Demo Class - Approximation

The Fourier Transform

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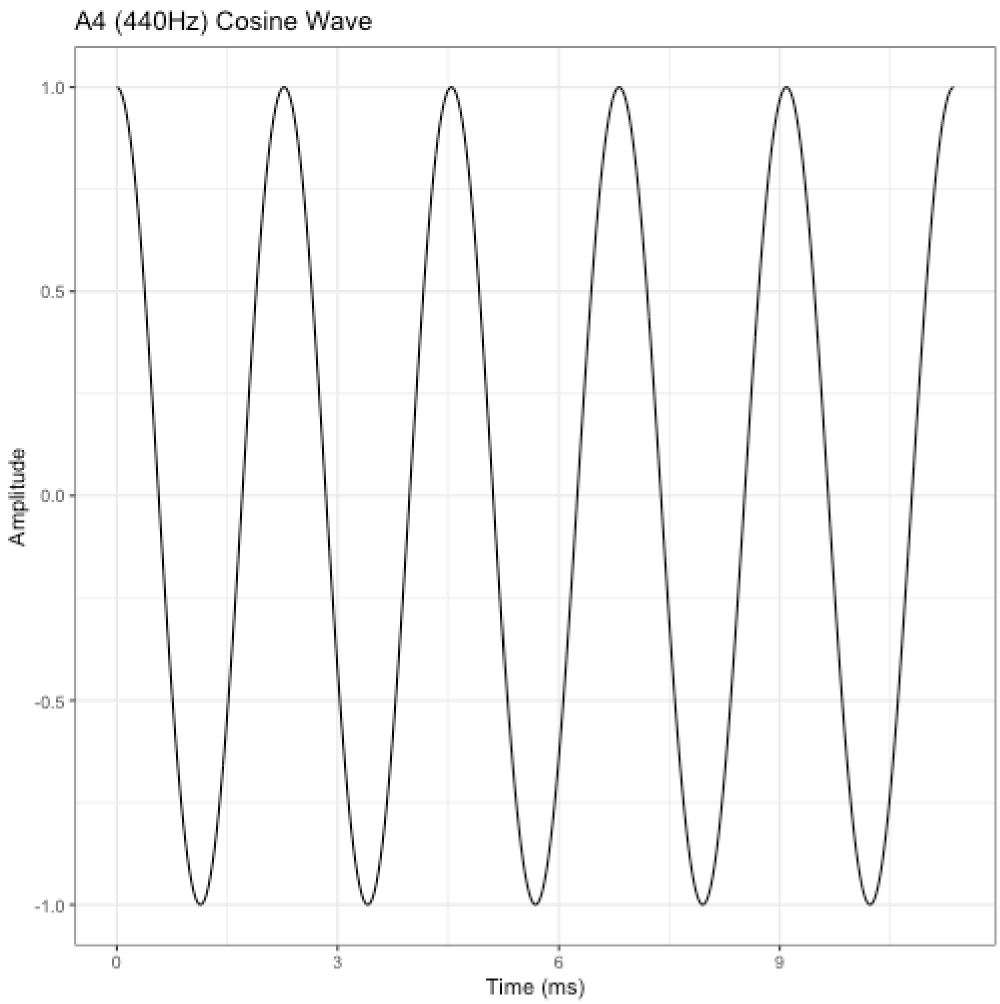
2024-04-20

How does an A4 sound wave look like?

