H 7.12 Solution

(a)

$$\hat{A} = -\frac{\hat{\alpha}^2}{2\hat{\beta}}$$

(b)

The key step for coming up with the confidence interval is figuring out the limiting distribution of $\sqrt{n}(\hat{A} - A)$. As a first step, our "usual" arguments for least squares regression imply that

$$\sqrt{n}\begin{pmatrix} \hat{\alpha} - \alpha \\ \hat{\beta} - \beta \end{pmatrix} \xrightarrow{d} N(0, \mathbf{V}_\beta)$$

where

$$\mathbf{V}_\beta = \mathrm{E}[XX']^{-1}\Omega\mathrm{E}[XX']^{-1}$$

and $\Omega = \mathrm{E}[XX'e^2]$ (and where, to keep the expressions from getting too long, I am taking X here to include an intercept, so that $\mathbf{V}_\beta$ is a 2×2 asymptotic variance matrix).

Next, notice that we can write $A = r(\alpha, \beta)$ and $\hat{A} = r(\alpha, \beta)$ where $r(a, b) = -a^2/2b$. This suggests using a delta method type of argument. In particular, using a mean value theorem argument, we can write

$$r(\hat{\alpha}, \hat{\beta}) = r(\alpha, \beta) + \nabla r(\bar{\alpha}, \bar{\beta})'\begin{pmatrix} \hat{\alpha} - \alpha \\ \hat{\beta} - \beta \end{pmatrix} \tag{1}$$

where

$$\begin{aligned} \nabla r(\bar{a}, \bar{b}) &:= \left.\begin{bmatrix} \frac{\partial r(a,b)}{\partial a} \\ \\ \frac{\partial r(a,b)}{\partial b} \end{bmatrix}\right|_{a=\bar{a}, b=\bar{b}} \\ &= \left.\begin{bmatrix} -\frac{a}{b} \\ \\ \frac{a^2}{2b^2} \end{bmatrix}\right|_{a=\bar{a}, b=\bar{b}} \end{aligned}$$

which is the vector of partial derivatives of $r(a, b)$ evaluated at $\bar{a}$ and $\bar{b}$. Plugging this back in to Equation 1 implies that

$$\hat{A} = A + \begin{bmatrix} -\frac{\bar{\alpha}}{\bar{\beta}} \\ \\ \frac{\bar{\alpha}^2}{2\bar{\beta}^2} \end{bmatrix}'\begin{pmatrix} \hat{\alpha} - \alpha \\ \hat{\beta} - \beta \end{pmatrix}$$

and, by multiplying by $\sqrt{n}$ and adding and subtracting terms, implies that

$$\begin{aligned}
\sqrt{n}(\hat{A} - A) &= \begin{bmatrix} -\frac{\alpha}{\beta} \\ \\ \frac{\alpha^2}{2\beta^2} \end{bmatrix}' \sqrt{n} \begin{pmatrix} \hat{\alpha} - \alpha \\ \hat{\beta} - \beta \end{pmatrix} + \underbrace{\left(\begin{bmatrix} -\frac{\bar{\alpha}}{\bar{\beta}} \\ \\ \frac{\bar{\alpha}^2}{2\bar{\beta}^2} \end{bmatrix} - \begin{bmatrix} -\frac{\alpha}{\beta} \\ \\ \frac{\alpha^2}{2\beta^2} \end{bmatrix} \right)'}_{=o_p(1)} \underbrace{\sqrt{n} \begin{pmatrix} \hat{\alpha} - \alpha \\ \hat{\beta} - \beta \end{pmatrix}}_{=O_p(1)} \\
&= \begin{bmatrix} -\frac{\alpha}{\beta} \\ \\ \frac{\alpha^2}{2\beta^2} \end{bmatrix}' \sqrt{n} \begin{pmatrix} \hat{\alpha} - \alpha \\ \hat{\beta} - \beta \end{pmatrix} + o_p(1) \\
&\xrightarrow{d} N(0, V)
\end{aligned}$$

where the $o_p(1)$ in the first equality arises because (i) $\bar{\alpha}$ is between $\hat{\alpha}$ and α and $\bar{\beta}$ is between $\hat{\beta}$ and β; (ii) $\hat{\alpha} \xrightarrow{p} \alpha$, $\hat{\beta} \xrightarrow{p} \beta$; and (iii) the continuous mapping theorem; and where

$$V = \begin{bmatrix} -\frac{\alpha}{\beta} \\ \\ \frac{\alpha^2}{2\beta^2} \end{bmatrix}' \mathbf{V}_\beta \begin{bmatrix} -\frac{\alpha}{\beta} \\ \\ \frac{\alpha^2}{2\beta^2} \end{bmatrix}$$

Moreover, we can estimate V by

$$\hat{V} = \begin{bmatrix} -\frac{\hat{\alpha}}{\hat{\beta}} \\ \\ \frac{\hat{\alpha}^2}{2\hat{\beta}^2} \end{bmatrix}' \hat{\mathbf{V}}_\beta \begin{bmatrix} -\frac{\hat{\alpha}}{\hat{\beta}} \\ \\ \frac{\hat{\alpha}^2}{2\hat{\beta}^2} \end{bmatrix}$$

and where we use the “usual” estimate of $\mathbf{V}_\beta$:

$$\hat{\mathbf{V}}_\beta = \left(\frac{1}{n} \sum_{i=1}^{n} X_i X_i' \right)^{-1} \frac{1}{n} \sum_{i=1}^{n} X_i X_i' \hat{e}_i^2 \left(\frac{1}{n} \sum_{i=1}^{n} X_i X_i' \right)^{-1}$$

Finally, we can construct a 95% confidence interval by

$$\hat{C} = \left[\hat{A} \pm 1.96 \frac{\sqrt{\hat{V}}}{\sqrt{n}} \right]$$