{g}\,g^{-1} \tag{3.vi}$$

Lifted action form of this conversion can be grouped into a single adjoint action

$$\begin{bmatrix} \overset{\leftarrow\circ}{g}_1 \\ \vdots \\ \overset{\leftarrow\circ}{g}_n \end{bmatrix} = \overbrace{(T_g R_{g^{-1}})(T_{\mathbf{e}} L_g)}^{Ad_g} \begin{bmatrix} \overset{\circ\rightarrow}{g}_1 \\ \vdots \\ \overset{\circ\rightarrow}{g}_n \end{bmatrix} \tag{3.vii}$$

Right velocity relative to left velocity is the inverse of the relationship in (3.vi),

$$\overset{\circ\rightarrow}{g} = g^{-1}\,\overset{\leftarrow\circ}{g}\,g \tag{3.viii}$$

Grouping the lifted actions associated with this conversion produces the inverse of the adjoint transformation,

$$\begin{bmatrix} \overset{\circ\rightarrow}{g}_1 \\ \vdots \\ \overset{\circ\rightarrow}{g}_n \end{bmatrix} = \overbrace{(T_g L_{g^{-1}})(T_{\mathbf{e}} R_g)}^{Ad_g^{-1}} \begin{bmatrix} \overset{\leftarrow\circ}{g}_1 \\ \vdots \\ \overset{\leftarrow\circ}{g}_n \end{bmatrix} \tag{3.ix}$$

from which the adjoint terms relating to the frame g have been dropped, as they do not directly affect the calculation of ξ_{h_i}.

If the relative positions of frames on a link are more convenient to use their absolute positions, a further reduction is possible, by evaluating (3.36) with the origin temporarily placed at g_i'. This change of coordinates transforms the position of each link frame by $(g_i')^{-1}$, so that g_i' and h_i respectively become $\mathbf{e}$ and $h_{i,g_i'}$. As the adjoint action at the origin is an identity matrix, the body velocity of the distal frame simplifies to the inverse adjoint action of that frame relative to g_i',

$$\xi_{h_i} = (Ad^{-1}_{h_{i,g_i'}})(\xi_{h_{i-1}} + v_i). \tag{3.37}$$

3.5.4 The Rotary-Prismatic Arm

The rotary-prismatic arm in Figure 3.4 provides an intuitive example of using the body-frame approach to iteratively calculating the Jacobian. As in the previous arm example, we have a fixed pivot at g_0, which thus has zero body velocity,

$$\xi_{g_0} = (0, 0, 0). \tag{3.38}$$

The body velocity of frame g_1', attached to the first link, but aligned with g_0 is then found by adding in its velocity relative to g_0,

$$\xi_{g_1'} = \xi_{g_0} + \overbrace{(0, 0, \dot{\alpha})}^{v_1} = (0, 0, \dot{\alpha}), \tag{3.39}$$

and can be converted into the body velocity of frame g_1 by applying the relative inverse adjoint action from (3.37),

$$\xi_{g_1} = Ad^{-1}_{g_{1,g_1'}} \xi_{g_1'} = \overbrace{\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{g_{1,g_1'}=(0,0,\alpha)}} \overbrace{\begin{bmatrix} 0 \\ 0 \\ \dot{\alpha} \end{bmatrix}}^{\xi_{g_1'}} = \begin{bmatrix} 0 \\ 0 \\ \dot{\alpha} \end{bmatrix} = \overbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}^{J^b_{g_1}} \dot{\alpha}, \tag{3.40}$$

Change of Basis

In treatments of rigid body motion, it is typical to encounter "change of basis" operations that move vectors between coordinate representations. On a casual level, these changes of basis are commonly identified with simple rotations of the coordinates. For example, consider a pair of frames g and h at the same location but with orientations differing by α, as in the illustration below. A vector v at this location may be represented as v_g with respect to frame g, and as v_h with respect to frame h. The relationship between these to vectors would then be described by a rotation matrix of $-\alpha$,

$$v_h = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} v_g. \tag{3.x}$$

While useful, this rotation is not well-defined in terms of $SE(2)$ operations. For instance, if we consider v as the velocity of frames g and h, it quickly becomes apparent that v in fact represents two separate vectors $\dot{g} \in T_g G$ and $\dot{h} \in T_h G$ in separate tangent spaces, and that a rigorous definition of change-of-basis operations must take this structure into account.

There are in fact at least two (mutually compatible) interpretations of $SE(2)$ that lead to the change of basis formulation in (3.x). The first we have already seen in §**??**, where transforming the body to the origin was equivalent to bringing the coordinate frame to the body, and the associated lifted action corresponded to the change of basis. More generally, the change of basis from frame g to frame h is given by

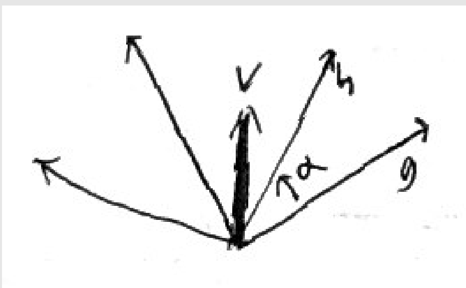
$$v_h = (T_g L_{h_g^{-1}}) v_g, \tag{3.xi}$$

which evaluates to the rotation matrix in (3.x).

The second source for the change of basis equation is based on the observation that as locked frames, g and h have a common spatial velocity, and thus

$$v_h = Ad_h^{-1} Ad_g v_g = Ad_{h_g}^{-1} v_g. \tag{3.xii}$$

When the x and y values in g and h are the same, they nullify each other in the adjoint product, leaving only the rotation component.



where $J^b_{g_1}$ is the Jacobian for the proximal end of the link, expressed in its own body frame. Moving to the end of the first link, the distal frame's body velocity is found in the same manner,

$$\xi_{h_1} = Ad^{-1}_{h_1,g_1}\xi_{g_1} = \overbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & \ell_1\\ 0 & 0 & 1\end{bmatrix}}^{Ad^{-1}_{h_1,g_1=(\ell_1,0,0)}} \overbrace{\begin{bmatrix}0\\0\\\dot\alpha\end{bmatrix}}^{\xi_{g_1}} = \begin{bmatrix}0\\ \ell_1\dot\alpha\\ \dot\alpha\end{bmatrix} = \overbrace{\begin{bmatrix}0\\ \ell_1\\ 1\end{bmatrix}}^{J^b_{h_1}}\dot\alpha, \tag{3.41}$$

confirming our intuition that the tip of a rigid link should move laterally when pivoted around its base.

Up to this point in the example, working with the frames' body velocities has allowed us to build the system Jacobian using only the frames' relative positions, without incorporating the full forward kinematics of the system. This locality becomes even more useful when we apply it to prismatic joints, for which the relative motion of frames g'_i and h_{i-1} is typically represented in the local frame. While we can of course convert this velocity into a global frame and insert it into (3.31) when calculating the Jacobian, it is significantly more convenient to use it directly in (3.37). Continuing with our rotary-prismatic arm example, the relative velocity at the prismatic joint is

$$v_2 = (\dot\delta, 0, 0), \tag{3.42}$$

giving the velocity of the distal link at the joint as

$$\xi_{g'_2} = \xi_{h_1} + v_2 = (\dot\delta, \ell_1\dot\alpha, \dot\alpha). \tag{3.43}$$

The body-frame Jacobians for the midpoint and distal end of the second link then follow naturally,

$$\xi_{g_2} = Ad^{-1}_{g_2,g'_2}\xi_{g'_2} = \overbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & \delta\\ 0 & 0 & 1\end{bmatrix}}^{Ad^{-1}_{g_2,g'_2=(\delta,0,0)}} \overbrace{\begin{bmatrix}\dot\delta\\ \ell_1\dot\alpha\\ \dot\alpha\end{bmatrix}}^{\xi_{g'_2}} = \begin{bmatrix}\dot\delta\\ (\ell_1+\delta)\dot\alpha\\ \dot\alpha\end{bmatrix} = \overbrace{\begin{bmatrix}0 & 1\\ \ell_1+\delta & 0\\ 1 & 0\end{bmatrix}}^{J^b_{g_2}}\begin{bmatrix}\dot\alpha\\ \dot\delta\end{bmatrix}, \tag{3.44}$$

and

$$\xi_{h_2} = Ad^{-1}_{h_2,g'_2}\xi_{g'_2} = \overbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & \delta+\ell_2\\ 0 & 0 & 1\end{bmatrix}}^{Ad^{-1}_{h_2,g'_2=(\delta+\ell_2,0,0)}} \overbrace{\begin{bmatrix}\dot\delta\\ \ell_1\dot\alpha\\ \dot\alpha\end{bmatrix}}^{\xi_{g'_2}} = \begin{bmatrix}\dot\delta\\ (\ell_1+\delta+\ell_2)\dot\alpha\\ \dot\alpha\end{bmatrix} = \overbrace{\begin{bmatrix}0 & 1\\ \ell_1+\delta+\ell_2 & 0\\ 1 & 0\end{bmatrix}}^{J^b_{h_2}}\begin{bmatrix}\dot\alpha\\ \dot\delta\end{bmatrix}. \tag{3.45}$$

3.6 Jacobians for Mobile Systems

3.6.1 Three-link Systems

The local approach to calculating the Jacobian further shows its worth in the analysis of mobile articulated systems. A common operation when working with such systems is to find the body velocity of each link as a function of the systems overall body velocity ξ and its shape velocity $\dot r$; the resulting Jacobians underly, for instance, the locomotive relationships explored in Chapter 4.

For a three-link system such as that shown in Figure 3.8, with link lengths ℓ_i and proximal, medial, and distal frames f_i, g_i. and h_i, we can start building the Jacobians by selecting the middle link to define the system's body frame.[3] This choice gives the middle link a trivial Jacobian,

$$\xi_{g_2} = \xi = \begin{bmatrix}I^{3\times3} & \mathbf{0}^{3\times2}\end{bmatrix}\begin{bmatrix}\xi\\ \dot r\end{bmatrix}, \tag{3.46}$$

[3] In §3.6.2, we consider the kinematics under different choices of body frame, with the computation of these kinematics for a center-of-mass frame provided as an exercise for the reader. Here, using the middle link simplifies our initial computations, without introducing any loss of generality.