An A4 sound wave



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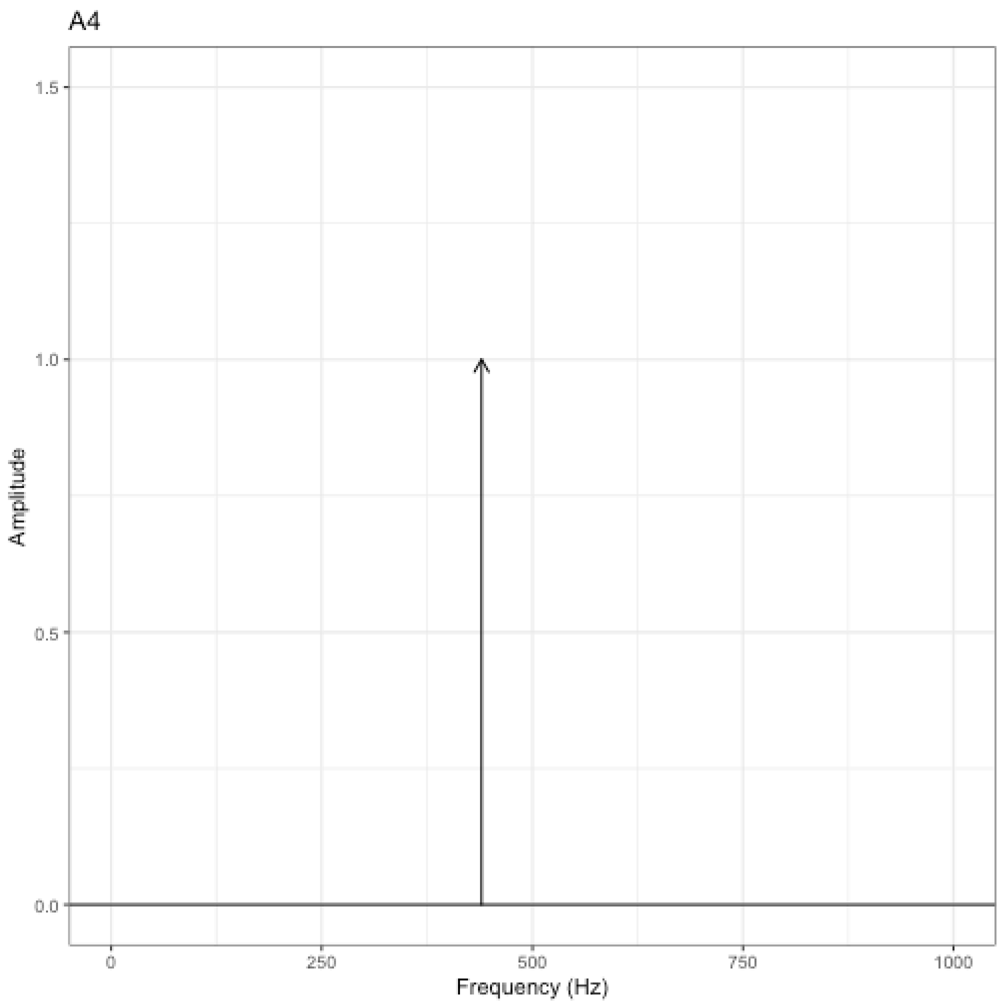
The representation of the 440 Hz (A4) sound wave is thus:

$$x(t) = \cos(2\pi f_1 t) = \cos(\omega_1 t)$$

An A4 sound wave in the frequency domain



Dirac Delta Function

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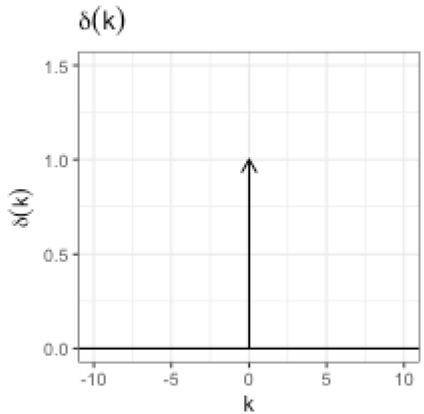

