EE5027 Adaptive Signal Processing
Homework Assignment #1

Notice

- **Due at 9:00pm, October 18, 2021 (Monday)** $= T_d$ for the electronic copy of your solution.
- Please submit your solution to NTU COOL (`https://cool.ntu.edu.tw/courses/7920`)
- ***All answers have to be fully justified***.
- No extensions, unless granted by the instructor one day before T_d.

Problems

1. (Complex Gaussian random vectors) Assume that the complex random vector $\mathbf{Z} = [Z_1, Z_2, Z_3]^T$ follows the complex multivariate circularly-symmetric Gaussian distribution with mean $\mathbf{0}$ and correlation matrix $\mathbf{R_z}$. The correlation matrix $\mathbf{R_z}$ has the following form

$$\mathbf{R_z} = \begin{bmatrix} 6 & j & 0 \\ r_{2,1} & 1 & -2-j \\ r_{3,1} & r_{3,2} & r_{3,3} \end{bmatrix}. \tag{1}$$

   (a) (2 points) What is the probability density function of Z_1?

   (b) (8 points) What are the values of $r_{2,1}$, $r_{3,1}$, $r_{3,2}$, and $r_{3,3}$, such that $\mathbf{R_z}$ is a valid correlation matrix? If there are multiple values, give the range of these values.

   (c) (5 points) Now let us consider another random vector $\mathbf{W} \triangleq \mathbf{AZ}$, where the matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}. \tag{2}$$

   Evaluate the correlation matrix of $\mathbf{W}$.

   (d) (5 points) Calculate $\mathbb{E}\left[\left(Z_1 - \frac{1}{\sqrt{5}}Z_2 + 4Z_3 - 1\right)^2\right]$.

2. (Autocorrelation functions) Determine whether the following functions are valid autocorrelation functions. If the function is a valid autocorrelation function $r_x(k)$, specify a system equation whose input signal is $v(n)$ and the output signal is $x(n)$. If the function is not an autocorrelation function, state why not.

   (a) (5 points) $f_1(k) = 2\delta(k-2) + \delta(k) + 2\delta(k+2)$.

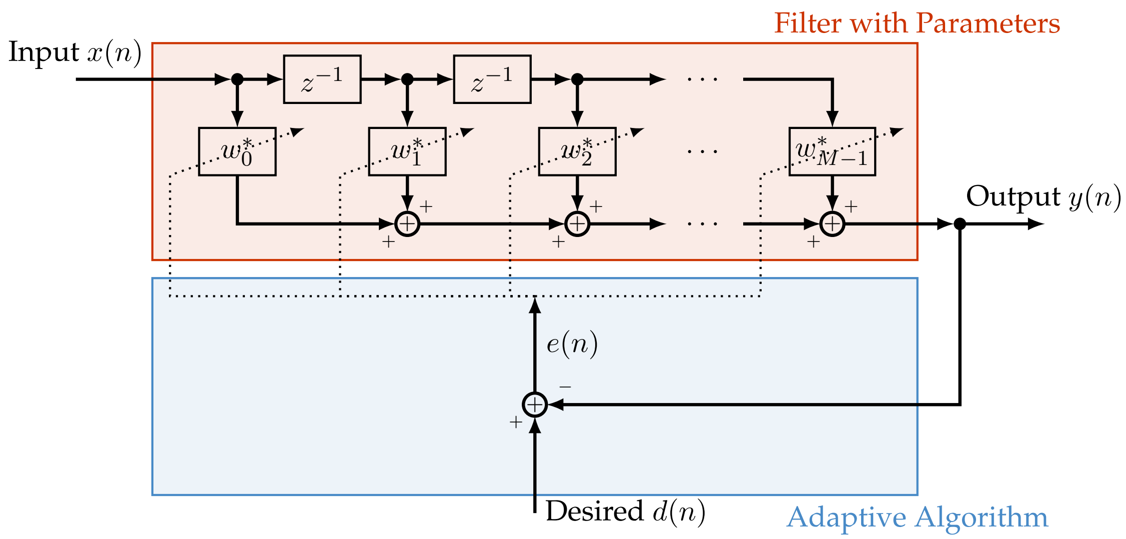

Figure 1: A block diagram for Problem 4.

(b) (5 points) $f_2(k) = 2\delta(k-2) + 3\delta(k) + 2\delta(k+2)$.

(c) (5 points) $f_3(k) = 2\delta(k-2) + 3\delta(k) + 3\delta(k+2)$.

3. (Stochastic models) Consider an AR process with the following difference equation

$$u(n) = 0.6u(n-1) + 0.67u(n-2) + 0.36u(n-3) + v(n), \tag{3}$$

where $v(n)$ is a zero-mean, circularly-symmetric complex Gaussian, white, wide-sense stationary random process with unit variance.

(a) (5 points) Find the autocorrelation function of $v(n)$.

(b) (5 points) Find the transfer function $H(z)$ which relates $v(n)$ and $u(n)$.

(c) (5 points) Find the power spectral density of $u(n)$.

(d) (5 points) Find the autocorrelation function of $u(n)$.

4. (Wiener filters) We consider a block diagram associated with the Wiener filter in Figure 1, where the number of taps M is 2. The input signal $x(n)$ is a WSS random process with zero mean and the autocorrelation function

$$r_x(k) = \left(\frac{1}{2}\right)^{|k|}. \tag{4}$$

The desired signal is given by

$$d(n) = x(n+1). \tag{5}$$

In the lecture, it was shown that the optimal weight vector of the Wiener filter is a solution to the Wiener-Hopf equation

$$\mathbf{R}\mathbf{w}_{\text{opt}} = \mathbf{p}, \tag{6}$$

where $\mathbf{R} = \mathbb{E}[\mathbf{x}(n)\mathbf{x}^H(n)]$ and $\mathbf{p} = \mathbb{E}[\mathbf{x}(n)d^*(n)]$.

