Demo Class - Approximation

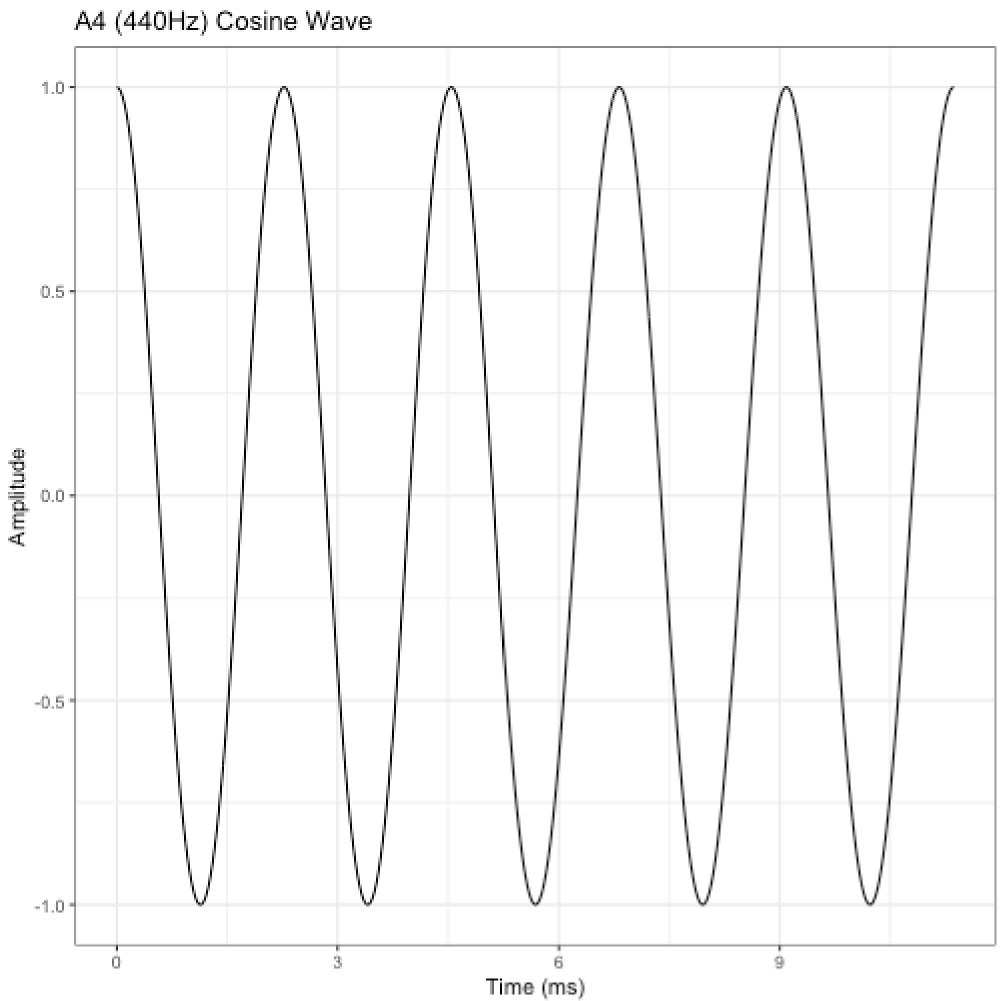
Introduction to the Fourier Transform

Borja Cadenato

2024-04-24

How does an A4 sound wave look like?

An A4 sound wave



Representation of a wave

Frequency is measured in Hertz, number events per second.

$$f_1 = 440 \frac{\text{periods}}{\text{s}} = 440\text{Hz}$$

Representation of a wave

Frequency is measured in Hertz, number events per second.

$$f_1 = 440 \frac{\text{periods}}{\text{s}} = 440\text{Hz}$$

Frequency in the context of sinusoidal waves is usually measured in radians:

$$\omega_1 = 2\pi \cdot 440 \frac{\text{radians}}{\text{s}}$$

Representation of a wave

Frequency is measured in Hertz, number events per second.

