ASP Final Project

R10942129 Lyu, Guan-Huei

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1. Bemformers review (reference from [1])

- Introduction

  Consider the array data model

$$\mathbf{x}(t) = \mathbf{A}(\theta)\mathbf{s}(t) + \mathbf{n}(t) \tag{1}$$

the output of a beamformer is

$$y(t) = \mathbf{w}^H \mathbf{x}(t) \tag{2}$$

where $\mathbf{x}(t)$ is the input signal and $\mathbf{s}(t)$ is the desired signal.The matrix $\mathbf{A}(\theta)$ is composed by columns of steering vector $\mathbf{a}(\theta_i)$ with different source i. θ_i means the angle of incidence of the $i-th$ source signal and $\mathbf{a}(\theta_i)$ is defined as

$$\mathbf{a}(\theta_i) = [1, e^{j2\pi(\frac{1}{\lambda}sin\theta_i)\times(d_0)}, ..., e^{j2\pi(\frac{1}{\lambda}sin\theta_i)\times((N-1)d_{N-1})}]^T \tag{3}$$

and N is number of sensing elements.

- The beamformer with uniform weights

  Let's assume the number of source is one (i.e. $i = 1$) and the source waveform is $s_1(t) = Ae^{j2\pi ft}$. If we consider uniform weightings $\mathbf{w} = \mathbf{1}/N$, the beamformer output becomes

$$\begin{aligned} y(t) &= \mathbf{w}^H \mathbf{x}(t) \\ &= \left(\frac{1}{N}\right)(Ae^{j2\pi ft}) + \left(\frac{1}{N}\mathbf{1}^H \mathbf{n}(t)\right) \\ &= (Ae^{j2\pi ft}) + \left(\frac{1}{N}\mathbf{1}^H \mathbf{n}(t)\right) \end{aligned} \tag{4}$$

with Direction-Of-Arrival (DOA) is zero. It turns out the SNR will become

$$\mathsf{SNR} = \frac{\mathbb{E}[|Ae^{j2\pi ft}|^2]}{(\mathbf{1}^H \mathbb{E}[\mathbf{n}(t)\mathbf{n}^H(t)]\mathbf{1})/N^2} = \frac{N\sigma_1^2}{\sigma_n^2} \tag{5}$$

Hence, the SNR is enhanced by N-times.

- The beamformer with array steering

  In the previous discussion, we assume the source DOA is 0^o. Next, we assume the source DOA is θ and the weightings are still uniform. The output of the beamformer becomes

$$\begin{aligned} y(t) &= \mathbf{w}^H \mathbf{x}(t) \\ &= \mathbf{w}^H (\mathbf{a}(\theta) s_1(t) + \mathbf{n}(t)) \\ &= (\mathbf{w}^H \mathbf{a}(\theta)) s_1(t) + (\mathbf{w}^H \mathbf{n}(t)) \end{aligned} \tag{6}$$

  and the SNR of this beamformer is

$$\mathsf{SNR} = \frac{\mathbb{E}[|\mathbf{w}^H \mathbf{a}(\theta) s_1(t)|^2]}{\mathbb{E}[|\mathbf{w}^H \mathbf{n}(t)|^2]} = |B_\theta(\theta)|^2 \times \frac{N\sigma_1^2}{\sigma_n^2} \tag{7}$$

  where $B_\theta(\theta) = \mathbf{w}^H \mathbf{a}(\theta)$ is an important factor for SNR enhancement. The figure below shows the relation between $|B_\theta(\theta)|$ and θ, it turns out that with the angles which are different from source, the SNR will decay a lot. For DOA $\theta_1 = \theta$, SNR depending on $B_\theta(\theta)$. Hence, we will