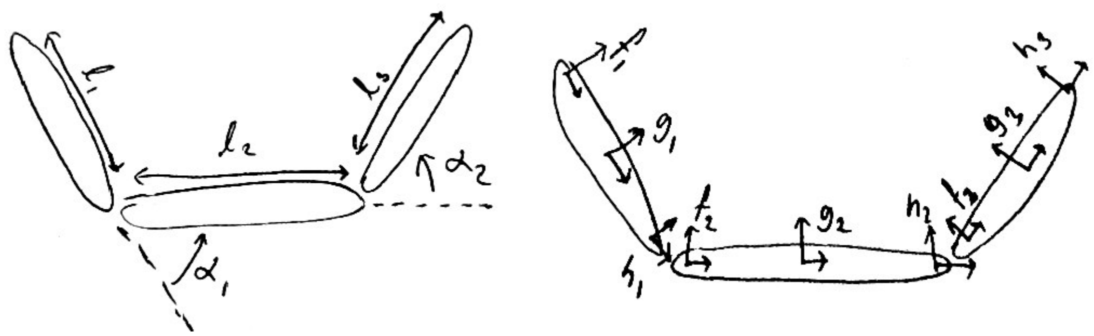


Figure 3.8 Parameterization and frame placement for a generic three-link mobile system.

and facilitates calculation of the velocity of the other links. To calculate the body velocity of link 1, we first find the body velocity of the proximal end of link 2,

$$\xi_{f_2} = Ad^{-1}_{f_{2,g_2}} \xi_{g_2} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\ell_2/2 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{f_{2,g_2}=(-\ell_2/2,0,0)}} \overbrace{\begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \end{bmatrix}}^{\xi_{g_2}} = \begin{bmatrix} \xi^x \\ \xi^y - (\xi^\theta \ell_2)/2 \\ \xi^\theta \end{bmatrix}. \tag{3.47}$$

and use it to calculate the body velocity of the distal end of link 1, which has rotational velocity of $-\dot{\alpha}_1$ with respect to f_2,

$$\xi_{h_1} = Ad^{-1}_{h_{1,h'_1}} \xi_{h'_1} = \overbrace{\begin{bmatrix} \cos\alpha_1 & -\sin\alpha_1 & 0 \\ \sin\alpha_1 & \cos\alpha_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{h_{1,h'_1}=(0,0,-\alpha_1)}} \overbrace{\left(\underbrace{\begin{bmatrix} \xi^x \\ \xi^y - (\xi^\theta \ell_2)/2 \\ \xi^\theta \end{bmatrix}}_{\xi_{f_2}} + \begin{bmatrix} 0 \\ 0 \\ -\dot{\alpha}_1 \end{bmatrix} \right)}^{\xi_{h'_1}} \tag{3.48}$$

$$= \begin{bmatrix} \xi^x \cos\alpha_1 - (\xi^y - (\xi^\theta \ell_2)/2) \sin\alpha_1 \\ \xi^x \sin\alpha_1 + (\xi^y - (\xi^\theta \ell_2)/2) \cos\alpha_1 \\ \xi^\theta - \dot{\alpha}_1 \end{bmatrix}. \tag{3.49}$$

Finally, we arrive at the body velocity of g_1 by moving along the link by $\ell_1/2$,

Adjoint is wrong; fix and check product in following line.

$$\xi_{g_1} = Ad^{-1}_{g_{1,h_1}} \xi_{h_1} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\ell_1/2 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{g_{1,h_1}=(0,-\ell_1/2,0)}} \overbrace{\begin{bmatrix} \xi^x \cos\alpha_1 - (\xi^y - (\xi^\theta \ell_2)/2) \sin\alpha_1 \\ \xi^x \sin\alpha_1 + (\xi^y - (\xi^\theta \ell_2)/2) \cos\alpha_1 \\ \xi^\theta - \dot{\alpha}_1 \end{bmatrix}}^{\xi_{h_1}} \tag{3.50}$$

$$= \begin{bmatrix} \xi^x \cos\alpha_1 - (\xi^y + (\xi^\theta \ell_2)/2) \sin\alpha_1 \\ \xi^x \sin\alpha_1 + (\xi^y - (\xi^\theta \ell_2)/2) \cos\alpha_1 - (\ell_1/2)(\xi^\theta - \dot{\alpha}_1) \\ \xi^\theta - \dot{\alpha}_1 \end{bmatrix}. \tag{3.51}$$

Similarly, we calculate the body velocity of link 3 by first taking the body velocity of the distal end of link 2,

$$\xi_{h_2} = Ad^{-1}_{h_{2,g_2}} \xi_{g_2} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \ell_2/2 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{f_{2,g_2}=(\ell_2/2,0,0)}} \overbrace{\begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \end{bmatrix}}^{\xi_{g_2}} = \begin{bmatrix} \xi^x \\ \xi^y + (\xi^\theta \ell_2)/2 \\ \xi^\theta \end{bmatrix}, \tag{3.52}$$

converting it into the proximal velocity of link 3,

$$\xi_{f_3} = Ad^{-1}_{f_3,f'_3}\xi_{f'_3} = \overbrace{\begin{bmatrix} \cos\alpha_2 & \sin\alpha_2 & 0 \\ -\sin\alpha_2 & \cos\alpha_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{f_3,f'_3=(0,0,\alpha_2)}} \overbrace{\left(\underbrace{\begin{bmatrix} \xi^x \\ \xi^y + (\xi^\theta \ell_2)/2 \\ \xi^\theta \end{bmatrix}}_{\xi_{h_2}} + \begin{bmatrix} 0 \\ 0 \\ \dot{\alpha}_2 \end{bmatrix} \right)}^{\xi_{f'_3}} \tag{3.53}$$

$$= \begin{bmatrix} \xi^x \cos\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\sin\alpha_2 \\ -\xi^x \sin\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\cos\alpha_2 \\ \xi^\theta + \dot{\alpha}_2 \end{bmatrix}, \tag{3.54}$$

Adjoint is wrong; fix and check product in following line.

and moving to the center of the link,

$$\xi_{g_3} = Ad^{-1}_{g_3,f_3}\xi_{f_3} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \ell_3/2 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{g_3,f_3=(0,\ell_3/2,0)}} \overbrace{\begin{bmatrix} \xi^x \cos\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\sin\alpha_2 \\ -\xi^x \sin\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\cos\alpha_2 \\ \xi^\theta + \dot{\alpha}_2 \end{bmatrix}}^{\xi_{f_3}} \tag{3.55}$$

$$= \begin{bmatrix} \xi^x \cos\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\sin\alpha_2 \\ -\xi^x \sin\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\cos\alpha_2 + (\ell_3/2)(\xi^\theta + \dot{\alpha}_2) \\ \xi^\theta + \dot{\alpha}_2 \end{bmatrix}. \tag{3.56}$$

Expressed as Jacobian products, the body velocities of the links are

$$\xi_{g_1} = \overbrace{\begin{bmatrix} \cos\alpha_1 & -\sin\alpha_1 & (\ell_2 \sin\alpha_1)/2 & 0 & 0 \\ \sin\alpha_1 & \cos\alpha_1 & -(\ell_2\cos\alpha_1 + \ell_1)/2 & \ell_1/2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}}^{J^b_{g_1}} \begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \\ \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}, \tag{3.57}$$

$$\xi_{g_2} = \overbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}}^{J^b_{g_2}} \begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \\ \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix} \tag{3.58}$$

$$\xi_{g_3} = \overbrace{\begin{bmatrix} \cos\alpha_2 & \sin\alpha_2 & (\ell_2 \sin\alpha_2)/2 & 0 & 0 \\ -\sin\alpha_2 & \cos\alpha_2 & (\ell_2\cos\alpha_2 + \ell_3)/2 & 0 & \ell_3/2 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}}^{J^b_{g_3}} \begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \\ \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}. \tag{3.59}$$

3.6.2 Jacobians and Alternative Body Frames

What happens to the three-link system's Jacobians when we use a frame that is kinematically linked to the system, such as the alternative body frame options described in §3.4?

This is really two questions:

1. If we keep the middle link as the system body frame, what is the body velocity of frame g in Figure 3.9?
2. If we take frame g as the system body frame, what are the body velocities of the links?

Answers:

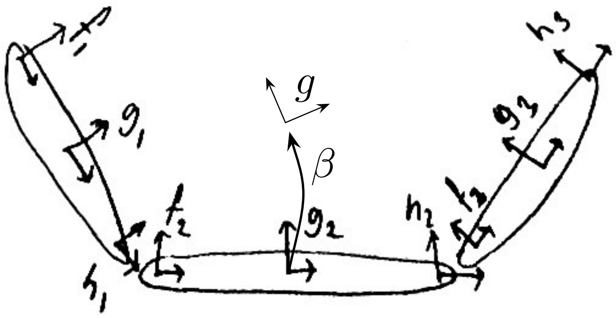


Figure 3.9 Three-link mobile system with a frame g a transformation β away from the middle link.

1. Taking our body-frame iterative Jacobian formula from (3.37), we get

$$\xi_g = Ad_\beta^{-1}(\xi_{g_2} + v_\beta). \tag{3.60}$$

This leaves the question, however, of calculating v_β. We don't just want $\dot\beta = (\partial\beta/\partial\alpha)\ \dot\alpha$, we want *the velocity with respect to* g_2 *of the frame rigidly attached to* g *and coincident with* g_2. Taking advantage of properties of the spatial velocity, we can use a right action to find this velocity,

$$v_\beta = T_\beta R_{\beta^{-1}}\dot\beta = T_\beta R_{\beta^{-1}}\tfrac{\partial\beta}{\partial\alpha}\dot\alpha, \tag{3.61}$$

and thus the body velocity of g,

$$\xi_g = Ad_\beta^{-1}(\xi_{g_2} + T_\beta R_{\beta^{-1}}\tfrac{\partial\beta}{\partial\alpha}\dot\alpha). \tag{3.62}$$

The Jacobian for this frame can then be found by expanding this expression to

$$\xi_g = \begin{bmatrix} \cos\beta^\theta & \sin\beta^\theta & \beta^x\sin\beta^\theta - \beta^y\cos\beta^\theta \\ -\sin\beta^\theta & \cos\beta^\theta & \beta^x\cos\beta^\theta + \beta^y\sin\beta^\theta \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} \xi_{g_2}^x \\ \xi_{g_2}^y \\ \xi_{g_2}^\theta \end{bmatrix} + \begin{bmatrix} 1 & 0 & \beta^y \\ 0 & 1 & -\beta^x \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial\beta^x}{\partial\alpha_1} & \frac{\partial\beta^x}{\partial\alpha_2} \\ \frac{\partial\beta^y}{\partial\alpha_1} & \frac{\partial\beta^y}{\partial\alpha_2} \\ \frac{\partial\beta^\theta}{\partial\alpha_1} & \frac{\partial\beta^\theta}{\partial\alpha_2} \end{bmatrix} \begin{bmatrix} \dot\alpha_1 \\ \dot\alpha_2 \end{bmatrix} \right), \tag{3.63}$$

evaluating the matrix multiplications, and collecting terms.

Alternatively, we can reform (3.62) to simplify the matrix algebra by separating the adjoint action into its components,

$$\xi_g = \overbrace{(T_\beta L_{\beta^{-1}})(T_\mathbf{e} R_\beta)}^{Ad_\beta^{-1}}(\xi_{g_2} + T_\beta R_{\beta^{-1}}\tfrac{\partial\beta}{\partial\alpha}\dot\alpha). \tag{3.64}$$

and distributing the right lifted action over the velocity terms,

$$\xi_g = T_\beta L_{\beta^{-1}}(T_\mathbf{e} R_\beta \xi_{g_2} + \overbrace{(T_\mathbf{e} R_\beta)(T_\beta R_{\beta^{-1}})}^{I}\tfrac{\partial\beta}{\partial\alpha}\dot\alpha). \tag{3.65}$$

This form of the equation expands as

$$\xi_g = \begin{bmatrix} \cos\beta^\theta & \sin\beta^\theta & 0 \\ -\sin\beta^\theta & \cos\beta^\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & -\beta^y \\ 0 & 1 & \beta^x \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_{g_2}^x \\ \xi_{g_2}^y \\ \xi_{g_2}^\theta \end{bmatrix} + \begin{bmatrix} \frac{\partial\beta^x}{\partial\alpha_1} & \frac{\partial\beta^x}{\partial\alpha_2} \\ \frac{\partial\beta^y}{\partial\alpha_1} & \frac{\partial\beta^y}{\partial\alpha_2} \\ \frac{\partial\beta^\theta}{\partial\alpha_1} & \frac{\partial\beta^\theta}{\partial\alpha_2} \end{bmatrix} \begin{bmatrix} \dot\alpha_1 \\ \dot\alpha_2 \end{bmatrix} \right), \tag{3.66}$$

which, in addition to being more compact than (3.63) also brings us back to how we might have informally described the motion of g if we did not have the benefit of the more general adjoint formulation: ξ_g is a combination of the body velocity of the reference link (with the right lifted action providing the "cross product" term for motion away from the center of that link's rotation), the relative velocity of g with respect to the reference link, and a coordinate rotation (provided by the left lifted action) to bring the vector's components into the new body frame,

$$\xi_g = \overbrace{T_\beta L_{\beta^{-1}}}^{\text{coordinate transformation}}(\underbrace{T_{\mathbf{e}} R_\beta \xi_{g_2}}_{\text{motion of original frame}} + \overbrace{\tfrac{\partial \beta}{\partial \alpha}\dot{\alpha}}^{\text{relative motion}}). \tag{3.67}$$

2. Building the link Jacobians based on the body velocity of g is accomplished by inverting (3.60) to provide ξ_{g_2} as a function of ξ_g,

$$\xi_{g_2} = Ad_\beta \xi_g - v_\beta, \tag{3.68}$$

then following an expansion similar to that presented above. The new expressions for ξ_{g_2} can then be inserted into the iterative development of the Jacobian as the velocity of the second link.

