EE5027 Adaptive Signal Processing
Homework Assignment #2

Notice

- **Due at 9:00 pm, November 15, 2021 (Monday)** $= T_d$ for the electronic copy of your solution.
- Please submit your solution to NTU COOL (`https://cool.ntu.edu.tw/courses/7920`)
- ***Please justify your answers***.
- All the figures should include labels for the horizontal and vertical axes, a title for a short description, and grid lines. Add legends and different line styles if there are multiple curves in one plot.
- No extensions, unless granted by the instructor one day before T_d.

Problems

1. (Linear prediction, 35 points) Consider a wide-sense stationary random process $x(n)$ whose autocorrelation function satisfies

$$r(0) = 1, \qquad r(1) = -0.6, \qquad r(2) = 0.3, \qquad r(3) = -0.2. \tag{1}$$

Find the following quantities

(a) (6 points) The reflection coefficients κ_1 κ_2, and κ_3.

(b) (6 points) The quantities Δ_0, Δ_1, and Δ_2.

(c) (8 points) The tap-weight vector of the forward prediction-error filter $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

(d) (8 points) The minimum mean-square prediction error P_0, P_1, P_2, and P_3.

(e) (7 points) Draw a signal-flow graph for the lattice predictors of order $M = 3$. The input is $x(n)$ and the outputs are $f_M(n)$ and $b_M(n)$.

2. (Prediction error bound and spectral flatness, 15 points) In the lecture, we studied a lower bound for the the minimum forward prediction error power for a one-step forward linear prediction with order m, denoted by P_m, as follows:

$$P_m \geq \exp\left(\int_{-1/2}^{1/2} \log S_x(e^{j2\pi f})\, \mathrm{d}f \right). \tag{2}$$

The right-hand side of (2) is the prediction error bound. We also define the spectral flatness measure γ_x^2 as

$$\gamma_x^2 \triangleq \frac{\exp\left(\int_{-1/2}^{1/2} \log S_x(e^{j2\pi f})\,\mathrm{d}f\right)}{\int_{-1/2}^{1/2} S_x(e^{j2\pi f})\,\mathrm{d}f}. \tag{3}$$

Find the spectral flatness measure γ_x^2 for the following cases

(a) (5 points) A WSS random process $x(n)$ with zero mean and autocorrelation function $r_x(k) = \delta(k)$.

(b) (10 points) A WSS random process $x(n)$ with zero mean and autocorrelation function $r_x(k) = \alpha^{|k|}$. The parameter α satisfies $0 < \alpha < 1$.
Hints: You can start with the z-transform of $r_x(k)$ and then apply the following Taylor expansion

$$\log(1-z) = \sum_{n=1}^{\infty} \left(\frac{-1}{n}\right) z^n \qquad \text{for } |z| < 1,$$

where $\log$ denotes the principal branch of the complex logarithm.

3. (Properties of prediction error filters, 25 points) Show the following properties

(a) (2 points) $\mathbb{E}[f_m(n)x^*(n-k)] = 0$ for all $1 \leq k \leq m$.

(b) (3 points) $\mathbb{E}[b_m(n)x^*(n-k)] = 0$ for all $0 \leq k \leq m-1$.

(c) (5 points) $\mathbb{E}[f_m(n)x^*(n)] = \mathbb{E}[b_m(n)x^*(n-m)] = P_m$.

(d) (5 points) $\mathbb{E}[b_m(n)b_i^*(n)] = \begin{cases} P_m, & m = i, \\ 0, & m \neq i. \end{cases}$

(e) (5 points) $\mathbb{E}[f_m(n)f_i^*(n-\ell)] = \mathbb{E}[f_m(n+\ell)f_i^*(n)] = 0$ for $1 \leq \ell \leq m-i$ and $m > i$.

(f) (5 points) $\mathbb{E}[b_m(n)b_i^*(n-\ell)] = \mathbb{E}[b_m(n+\ell)b_i^*(n)] = 0$ for all $0 \leq \ell \leq m-i-1$ and $m > i$.

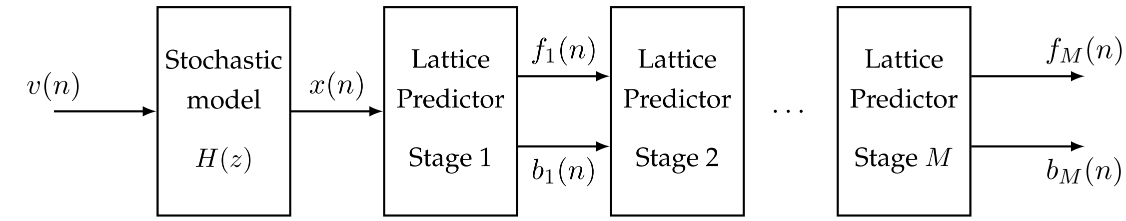
4. (Monte-Carlo experiments on prediction error filters, 25 points) We consider a system diagram in Figure 1. The random process $v(n)$ is a zero-mean, circularly-symmetric complex Gaussian, white, wide-sense stationary random process with unit variance ($\sigma_v^2 = 1$). The transfer function $H(z)$ is given by

$$H(z) = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}. \tag{4}$$

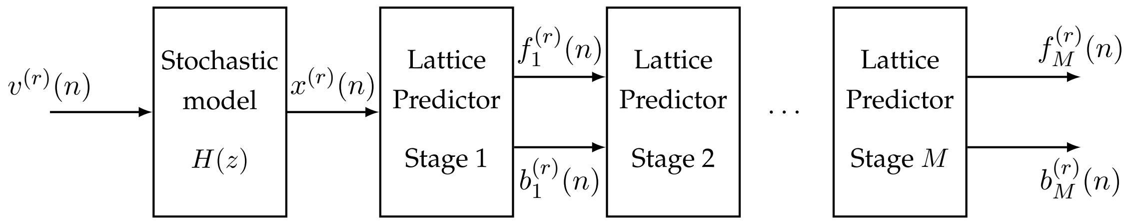
The region of convergence is $|z| > \frac{1}{2}$.