$$f_1 = 440 \frac{\text{periods}}{\text{s}} = 440\text{Hz}$$

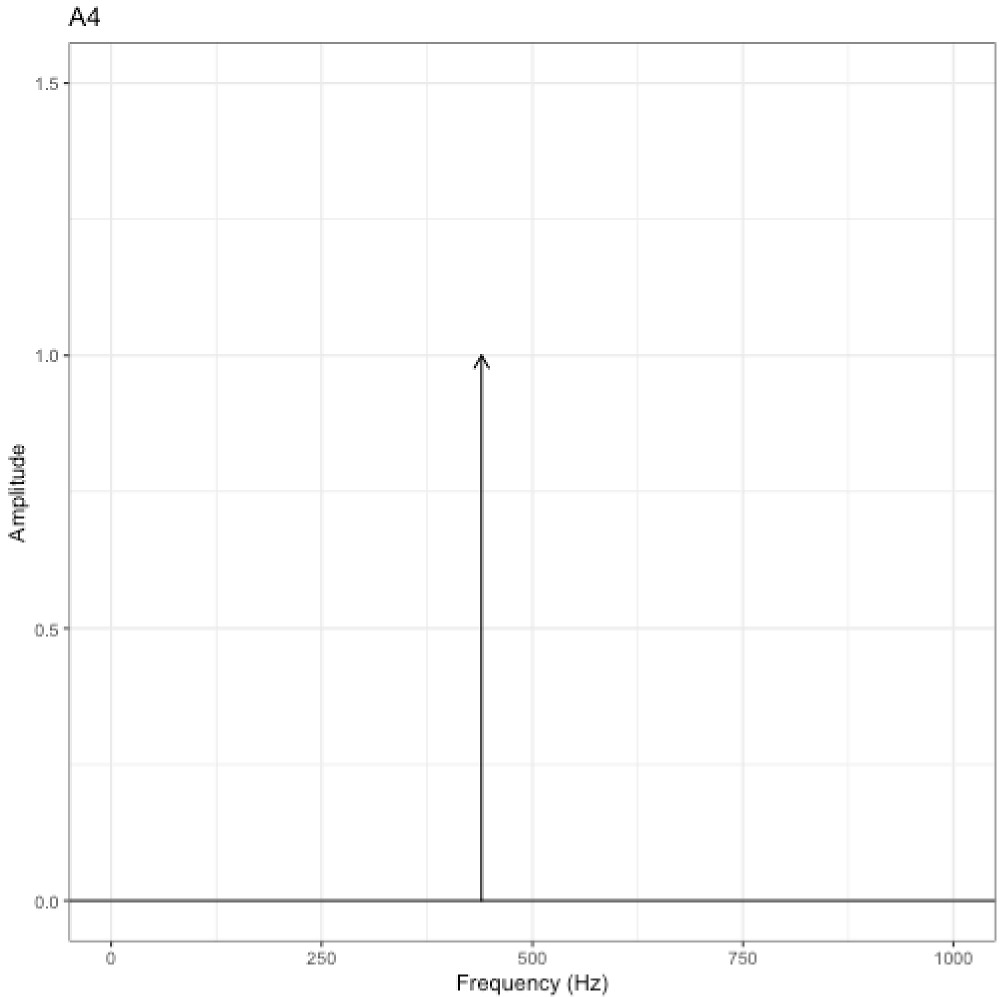
Frequency in the context of sinusoidal waves is usually measured in radians:

$$\omega_1 = 2\pi \cdot 440 \frac{\text{radians}}{\text{s}}$$

The representation of the 440 Hz (A4) sound wave is thus:

$$x(t) = \cos(2\pi f_1 t) = \cos(\omega_1 t)$$

An A4 sound wave in the frequency domain



Dirac Delta Function

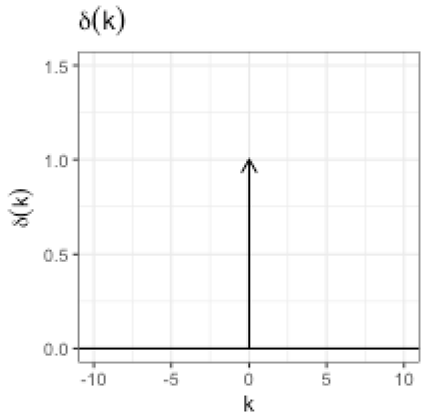
$$\delta(k) = \begin{cases} \infty & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(k) dk = 1$$

Dirac Delta Function

$$\delta(k) = \begin{cases} \infty & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

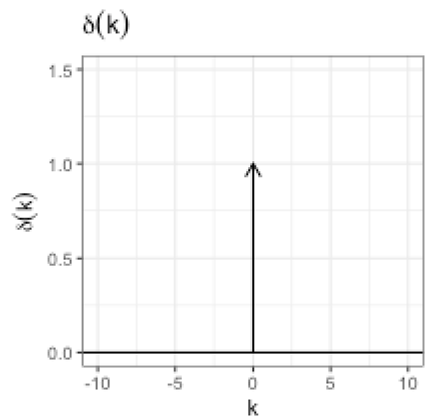
$$\int_{-\infty}^{\infty} \delta(k) dk = 1$$