3.7 Modal Kinematics

- Joint angles are convenient parameters for describing system shape, but are not the only ones
- For a differential drive car, it is often more useful to think about how much the wheels are turning together, verses how much they are turning oppositely, rather than how much wheel 1 is turning and how much wheel 2 is turning
- wheels moving together corresponds to driving forward, wheels moving oppositely is turning
- this lets us align our coordinate axes with our control directions
- Similarly in figure 3.6, we talked about "even" and "odd" combinations of the joint angles
- Instead of representing shape as a pair of joint angles, we can make a change of coordinates to represent the shape as amplitudes of these odd and even combinations, (a_1, a_2) for

$$a_1 = (\alpha_1 + \alpha_2)/2 \tag{3.69}$$

$$a_2 = (\alpha_1 - \alpha_2)/2 \tag{3.70}$$

- This represents taking a *modal decomposition* of the shape, declaring a new set of coordinate bases
- Figure of a basis rotation on the shape space
- Identifying useful system modes can clarify control like for the differential drive car
- It can also help to separate the ideas of "degree of freedom" and "point of articulation"
- Up until now, we have been considering each joint as an individual degree of freedom
- Mechanism design may directly couple multiple points of articulation into a mode
- Alternately, we may choose to combine multiple points of articulation into a single "virtual joint" to gain more complex kinematics without increasing planning dimensionality
- Example with a three-joint system and two modes getting a reduced-dimensional planning space

3.8 Continuous Curvature

- Modes also let us apply finite-dimensional analysis to systems with continuous deformation
- Same concept as appears in modal analysis for things like vibration of continuous systems: if we know ways in which system will deform (or can command it to deform in certain ways), then can use a series of modes to represent these deformations.
- For practical systems, where actuation can be expected to occur at constant points along the body, curvature modes are a natural choice for describing shape
- Generalization of point articulations – joint angles are discretized curvature
- Greybox: curvature (of a 1-d structure)
- Examples with polynomial and sinusoidal curvature

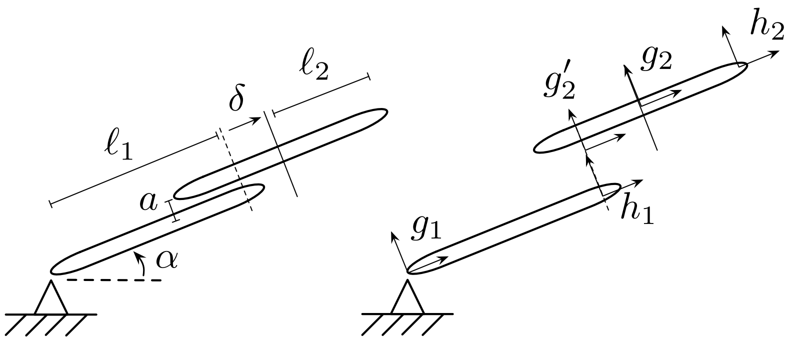


Figure 3.10 Rotary-prismatic arm with links laterally offset by a.

Exercises

3.1 Find a set of holonomic constraints that produce the same accessible manifold as that produced by the constraints in (3.i) and (3.ii), but for which the first constraint restricts the system to points that are a fixed distance from $(0,0,0) \in \mathbb{R}^3$.

3.2 Find the Jacobian for the end-effector of the offset rotary-prismatic arm shown in Figure 3.10, using

a. the absolute-position iterative Jacobian method in §3.5.2
b. the relative-position iterative Jacobian method in §3.5.3

3.3 Three-link systems:

a. For the three-link system shown in Figure 3.9, calculate the transform

$$\beta(\alpha) = g_{g_2} \tag{3.71}$$

that makes g the body frame defined by the mean center of mass position of the three links (assume uniform mass distribution along their lengths) and their mean orientations.

b. Calculate the (body-frame) Jacobian of this center-of-mass frame, taking the middle link as the system body frame.

c. Calculate the (body-frame) Jacobians for the three links, taking the center-of-mass frame as the system body frame.

3.4 Show (in coordinates) that $Ad_h^{-1} Ad_g$ where the x and y values of g and h are equal produces the rotational change of basis operation in $SE(2)$

4
Kinematic Locomotion

A fundamental difference between the motions of fixed-base and mobile articulated systems lies in the degree to which they are actuated. As a general principle, motors or muscles can be attached to an articulated system at the joints, and can then be controlled to specify the relative positions of the links. For a fixed-base system, in which mechanical ground acts as a link, controlling the joint angles completely specifies the configuration of the system, making it *fully actuated*. Mobile systems, in contrast, are almost exclusively *underactuated*—controlling their joint angles specifies their shapes, but does not directly influence their positions.[1]

Despite this underactuation, many mobile articulated systems *can* propel themselves through their environments, by exploiting physical constraints on their motions. As a system changes shape, it may be subject to restrictions on the velocities of points on the body, momentum conservation laws, or other effects, such as fluid or frictional drag. These interactions produce reaction forces on the system that ultimately dictate the motion of its body frame. The process of using these reaction forces to turn internal shape changes into external position changes is *locomotion*.

4.1 Kinematic Locomotion

As a general rule, any configuration trajectory executed by a locomoting system over a time interval $[0, T]$ can be decomposed into a *shape trajectory* ψ in which each shape variable is defined as a function of time,

$$\begin{aligned} \psi : [0, T] &\to M \\ t &\mapsto r, \end{aligned} \tag{4.1}$$

and an associated *induced a position trajectory* g^ψ

$$\begin{aligned} g^\psi : [0, T] &\to G \\ t &\mapsto g, \end{aligned} \tag{4.2}$$

defined by the interaction between the shape change, the system constraints, and the initial state of the system.

Kinematic locomoting systems are those for which changing the *pacing*, or time parameterization, of a shape trajectory changes the pacing of its induced position trajectory in the same fashion, without affecting the path it traces out: given two shape changes ψ and ψ' defined such that

$$\psi(t) = \psi'(\tau(t)), \tag{4.3}$$

i.e. such that ψ' is a continuous reparameterization of ψ with respect to time, the induced position trajectories of a kinematic system have the same relationship,

$$g^\psi(t) = g^{\psi'}(\tau(t)), \tag{4.4}$$

so that $g^{\psi'}$ can be found equivalently by repacing g^ψ or by evaluating the system constraints during the execution of ψ', as indicated in the commutative diagram in Figure 4.1. This geometric reparameterizability then

[1] An articulated system may also be underactuated in the sense that one or more joints are uncontrolled, but we do not consider this case here.

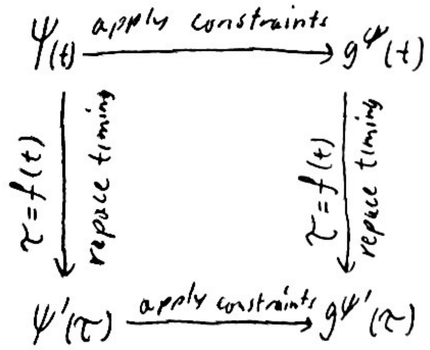


Figure 4.1 Commutative diagram showing relationship between shape changes, induced position changes, and time reparameterizations that defines a kinematic system.

means that the displacement induced by a given shape trajectory is entirely a function of the geometric path it follows, and not the rate with it is executed.

Kinematic systems both provide useful mathematical examples, embodying many fundamental concepts in geometric mechanics and differential geometry, and are important systems in their own rights, appearing independently across several areas of applied mechanics research. Here and in the following chapters, we focus our attention on a particularly interesting class of such systems, *symmetric linear-kinematic locomotors*, over a range of physical domains.

At a differential level, kinematic locomotion corresponds to a configuration-dependent *directionally linear*[2] relationship between a system's shape velocity and induced position velocity,

$$\dot{g} = f(q, \hat{\dot{r}}) \|\dot{r}\|, \tag{4.5}$$

with the effect that scaling the shape velocity proportionally scales the position velocity, without altering its direction. In a *linear-kinematic* locomoting system, the mapping between the shape velocity and position velocity is fully linear at each configuration,

$$\dot{g} = f(q)\dot{r}. \tag{4.6}$$

Linear-kinematic effects feature in the locomotion of many engineered and biological systems, to the extent that the existence of the broader class of general kinematic systems is seldom mentioned, and the "linear" qualifier is consequently dropped in much of the literature. Following this section, we will adopt this convention, except where linearity or the lack thereof needs to be clarified or emphasized.

In a uniform environment, the dynamics of many locomoting systems are *symmetric*[3] with respect to the position and orientation of the body frame: if the equations of motion are expressed in the body frame, then they are independent of the position variables. For a symmetric system that is also linear-kinematic, the body velocity can thus be expressed as a shape-dependent linear function of the shape velocity,

$$\xi = f(r)\dot{r}. \tag{4.7}$$

Exploiting the symmetric nature of such systems offers several analytical advantages. First, removing the position variables from the equations of motion both reduces their dimensionality and provides an intuitive description of how changes in the system shape move the system "forward" or "sideways." Second the dual interpretation of the body velocity as a vector in the tangent space at the origin (§**??**) allows us to combine (4.7) with powerful differential geometric tools that operate on $T_{\mathbf{e}}G$, such as the Lie bracket in Chapter **??**.

Two familiar, concrete examples of symmetric linear-kinematic locomotors are the differential drive and Ackerman-steered cars. As depicted in Figure 4.2, both of these systems move in the plane, with positions $g \in SE(2)$ defining body frames aligned with the vehicle chassis and located at the axle. In these frames, the

[2] See the box on the following page.
[3] See the box on page 73.

