Solutions to Additional Practice 1 Questions

Hansen 4.1

Part (a):

$$\hat{\mu}_k = \frac{1}{n}\sum_{i=1}^{n} Y_i^k$$

Part (b):

$$\begin{aligned}
\mathrm{E}\left[\hat{\mu}_k\right] &= \mathrm{E}\left[\frac{1}{n}\sum_{i=1}^{n} Y_i^k\right] \\
&= \frac{1}{n}\sum_{i=1}^{n} \mathrm{E}[Y_i^k] \\
&= \frac{1}{n}\sum_{i=1}^{n} \mathrm{E}[Y^k] \\
&= \mathrm{E}[Y^k]
\end{aligned}$$

where the third equality holds because the Y_i are identically distributed (implying the mean is the same across i). This result implies that $\hat{\mu}_k$ is unbiased for μ_k.

Part (c):

$$\begin{aligned}
\mathrm{var}\left(\hat{\mu}_k\right) &= \mathrm{var}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i^k\right) \\
&= \frac{1}{n^2}\mathrm{var}\left(\sum_{i=1}^{n} Y_i^k\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{var}(Y^k) \\
&= \frac{\mathrm{var}(Y^k)}{n}
\end{aligned}$$

where the second equality holds because $1/n$ is a constant and it should be squared to come out of the variance, the third equality holds by passing the variance through the sum (in order for their not to be any covariance terms introduced here, it requires the "independence" part of iid; for this variance to be the same across all units requires the "identically distributed" part of iid), and the last equality holds because summing a constant n times cancels one of the n's from the denominator.

For $\mathrm{var}(\hat{\mu}_k)$ to exist, we need for $\mathrm{var}(Y^k)$ to exist. Notice that,

$$\mathrm{var}(Y^k) = \mathrm{E}[(Y^k)^2] - \mathrm{E}[Y^k]^2$$

Thus, the condition that we need is that $\mathrm{E}[(Y^k)^2] = \mathrm{E}[Y^{2k}] < \infty$.

Part (d): We can estimate by

$$\widehat{\mathrm{var}}(\hat{\mu}_k) = \frac{\widehat{\mathrm{var}}(Y^k)}{n} = \frac{\frac{1}{n}\sum_{i=1}^{n} Y_i^{2k} - \left(\frac{1}{n}\sum_{i=1}^{n} Y_i^k\right)^2}{n}$$

Hansen 4.5

First (and notice that this is exactly the same as what we showed in class... because unbiasedness did not rely on homoskedasticity),

$$\begin{aligned}\mathrm{E}[\hat{\beta}|\mathbf{X}] &= \mathrm{E}[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}|\mathbf{X}] \\ &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathrm{E}[\mathbf{Y}|\mathbf{X}] \\ &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}\beta \\ &= \beta\end{aligned}$$

For thinking about the sampling variance, first notice that

$$\begin{aligned}\mathrm{var}(\mathbf{Y}|\mathbf{X}) &= \mathrm{var}(\mathbf{X}\beta + \mathbf{e}|\mathbf{X}) \\ &= \mathrm{var}(\mathbf{e}|\mathbf{X}) \\ &= \sigma^2\mathbf{\Sigma}\end{aligned}$$

where the first equality holds by plugging in for $\mathbf{Y}$, the second equality holds because we are conditioning on $\mathbf{X}$, and the third equality holds from the way that $\mathbf{\Sigma}$ is defined in the textbook. Next,

$$\begin{aligned}\mathrm{var}(\hat{\beta}|\mathbf{X}) &= \mathrm{var}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}|\mathbf{X}) \\ &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathrm{var}(\mathbf{Y}|\mathbf{X})\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\ &= \sigma^2(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Sigma}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\end{aligned}$$

where the first equality holds by plugging in for $\hat{\beta}$, the second equality holds because the variance is conditional on $\mathbf{X}$ (so the terms involving $\mathbf{X}$ can come out but need to be "squared"), and the last equality holds by plugging in the expression for $\mathrm{var}(\mathbf{Y}|\mathbf{X})$ that we derived above. This is the result we were trying to show.