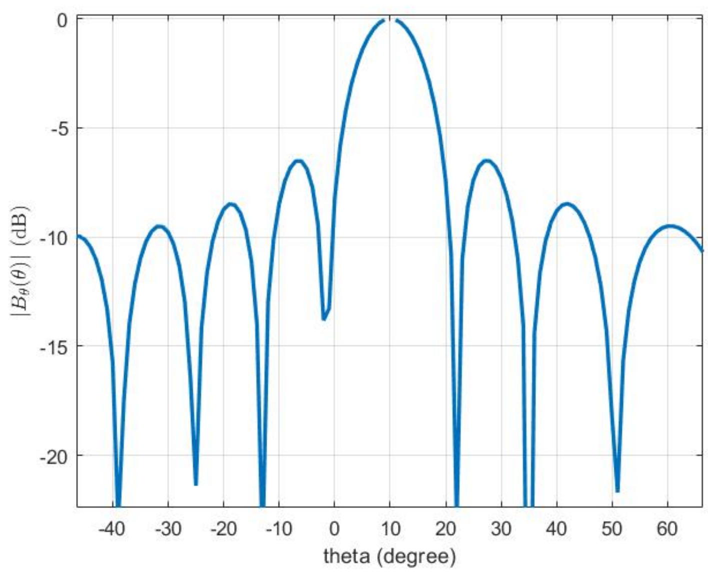

Figure 1: $|B_\theta(\theta)|$

  need to design the weight vector $\mathbf{w}$ for the signal waveform near $\theta = \theta_s$ is kept and the others which away from θ_s must be rejected. Assumed the θ_s is already known, we can select the weight vector to be

$$\mathbf{w} = \frac{1}{N}\mathbf{a}(\theta_s) \tag{8}$$

  and the pattern for this weight is

$$\begin{aligned} B_\theta(\theta) &= \mathbf{w}^H \mathbf{a}(\theta) \\ &= e^{j\frac{N-1}{2}(\pi \sin\theta - \pi \sin\theta_s)} \times \frac{1}{N} \times \frac{\sin[N/2(\pi \sin\theta - \pi \sin\theta_s)]}{\sin[1/2(\pi \sin\theta - \pi \sin\theta_s)]} \end{aligned} \tag{9}$$

  It can be shown that the maximum of $|B_\theta(\theta)|$ occurs at $\theta = \theta_s$

- The MVDR Beamformer

  The definition of Minimum Variance Distortionless Response (MVDR) beamforming is defined as the following optimization problem.

$$\mathbf{w}_{\mathsf{MVDR}} = \arg\min_{\mathbf{w}} \mathbb{E}[|y(t)|^2] \qquad \textbf{subject to} \qquad \mathbf{w}^H \mathbf{a}(\theta_s) = 1 \tag{10}$$

  where $y(t) = \mathbf{w}^H \mathbf{x}(t)$. By solving this optimization problem, the solution for $\mathbf{w}_{\mathsf{MVDR}}$ is

$$\mathbf{w}_{\mathsf{MVDR}} = \frac{\mathbf{R}^{-1}\mathbf{a}(\theta_s)}{\mathbf{a}^H(\theta_s)\mathbf{R}^{-1}\mathbf{a}(\theta_s)} \tag{11}$$

  By letting $\theta_s = \theta$, the MVDR spectrum is defined as

$$P_{\mathsf{MVDR}}(\theta) = \mathbf{w}_{\mathsf{MVDR}}^H \mathbf{R} \mathbf{w}_{\mathsf{MVDR}} = \frac{1}{\mathbf{a}^H(\theta)\mathbf{R}^{-1}\mathbf{a}(\theta)} \tag{12}$$

  Since P is the function of θ. We can do DOA estimation by finding the peak on this spectrum, which is useful to estimate the unknown DOA of the source.

- The LCMV Beamformer

  The definition of Linearly Constrained Minimum Variance (LCMV) beamformer is defined as the following optimization problem

$$\mathbf{w}_{\mathsf{LCMV}} = \arg\min_{\mathbf{w}} \mathbb{E}[|y(t)|^2] \qquad \textbf{subject to} \qquad \mathbf{C}^H \mathbf{w} = \mathbf{g} \tag{13}$$

  where $\mathbf{C}$ is a matrix about linear constraints, each column of the matrix $\mathbf{C}$ represents a single linear constraint. $\mathbf{g}$ is the gain vector, which represents the signal you want to preserve. For example, with

$$[a(\theta_0), a(\theta_1)]^H \mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{14}$$

  the narrowband beamformer is constrained to preserve a signal of interest impinging on the array along the electrical angle θ_0 and, at the same time, to suppress an interference known to originate along the electrical angle θ_1. If $\mathbf{C} = \mathbf{a}(\theta_s)$ and $\mathbf{g} = 1$, then LCMV is equivalent to MVDR.