Directional Linearity

A function

$$\begin{aligned} f : A &\to B \\ a &\mapsto b, \end{aligned} \tag{4.i}$$

where A is a vector space, is *directionally linear* if each element of b is a linear function of a along each line passing through the origin, *i.e.* it can be written in the form

$$b_i = f_i(\hat{a})\|a\|, \tag{4.ii}$$

where $\hat{a}$ is the unit vector $a/\|a\|$, and satisfies the condition

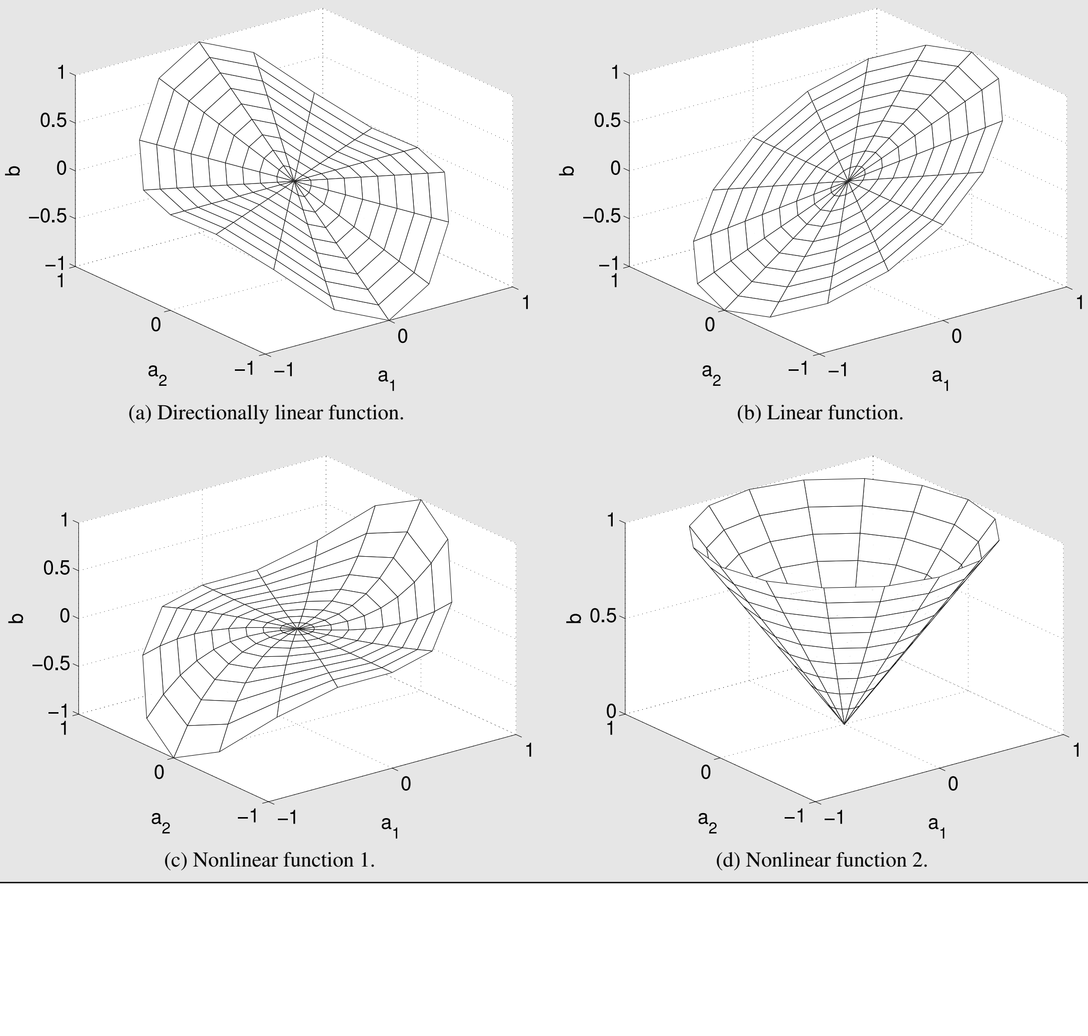
$$b(-a) = -b(a). \tag{4.iii}$$

The four functions shown below illustrate characteristics of functions that satisfy this definition, and how they relate to more commonly encountered fully-*linear* functions. The directionally linear function in (a) is linear along each azimuth (characterized by its unit vector $\hat{a}$), and the slope is maintained across the origin, satisfying $b(-a) = -b(a)$. Linear functions, as shown in (b), are a special case of directionally linear functions in which the slopes along the azimuths are linearly related, and (4.ii) takes the form

$$b = \overbrace{\mathcal{M}\hat{a}}^{f(\hat{a})}\|a\| = \mathcal{M}a, \tag{4.iv}$$

where $\mathcal{M}$ is a matrix that multiplies a to produce b.

The functions in (c) and (d) are *not* directionally linear. The first function's magnitude increases nonlinearly along the azimuths, violating the condition in (4.ii). The second function is linear along the azimuths, but does not meet the second linearity condition in (4.iii).



(a) Directionally linear function.

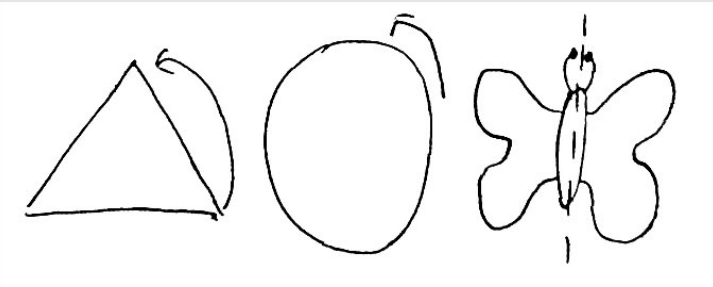
(b) Linear function.

(c) Nonlinear function 1.

(d) Nonlinear function 2.

Symmetry and Groups

When we say that something has *symmetry*, we mean that there is some action we can perform on it so that its starting and ending configurations are indistinguishable. For example, an equilateral triangle is symmetric with respect to rotations by $120°$, a circle is symmetric with respect to any rotation, and a butterfly is symmetric with respect to reflections across its central axis.



Symmetry is closely linked to mathematical groups structure—the set of actions under which an object is symmetric form a group, satisfying the four basic properties outlined on on page 9:

1. **Closure**: Two symmetry-preserving actions can be concatenated into a third action that is also symmetry-preserving. Example: two rotations by $120°$ form a $240°$ rotation, which is also a symmetric action on the equilateral triangle.
2. **Associativity**: If a series of transformations is conducted, the order of operations does not matter (though left-right order may still be important). Example: rotating the triangle twice by $120°$ is equivalent to concatenating the rotations into a single $240°$ rotation and applying it to the triangle.
3. **Identity**: Objects are trivially symmetric under null actions, which can be incorporated as zero-magnitude members of the symmetry group. Example: the rotational symmetry group for the triangle is rotations by $k \cdot 120°$ for integer values of k, and the identity element (no rotation) is the element for whcih $k = 0$.
4. **Inverse**: Any symmetry-preserving transformation may be "undone" or reversed, and concatenating this reversal with the original action produces a null action. Example: the inverse of a $120°$ rotation is a $-120°$ rotation, and together these actions produce a $0°$ rotation.

In many cases, we can use symmetries in a system to simplify its representation. Several examples of this principle have implicitly appeared in previous chapters; for instance, when we describe the configuration space of a rotary joint as $\mathbb{S}^1$, we are recognizing that the configuration of the attached link is symmetric with respect to complete revolutions of the joint, and thus (unless the number of revolutions is important for other reasons) that we can treat such configurations as equivalent. Similarly, our definitions of rigid and articulated bodies are fundamentally based on their symmetry with respect to $SE(2)$—under any $SE(2)$ transformation of the system (which may be applied directly to the particles using (1.20)), the body-frame positions of particles in the system are unchanged, allowing us to characterize the system by its shape in the body frame.

More generally, link between symmetries and rigid bodies leads us to an important aspect of symme

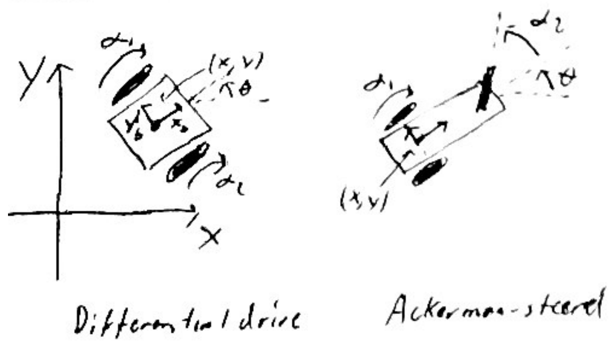


Figure 4.2 Differential drive and Ackerman-steered systems

components of the body velocity, ξ^x, ξ^y, and ξ^θ, correspond respectively to the vehicle's longitudinal, lateral, and rotational velocities. The "shape" of these systems describes their internal degrees of freedom. For the differential drive car, this shape is the rotation of each wheel around its axle (relative to a reference angle); for the Ackerman car (also known as the "kinematic bicycle"), the shape variables are the rotation of the drive axle and the steering angle.

The differential drive car moves forward when its wheels are turned together, and rotates when they are turned oppositely; for appropriately normalized wheel diameters and separations, we can encode these rules in an equation matching the form of (4.7),

$$\begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}. \tag{4.8}$$

Similarly, the Ackerman car moves forward when the drive axle is rotated, and simultaneously rotates at a rate dictated by the steering angle,

$$\begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \tan\alpha_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}. \tag{4.9}$$

4.2 Pfaffian Constraints and the Local Connection

In geometric mechanics, the equations of motion for symmetric linear-kinematic systems (which, from now on we will refer to as "kinematic locomotors") are generally expressed as a *kinematic reconstruction equation*,

$$\xi = -\mathbf{A}(r)\dot{r}, \tag{4.10}$$

where $\mathbf{A}(r)$ is the *local connection*[4] associated with the system constraints, and the negative sign appears for historical reasons. As a general principle, a system's local connection corresponds to, and can be derived from, a *Pfaffian constraint*[5] on its configuration velocities,

$$\mathbf{0} = \omega(r) \begin{bmatrix} \xi \\ \dot{r} \end{bmatrix}. \tag{4.11}$$

For an $SE(2)$ system with m individual Pfaffian constraints and n shape variables, the overall Pfaffian constraint on the system is encoded by an $m \times (3+n)$ element matrix,

$$\mathbf{0}^{m\times 1} = \omega^{m\times(3+n)} \begin{bmatrix} \xi^{3\times 1} \\ \dot{r}^{n\times 1} \end{bmatrix}, \tag{4.12}$$

[4] For the origin of this term, see the discussion on fiber bundles in Chapter. **??**.
[5] See the box on the next page.

Nonholonomic Constraints

Nonholonomic constraints restrict the velocity with which a system can move, but *without* restricting the accessible configurations. Formally, a nonholonomic constraint is defined by a (possibly time-varying) function c on the system's configuration tangent bundle, TQ. The zero set of this function (the set of velocities $\dot{Q}_0 \in T_qQ$ at each point q in the configuration space such that $c(q, \dot{q}, t) = 0$) defines the system's allowable velocities at each configuration that satisfy the constraint.

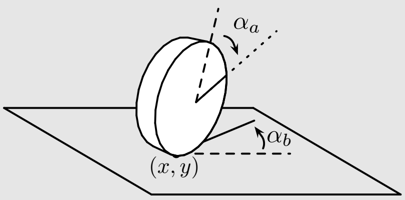
Rolling disk. The classic example of a nonholonomically constrained system is a rolling disk or coin. Viewed from the top as an $SE(2)$ system, the coin is free to roll forward and backward or turn, but is unable to slide sideways; this no-slide condition can be expressed as a nonholonomic constraint either in world coordinates, as

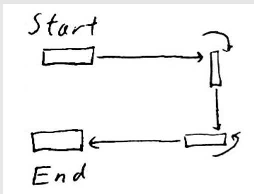
$$c_{\text{world}} = \dot{y}\cos\theta - \dot{x}\sin\theta, \tag{4.v}$$

or with respect to the body velocity of the disk,

$$c_{\text{body}} = \xi^y. \tag{4.vi}$$

This constraint prevents lateral motion of the disk, but does not prevent it from achieving arbitrary net lateral displacement. By combining rolling and turning actions into a "parallel parking" motion, the disk can access any point in the $SE(2)$ space despite the constraint on its velocity.