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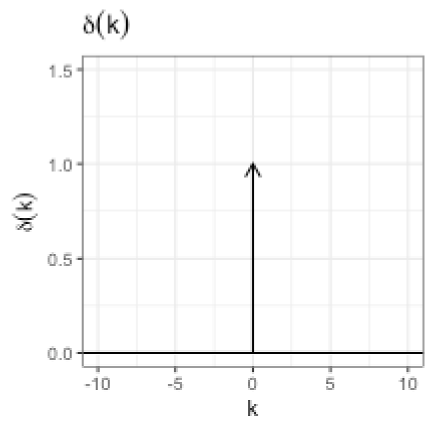
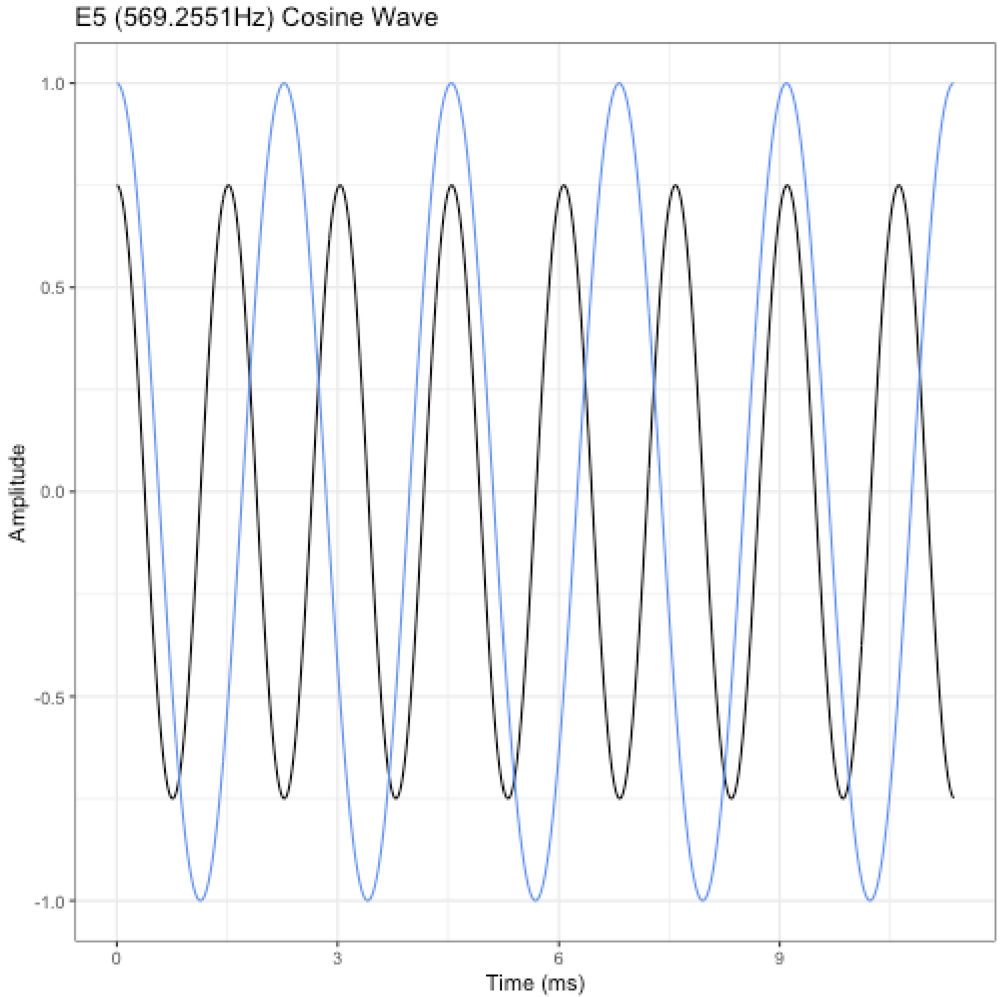
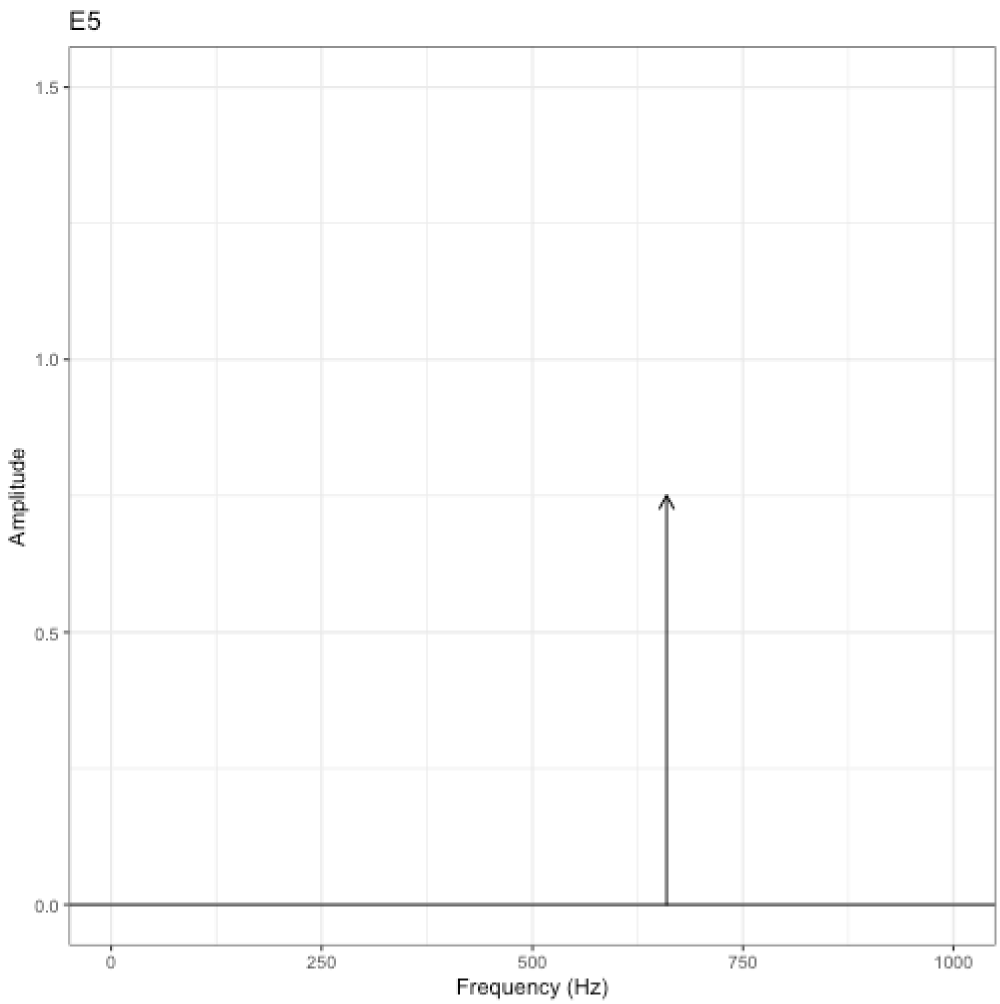