  The solution to equation 14 is

$$\mathbf{w}_{\mathsf{LCMV}} = \mathbf{R}^{-1}\mathbf{C}(\mathbf{C}^H\mathbf{R}^{-1}\mathbf{C})^{-1}\mathbf{g} \tag{15}$$

  where $\mathbf{R} = \mathbb{E}[\mathbf{x}(t)\mathbf{x}^H(t)]$.

2. DOA estimation

   To estimate the DOA, I use the MVDR spectrum defined as equation 12 by finding the peak of the P with different θ. To implement the equation 12, the only unknown is the inverse of correlation matrix $\mathbf{R}$. Note the θ will be varying with time, so I can't simply take average of the $\mathbf{R}$. In my implementation, the correlation matrix $\mathbf{R}$ is calculated recursively by

$$\mathbf{R}(n) = \lambda \mathbf{R}(n-1) + \tilde{\mathbf{x}}(n-1)\tilde{\mathbf{x}}^H(n-1) \tag{16}$$

   where $\tilde{\mathbf{x}}(n)$ is given and λ is like "memory". If we estimate the $\mathbf{R}(n)$, then the information of $\mathbf{R}(1)$ is less important than the information of $\mathbf{R}(n-1)$. Hence, the value $\lambda = 0.9$ is set in my implementation to forget the information that is less important to $\mathbf{R}(n)$, but also consider the information around the $\mathbf{R}(n)$. This concept is corresponding to RLS algorithm. Hence, we use the recursive equations in RLS to calculate the inverse of $\mathbf{R}$

$$\begin{aligned}\mathbf{k}(n) &= \frac{\lambda^{-1}\mathbf{P}(n-1)\mathbf{x}(n)}{1+\lambda^{-1}\mathbf{x}^H(n)\mathbf{P}(n-1)\mathbf{x}(\mathbf{n})} \\ \mathbf{P}(n) &= \lambda^{-1}\mathbf{P}(n-1) - \lambda^{-1}\mathbf{k}(n)\mathbf{x}^H\mathbf{P}(n-1)\end{aligned} \tag{17}$$

where $\mathbf{P} = \mathbf{R}^{-1}$ and the initial condition is $\mathbf{P}(0) = \delta^{-1}\mathbf{I}$ with $\delta = 0.01$ in my implementation.

When the MVDR spectrum is derived, we can find the source DOA by the condition $0^\circ \leq \theta_s \leq 10^\circ$. In Figure 2, we can tell the angle of left peak is corresponding to the interference DOA, and the angle of right peak is corresponding to the source DOA.

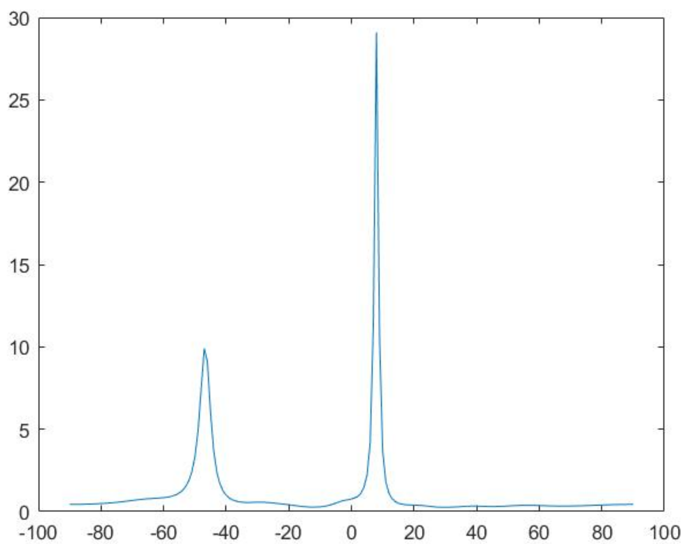

Figure 2: MVDR spectrum

3. The estimated angles result

   The DOA estimation result is shown as Figure 3 and Figure 4.

