Bergen County Academies Math Competition
Sample Solutions
Prepared By: Veena Ramakrishnan

Fourth Grade

1. A problem author for a math competition wrote 44 problems so far, but he needs 50. How many more problems must he write?

   He must write $50 - 44 = 6$ more problems.

2. Express $\frac{13}{4}$ in decimal form.

   First Solution. Note that $\frac{13}{4} = \frac{12+1}{4} = \frac{12}{4} + \frac{1}{4} = 3 + \frac{1}{4} = 3 + 0.25 = 3.25$.

   Second Solution. Dividing directly also yields $13 \div 4 = 3.25$.

3. If a bicycle costs \$35.55 and Tom has \$29.55, how many more dollars does he have to save in order to buy the bike?

   Tom needs to save $\$35.55 - \$29.55 = \$6.00$ more dollars.

4. Calculate $1 + 2 \times 3$.

   By order of operations, $1 + 2 \times 3 = 1 + (2 \times 3) = 1 + 6 = 7$.

5. Calculate $\frac{1}{2} - \frac{1}{3}$.

   Using the common denominator 6, we have $\frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6} = \frac{1}{6}$.

6. What is the product of all the numbers on the buttons of a standard telephone?

   Note that 0 is included in the product, so the answer is 0.

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Typical Questions
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Fifth Grade

1. Evaluate $17 - 16 + 15 - 14 + \cdots + 3 - 2 + 1$.

   We group the numbers into eight pairs and one leftover: $(17 - 16) + (15 - 14) + (13 - 12) + \cdots + (5 - 4) + (3 - 2) + 1$. The quantity inside each pair of parentheses is 1, and there are eight of them, so the expression is equal to $8 + 1 = 9$.

2. The Academy Math Team wants to purchase some tee shirts. The first tee shirt costs \$182, and each subsequent shirt costs \$2. If there are 90 people on the team, and the team equally distributes the cost amongst its members, how much does each member have to pay?

   The total cost is $182 + (89 \times 2) = 360$ dollars, so each member needs to pay $360 \div 90 = 4$ dollars.

3. On a warm day, the temperature was $77\,^\circ\mathrm{F}$. The conversion between Centigrade and Fahrenheit is: $^\circ\mathrm{C} = \frac{5}{9} \times (\,^\circ\mathrm{F} - 32)$. What was the temperature in degrees Centigrade?

   Substituting directly into the given expression, we compute that the temperature was $\frac{5}{9} \times (77 - 32) = 25$ degrees Centigrade.

   Note. A question worth \$125,000 on the game show *Who Wants to Be a Millionaire?* asked what $-40\,^\circ\mathrm{F}$ is equivalent to in degrees Centigrade. (The conversion expression was not given) What is the answer?

4. A painter mixes 4 gallons of white paint with 1 gallon of red paint to make 5 gallons of her signature pink paint. Each gallon of white paint costs \$2 and each gallon of red paint costs \$3. How much money does the painter need to make 400 gallons of pink paint?

   Since the painter mixes white paint and red paint in a 4:1 ratio, to produce 400 gallons of her pink paint, she needs $400 \times \frac{4}{5} = 320$ gallons of white paint and $400 \times \frac{1}{5} = 80$ gallons of red paint. The white paint will cost $320 \times 2 = 640$ dollars and the red paint will cost $80 \times 3 = 240$ dollars. Therefore, the painter needs $640 + 240 = 880$ dollars.

5. What two-digit number evenly divides both 323 and 391?

   First Solution. By trial and error, we find that 323 is divisible by 17, and also $323 = 17 \times 19$. It is now easy to show that 391 is divisible by 17, but not by 19, and therefore, the answer is 17.

   Second Solution. If two numbers are divisible by n, then their difference is also divisible by n. To prove this, we note that the two multiples of n can be expressed as nx and ny for some integers x and y. Then, their difference $nx - ny = n(x - y)$ is clearly a multiple of n as well. Applying this fact to the problem, we see that the number in question also divides $391 - 323 = 68$. Since $68 = 4 \times 17$, we see that 17 is the only possible candidate, and thus the answer.

   Note. The method used in the second solution is a fundamental method of solving *number theory* problems, which are problems that involve integers and divisibility.

6. Compute 1.55×21.4.

   Multiplying directly, we have $1.55 \times 21.4 = 33.17$.

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Sixth Grade

1. In the game of *Mafball*, points can only be scored in 3 points or 5 points. What is the largest unattainable score in *Mafball*?