In general, a wave of amplitude c_1 and angular frequency ω_1 ($c_1 \cdot \cos(\omega_1 t)$) can be represented as $c_1 \cdot \delta(\omega - \omega_1)$

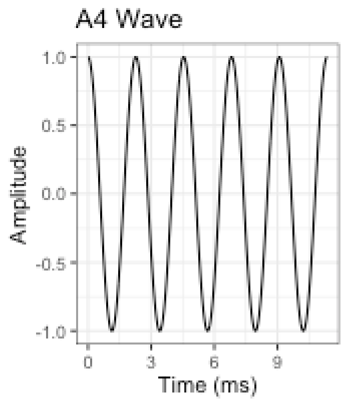
An E5 sound wave



An E5 sound wave in the frequency domain



A4 and E5 sound waves added up



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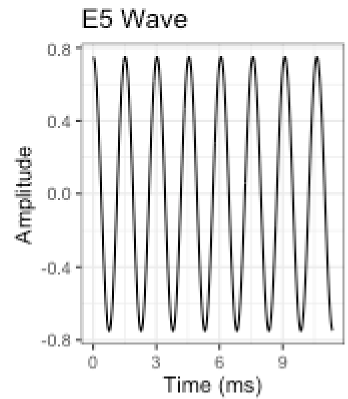