Dirac Delta Function

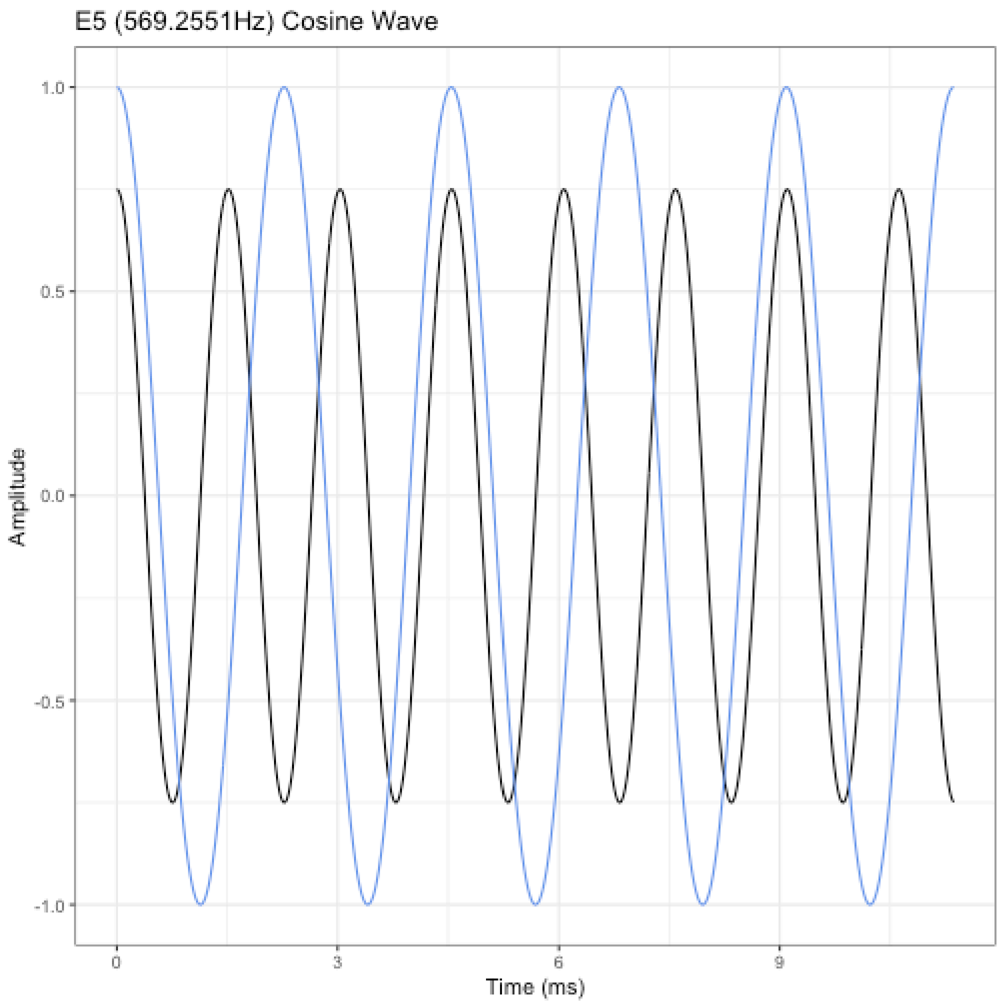
$$\delta(k) = \begin{cases} \infty & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(k) dk = 1$$