   The answer is 7, which is unattainable. Note that 8 points can be achieved by scoring one 3-pointer and one 5-pointer, 9 points can be achieved by scoring three 3-pointers, and 10 points can be achieved by scoring two 5-pointers. Now, note that if n points can be attained, so can $n+3$, since one only needs to score one additional 3-pointer. Since 8, 9, and 10 are attainable, so are 11, 12, and 13, and thus, 14, 15, and 16 are as well, and etc. Therefore, every score higher than 7 can be attained.

   Note. In general, if points can only be scored in a points or b points, where a and b are relatively prime, the largest unattainable score is $ab - a - b$.

2. Andy can paint a fence in 1 hour by himself, and Bobby can paint a fence in 2 hours by himself. How many minutes does it take Andy and Bobby to paint a fence together?

   First Solution. From the given, Andy can paint a fence per hour, whereas Bobby can paint half a fence per hour. Therefore, if they work together, they can paint $1 + \frac{1}{2} = \frac{3}{2}$ of a fence per hour, so it would take them $\frac{2}{3}$ hours to paint a fence. Since $\frac{2}{3}$ hours is equal to $\frac{2}{3} \times 60 = 40$ minutes, the answer is 40.

   Second Solution. Often, it is easier to understand the problem by assigning numbers. Let the total amount of fence to paint be $2m^2$. Then, Andy paints at a speed of $2m^2/hr$, and Bobby paints at a speed of $1m^2/hr$. Therefore, the two of them together can paint at a speed of $3m^2/hr$. Since one fence is $2m^2$, painting it would take them $2m^2 \times \frac{1hr}{3m^2} = \frac{2}{3}hr$, or $\frac{2}{3} \times 60 = 40$ minutes.

3. If A, B, C are three distinct points such that all three do not lie on one line, how many parallelograms can be formed using A, B, C, and a fourth point?

   In any quadrilateral with A, B, and C as vertices, either $\overline{AB}$, $\overline{BC}$, or $\overline{CA}$ must be a diagonal. If $\overline{AB}$ is a diagonal, reflect C across the midpoint of $\overline{AB}$ to obtain the fourth point. We can construct two other points in a similar fashion. Therefore, the answer is 3.

4. A fruit company orders 4800 pounds of oranges at \$1.80 per pound. The shipping cost is \$3000. Suppose 10% of the oranges are spoiled during the shipping and the remaining oranges are all sold. What should the selling price per pound be, given that the fruit company wants to make a net 8% profit?

   The total cost for the company is $(4800 \times 1.80) + 3000 = 11640$ dollars, so to make a net 8% profit, the company wishes to earn $11640 \times 1.08 = 12571.20$ dollars. Since only $4800 \times 0.90 = 4320$ pounds of oranges remain, the company needs to sell them at $12571.20 \div 4320 = 2.91$ dollars per pound.

   Note. Though the above solution is straightforward, there are two major obstacles: Multiplying 11640 and 1.08, and dividing 12571.20 by 4320. To avoid performing such long arithmetic, it is often necessary to simplify *less*. To illustrate this method, we note that the final answer may be written as $\frac{11640 \times 1.08}{4800 \times 0.90}$. We can separate all the powers of ten, leaving: $\frac{1164 \times 10 \times 108 \times 0.01}{48 \times 100 \times 90 \times 0.01} = \frac{1164}{48} \times \frac{1}{100} \times \frac{10 \times 108}{90} = \frac{97}{4} \times \frac{1}{100} \times 12 = \frac{291}{100} = 2.91$.

5. Let $\lfloor x \rfloor$ denote the greatest whole number less than or equal to x. For example, $\lfloor 4.6 \rfloor = 4$, $\lfloor \frac{16}{7} \rfloor = 2$, and $\lfloor 5 \rfloor = 5$. Calculate $\lfloor \frac{1}{3} \rfloor + \lfloor \frac{2}{3} \rfloor + \lfloor \frac{3}{3} \rfloor + \cdots + \lfloor \frac{97}{3} \rfloor + \lfloor \frac{98}{3} \rfloor + \lfloor \frac{99}{3} \rfloor$.

   Note that $\lfloor \frac{1}{3} \rfloor = \lfloor \frac{2}{3} \rfloor = \lfloor \frac{3}{3} \rfloor = 1$, $\lfloor \frac{4}{3} \rfloor = \lfloor \frac{5}{3} \rfloor = \lfloor \frac{6}{3} \rfloor = 2$, $\lfloor \frac{7}{3} \rfloor = \lfloor \frac{8}{3} \rfloor = \lfloor \frac{9}{3} \rfloor = 3$, $\ldots$, $\lfloor \frac{94}{3} \rfloor = \lfloor \frac{95}{3} \rfloor = \lfloor \frac{96}{3} \rfloor = 32$, and $\lfloor \frac{97}{3} \rfloor = \lfloor \frac{98}{3} \rfloor = \lfloor \frac{99}{3} \rfloor = 33$. Therefore, the given sum is equal to $(3 \times 1) + (3 \times 2) + (3 \times 3) + \cdots + (3 \times 32) + (3 \times 33) = 3 \times (1 + 2 + 3 + \cdots + 32 + 33)$. It is well known that the sum of the first n positive integers is $\frac{1}{2} \times n \times (n+1)$, so $1 + 2 + 3 + \cdots + 32 + 33 = \frac{1}{2} \times 33 \times 34 = 561$. The answer is thus $3 \times 561 = 1683$.

