Solutions to Additional Practice 2 Questions

Question 1

a) Yes,

$$
\begin{aligned}
CATE(X) &= \mathbb{E}[Y(1)|X] - \mathbb{E}[Y(0)|X] \\
&= \mathbb{E}[Y(1)|X, D=1] - \mathbb{E}[Y(0)|X, D=0] \\
&= \mathbb{E}[Y|X, D=1] - \mathbb{E}[Y|X, D=0]
\end{aligned}
$$

where the first equality is just the definition of $CATE(X)$, the second equality uses unconfoundedness, the third equality holds because treated potential outcomes are observed for the treated group and untreated potential outcomes are observed for the untreated group, and this line implies that $CATE(X)$ is identified.

b) Yes,

$$
\begin{aligned}
CATT(X) &= \mathbb{E}[Y(1)|X, D=1] - \mathbb{E}[Y(0)|X, D=1] \\
&= \mathbb{E}[Y(1)|X, D=1] - \mathbb{E}[Y(0)|X, D=0] \\
&= \mathbb{E}[Y|X, D=1] - \mathbb{E}[Y|X, D=0]
\end{aligned}
$$

where the first equality is just the definition of $CATT(X)$, the second equality holds by unconfoundedness, and the third equality holds because treated potential outcomes are observed for the treated group and untreated potential outcomes are observed for the untreated group, and this line implies that $CATT(X)$ is identified.

c) The expression for $CATE(X)$ and $CATT(X)$ are the same under unconfoundedness. Notice that this does not imply that ATE and ATT are equal, as recovering ATE would involve integrating over the overall distribution of X while recovering ATT would involve integrating over the distribution of X for the treated group. Thus, if $CATE(X)$ varied over X, then this would not, in general, be equal.

Question 2

Notice that

$$
\begin{aligned}
\mathbb{E}[g(X)|D=1] &= \int g(x) f(x|D=1)\, dx \\
&= \int g(x) \underbrace{\frac{f(x|D=1)}{f(x|D=0)}} f(x|D=0)\, dx
\end{aligned}
$$

Now, by repeatedly applying the definition of conditional probability, we have that

$$
f(x|D=1) = \frac{p(x) f(x)}{p} \quad \text{and} \quad f(x|D=0) = \frac{(1-p(x)) f(x)}{1-p}
$$

Plugging these expressions into the underlined term in the previous display, we have that

$$
\begin{aligned}
\mathbb{E}[g(X)|D=1] &= \int g(x)\frac{p(x)f(x)(1-p)}{(1-p(x))f(x)p}\,f(x|D=0)\,dx \\
&= \mathbb{E}\left[\frac{p(X)(1-p)}{(1-p(X))p}g(X)|D=0\right]
\end{aligned}
$$

as we were trying to show.

Question 3

To show the result, I'll start from the second decomposition that we discussed in class, where we showed that

$$
\begin{aligned}
\alpha = \mathbb{E}\Big[w(D,X)\Big(\mathbb{E}[Y|X,D=1]-\mathbb{E}[Y|X,D=0]\Big)\Big] \qquad (1)\\
+\,\mathbb{E}\Big[w(D,X)\Big(\mathbb{E}[Y|X,D=0]-\mathrm{L}_0(Y|X)\Big)\Big] \qquad (2)
\end{aligned}
$$

First, recall that, under unconfoundedness and overlap, $CATE(X)=\mathbb{E}[Y|X,D=1]-\mathbb{E}[Y|X,D=0]$. This implies that the right hand side of Eq.(1) is equal to $\mathbb{E}[w(D,X)CATE(X)]$. Then, to show the result, we show that the term in Eq.(2) is equal to 0 under either of the two conditions in the problem. In the second case, where $\mathbb{E}[Y|X,D=0]=\mathrm{L}_0(Y|X)$, it immediately holds that Eq.(2) is equal to 0.

Next, suppose that $p(X)=\mathrm{L}(D|X)$ while allowing for the second condition not holding. Consider each component of the expression in Eq.(2) individually. First,

$$
\begin{aligned}
\mathbb{E}\Big[D(1-\mathrm{L}(D|X))\mathbb{E}[Y|X,D=0]\Big] &= \mathbb{E}\Big[(1-\mathrm{L}(D|X))\mathbb{E}[Y|X,D=0]\big|D=1\Big]\,p \\
&= \mathbb{E}\left[\frac{p(X)}{1-p(X)}(1-\mathrm{L}(D|X))\mathbb{E}[Y|X,D=0]\big|D=0\right]\,(1-p) \\
&= \mathbb{E}\left[\frac{p(X)}{1-p(X)}(1-\mathrm{L}(D|X))Y|D=0\right]\,(1-p) \\
&= \mathbb{E}\Big[p(X)Y|D=0\Big]\,(1-p)
\end{aligned}
$$

where the first three equalities hold from the law of iterated expectations and where the last equality uses the condition in the problem. Next,

$$
\begin{aligned}
\mathbb{E}\Big[D(1-\mathrm{L}(D|X))\mathrm{L}_0(Y|X)\Big] &= \mathbb{E}\Big[(1-\mathrm{L}(D|X))\mathrm{L}_0(Y|X)\big|D=1\Big]\,p \\
&= \mathbb{E}\left[\frac{p(X)}{1-p(X)}(1-\mathrm{L}(D|X))\mathrm{L}_0(Y|X)\big|D=0\right]\,(1-p) \\
&= \mathbb{E}\Big[p(X)\mathrm{L}_0(Y|X)\big|D=0\Big]\,(1-p) \\
&= \mathbb{E}\Big[\mathrm{L}(D|X)\mathrm{L}_0(Y|X)\big|D=0\Big]\,(1-p) \\
&= \mathbb{E}\Big[\mathrm{L}(D|X)Y|D=0\Big]\,(1-p) \\
&= \mathbb{E}\Big[p(X)Y|D=0\Big]\,(1-p)
\end{aligned}
$$

where the first and second equalities hold by the law of iterated expectations, the third, fourth, and sixth equalities hold when $p(X) = \mathrm{L}(D|X)$, and the fifth equality holds by the result in class that $\mathbb{E}[\mathrm{L}(D|X)\mathrm{L}_d(Y|X)|D = d] = \mathbb{E}[\mathrm{L}(D|X)Y|D = d]$. That the last two displays are equal to each other when the propensity score is linear completes the proof.