Hansen 4.6

Recall that the restriction to linear estimators implies that we can write any estimator in this class as $\tilde{\beta} = \mathbf{A}'\mathbf{Y}$ for an $n \times k$ matrix $\mathbf{A}$ that is a function of $\mathbf{X}$. Unbiasedness implies that, it must be the case that $\mathrm{E}[\tilde{\beta}|\mathbf{X}] = \beta$. Then, notice that under linearity, we have that

$$\mathrm{E}[\tilde{\beta}|\mathbf{X}] = \mathrm{E}[\mathbf{A}'\mathbf{Y}|\mathbf{X}] = \mathbf{A}'\mathrm{E}[\mathbf{Y}|\mathbf{X}] = \mathbf{A}'\mathbf{X}\beta$$

where the second equality holds because $\mathbf{A}$ is a function of $\mathbf{X}$. Therefore, together linearity and unbiasedness imply that $\mathbf{A}'\mathbf{X} = \mathbf{I}_k$. Next, notice that

$$\mathrm{var}(\tilde{\beta}|\mathbf{X}) = \mathrm{var}(\mathbf{A}'\mathbf{Y}|\mathbf{X}) = \mathbf{A}'\mathrm{var}(\mathbf{Y}|\mathbf{X})\mathbf{A} = \sigma^2\mathbf{A}'\mathbf{\Sigma}\mathbf{A}$$

We aim to show that $\text{var}(\tilde{\beta}|\mathbf{X}) - \sigma^2(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1} \geq \mathbf{0}$. Notice that

$$\begin{aligned}
\text{var}(\tilde{\beta}|\mathbf{X}) - \sigma^2(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1} &= \sigma^2\left(\mathbf{A}'\mathbf{\Sigma}\mathbf{A} - (\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\right) \\
&= \sigma^2\left(\mathbf{A}'\mathbf{\Sigma}\mathbf{A} - \mathbf{A}'\mathbf{\Sigma}^{1/2}\mathbf{\Sigma}^{-1/2}\mathbf{X}(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1/2}\mathbf{\Sigma}^{1/2}\mathbf{A}\right) \\
&= \sigma^2\mathbf{A}'\mathbf{\Sigma}^{1/2}\underbrace{\left(\mathbf{I} - \mathbf{\Sigma}^{-1/2}\mathbf{X}\left((\mathbf{\Sigma}^{-1/2}\mathbf{X})'\mathbf{\Sigma}^{-1/2}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1/2}\right)}_{=:\mathbf{M}_{\Sigma^{-1/2}X}}\mathbf{\Sigma}^{1/2}\mathbf{A} \\
&= \sigma^2\mathbf{A}'\mathbf{\Sigma}^{1/2}\mathbf{M}_{\Sigma^{-1/2}X}\mathbf{\Sigma}^{1/2}\mathbf{A} \\
&= \sigma^2\left(\mathbf{M}_{\Sigma^{-1/2}X}\mathbf{\Sigma}^{1/2}\mathbf{A}\right)'\mathbf{M}_{\Sigma^{-1/2}X}\mathbf{\Sigma}^{1/2}\mathbf{A} \\
&\geq 0
\end{aligned}$$

where the above result repeatedly uses $\mathbf{\Sigma}$ is positive definite and symmetric (which implies that it has a positive definite and symmetric inverse, and that it has a positive definite and symmetric square root matrix, and so does its inverse). In particular, the second equality holds because (i) $\mathbf{\Sigma}^{-1/2}\mathbf{\Sigma}^{1/2} = \mathbf{I}_n$, and $\mathbf{A}'\mathbf{X} = \mathbf{I}_k$ (due to linearity and unbiasedness as discussed above); the third equality holds by factoring out $\mathbf{A}'\mathbf{\Sigma}^{1/2}$ and from a slight manipulation of the inside term; the fourth equality holds by the definition of $\mathbf{M}_{\Sigma^{-1/2}X}$ (which is an annihilator matrix); the fifth equality holds because $\mathbf{M}_{\Sigma^{-1/2}X}$ is idempotent and symmetric; and the last equality holds because the previous expression is a quadratic form.

Hansen 4.23

Notice that

$$\begin{aligned}
\text{E}[\hat{\beta}_{ridge}|\mathbf{X}] &= \text{E}\left[(\mathbf{X}'\mathbf{X} + \mathbf{I}_k\lambda)^{-1}\mathbf{X}'\mathbf{Y}\right] \\
&= (\mathbf{X}'\mathbf{X} + \mathbf{I}_k\lambda)^{-1}\mathbf{X}'\text{E}[\mathbf{Y}|\mathbf{X}] \\
&= (\mathbf{X}'\mathbf{X} + \mathbf{I}_k\lambda)^{-1}\mathbf{X}'\mathbf{X}\beta \\
&\neq \beta
\end{aligned}$$

This implies that $\hat{\beta}_{ridge}$ is not unbiased for β.