Pfaffian constraints. A particularly important class of nonholonomic constraints are linear *Pfaffian* constraints, which take the form

$$c(q, \dot{q}) = \omega(q)\dot{q}, \tag{4.vii}$$

where ω is a matrix with as many rows as there are independent constraints in the Pfaffian, and as many columns as there are dimensions in the configuration space. At each configuration, the allowable velocities for the system are in the nullspace of $\omega(q)$, forming a vector space. The collection of these vector spaces is the *distribution* of the system's allowable velocities.

An unconstrained system with an n-dimensional configuration manifold has an n-dimensional distribution at each point in the configuration space, corresponding to the whole of T_qq. Each independent Pfaffian constraint added to the system removes one dimension from the distribution, reducing the local degrees of freedom of the system in much the same way that multiple holonomic constraints reduce the global degrees of freedom.

continued ...

Nonholonomic Constraints, *continued*

Nonintegrablity. To qualify as a nonholonomic constraint function, c must not be integrable into a holonomic constraint $f(q,t) = 0$. For instance, the constraint function

$$c = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \dot{x}x + \dot{y}y, \tag{4.viii}$$

applied to a point moving on the (x, y) plane dictates that the point cannot move radially with respect to the origin. Although written in the form of a Pfaffian constraint, it is not a nonholonomic constraint: the restriction against radial velocities means that we can rewrite the function as the holonomic constraint

$$f = \sqrt{x^2 + y^2} - R, \tag{4.ix}$$

where R is the initial distance of the point from the origin, and the corresponding accessible manifold is the circle of radius R. forward reference to using Lie brackets to check integrability

each row of which represents a restriction on the motion of the system by mapping the configuration velocity (the body and shape velocities) to zero. Kinematic locomoting systems have Pfaffians composed of at least three independent constraints, one per degree of freedom in the position space. A system with fewer constraints gains the ability to *drift* through the position space without changing shape, and thus is not fully kinematic. With exactly as many constraints as position variables, specifying the system's shape velocity "uses up" all of the local degrees of freedom, so the body velocity becomes a (linear) function of the shape velocity, making the system kinematic. If there are more constraints present than positional degrees of freedom, the system is overconstrained from a controls perspective, and only certain shape trajectories can be executed.

The systems we consider below all have three independent constraints, and thus are kinematic without being overconstrained. In part, this simplifies our subsequent analysis, in that we are able to make unconstrained choices of shape trajectories when designing locomotive gaits for the system in Chapters **??**, **??**, and **??**. More saliently, however, the physical principles imposing the constraints often intrinsically match the number of constraints to the dimension of the position space. For example, the inertial systems in §4.4.3 are constrained to conserve linear and angular momentum; these quantities together have as many components as there are in G, and thus provide a corresponding number of constraints (in the case of SE(2), two for linear momentum and one for the angular term).

Given a three-constraint Pfaffian ($m = 3$), we can directly calculate the system's local connection by making the natural decomposition of ω into ω_ξ and $\omega_{\dot{r}}$, the submatrices that are respectively multiplied by the body and shape velocity terms,

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \omega_\xi^{3\times 3} & \omega_{\dot{r}}^{3\times n} \end{bmatrix} \begin{bmatrix} \xi \\ \dot{r} \end{bmatrix}, \tag{4.13}$$

then separating the constraint equation into the sum of two smaller matrix multiplications,

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \omega_\xi \xi + \omega_{\dot{r}} \dot{r}. \tag{4.14}$$

Bringing the ξ terms to the left hand side of the equation,

$$\omega_\xi \xi = -\omega_{\dot{r}} \dot{r}, \tag{4.15}$$

and multiplying both sides by ω_ξ^{-1},

$$\xi = -\omega_\xi^{-1} \omega_{\dot{r}} \dot{r}, \tag{4.16}$$

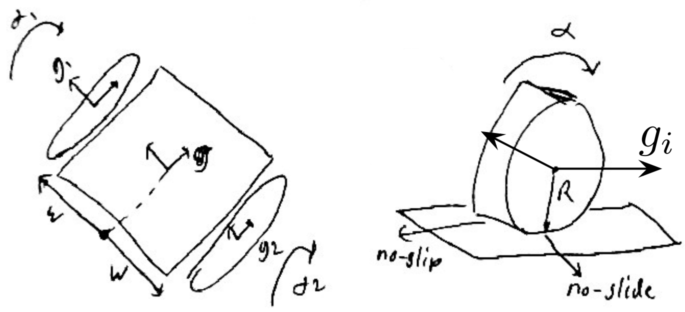


Figure 4.3 Parameterized planar model for a differential-drive car. At left, the (out-of-plane) no-slip, no-slide constraint on the wheels.

puts the relationship between the body and shape velocities into the form of (4.10), revealing the local connection as

$$\mathbf{A} = \omega_{\xi}^{-1}\omega_{\dot{r}}. \tag{4.17}$$

As an example of this process, consider a differential-drive car modeled as a three-body planar system as in Figure 4.3. The body frame of the system, g is located at the center of the axle-line and fixed to the vehicle chassis. The two wheels, with body frames g_1 and g_2, have a radius R and are at a distance w from the center of the chassis. Under the conventional no-slip, no-slide condition for a rolling wheel, the wheel's point of contact with the ground has zero velocity; these conditions thus impose a pair of nonholonomic constraints on each wheel,

$$\xi^x_{g_i} - R\dot{\alpha}_i = 0 \qquad \text{(no-slip)} \tag{4.18}$$
$$\xi^y_{g_i} = 0 \qquad \text{(no-slide)} \tag{4.19}$$

To turn these constraints on the wheels into constraints on the system as a whole, we can employ the Jacobian generation techniques in §3.5.2 to express ξ_{g_1} and ξ_{g_2} in terms of the system shape and body velocities. Here, as the wheels have fixed positions with respect to the system body frame,

$$g_{1,g} = (0, w, 0) \qquad \text{and} \qquad g_{2,g} = (0, -w, 0), \tag{4.20}$$

finding these expressions reduces to applying a single inverse adjoint action to the body velocity to find the body velocity of each wheel,

$$\xi_{g_1} = Ad^{-1}_{g_{1,g}}\xi = \begin{bmatrix} \xi^x - w\xi^\theta \\ \xi^y \\ \xi^\theta \end{bmatrix} \tag{4.21}$$

and

$$\xi_{g_2} = Ad^{-1}_{g_{2,g}}\xi = \begin{bmatrix} \xi^x + w\xi^\theta \\ \xi^y \\ \xi^\theta \end{bmatrix}. \tag{4.22}$$

Substituting the components of ξ_{g_1} and $\xi_{g_{g_2}}$ from (4.21) and (4.22) into the no-slip and no-slide constraints from (4.18) and (4.19) produces three independent nonholonomic constraints on the car,

$$\xi^x - w\xi^\theta - R\dot{\alpha}_1 = 0 \qquad \text{(no-slip wheel 1)} \tag{4.23}$$
$$\xi^x + w\xi^\theta - R\dot{\alpha}_2 = 0 \qquad \text{(no-slip wheel 2)} \tag{4.24}$$
$$\xi^y = 0 \qquad \text{(no-slide wheels 1 \& 2)}, \tag{4.25}$$

with the no-slide constraints on the two wheels each representing the same constraint on the whole vehicle.

These constraints are linear in the shape and body velocities, and so can be expressed in Pfaffian form as

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix} \overbrace{\begin{matrix}1 & 0 & -w\\ 1 & 0 & w\\ 0 & 1 & 0\end{matrix}}^{\omega_\xi} & \overbrace{\begin{matrix}-R & 0\\ 0 & -R\\ 0 & 0\end{matrix}}^{\omega_{\dot r}}\end{bmatrix} \begin{bmatrix}\xi^x\\ \xi^y\\ \xi^\theta\\ \dot\alpha_1\\ \dot\alpha_2\end{bmatrix} \begin{matrix}\left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\}\xi\\ \left.\vphantom{\begin{matrix}1\\1\end{matrix}}\right\}\dot r\end{matrix}. \tag{4.26}$$

The reconstruction equation and local connection for the system can then be directly calculated as

$$\xi = -\overbrace{\begin{bmatrix}1 & 0 & -w\\ 1 & 0 & w\\ 0 & 1 & 0\end{bmatrix}}^{\omega_\xi}{}^{-1} \overbrace{\begin{bmatrix}-R & 0\\ 0 & -R\\ 0 & 0\end{bmatrix}}^{\omega_{\dot r}} \dot r \tag{4.27}$$

$$= -\begin{bmatrix}1/2 & 1/2 & 0\\ 0 & 0 & 1\\ -1/(2w) & 1/(2w) & 0\end{bmatrix}\begin{bmatrix}-R & 0\\ 0 & -R\\ 0 & 0\end{bmatrix}\dot r \tag{4.28}$$

$$= -\underbrace{\begin{bmatrix}-R/2 & -R/2\\ 0 & 0\\ R/(2w) & -R/(2w)\end{bmatrix}}_{\mathbf{A}}\dot r, \tag{4.29}$$

which, for $R = 2$ and $w = 1$, exactly matches (4.8).

4.3 Connection Vector Fields

How can we visualize the local connection?

- First, consider what the rows and columns mean.
- Columns: jth column is the body velocity the system will have if the jth joint is moving with unit velocity and all other joints are fixed.
- Rows: ith row is the *local gradient* of the position with respect to the shape – its dot product with the shape velocity givesif we plot it as a vector field
-

show examples of plane fields and local contour plots to demonstrate why they make poor visualizations

Will show diffdrive and ackerman connection vector fields here as examples

4.4 Full-body Locomotion

Wheeled vehicles, such as the differential-drive and Ackerman-steered cars, tend to have well-defined “drive” and “steering” inputs. In many other systems, the notions of drive and steering are less distinct. For example, snakes, fish, and microorganisms locomote by undulating their bodies in traveling-wave patterns. During these motions, sections of the body alternately take on roles as thrust elements or control surfaces guiding the motion of the system, and applying control abstractions like “drive forward” or “turn with a given radius” becomes less intuitive.

The local connection and other tools from differential geometry offer a systematic overview of the kinematics

Vectors, Covectors, and One-forms

In vector calculus, the term "vector" applies to any directional quantity, such as the velocity of a point, a flow rate, or the gradient of a function. In differential geometry, there is a distinction between *tangent vectors* (often referred to simply as vectors), which are velocity-like terms that describe motion *through* the underlying space, and *cotangent vectors* (often shortened to *covectors*) that represent gradient-like terms describing how a quantity varies *across* the space.

Tangent and cotangent vectors share many characteristics. Just as (tangent) vectors are elements of a manifold's tangent bundle, TQ, covectors are elements of its *cotangent bundle*, T^*Q, which is *dual* to TQ. A *covector field* assigns a covector to each cotangent space T_q^*Q in a subspace of T^*Q, in the same manner as a vector field assigns vectors to a set of tangent spaces.