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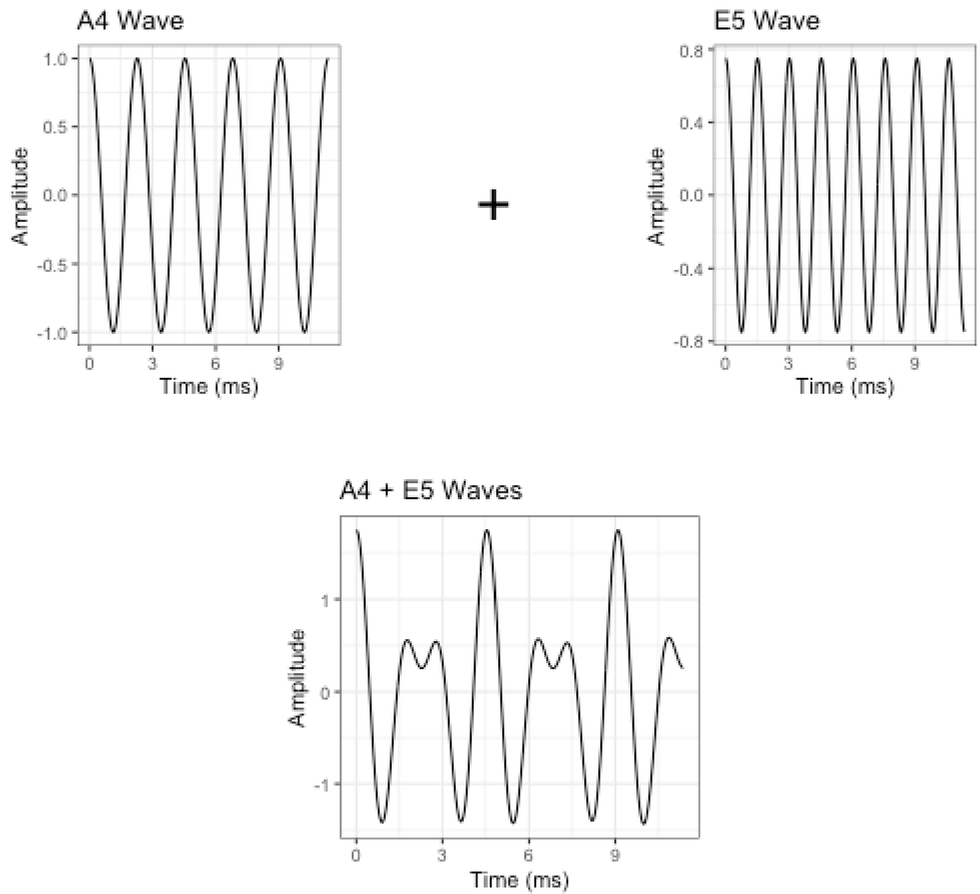
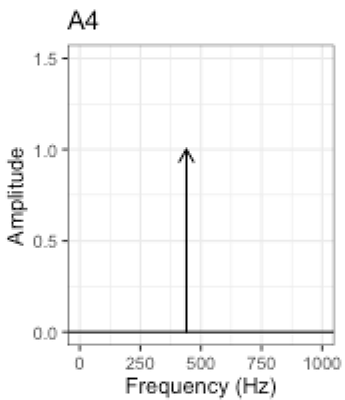


A4 and E5 sound waves in the frequency domain



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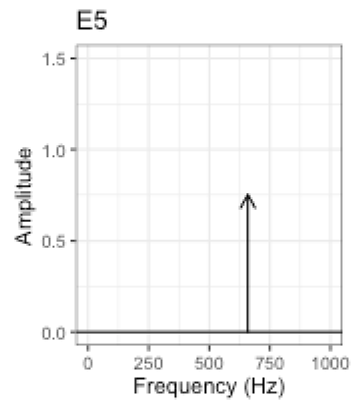