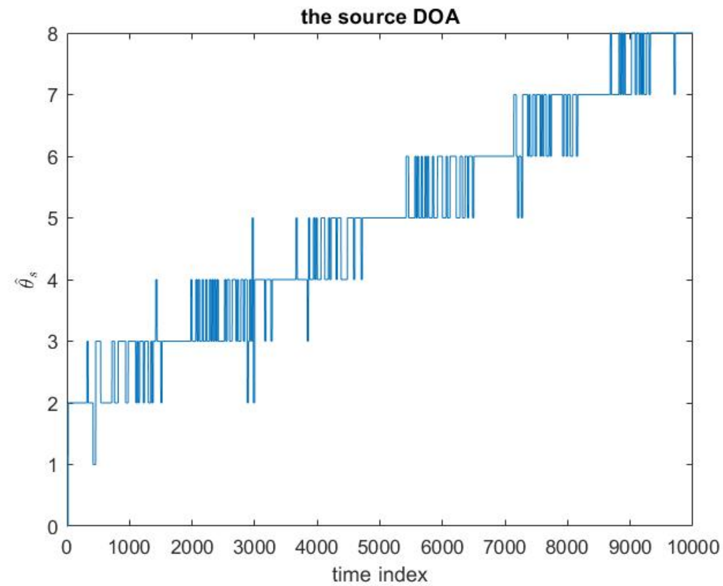

Figure 3: The source DOA

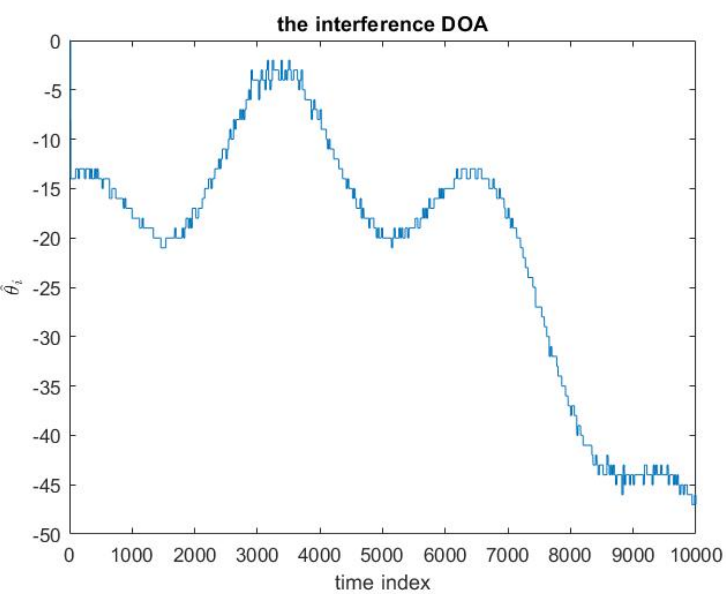

Figure 4: The interference DOA

4. Beamformer design (reference from [2])

   My beamformer design is based on the concept of LMS and LCMV. In LCMV, there is a linear constraint matrix $\mathbf{C}$ and a gain vector $\mathbf{g}$. In my implementation, the $\mathbf{C}$ is adaptive with the estimated DOA

$$\mathbf{C} = [\mathbf{a}(\theta_s(t)) \quad \mathbf{a}(\theta_i(t))] \tag{18}$$

and the linear constraint will become

$$\mathbf{C}^H\mathbf{w} = \mathbf{g} = \begin{bmatrix}1\\0\end{bmatrix} \tag{19}$$

because I want to preserve the signal from the direction of source and suppress the signal from the direction of interference. Where $\mathbf{w}$ is the weight vector.