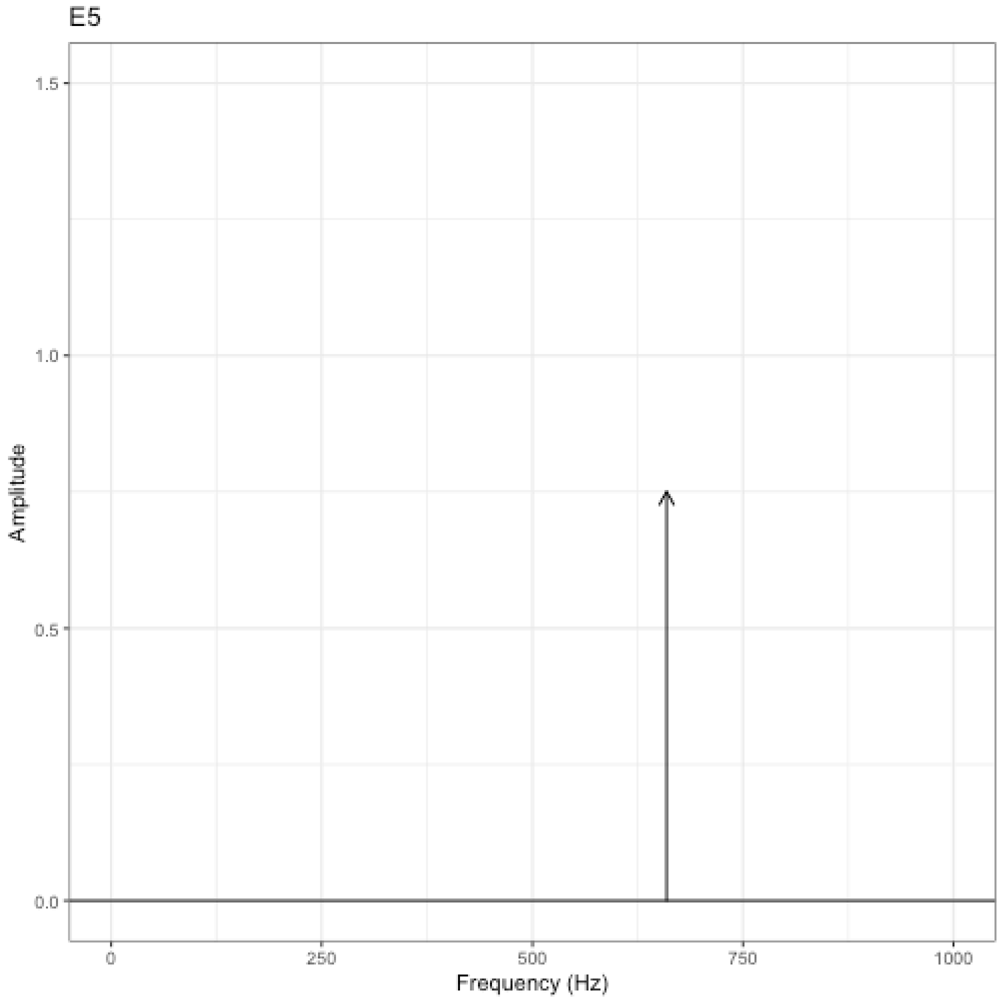


In general, a wave $c_1 \cdot \cos(\omega_1 t)$ of amplitude c_1 and angular frequency ω_1 can be represented as $c_1 \cdot \delta(\omega - \omega_1)$

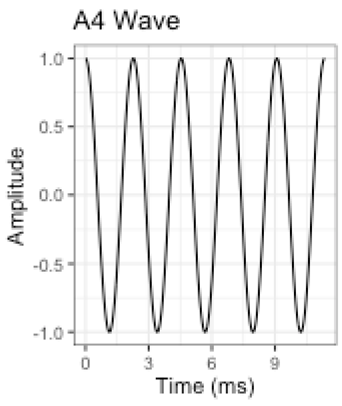
An E5 sound wave



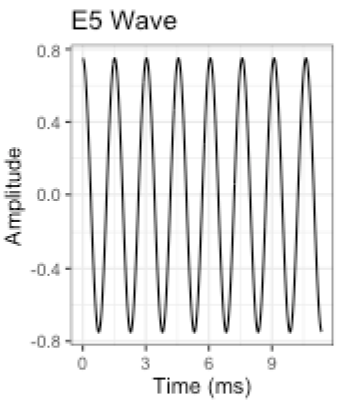
An E5 sound wave in the frequency domain