Notation. Components of a vector are identified with superscripts,

$$v = (v^1, v^2, \ldots, v^n) \in T_qQ \tag{4.x}$$

and components of a covector with subscripts,

$$\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in T_q^*Q. \tag{4.xi}$$

When equivalent bases are used for the tangent and cotangent spaces, vectors and covectors with matched coefficients $\omega_i = v^i$ are *natural duals* to each other. To convert a vector into its natural dual covector, or vice versa, we use *musical notation* to "lower" or "raise" the indices,

$$v^\flat = \omega \quad \text{and} \quad \omega^\sharp = v. \tag{4.xii}$$

Products Vectors and covectors have a natural product corresponding to the vector-calculus notion of a dot product between two vectors,

$$\omega v = \langle \omega, v \rangle = \sum_i \omega_i v^i = \begin{bmatrix} \omega_1 & \ldots & \omega_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \omega^\sharp \cdot v, \tag{4.xiii}$$

meaning that we can consider a covector $\omega \in T_q^*Q$ to be a mapping from vectors to real numbers,

$$\begin{aligned} \omega : T_qQ &\to \mathbb{R} \\ v &\mapsto \langle \omega, v \rangle. \end{aligned} \tag{4.xiv}$$

If ω is a function of the configuration (*i.e.* is a covector field, rather than an isolated covector), it becomes a map from configuration, velocity pairs to real numbers,

$$\begin{aligned} \omega : TQ &\to \mathbb{R} \\ (q, v) &\mapsto \langle \omega(q), v \rangle, \end{aligned} \tag{4.xv}$$

that is linear with respect to v, but may vary nonlinearly with q. Functions of this type are also known as *(differential) one-forms*. Differential forms play an important role in geometric mechanics, and we will examine them in greater depth in Chapter **??**.

continued ...

Vectors, Covectors, and One-forms, *continued*

Vector-valued output. Multiple covector fields may be combined to form a *vector-valued one-form*, in which each element of the output is the product of one of the component covectors with the input vector. A vector-valued one-form ω with m components, $(\omega^1, \ldots, \omega^m)$,[a] can be encoded as an $m \times n$ matrix,

$$\omega(q)v = \begin{bmatrix} \omega_1^1(q) & \ldots & \omega_n^1(q) \\ \vdots & \ddots & \vdots \\ \omega_1^m(q) & \ldots & \omega_n^m(q) \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}. \tag{4.xvi}$$

Many familiar mathematical objects have this structure, including the Jacobians in §3.5 and the Pfaffians and local connections in §4.2. In the following chapters, we combine the physical nature of these objects with differential-geometric properties of differential forms to arrive at a rigorous mathematical description of locomotion.

Directional derivative. Above, we commented that covectors represent "gradient-like" terms that describe the rates of change of functions with respect to the underlying space. More specifically, any covector ω locally represents the derivative[b] of an implied function f with respect to the configuration space,

$$\omega = \mathbf{d}f = \left(\frac{\partial f}{\partial q^1}, \ldots, \frac{\partial f}{\partial q^n} \right) \tag{4.xvii}$$

Under this interpretation, the product $\langle \omega, v \rangle$ represents the *directional derivative* of f along v,

$$D_v f = \langle \mathbf{d}f, v \rangle = \frac{\partial f}{\partial q} v, \tag{4.xviii}$$

i.e., the rate of change of f along v. In some sources, the directional derivative is normalized by the magnitude of v,

$$D_v f = \frac{\langle \mathbf{d}f, v \rangle}{\|v\|} = \frac{\partial f}{\partial q} \hat{v}. \tag{4.xix}$$

By removing the speed of v from the calculation, this definition better matches the notion of "directionality" than does the definition in (4.xviii), but is ill-defined when v is a zero-vector; consequently the first definition is preferred in differential-geometric contexts.

Note that the function f may be defined over the whole space, in which case

$$\nabla f = \omega^\sharp \tag{4.xx}$$

is its *gradient vector field*,[c] pointing in the direction in which f increases the most quickly. Alternately, it may be only locally definable – the local connection is the derivative of the system's position with respect to its shape, $\partial g / \partial r$, but the output is represented in the system's instantaneous body coordinates, which may change as the system moves. The ability to define f globally is closely related to the *conservativity* of ω and to the *integrability* of nonholonomic constraints. We will return to these subjects in Chapters **??** and **??**.

[a] While elements of a covector are called out by subscripts, here we are in effect dealing with a "vector of covectors," and so indicate individual covectors with a superscript.

[b] Formally, the *exterior derivative* discussed on on page ??.

[c] An alternate definition of the gradient additionally incorporates the *distance metrics* on a space, as discussed in Chapter **??**.

of such full-body locomotion modes. This overview is especially powerful in that it unifies the analysis of kinematic systems across a wide range of physical domains and body articulations. As an introduction to this analysis, in this section we consider the local connections of *three-link locomotors* in several physical regimes. Three-link systems play an important role in locomotion research; as discussed in §**??**, they are among the "simplest" systems capable of effective locomotion, and so readily lend themselves as minimal examples of locomotive principles. The physical regimes we consider, based on no-slide, inertial, and fluid constraints, are likewise representative of research in geometric mechanics, reflecting the communities that have driven development of these tools.

4.4.1 Three-link Locomotors

Three-link locomoting systems
 §3.6.1
 velocities from (3.46), (3.51), and (3.56)

$$\xi_{g_1} = \begin{bmatrix} \xi^x \cos\alpha_1 - (\xi^y + (\xi^\theta \ell_2)/2)\sin\alpha_1 \\ \xi^x \sin\alpha_1 + (\xi^y - (\xi^\theta \ell_2)/2)\cos\alpha_1 - (\ell_1/2)(\xi^\theta - \dot\alpha_1) \\ \xi^\theta - \dot\alpha_1 \end{bmatrix} \tag{4.30}$$

$$\xi_{g_2} = \begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \end{bmatrix} \tag{4.31}$$

$$\xi_{g_3} = \begin{bmatrix} \xi^x \cos\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\sin\alpha_2 \\ -\xi^x \sin\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\cos\alpha_2 + (\ell_3/2)(\xi^\theta + \dot\alpha_2) \\ \xi^\theta + \dot\alpha_2 \end{bmatrix} \tag{4.32}$$

4.4.2 No-slide Constraints

For

$$\begin{bmatrix} \xi^y_{g_1} \\ \xi^y_{g_2} \\ \xi^y_{g_3} \end{bmatrix} = \begin{bmatrix} \xi^x \sin\alpha_1 + (\xi^y + (\xi^\theta \ell_2)/2)\cos\alpha_1 - (\ell_1/2)(\xi^\theta - \dot\alpha_1) \\ \xi^y \\ -\xi^x \sin\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\cos\alpha_2 + (\ell_3/2)(\xi^\theta + \dot\alpha_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.33}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sin\alpha_1 & \cos\alpha_1 & (\ell_2\cos\alpha_1 - \ell_1)/2 & \ell_1/2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -\sin\alpha_2 & \cos\alpha_2 & (\ell_2\cos\alpha_2 + \ell_3)/2 & 0 & \ell_3/2 \end{bmatrix} \begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \\ \dot\alpha_1 \\ \dot\alpha_2 \end{bmatrix} \tag{4.34}$$

$$\xi = -\overbrace{\begin{bmatrix} \sin\alpha_1 & \cos\alpha_1 & (\ell_2\cos\alpha_1 - \ell_1)/2 \\ 0 & 1 & 0 \\ -\sin\alpha_2 & \cos\alpha_2 & (\ell_2\cos\alpha_2 + \ell_3)/2 \end{bmatrix}}^{\omega_\xi}{}^{-1} \overbrace{\begin{bmatrix} -\ell_1/2 & 0 \\ 0 & 0 \\ 0 & \ell_3/2 \end{bmatrix}}^{\omega_{\dot r}} \dot\alpha \tag{4.35}$$

$$\tag{4.36}$$

$$\xi = -\overbrace{\frac{1}{D}\begin{bmatrix} -\ell_1(\ell_3 + \ell_2\cos\alpha_2)/2 & \ell_3(\ell_1 - \ell_2\cos\alpha_1)/2 \\ 0 & 0 \\ -\ell_1\sin\alpha_2 & \ell_3\sin\alpha_1 \end{bmatrix}}^{\mathbf{A}} \dot\alpha \tag{4.37}$$

Noether's Theorem

Symmetry and Noether's Theorem, with momentum-conservation example

where $D = \ell_2 \sin(\alpha_1 + \alpha_2) - \ell_1 \sin\alpha_2 + \ell_3 \sin\alpha_3$.

4.4.3 Inertial Constraints

The position and shape of the floating snake are chosen similarly to those for the kinematic snake, with $g = (x, y, \theta) \in SE(2)$ and $r = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. In this case, however, the center of mass is a more natural choice for the x and y coordinates; conservation of linear momentum ensures that the velocity of the center of mass does not change, so the first two rows of the local connection are correspondingly zero. The third row of the local connection identifies rotational velocities which preserve a net angular momentum of zero in response to specified joint velocities Shammas et al. (2007). Taking the rows together, the reconstruction equation for a floating snake with equal link lengths and inertias is

$$\xi = -\frac{1}{D}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ a_{31} & a_{32} \end{bmatrix}\begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}, \tag{4.38}$$

where

$$\begin{aligned} a_{31} &= 5 + 3\cos(\alpha_2) + \cos(\alpha_1 - \alpha_2) \\ a_{32} &= 5 + 3\cos(\alpha_1) + \cos(\alpha_1 - \alpha_2) \\ D &= 19 + 6(\cos(\alpha_1) + \cos(\alpha_2)) + 2\cos(\alpha_1 - \alpha_2). \end{aligned}$$

The derivation of (4.38) corresponds naturally to a Pfaffian in which the output of ω is the net linear and angular momentum of the system. In some cases, calculation of this quantity is made easier by an alternate, energy-based calculation Ostrowski and Burdick (1998); Shammas et al. (2007). For a momentum-conserving system whose Lagrangian is equal to its kinetic energy (*i.e.* that has no means of storing potential energy), and whose kinetic energy can be expressed as

$$KE = \frac{1}{2}\begin{bmatrix} \xi & \dot{r} \end{bmatrix} \mathbb{M}(r) \begin{bmatrix} \xi \\ \dot{r} \end{bmatrix}, \tag{4.39}$$

the mass matrix $\mathbb{M}$ contains within itself the local connection. Specifically, $\mathbb{M}$ is of the form

$$\mathbb{M} = \begin{bmatrix} \mathbb{I}(r) & \mathbb{I}(r)\mathbf{A}(r) \\ (\mathbb{I}(r)\mathbf{A}(r))^T & m(r) \end{bmatrix}, \tag{4.40}$$

from which $\mathbf{A}$ is easily extracted.

4.4.4 Low Reynolds Number Swimming

At very low Reynolds numbers, viscous drag forces dominate the fluid dynamics of swimming and any inertial effects are immediately damped out. This effect has two consequences, which we can combine to find the local connection. First, the drag forces on the swimmer are linear functions of the body and shape velocities. Second, the net drag forces and moments on an isolated system interacting with the surrounding fluid go to zero.

For an illustration of the first consequence, consider a three-link swimmer with links modeled as slender members according to Cox theory (Cox, 1970). For simplicity here, we regard the flows around each link as independent, per resistive force theory, but the solution for coupled flows is of the same form. The drag forces and moments on the ith link are based on a principle of lateral drag forces being approximately twice the longitudinal forces (Cox, 1970), with the moment around the center of the link found by assuming the lateral drag forces to be linearly distributed along the link according to its rotational velocity, *i.e.*,

Note that L here is a link half-length. Correct the moment terms accordingly (this is a holdover from an old paper, and will be fixed in the book).

$$F_{i,x} = \int_{-L}^{L} \frac{1}{2} k \xi_{i,x} d\ell = kL\xi_{i,x} \tag{4.41}$$

$$F_{i,y} = \int_{-L}^{L} k \xi_{i,y} d\ell = 2kL\xi_{i,y} \tag{4.42}$$

$$M_i = \int_{-L}^{L} k\ell(\ell \xi_{i,\theta}) d\ell = \frac{2}{3} kL^3 \xi_{i,\theta}, . \tag{4.43}$$

where $F_{i,x}$ and $F_{i,y}$ are respectively the longitudinal and lateral forces, M_i the moment, k the differential viscous drag constant, and $\xi_i = [\xi_{i,x}, \xi_{i,y}, \xi_{i,\theta}]^T$ is the body velocity of the center of the ith link.[a] The link body velocities are readily calculated from the system body and shape velocities as

[a] Note that by "body velocity", we mean the longitudinal, lateral, and rotational velocity of the link, and not its velocity with respect to the body frame of the system.