(a) (4 points) Find $\mathbf{R}$ and $\mathbf{p}$.

(b) (3 points) Find $\sigma_d^2 = \mathbb{E}[d(n)d^*(n)]$.

(c) (3 points) Find the optimal weight vector $\mathbf{w}_{\text{opt}}$.

(d) (5 points) Find the autocorrelation of $y(n)$.

(e) (5 points) Plot the power spectral density of $y(n)$ using MATLAB.

(f) (2 points) Find $J_{\min}$.

(g) (3 points) Let $e_{\text{opt}}(n)$ be the error signal when $\mathbf{w} = \mathbf{w}_{\text{opt}}$. Find $\mathbb{E}\left[x(n+1)e^*_{\text{opt}}(n)\right]$

5. (The error-performance surface) Consider a Wiener filter with a complex input process $x(n)$, a complex desired process $d(n)$, and the weight vector $\mathbf{w} = [w_0, w_1, w_2]^T$. Suppose the covariance matrix $\mathbf{R}$, the cross-correlation vector $\mathbf{p}$, and the variance of $d(n)$ are given by

$$\mathbf{R} = \begin{bmatrix} 2 & 0.8 & -0.4j \\ 0.8 & 2 & 0.8 \\ 0.4j & 0.8 & 2 \end{bmatrix}, \qquad \mathbf{p} = \begin{bmatrix} 1.6 \\ -1.9 \\ 1.8 \end{bmatrix}, \qquad \sigma_d^2 = 12. \tag{7}$$

In this problem, we will generate the error-performance surface using MATLAB. The real part and the imaginary part of a complex number are denoted by $\text{Re}\{z\}$ and $\text{Im}\{z\}$, respectively.

(a) (3 points) Write a MATLAB function that computes the mean square error (MSE) J given the covariance matrix $\mathbf{R}$, the weight vector $\mathbf{w}$, the cross-correlation vector $\mathbf{p}$, and σ_d^2 (`sd2`). The syntax is as follows:

```
J = ASP_Wiener_MSE(R, w, p, sd2);
```
(8)

This function throws error messages if any of the following occurs: 1) $\mathbf{R}$ is not positive semidefinite, 2) The dimensions of these input arguments are not suitable, or 3) `sd2` is negative or complex-valued.

Note: **We will have additional test data for these exceptions**.

(b) (2 points) Use the MATLAB function in (8) to compute the MSE $J_{\min}$.

(c) (5 points) Use the MATLAB function `plot` to generate a 1-D plot of the MSE with the following specifications

- The horizontal axis is $\text{Re}\{w_0\}$, consisting of 201 uniform samples from -4 to 4.
- The vertical axis is the MSE J.
- We assume that $\text{Im}\{w_0\} = 1$, $w_1 = -0.5 + j$, and $w_2 = -1$.
- Include appropriate labels for the x-axis and the y-axis.
- Mark the extreme point and the associated J on this plot.

(d) (5 points) Use the MATLAB function `surf` to generate a surface plot of the MSE with the following specifications:

- The parameter $w_1 = -2.5 + 0.1j$ and $w_2 = 2 - 0.4j$.
- The x-axis is the real part of w_0, taking 201 uniform samples from -4 to 4.
- The y-axis is the imaginary part of w_0, taking 201 uniform samples from -4 to 4.
- The z-axis is the MSE J.
- Include the axis labels and the color bar in your plot.
- Mark the extreme point and the associated J on this plot.

(e) (5 points) Use the MATLAB function `contour` to generate a contour plot of the MSE. The specifications of this plot are as follows.

- The imaginary parts of w_0, w_1, and w_2 are $\mathrm{Im}\{w_0\} = 0.4$, $\mathrm{Im}\{w_1\} = -0.0125$, and $\mathrm{Im}\{w_2\} = -0.37$, respectively.
- The real part of w_2 is $\mathrm{Re}\{w_2\} = 2$.
- The x-axis is $\mathrm{Re}\{w_0\}$. We take 201 uniform samples from -3 to 3
- The y-axis is $\mathrm{Re}\{w_1\}$. We take 201 uniform samples from -3 to 3.
- The contour levels are $0.8, 0.9, 1, 2, 3, 4, 5$.
- Also include labels on the contour lines and labels for the axes.
- Mark the extreme point and the associated J on this plot.

Note: In localizing the extreme points in Problems 5c, 5d, and 5e, there are two approaches in general.

- The first approach is *grid-based*. For instance, in the plot of Problem 5c, we consider all these 201 samples of the MSE, find the minimum value among these samples, and finally identify the associated $\mathrm{Re}\{w_0\}$.
- The second approach is *off-grid*. Instead of taking discrete grid points, we consider the explicit form of the function and find the exact extreme point(s) of this function.

In Problems 5c, 5d, and 5e, **please implement the second approach (off-grid) for the extreme points**. You can use the MATLAB command `text` to mark these points.

Note: Please include the following MATLAB scripts and figure files in your submission

(a) **`ASP_Wiener_MSE.m`**

(b) **`ASP_HW1_Problem_5b.m`** for the MATLAB codes.

(c) **`ASP_HW1_Problem_5c.m`** for the MATLAB codes and **`ASP_HW1_Problem_5c.fig`** for the plot.

(d) **`ASP_HW1_Problem_5d.m`** for the MATLAB codes and **`ASP_HW1_Problem_5d.fig`** for the plot.

(e) **`ASP_HW1_Problem_5e.m`** for the MATLAB codes and **`ASP_HW1_Problem_5e.fig`** for the plot.

Make sure that the MATLAB codes in Problems 5c to 5e match the figure files.

Last updated October 9, 2021.