   Note. We will prove here that $1 + 2 + 3 + \cdots + (n-2) + (n-1) + n = \frac{1}{2} \times n \times (n+1)$. Let $S = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$, and then write $S = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1$. If we add these two equations term by term, we have $2S$ on the left side and $(n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1)$ on the right side. Since there are n of these $(n+1)$'s, the right side is equal to $n \times (n+1)$. Therefore, we have $2S = n \times (n+1)$, so $S = \frac{1}{2} \times n \times (n+1)$.

6. Compute $664.02 \div 9.3$.

   Dividing directly, we have $664.02 \div 9.3 = 71.4$

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Typical Questions
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Seventh Grade

1. A *palindrome* is a number such that it is read the same regardless of whether the digits are read forwards or backwards. For example, 141, 7007, and 8888 are *palindromes*, whereas 345 and 5959 are not. How many even, four-digit numbers are *palindromes*?

   First, note that every four-digit *palindrome* is defined by the number formed by its first two digits. For example, to obtain the *palindrome* 1331, one can take the number 13, reverse the digits to obtain 31, and attach this to the end of the original number. Every *palindrome* can clearly be formed this way. Since a number is even if and only if its last digit is even, a *palindrome* is even if and only if its first digit is even. Therefore, the number of even, four-digit *palindromes* is equal to the number of two-digit numbers whose tens digit is nonzero and even. There are 10 two-digit numbers that start with 2, 10 that start with 4, 10 that start with 6, and 10 that start with 8, so there are 40 total even, four-digit *palindromes*.

2. Philip has 3 triangles and 6 pentagons. Let S be the total number of sides of the shapes he has. Let N be the number of shapes he has. What is $S + N$?

   Since a triangle has 3 sides and a pentagon has 5 sides, $S = (3 \times 3) + (6 \times 5) = 39$. Also, since Philip has $3 + 6 = 9$ total shapes, $N = 9$. Therefore $S + N = 39 + 9 = 48$.

3. Yao Ming is 7 feet 5 inches tall. A typical basketball hoop is 10 feet from the ground. If there are 12 inches in a foot, how many inches must Yao jump to touch the hoop with his head?

   Yao is $(7 \times 12) + 5 = 89$ inches tall, but the hoop is $10 \times 12 = 120$ inches from the ground. Therefore, he must jump $120 - 89 = 31$ inches.

4. In triangle ABC, $\overline{BC} = 4$ and $\overline{CA} = 6$. If the perimeter of the triangle is 4 times the length of side $\overline{BC}$, what is the length of $\overline{AB}$?

   The perimeter is obviously $4 \times 4 = 16$, so $\overline{AB} = 16 - 4 - 6 = 6$.

5. Find the height of a triangle with base 20 and area 60.

The area A of a triangle with base b and height h can be obtained using the formula $A = \frac{1}{2}bh$. Substituting the given values yields the equation $60 = \frac{1}{2}(20)h$, and solving for h, we have $h = 6$.

6. What is the area of a square in square feet, if each of its diagonals is 4 feet long?

First Solution. Label the square $ABCD$, and let its side have length s. Applying the Pythagorean Theorem in right triangle ABC, we have $s^2 + s^2 = 4^2$, or $s^2 = 8$. Since the area is equal to s^2, the answer is 8.

Second Solution. Regard the square as a rhombus with diagonals whose lengths are both 4. The area of a rhombus is equal to half the product of its diagonals, so the area of the square is simply $\frac{1}{2}(4)(4) = 8$.

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Eighth Grade

1. How many of the following are always true:
   I: the square of an integer is a whole number
   II: if $a \neq b$ and $b \neq c$, then $a \neq c$
   III: every integer has a rational inverse in multiplication
   IV: the square root of a positive integer is real

   Two of the above statements, namely I and IV, are true. A counterexample for statement II can be obtained by letting $a = 1$, $b = 2$, and $c = 1$. A counterexample (in fact, the only counterexample) for statement III is zero, which does not have an inverse in multiplication.

2. A number is *strictly decreasing* if each digit is strictly less than the digit to its left. For example, 543, 531, and 962 are *strictly decreasing*, whereas 562, 537, and 322 are not. How many integers between 100 and 600 are *strictly decreasing*?