By extension, the forces in (4.41)–(4.43), which are linearly dependent on the link body velocities, are also linear functions of ξ and $\dot{\alpha}$ and nonlinear functions of α. Summing these forces into the net force and moment on the system (as measured in the system's body frame),

Note that L here is a link half-length. Correct the moment terms accordingly (this is a holdover from an old paper, and will be fixed in the book to denote L as the total body length and some other value as the link length).

$$\begin{bmatrix} F_x \\ F_y \\ M \end{bmatrix} = \begin{bmatrix} \cos\alpha_1 & \sin\alpha_1 & 0 \\ -\sin\alpha_1 & \cos\alpha_1 & 0 \\ L\sin\alpha_1 & -L(1+\cos\alpha_1) & 1 \end{bmatrix} \begin{bmatrix} F_{1,x} \\ F_{1,y} \\ M_1 \end{bmatrix} + \begin{bmatrix} F_{2,x} \\ F_{2,y} \\ M_2 \end{bmatrix} + \begin{bmatrix} \cos\alpha_2 & -\sin\alpha_2 & 0 \\ \sin\alpha_2 & \cos\alpha_2 & 0 \\ L\sin\alpha_2 & L(1+\cos\alpha_2) & 1 \end{bmatrix} \begin{bmatrix} F_{3,x} \\ F_{3,y} \\ M_3 \end{bmatrix}, \tag{4.44}$$

preserves the linear relationship with the velocity terms while only adding further nonlinear dependence on α, such that the net forces $F = [F_x, F_y, M]^T$ can be expressed with respect to the velocities as

$$F = \omega(\alpha) \begin{bmatrix} \xi \\ \dot{\alpha} \end{bmatrix}, \tag{4.45}$$

where ω is a 3×5 matrix.

We now turn to the second consequence of being at low Reynolds number, that the net forces and moments on an isolated system should be zero, *i.e.* $F = [0, 0, 0]^T$. Applying this rule turns (4.45) into a Pfaffian constraint equation, and thus provides the system's local connection.

4.4.5 High Reynolds Number Swimming

At very large Reynolds numbers, viscous drag is negligible and inertial effects dominate the swimming dynamics. While these conditions appear to be the direct opposite of those in the low Reynolds number case, they also result in the system equations of motion forming a kinematic reconstruction equation. This property can be demonstrated via several approaches of varying technical depth Kelly (n.d.); Melli et al. (2006), but to maximize the physical intuition associated with this derivation, we give here a novel presentation based on the that used for the floating snake.

As discussed in the case of the floating snake, for a momentum-conserving system whose Lagrangian is equal to its kinetic energy (*i.e.* it has no means of storing potential energy), the local connection can be easily extracted from the inertial matrix. To apply this principle to the high Reynolds number system, it just remains to be shown that the three-link swimmer at high Reynolds number meets the afore-mentioned conditions. The first condition, that momentum is conserved in the system, follows from the lack of dissipative forces in the high Reynolds number regime. The second condition, that the Lagrangian equal the kinetic energy, can be easily seen by observing that for a planar system with no gravity effects in the plane, there is no mechanism for storing potential energy, leaving only the kinetic term in the Lagrangian. The third condition is more subtle, and as above, we will use a hydrodynamically decoupled example while noting the existence of an equivalent coupled solution.

An object immersed in a fluid displaces this fluid as it moves. In an ideal inviscid fluid, the drag forces on the object are entirely due to this displacement, and act as directional *added masses* $\mathcal{M}$ on the object that sum with the actual inertia of the object to produce the effective inertia of the combined system. The added masses of single rigid bodies (and elements of articulated bodies when the inter-body fluid interactions are neglected) are solely functions of the geometries of the bodies. For example, the added mass tensor of an ellipse with semi-major axis a and semi-minor axis b in a fluid of density ρ is

$$\mathcal{M} = \begin{bmatrix} \mathcal{M}_x & 0 & 0 \\ 0 & \mathcal{M}_y & 0 \\ 0 & 0 & \mathcal{M}_\theta \end{bmatrix} = \begin{bmatrix} \rho\pi b^2 & 0 & 0 \\ 0 & \rho\pi a^2 & 0 \\ 0 & 0 & \rho(a^2 - b^2)^2 \end{bmatrix}, \tag{4.46}$$

with $\mathcal{M}_x$, $\mathcal{M}_y$, and $\mathcal{M}_\theta$ respectively corresponding to the added mass for longitudinal, lateral, and rotational motion.

Returning to the three-link swimmer, the kinetic energy associated with motion of the ith link through the fluid is

$$KE_i = \frac{1}{2}\xi_i^T (I_i + \mathcal{M}_i)\xi_i, \tag{4.47}$$

where I_i is the link's inertia tensor and ξ_i is its body velocity, as calculated in (**??**)–(**??**). Using the same linear dependence of ξ_i on ξ and $\dot{r}$ as we exploited in the low Reynolds number case, it is relatively easy to transform (4.47), and thus $KE = \sum KE_i$, into the form of (4.39), and from there to extract the local connection $\mathbf{A}$. The derivation for the hydrodynamically coupled case is essentially similar, with the chief difference being the additional dependence of $\mathcal{M}$ on r, which captures the distortion of the flow around each link caused by the proximity of the other links Kelly (n.d.).

Exercises

4.1 Show that the nonholonomic constraints on the differential drive car induce a *holonomic* constraint between its orientation and its wheel angles.

4.2 What would change in the calculation of the local connection if the Pfaffian had < 3 rows? How about > 3?

4.3 Find the Pfaffian constraints on an Ackerman car (modeled as a kinematic bicycle), and use them to calculate the system's local connection

a. For the standard rear-wheel drive model
b. For a front wheel drive with passive rear wheels

4.4 Find the local connection for two link swimmers at low and high Reynolds numbers. Assume

- links of equal length ℓ
- a body frame located at the hinge and oriented as the mean angle of the two links
- a $2:1$ drag force ratio for lateral/longitudinal drag on the low Re swimmer
- elliptical links with a $10:1$ aspect ratio for the high reynolds number system

4.5 Generate the θ connection vector field for a three-link pinned system Illustrate this system in a figure

5
Gaits

The connection vector fields illustrate the structure of the local connection matrix and give insight into how shape changes move a system differentially. When studying locomotion, however, we are generally most interested in how shape changes induce net displacements over longer distances and timescales.

- Ability to effectively locomote over long distances rooted in local dynamics – if you can't move over short distances, you can't move over long distances
- For many systems, the way the system dynamics change over the shape space plays an equally important role in long-range motion
- These changes affect how the local motions are integrated together – emergent effects
- Especially relevant when there are some local directions in which the system cannot move, or when there are joint limits
- In these cases, coordinating shape changes to take advantage of changes in system dynamics can allow the system to move in directions that are locally unavailable, or execute cyclic shape changes that produce net displacement over each cycle, while respecting joint limits
- In this chapter, we first look at how local motion integrates into global displacements
- then spec

the net displacements that the system can achieve over longer time periods.

Swimmers' *available* shape spaces are bounded by limits on the extent to which they can bend, but a finite ratio generally exists between the amount the system changes shape and the distance it travels (this ratio is explicitly encoded here by the local connection $\mathbf{A}$, but the principle holds even if the system has a (dissipative) "drifting" or "coasting" term in its equations of motion). To fit arbitrarily-long changes in shape into the bounded shape spaces, therefore, swimmers and other locomoting systems tend to move in cyclic patterns, which can then be characterised by the motion over a single cycle.

5.1 Shape changes

The connection vector fields illustrate the structure of the local connection matrix and give insight into how shape changes move a system differentially. When studying locomotion, however, we are primarily concerned in the net displacement that the system can achieve over longer time periods T,

$$g(T) = \int_0^T \dot{g}(t)\,\mathrm{d}t = \int_0^T \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) & 0 \\ \sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \xi(t)\,\mathrm{d}t, \tag{5.1}$$

where we take the system as starting at the origin of the space the matrix in the rightmost expression rotates the body velocity into the global frame for integration. Substituting in the relationship between shape and body velocity in (4.10) finds the net displacement over a given change in shape (a trajectory $r(t)$ through the shape space with an associated velocity function $\dot{r}(t)$),

$$g(T) = -\int_0^T \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) & 0 \\ \sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{A}(r)\,\dot{r}\,\mathrm{d}t. \tag{5.2}$$

Because the local dynamics, and therefore (5.2), are linear in the shape velocity, increasing the rate at which the shape trajectory is followed proportionally increases the rate at which the swimmer moves along the corresponding position trajectory. Therefore, the time integral in (5.2) can be converted into a line integral over the shape space,

$$g(r(T)) = -\int_{r(0)}^{r(T)} \begin{bmatrix} \cos\theta(r) & -\sin\theta(r) & 0 \\ \sin\theta(r) & \cos\theta(r) & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{A}(r)\ \mathrm{d}r, \tag{5.3}$$

in which the time-scaling of the motion drops out. This conversion embodies the essence of kinematic locomotion—the displacement we get is a function only of the path through the shape space, $r(t)$, and is independent of the pacing with which it is followed.

Equation (**??**) maps changes in the swimmer's shape to the displacements they induce. When considering the motion of locomoting systems we are often concerned with the inverse of this mapping: finding shape changes that produce desired net displacements. In general, such an inversion requires either a parametric "shooting method" optimization of the $r(t)$ trajectory (**?**) or a closed-form solution to the integral in (**??**). For most combinations of systems and shape inputs, such a closed form solution does not exist—the θ in the integrand's rotation matrix is itself a component of the integrated displacement g. A useful approximation of a closed form does exist, however, for an important class of shape changes: *strokes* or *gaits*, which form closed curves in the shape space.

Covariant Derivative

Exterior Derivative

5.2 Gaits

Strokes are interesting for system analysis because they capture a swimmer's ability to transform internal shape changes into net displacement over large scales. Swimmers' *available* shape spaces are bounded by limits on the extent to which they can bend, but a finite ratio generally exists between the amount the system changes shape and the distance it travels (this ratio is explicitly encoded here by the local connection $\mathbf{A}$, but the principle holds even if the system has a (dissipative) "drifting" or "coasting" term in its equations of motion). To fit arbitrarily-long changes in shape into the bounded shape spaces, therefore, swimmers and other locomoting systems tend to move in cyclic patterns, which can then be characterised by the motion over a single cycle.

Strokes' cyclic nature also makes them easier to analyse than open-ended shape changes, in that they allow us to apply a family of tools called *curvature methods* to find the displacements they induce. These methods are based on the principle that to find the net displacement over a cycle, we do not have to explicitly calculate the intermediate displacements, but only *their failure to cancel themselves out over the course of a cycle*. In general, this failure to self-cancel corresponds to the change in the system dynamics across the gait. If the dynamics remain the same as the swimmer moves away from and then returns to the starting shape, then the translations induced by the return will "undo" the effects of the outbound motion. If, however, the return is executed with different dynamics, then there will be a residual net displacement commensurate with the change in dynamics. For kinematic systems, the change in system dynamics is measured by the *curvature* of the system constraints encoded in the local connection.