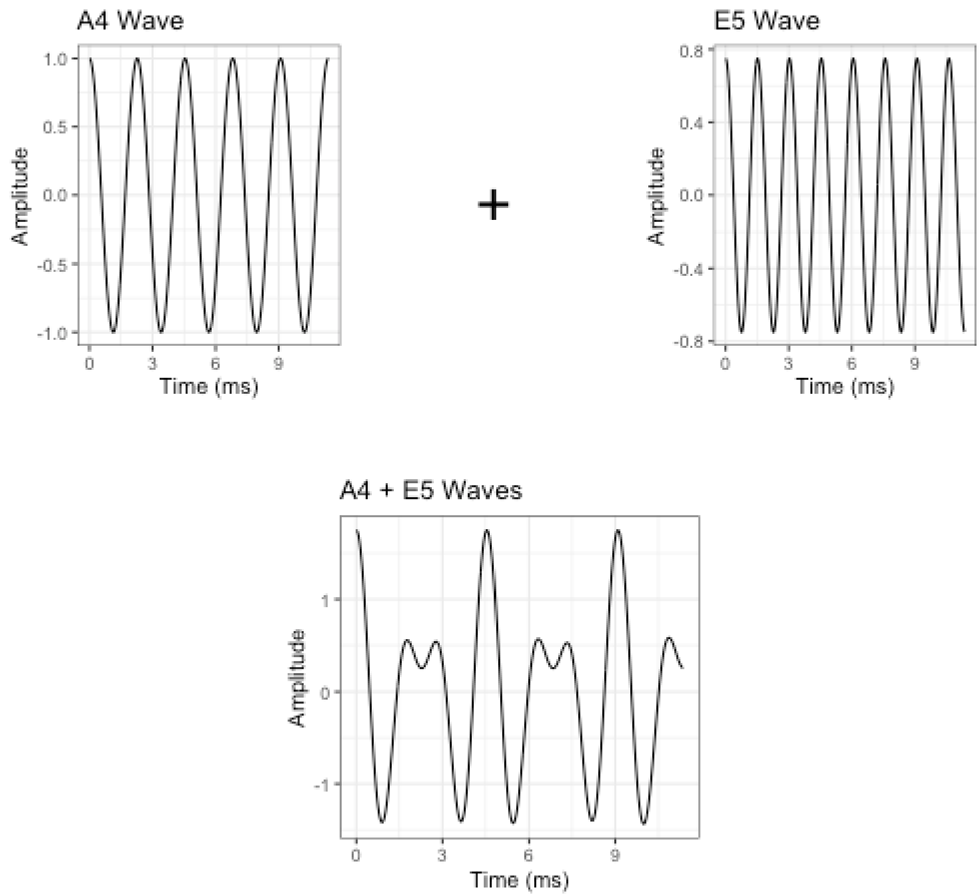
A4 and E5 sound waves added up



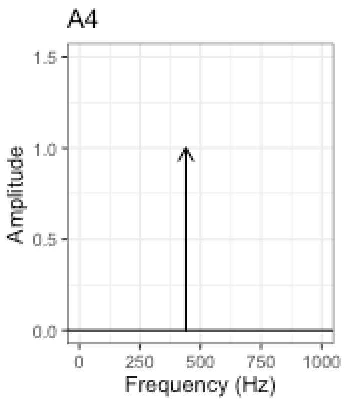
+



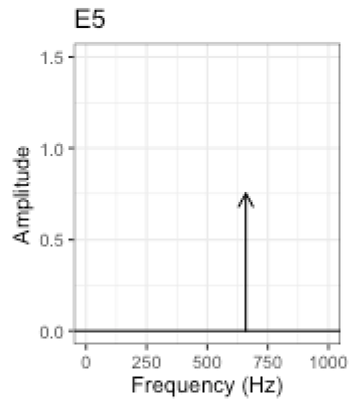
A4 and E5 sound waves added up



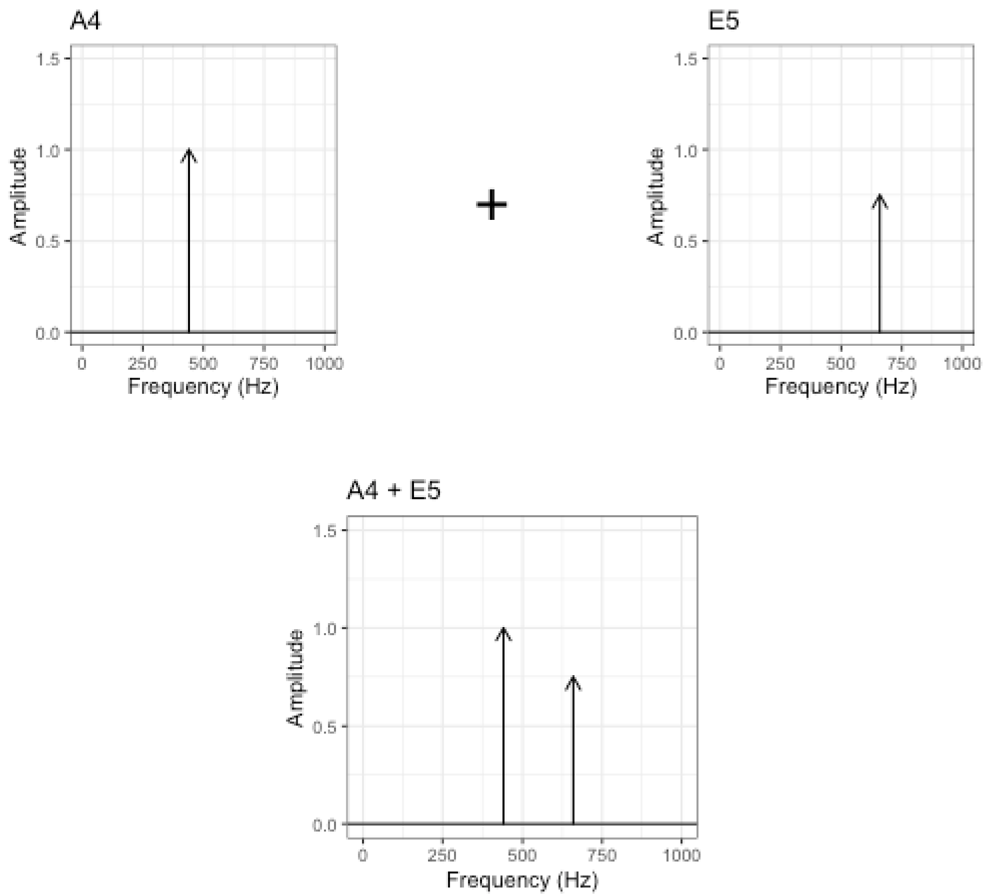
A4 and E5 sound waves in the frequency domain



+



A4 and E5 sound waves in the frequency domain



What is a Fourier Transform?

What is a Fourier Transform?

If a function can be represented as a sum of sinusoidal functions:

$$x(t) = c_1 \cdot \cos(\omega_1 t) + c_2 \cdot \cos(\omega_2 t) + \ldots$$

What is a Fourier Transform?