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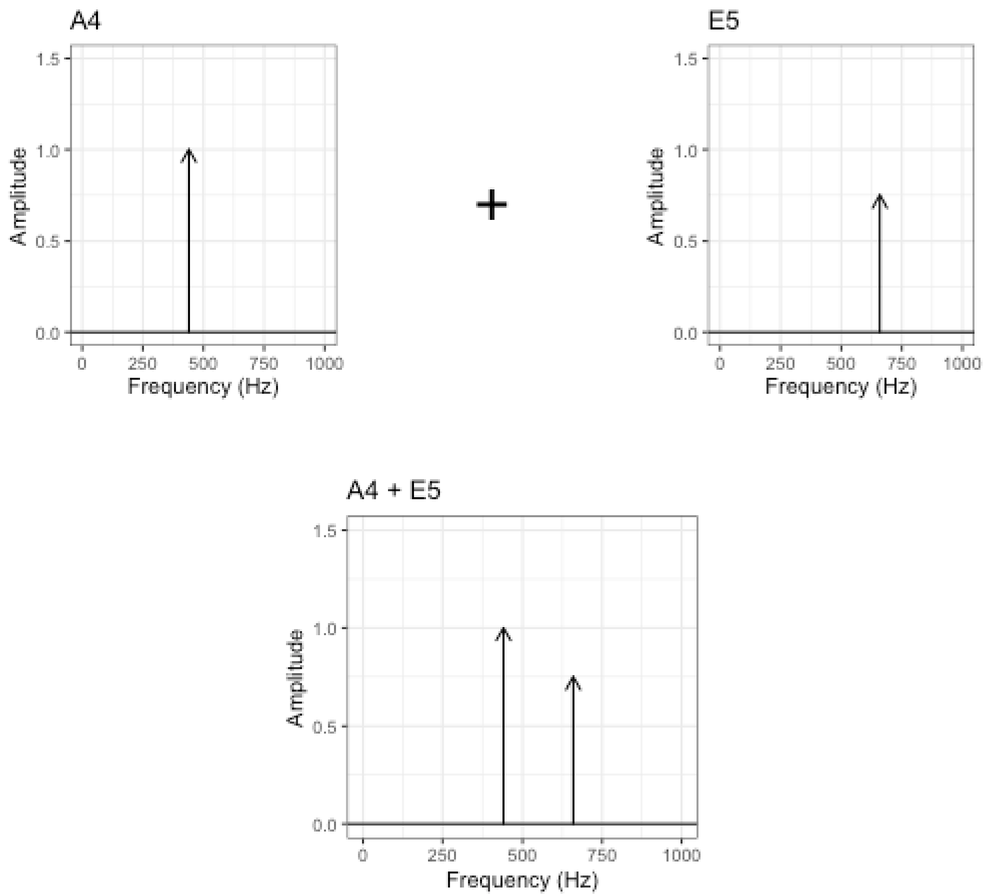

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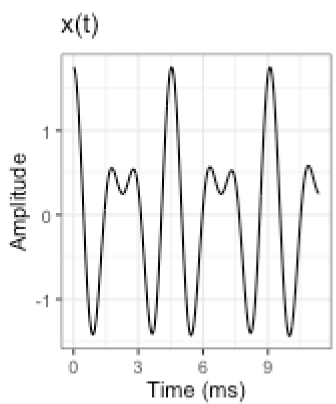
We say that $X(\omega)$ is the Fourier Transform of $x(t)$