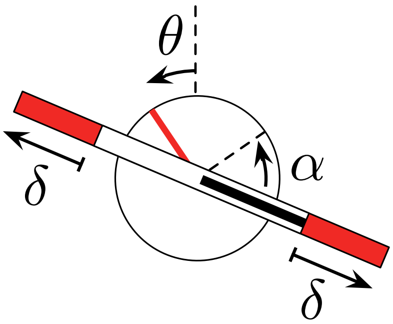
The constraint curvature is a derivative of $\mathbf{A}$; the designation "curvature" derives from $\mathbf{A}$ being the derivative $\partial g/\partial r$ after the constraints are applied, so that its own derivative is the second derivative, or curvature, of the constraints. This derivative has two components: a *nonconservative* part—the curl—that captures the change in the local connection across the shape space, and a second, *noncommutative*, part—the local Lie bracket—that captures the effects of $\mathbf{A}$ being defined in a moving body frame.

5.3 Nonconservativity

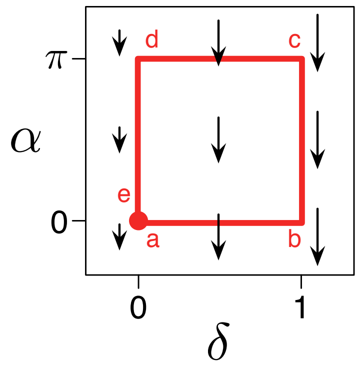
- If the local connection changes over the shape space, a system may be able to change its shape in a way that generates some net displacement, then return to the original shape without undoing (all) of the displacement.
- example: elroy's extensible beanie

$$\xi = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -I_{\text{rotor}}(\delta)/I_{\text{body}} \end{bmatrix} \begin{bmatrix} \dot{\delta} \\ \dot{\alpha} \end{bmatrix} \tag{5.4}$$

- Only some kinds of change in the local connection count
  - Only changes in a component along directions orthogonal to that component matter
  - sometimes contribution from changes in x component cancels out that from changes in y component
- This notion is captured by the curl or exterior derivative of a vector field or one-form
- example: three-link floating snake

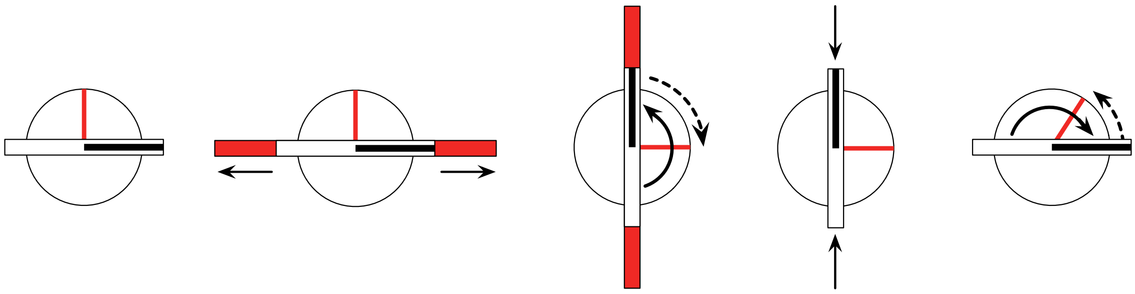


(a) Schematic of Elroy's extensible beanie. (Note that α is negative in the illustrated configuration.)



(b) $\mathbf{A}^{\theta}$ connection vector field for the extensible beanie, overlaid with a gait producing the motion shown in Fig. 5.2.

Figure 5.1



(a) Starting configuration

(b) Rotor arms extend

(c) Rotor turns counterclockwise, body rotates clockwise

(d) Rotor arms retract

(e) Rotor turns clockwise, body rotates counterclockwise

Figure 5.2 Elroy's extensible beanie (Fig. 5.1(a)) executing the gait shown in Fig. 5.1(b).

5.4 Noncommutativity

- even if the local connection is constant with respect to the shape space, it is still defined in a moving body frame.
- most importantly, as the frame rotates, the body translational basis is realigned with respect to the world axes
- example: differential drive car
  - show full solution for squares in the shape space (aligned with natural control modes)
  - small-angle approximation linearization
- this notion is captured by the ideas of Lie bracket averaging

5.5 Combining Nonconservativity and Noncommutativity

- If the local connection changes as a function of shape, and the motion of the body frame is relevant, then both nonconservativity and noncommutativity play a role in the system's net displacement.
- example: ackerman car parallel parking

Stokes' theorem

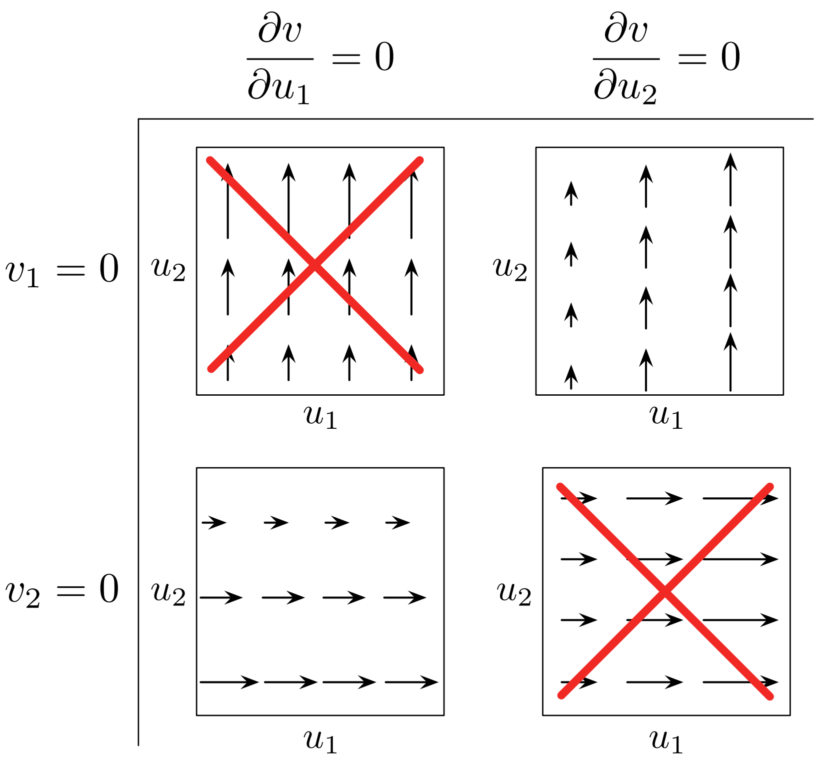


Figure 5.3 Vector fields that change along axes aligned with their components (upper left and lower right) have zero curl, but vector fields that change along axes orthogonal to their components (lower left and upper right) have non-zero curl.

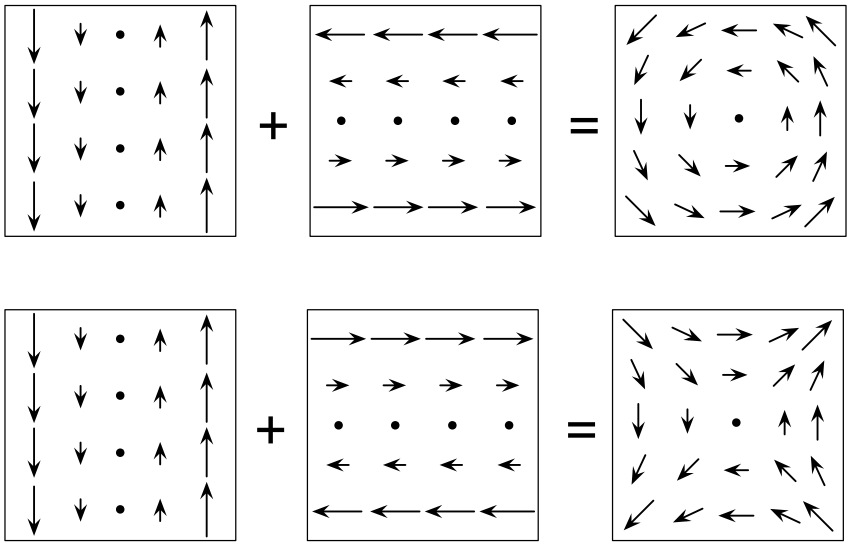
Lie Bracket

- Series approximation for exponential coordinates of net displacement
- full configuration space Lie bracket
- examples from full body locomotion

5.6 Minimum Perturbation Coordinates

- The division of the full configuration space Lie bracket into nonconservative and noncommutative components is not unique for a given system, but depends on the coordinates chosen
- example: differential drive car and south-pointing chariot
- this is important, because the higher-order terms in the series approximation are linked to the local Lie bracket
- If we can find coordinates that shift information into nonconservative component, then series truncation is a better representation of net displacement
- Accuracy of both magnitude of net displacement and direction of travel improve
- Goal is to find change of coordinates that minimizes local Lie bracket, subject to the constraint that change of coordinates must be from valid set
- This leads to Hodge-Helmholtz decomposition for rotation, and generalized hodge-helmholtz decomposition for translation.
- Examples from full body locomotion

$$\frac{\partial v_2}{\partial u_1} = k \qquad \frac{\partial v_1}{\partial u_2} = -k \qquad \frac{\partial v_2}{\partial u_1} - \frac{\partial v_1}{\partial u_2} = 2k$$



$$\frac{\partial v_2}{\partial u_1} = k \qquad \frac{\partial v_1}{\partial u_2} = k \qquad \frac{\partial v_2}{\partial u_1} - \frac{\partial v_1}{\partial u_2} = 0$$

Figure 5.4 Having vectors that change along directions orthogonal to their own heading is necessary, but not sufficient, to have non-zero curl. In the top row, the curls of the individual fields complement each other, producing a vector field with curl. In the bottom row, the curls are opposite to each other, so their sum has no curl.

Hodge-Helmholtz Decomposition

Exercises

5.1 Generate the local connection for Elroy's Extensible Beanie

6
Distance and Curvature

Grouped together, because distance metrics and curvature are closely linked

6.1 Cost to Move

6.2 Cartographic Projection

6.3 Geodesic Distances

6.4 Curvature (of 2-d structures)

Distance Metrics

Isomap

Geodesics

Curvature

7

Advanced Topics

Notes

References

Cox, R. G. 1970. The Motion of Long Slender Bodies in a Viscous Fluid Part 1. General Theory. *J. Fluid Mechanics*, **44**(4), 791–810. 4.4.4

Kelly, Scott D. *The Mechanics and Control of Driftless Swimming*. In press. 4.4.5, 4.4.5

Melli, Juan B., Rowley, Clarence W., and Rufat, Dzhelil S. 2006. Motion Planning for an Articulated Body in a Perfect Planar Fluid. *SIAM Journal of Applied Dynamical Systems*, **5**(4), 650–669. 4.4.5

Ostrowski, J., and Burdick, J. 1998. The Mechanics and Control of Undulatory Locomotion. *International Journal of Robotics Research*, **17**(7), 683 – 701. 4.4.3

Shammas, Elie A., Choset, Howie, and Rizzi, Alfred A. 2007. Geometric Motion Planning Analysis for Two Classes of Underactuated Mechanical Systems. *Int. J. of Robotics Research*, **26**(10), 1043–1073. 4.4.3, 4.4.3

Author index

Subject index