If a function can be represented as a sum of sinusoidal functions:

$$x(t) = c_1 \cdot \cos(\omega_1 t) + c_2 \cdot \cos(\omega_2 t) + \ldots$$

It can be represented in the frequency domain:

$$X(\omega) = c_1 \cdot \delta(\omega - \omega_1) + c_2 \cdot \delta(\omega - \omega_2) + \ldots$$

What is a Fourier Transform?

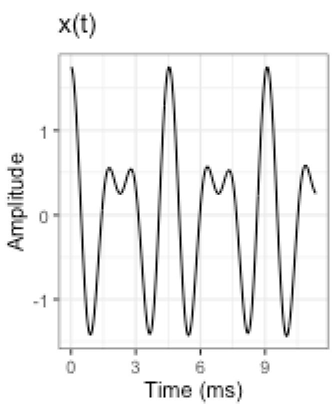
If a function can be represented as a sum of sinusoidal functions:

$$x(t) = c_1 \cdot \cos(\omega_1 t) + c_2 \cdot \cos(\omega_2 t) + \dots$$

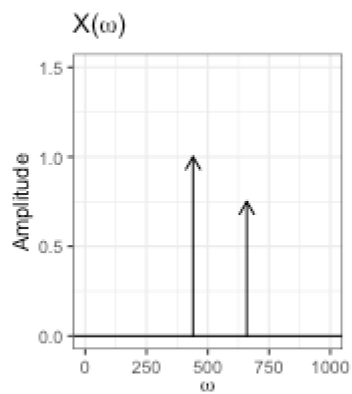
It can be represented in the frequency domain:

$$X(\omega) = c_1 \cdot \delta(\omega - \omega_1) + c_2 \cdot \delta(\omega - \omega_2) + \dots$$

We say that $X(\omega)$ is the Fourier Transform of $x(t)$



$\Leftrightarrow$



Angular Frequency Shifting

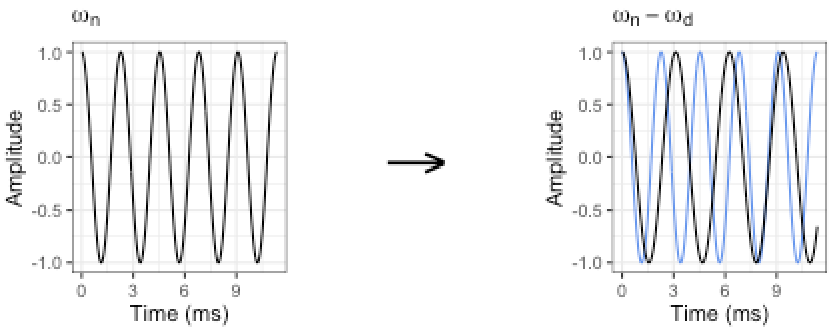
A function $x(t) = cos(\omega t)$ can be shifted in frequency by an angular frequency offset ω_d:

$$x(t)_{\omega_d} = \cos([\omega - \omega_d]t)$$

Angular Frequency Shifting

A function $x(t) = cos(\omega t)$ can be shifted in frequency by an angular frequency offset ω_d:

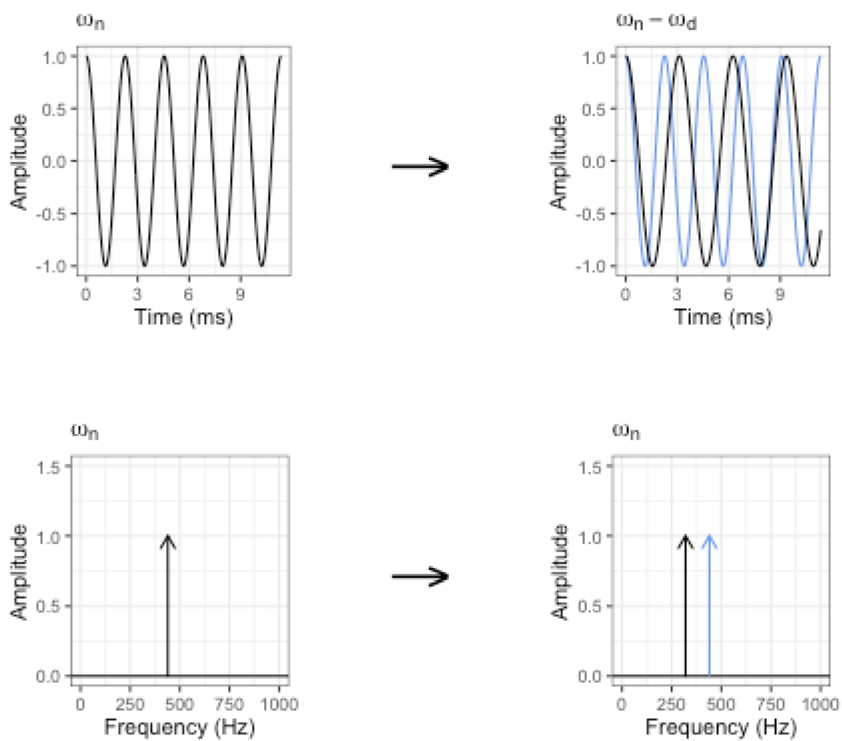
$$x(t)_{\omega_d} = \cos([\omega - \omega_d]t)$$



Angular Frequency Shifting

A function $x(t) = cos(\omega t)$ can be shifted in frequency by an angular frequency offset ω_d:

$$x(t)_{\omega_d} = \cos([\omega - \omega_d]t)$$



How to frequency shift a function?