(a) (Bonus problem, 5 points) Show that the autocorrelation function $r_x(k)$ is

$$r_x(k) = -\frac{128}{105}\left(\frac{1}{4}\right)^{|k|} + \frac{64}{21}\left(\frac{1}{2}\right)^{|k|}. \tag{5}$$



(a)



(b)

Figure 1: The system diagrams for lattice predictors. (a) The system diagram driven by the *random process* $v(n)$. (b) The same system diagram driven by the *realization* $v^{(r)}(n)$ in the rth Monte-Carlo experiment.

(b) (5 points) First we implement the Levinson-Durbin algorithm using MATLAB. The autocorrelation function is specified by a column vector `r` $= [r_x(0), r_x(1), \ldots, r_x(M)]^T$. Write the following MATLAB function for the Levinson-Durbin algorithm:

```
[a, P, kappa] = ASP_Levinson_Durbin(r);
```
(6)

The output arguments are specified as follows.

- `a` is a MATLAB cell array of size M. The entries in `a` contain the coefficients of the forward prediction error filter. More specifically, we have `a{1}` $= \mathbf{a}_1$, `a{2}` $= \mathbf{a}_2$, and `a{M}` $= \mathbf{a}_M$.
- `P` $= [P_0, P_1, P_2, \ldots, P_M]^T$ is an $(M+1)$-by-1 vector for the prediction errors.
- `kappa` $= [\kappa_1, \kappa_2, \ldots, \kappa_M]^T$ is an M-by-1 vector for the reflection coefficients.

(c) (4 points) Next, we conduct *Monte-Carlo experiments* for these random processes. You can read the file `ASP_HW2_Problem_4.mat` for the realization $v^{(r)}(n)$ in the rth Monte-Carlo experiment. In this mat file, the matrix `V` is defined as

$$\mathtt{V} \triangleq \begin{bmatrix} v^{(1)}(0) & v^{(1)}(1) & \ldots & v^{(1)}(L-1) \\ v^{(2)}(0) & v^{(2)}(1) & \ldots & v^{(2)}(L-1) \\ \vdots & \vdots & \ddots & \vdots \\ v^{(R)}(0) & v^{(R)}(1) & \ldots & v^{(R)}(L-1) \end{bmatrix}, \tag{7}$$

where R is the number of Monte-Carlo experiments and L is the length of the realizations. We set $v^{(r)}(n) = 0$ if $n < 0$ or $n \geq L$.

According to Fig. 1(b), we compute the associated $x^{(r)}(n)$ and $f_1^{(r)}(n)$. Finally, we estimate the average power of $f_1(n)$ by[1],

$$\widehat{P}_{f,1}(n) \triangleq \frac{1}{R}\sum_{r=1}^{R} |f_1^{(r)}(n)|^2, \tag{8}$$

where the subscript $f, 1$ denotes the association with $f_1(n)$. Plot $\widehat{P}_{f,1}(n)$ over the time index $0 \leq n \leq L-1$.

(d) (1 points) Since $\widehat{P}_{f,1}(n)$ is a sequence over time, we collect the data points from $n = N_1$ to N_2 and then average these results

$$\widehat{P}_{f,1} \triangleq \frac{1}{N_2 - N_1 + 1}\sum_{n=N_1}^{N_2} \widehat{P}_{f,1}(n). \tag{9}$$

We select $N_1 = L/4$ and $N_2 = 3L/4$. What is the value of $\widehat{P}_{f,1}$?

(e) (5 points) Repeat Problems 4c and 4d for the backward prediction-error filter $b_1(n)$.

(f) (10 points) Plot the prediction error power over the index $m = 1, 2, \ldots, 10$. This plot contains four curves:

i. One curve for the forward prediction error power $\widehat{P}_{f,m}$ (from the realizations $f_m^{(r)}(n)$).

ii. Another curve for the backward prediction error power $\widehat{P}_{b,m}$ (from the realizations $b_m^{(r)}(n)$).

iii. Another curve for the prediction error power P_m from the Levinson-Durbin algorithm.

iv. The other curve for the prediction error bound of $x(n)$. You may use numerical integration in MATLAB for the value of this bound.

Determine whether your results are consistent with Equation (14), Equation (48), and the theorem on Page 43 of `06_Linear_Prediction.pdf`. If not, state why.

Note: Please include the following MATLAB scripts and figure files in your submission

- **`ASP_Levinson_Durbin.m`**
- **`ASP_HW2_Problem_4.m`**
- The plots in fig files (Make sure that the MATLAB codes match these figure files.)
  - **`ASP_HW1_Problem_4c.fig`**
  - **`ASP_HW1_Problem_4e.fig`**
  - **`ASP_HW1_Problem_4f.fig`**

Last updated October 24, 2021.

[1] The right-hand side of (8) resembles the definition of the minimum forward prediction error power $\mathbb{E}\left[|f_1(n)|^2\right]$. More specifically, the expectation operator $\mathbb{E}$ is replaced with the average over R terms in (8). The random process $f_1(n)$ is replaced with its realizations $f_1^{(r)}(n)$.