   First Solution. It is easy to simply list all of them: 210, 310, 320, 321, 410, 420, 421, 430, 431, 432, 510, 520, 521, 530, 531, 532, 540, 541, 542, 543. Therefore, there are 20.

   Second Solution. Note that given three distinct digits, there is exactly one way to form a *strictly decreasing* three-digit number using them. Therefore, the number of *strictly decreasing* integers between 100 and 600 is equal to the number of ways of choosing three distinct numbers from the digits $\{0, 1, 2, 3, 4, 5\}$, which is ${}_6C_3 = 20$.

   Note. In the second solution, ${}_nC_k$ refers to "n choose k," a fundamental expression in combinatorics. It basically refers to the number of ways to choose k things from n choices without replacement, and can be computed using the formula ${}_nC_k = \frac{n!}{k!(n-k)!}$, where $x!$ denotes the product of the first x positive integers.

3. What is the units digit of $13^{17} + 17^{13}$?

   Note that in calculating the units digit of a sum or product, the other digits are irrelevant. In other words, 13^{17} and 3^{17} have the same last digit. Now, note that $3^2 = 9$ has last digit 9, and therefore, the last digit of $3^3 = 3^2 \times 3$ is the same as the last digit of 9×3, or 7. Furthermore, the last digit of $3^4 = 3^3 \times 3$ is the same as the last digit of 7×3, or 1. Continuing this process, we note that the last digit of

$3^5 = 3^4 \times 3$ is the same as the last digit of 1×3, or 3, and eventually, we see that there is a cycle of period 4. In other words, the last digits of 3^1, 3^2, 3^3, 3^4, 3^5, 3^6, 3^7, 3^8, 3^9, $\ldots$, form the cycle 3, 9, 7, 1, 3, 9, 7, 1, 3, and etc. Therefore, the last digit of 13^{17} is 3. Similarly, the last digits of 7^1, 7^2, 7^3, 7^4, 7^5, 7^6, 7^7, 7^8, 7^9, $\ldots$, form the cycle 7, 9, 3, 1, 7, 9, 3, 1, 7, and etc. We thus deduce that the last digit of 17^{13} is 7. Consequently, the last digit of $13^{17}+17^{13}$ is the same as the last digit of $3+7$, which is 0.

Note. This method can be used to calculate the units digit of any number of the form m^n, where m and n are positive integers. In other words, there always exists such a cycle from which the final digit may be determined.

4. There are ten lottery tickets in a hat, and four of them are winning tickets. First, Joe reaches in and takes a ticket. Then, Kim reaches in and takes a ticket from the remaining nine. What is the probability that Kim takes a winning ticket?

   There are two independent cases to consider: Either Joe takes a winning ticket, or he takes a losing ticket. The probability that Joe takes a winning ticket and Kim takes a winning ticket is $\frac{4}{10} \times \frac{3}{9} = \frac{12}{90} = \frac{2}{15}$. The probability that Joe takes a losing ticket and Kim takes a winning ticket is $\frac{6}{10} \times \frac{4}{9} = \frac{24}{90} = \frac{4}{15}$. Thus, the total probability that Kim takes a winning ticket is $\frac{2}{15} + \frac{4}{15} = \frac{6}{15} = \frac{2}{5}$.

   Note. Amazingly enough, the probability Kim wins is exactly the same as the probability she wins when she is the first to choose a ticket. This is not a coincidence, however, and in such a game of choosing lottery tickets without replacement, it can be proved that the order in which the people choose is irrelevant.

5. A *silly* number *ababab* is formed by repeating a two-digit number *ab* exactly three times. For example, 252525 is a *silly* number. What is the greatest common factor of all *silly* numbers?

   Let N denote the two-digit number *ab*. Then, the number *ababab* is equal to $10000N + 100N + N = 10101N$. In other words, all *silly* numbers are of the form $10101N$, where N is an arbitrary two-digit number. Since the greatest common factor of all possible values of N is obviously 1, (for instance, N can be 11 and 12, which are relatively prime) the greatest common factor of all *silly* numbers is 10101.

6. A number p yields a remainder of 3 when divided by 5, a remainder of 5 when divided by 7, and a remainder of 11 when divided by 13. If p is less than 1000, what is the maximum value of p?

The given implies that $p + 2$ is divisible by 5, 7, and 13. Therefore, $p + 2$ must be a multiple of $5 \times 7 \times 13 = 455$. Since p is less than 1000, $p+2$ is less than 1002, so the largest possible value of $p+2$ is $455 \times 2 = 910$. We thus obtain the answer 908.

Note. The numbers in this problem were carefully chosen to allow the above solution to work. In similar but more general situations, one needs to evoke the *Chinese Remainder Theorem*.