Euler's formula is:

$$e^{ix} = \cos x + i \sin x$$

If $x = \omega t$ then

$$e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$$

How to frequency shift a function?

Euler's formula is:

$$e^{ix} = \cos x + i \sin x$$

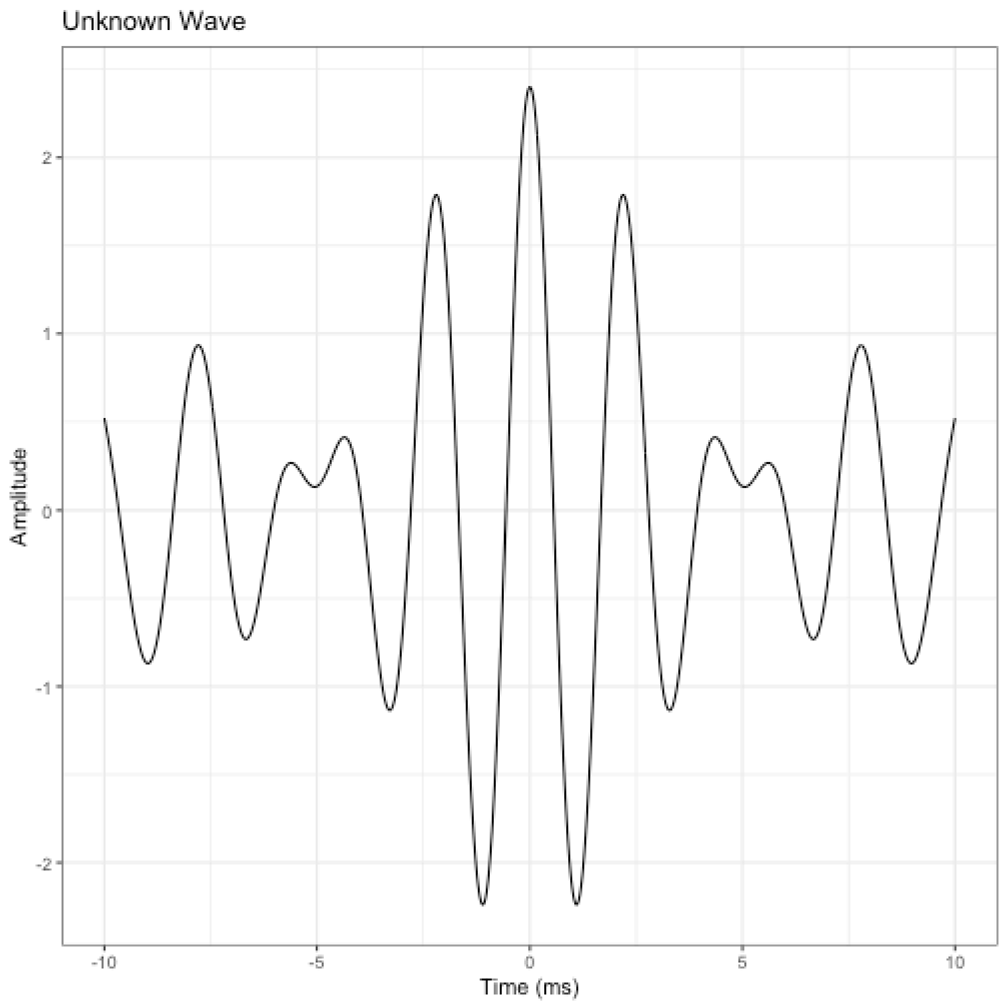
If $x = \omega t$ then

$$e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$$

Multiplying the previous expression by $e^{-i\omega_d t}$

$$e^{i\omega t} \cdot e^{-i\omega_d t} = e^{i(\omega - \omega_d)t} = \cos([\omega - \omega_d]t) + i \sin([\omega - \omega_d]t)$$

What is the Fourier Transform of this function?



How can we identify the wave parameters?

Let's assume that the function $x(t)$ is a sum of cosine functions:

$$x(t) = c_1 \cdot \cos(\omega_1 t) + c_2 \cdot \cos(\omega_2 t) + \cdots$$

How can we identify the wave parameters?