To separate the weight vector to the part of the weight vector affected by the constraints and the weight vector unaffected by the constraints, we can use partition like

$$\mathbf{w} = [\mathbf{C} \quad \mathbf{C}_a]\begin{bmatrix}\mathbf{v}\\-\mathbf{w}_a\end{bmatrix} = \mathbf{C}\mathbf{v} - \mathbf{C}_a\mathbf{w}_a \tag{20}$$

where the column of the matrix $\mathbf{C}_a$ is defined as a basis for the orthogonal complement of the space spanned by the columns of matrix $\mathbf{C}$ and the vector $\mathbf{w}_a$ is the portion of the weight vector $\mathbf{w}$ that is not affected by the constraints.

Now apply the above equation to equation 19, obtaining

$$\mathbf{C}^H\mathbf{C}\mathbf{v} - \mathbf{C}^H\mathbf{C}_a\mathbf{w}_a = \mathbf{g}. \tag{21}$$

Since the columns of $\mathbf{C}_a$ is orthogonal to the columns of $\mathbf{C}$, it means that $\mathbf{C}^H\mathbf{C}_a$ is zero, hence the above equation will reduce to

$$\mathbf{C}^H\mathbf{C}\mathbf{v} = \mathbf{g} \tag{22}$$

Solving the vector $\mathbf{v}$

$$\mathbf{v} = (\mathbf{C}^H\mathbf{C})^{-1}\mathbf{g} \tag{23}$$

which shows that the linear constraints do not affect $\mathbf{w}_a$.

Let's define $\mathbf{C}\mathbf{v} = \mathbf{w}_q$, which means the part of the weight vector $\mathbf{w}$ which satisfies the constraints. The equation 22 reduces to

$$\mathbf{C}^H\mathbf{w}_q = \mathbf{g} \tag{24}$$

Now the weight vector has been regarded as two part, $\mathbf{w}_q$ and $\mathbf{w}_a$. The $\mathbf{w}_q$ is designed to preserver an incident signal along a direction of interest. Hence, to reconstruct the signal precisely, we could use adaptive algorithm to minimize the effects of interference and noise at the beamformer output, by adjust the weight of $\mathbf{w}_a$.

According to the weight vector described as equation 20, the beamformer output is

$$\begin{aligned} y(n) &= \mathbf{w}^H(n)\mathbf{x}(n) \\ &= (\mathbf{w}_q - \mathbf{C}_a\mathbf{w}_a(n))^H\mathbf{x}(n) \\ &= \mathbf{w}_q^H\mathbf{x}(n) - \mathbf{w}_a(n)^H\mathbf{C}_a^H\mathbf{x}(n) \end{aligned} \tag{25}$$

It seems that the desired response is $\mathbf{w}_q^H\mathbf{x}(n)$, and we can adjust the $\mathbf{w}_a(n)$ to minimize the output $|y(n)|^2$, by performing LMS algorithm

$$\begin{aligned} \mathbf{w}_a(n+1) &= \mathbf{w}_a(n) + \mu\mathbf{C}_a^H\mathbf{x}(n)(\mathbf{w}_q^H\mathbf{x}(n) - \mathbf{w}_a^H(n)\mathbf{C}_a^H\mathbf{x}(n))^* \\ &= \mathbf{w}_a(n) + \mu\mathbf{C}_a^H\mathbf{x}(n)\mathbf{x}(n)^H(\mathbf{w}_q - \mathbf{C}_a\mathbf{w}_a(n)) \end{aligned} \tag{26}$$