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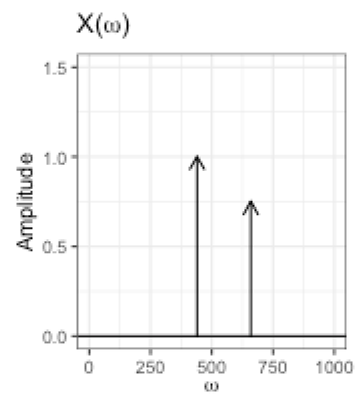
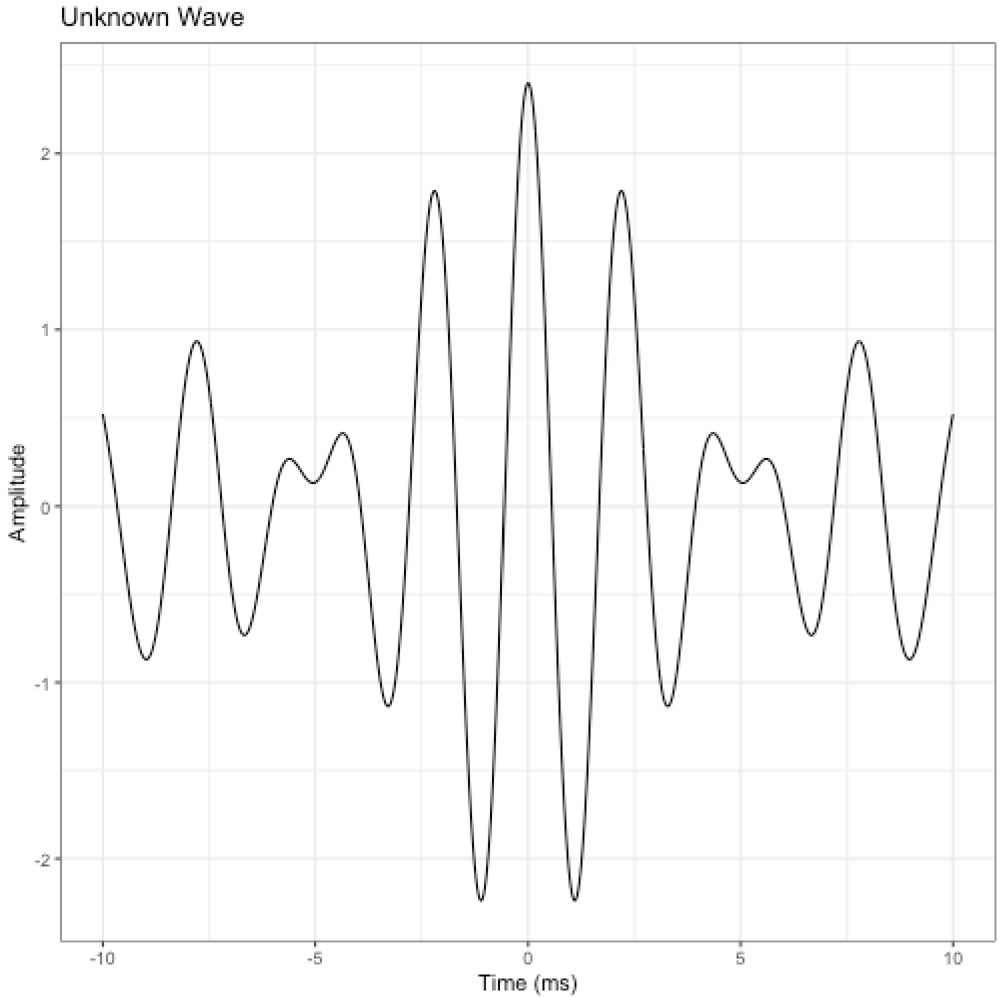


What is the Fourier Transform of this function?



How can we identify the wave parameters?

Let's assume that the function $x(t)$ is a sum of cosine functions:

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The Fourier Transform $X(\omega)$ would be

$$X(\omega) = c_1 \cdot \delta(\omega - \omega_1) + c_2 \cdot \delta(\omega - \omega_2) + \cdots$$

But now the parameters c_1, ω_1, c_2, ω_2, etc. are unknown.

Angular Frequency Shifting

A function $x(t) = cos(\omega t)$ can be shifted in frequency by an angular frequency ω_d:

$$x(t)_{\omega_d} = \cos([\omega - \omega_d]t)$$

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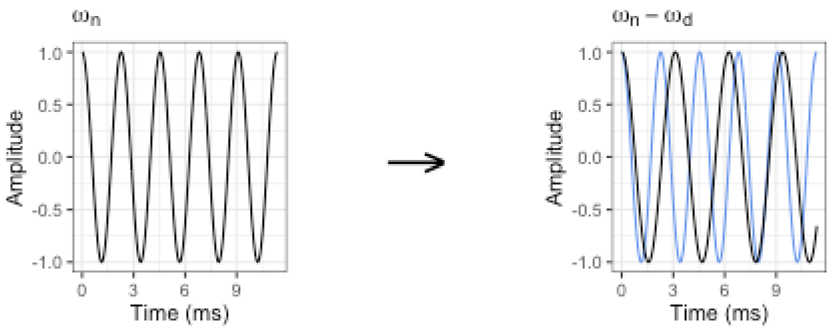