Let's assume that the function $x(t)$ is a sum of cosine functions:

$$x(t) = c_1 \cdot \cos(\omega_1 t) + c_2 \cdot \cos(\omega_2 t) + \cdots$$

The Fourier Transform $X(\omega)$ would be

$$X(\omega) = c_1 \cdot \delta(\omega - \omega_1) + c_2 \cdot \delta(\omega - \omega_2) + \cdots$$

But now the parameters c_1, ω_1, c_2, ω_2, etc. are unknown.

How to transform $\cos(\omega_1 t)$

Let $x(t) = \cos(\omega_1 t)$

Let's shift $\cos(\omega_1 t)$ by an angular frequency ω_d

$$\cos([\omega_1 - \omega_d]t)$$

How to transform $\cos(\omega_1 t)$

Let $x(t) = \cos(\omega_1 t)$

Let's shift $\cos(\omega_1 t)$ by an angular frequency ω_d

$$\cos([\omega_1 - \omega_d]t)$$

Let's call

$$F(\cos(\omega_1 t), \omega) = \int_{-\infty}^{\infty} \cos([\omega_1 - \omega]t) dt$$

Then, if $\omega_1 - \omega_d \neq 0$

$$F(\cos(\omega_1 t), \omega_d) = \int_{-\infty}^{\infty} \cos([\omega_1 - \omega_d]t) dt = 0$$

How to transform $\cos(\omega_1 t)$

Let $x(t) = \cos(\omega_1 t)$

Let's shift $\cos(\omega_1 t)$ by an angular frequency ω_d

$$\cos([\omega_1 - \omega_d]t)$$

Let's call

$$F(\cos(\omega_1 t), \omega) = \int_{-\infty}^{\infty} \cos([\omega_1 - \omega]t)dt$$

Then, if $\omega_1 - \omega_d \neq 0$

$$F(\cos(\omega_1 t), \omega_d) = \int_{-\infty}^{\infty} \cos([\omega_1 - \omega_d]t)dt = 0$$

If $\omega_1 - \omega_d = 0$

$$F(\cos(\omega_1 t), \omega_d) = \int_{-\infty}^{\infty} \cos([\omega_1 - \omega_d]t)dt = \int_{-\infty}^{\infty} dt = \infty$$

because $\cos([\omega_1 - \omega_d]t) = \cos(0) = 1$

How to transform $\cos(\omega_1 t)$

To summarise, if $x(t) = \cos(\omega_1 t)$:

$$F(x(t), \omega) = \begin{cases} 0 & \text{if } \omega \neq \omega_1 \\ \infty & \text{if } \omega = \omega_1 \end{cases} = \delta(\omega - \omega_1)$$

How to transform $\cos(\omega_1 t)$

To summarise, if $x(t) = \cos(\omega_1 t)$:

$$F(x(t), \omega) = \begin{cases} 0 & \text{if } \omega \neq \omega_1 \\ \infty & \text{if } \omega = \omega_1 \end{cases} = \delta(\omega - \omega_1)$$

Let's call

$$X(\omega) = F(x(t), \omega)$$

Then, in general, if $x(t) = c_1 \cdot \cos(\omega_1 t)$ then

$$X(\omega) = c_1 \cdot \delta(\omega - \omega_1)$$

How to transform $\cos(\omega_1 t)$

To summarise, if $x(t) = \cos(\omega_1 t)$:

$$F(x(t), \omega) = \begin{cases} 0 & \text{if } \omega \neq \omega_1 \\ \infty & \text{if } \omega = \omega_1 \end{cases} = \delta(\omega - \omega_1)$$

Let's call

$$X(\omega) = F(x(t), \omega)$$

Then, in general, if $x(t) = c_1 \cdot \cos(\omega_1 t)$ then

$$X(\omega) = c_1 \cdot \delta(\omega - \omega_1)$$

And if $x(t) = c_1 \cdot \cos(\omega_1 t) + c_2 \cdot \cos(\omega_2 t) + \ldots$ then

$$X(\omega) = c_1 \cdot \delta(\omega - \omega_1)\; c_2 \cdot \delta(\omega - \omega_2) + \ldots$$

Conclusion - Fourier Transform of $x(t)$

Finally, if $x(t)$ is a general function over time, and it can be described as a (possibly infinite) series of sinusoidal functions…

Conclusion - Fourier Transform of $x(t)$

Finally, if $x(t)$ is a general function over time, and it can be described as a (possibly infinite) series of sinusoidal functions…

… the Fourier Transform of $x(t)$ is a new function $X(\omega)$ in the angular frequency domain.

Conclusion - Fourier Transform of $x(t)$

Finally, if $x(t)$ is a general function over time, and it can be described as a (possibly infinite) series of sinusoidal functions…

… the Fourier Transform of $x(t)$ is a new function $X(\omega)$ in the angular frequency domain.

If $x(t)$ can be expressed as a series of complex exponential functions $c_k \cdot e^{i\omega_k t}$, since frequency shifting is just multiplying by $e^{-i\omega t}$, and following the same approach described before:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-i\omega t} dt$$

Slides: github.com/bcadenato/ie-demo