5. The estimated source signal

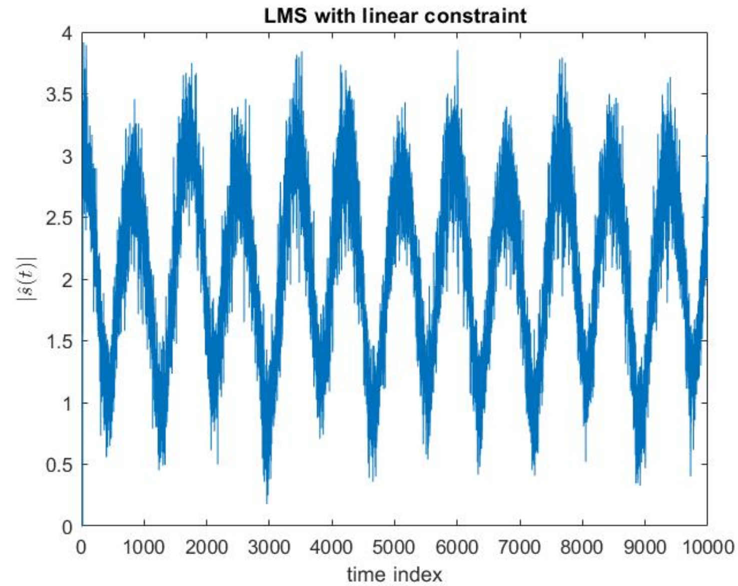

Figure 5: The estimated source signal with linear constrained LMS

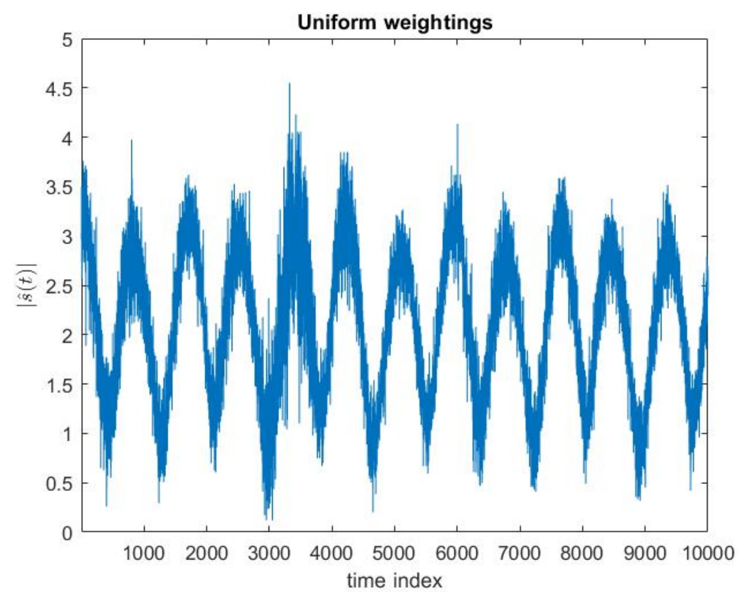

Figure 6: The estimated source signal with uniform weighting

The source signal estimated by the weight I proposed is depicted as Figure 5, compared to the beamformer with uniform weighting in Figure 6, defined as equation 8, where the θ_s in equation 8 is also adaptive with the DOA estimation in part 3. It's really similar because in the equation 25, we preserver the signal with desired angle by $\mathbf{w}_q^H \mathbf{x}(n)$, which is corresponding to the spirit of the beamformer with array steering. The subtle difference is in the range of time index 3000 to 4000, the θ_s and θ_i is very close inside this region, and we can see the resolution of the linear constrained LMS is quite better than the uniform weighting. This is because the signal blocking term $\mathbf{w}_a(n)^H \mathbf{C}_a^H \mathbf{x}(n)$ in equation 25. The function of this term is to cancel interference that leaks through the sidelobes of $\mathbf{w}_q^H \mathbf{x}(n)$, and it's also the reason why the resolution is good in linear constrained LMS.

Computational Cost

- DOA estimators

  The main cost in DOA estimators is calculating the inverse of correlation matrix $\mathbf{R}^{-1}$. In my implementation, the cost is reduced by using RLS algorithm.

- The beamformers The cost of the beamformer designed by myself could be higher than the uniform weighting, due to the computation of the complementary orthogonal column in $\mathbf{C}_a$.

References

[1] Handout from Professor Chun-Lin Liu: 13 Adaptive Beamforming

[2] S. Haykin,Adaptive Filter Theory, 5th Edition, Pearson, 2014