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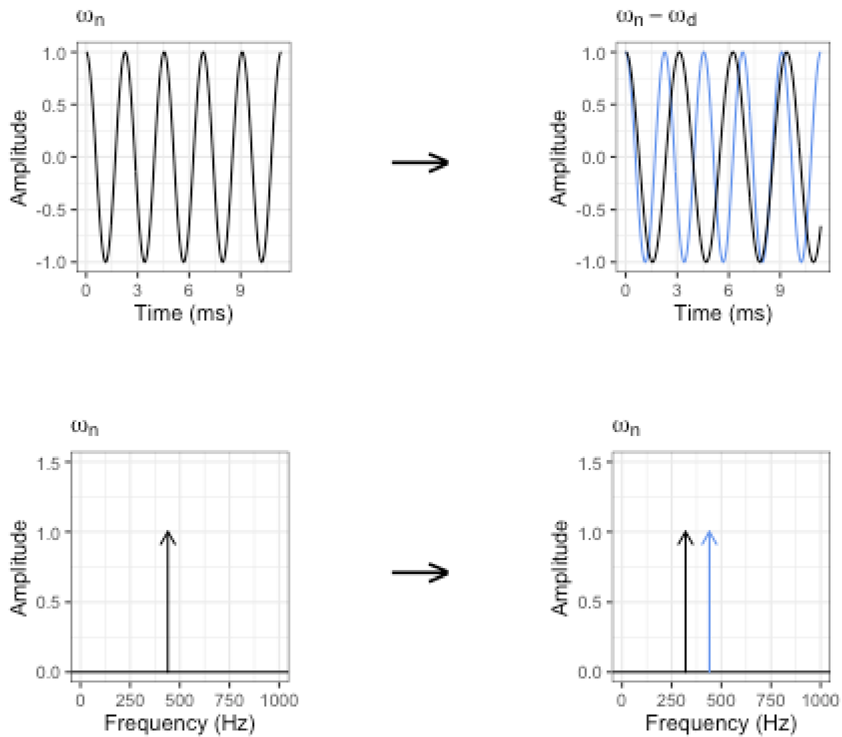

How to frequency shift a function?

An alternative expression of $\cos(x)$ is:

$$\cos(x) = \frac{e^{-ix} + e^{ix}}{2}$$

and thus

$$\cos(\omega t) = \frac{e^{-i\omega t} + e^{i\omega t}}{2}$$

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Multiplying the previous expression by $e^{-i\omega_d t}$

$$\cos(\omega t) \cdot e^{-i\omega_d t} = \frac{e^{-i(\omega-\omega_d)t} + e^{i(\omega-\omega_d)t}}{2} = \cos([\omega - \omega_d]t)$$

Strategy to identify parameters for $X(\omega)$

Assuming the function $x(t)$ can be represented as a sum of cosine functions, let's shift the frequency of those cosine functions by an angular frequency ω_d

$$x(t)_{\omega_d} = c_1 \cdot \cos([\omega_1 - \omega_d]t) + c_2 \cdot \cos([\omega_2 - \omega_d]t) + \cdots$$

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If $\omega_d = \omega_n$, the factor n would become a constant since $\cos(0) = 1$.

$$x(t)_{\omega_n} = \cdots + c_{n-1} \cdot \cos([w_{n-1} - w_n]t) + c_n + c_{n+1} \cdot \cos([w_{n+1} - w_n]t) + \cdots$$

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If the function $x(t)_{\omega_n}$ is integrated over time, all the cosine terms will become 0, and only the constant term c_n will be left.

$$\int_{-\infty}^{\infty} x(t)_{\omega_n} dt = \int_{-\infty}^{\infty} c_n dt$$

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It can be concluded that c_n is the amplitude corresponding to the term with angular frequency ω_n, or following our convention:

$$X(\omega_n) = c_n$$

The Fourier Transform Equation

The previous strategy can be followed for every ω to build the function $X(\omega)$:

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Putting all together the Fourier Transform Equation is

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