Algebraic Manipulations for AMC

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§1 Introduction

Algebra is a central subject on math contests, and with algebra comes algebraic manipulations. Many problems involve finding or minimizing values of certain expressions. Here are some of the most common types of problems that appear.

§2 Factorizations and algebraic identities

- Difference of squares: $a^2 - b^2 = (a-b)(a+b)$
- Difference of powers: $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1})$ (why?)
- Sum of odd powers: $a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + \ldots - ab^{n-2} + b^{n-1})$ (why?)
- Binomial expansions: $(a+b)^2 = a^2 + 2ab + b^2$ and $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
- $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab+bc+ca)$
- Sophie-Germain identity: $a^4 - 4b^4 = (a^2 - 2ab + 2b^2)(a^2 + 2ab + 2b^2)$ (why?)
- Simon's Favorite Factoring Trick: $ab + a + b = (a+1)(b+1) - 1$
- Symmetric sums: $(a-1)(b-1)(c-1) = abc - (ab+bc+ca) + (a+b+c) - 1$

Example 2.1
Solve in real numbers the equation $x^3 - 3x^2 + 3x + 3 = 0$.

Solution. Notice that $x^3 - 3x^2 + 3x + 3 = (x-1)^3 + 4$. Thus $(x-1)^3 + 4 = 0 \implies x = \boxed{1 - \sqrt[3]{4}}$. □

Example 2.2
Factor the expression $x^4 + x^2 + 1$ over the reals.

Solution. We try to complete the square, writing $x^4 + x^2 + 1 = (x^2+1)^2 - x^2$. Using difference of squares, we find $\boxed{(x^2 - x + 1)(x^2 + x + 1)}$. (Can you now prove Sophie-Germain?) □

Example 2.3

Prove that for all real numbers a, b, c

$$\left(\frac{2a+2b-c}{3}\right)^2 + \left(\frac{2b+2c-a}{3}\right)^2 + \left(\frac{2c+2a-b}{3}\right)^2 = a^2+b^2+c^2.$$

Solution. We expand the left hand side using the expansion of $(x+y+z)^2$ and collect terms. (Try to finish it.) $\square$

§3 Completing the square and quadratics

As we all know, the quadratic equation says that the solutions to $ax^2+bx+c=0$ are $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$. But where does this equation come from? To prove the quadratic equation, we use the technique of **completing the square**. We try to create the square of a binomial as follows:

$$ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) =$$

$$a\left(\left(x+\frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right) = a\left(x+\frac{b}{2a}\right)^2 - \frac{\Delta}{4a},$$

where $\Delta = b^2-4ac$ is the **discriminant** of the quadratic. Completing the square can sometimes be quicker than using the quadratic equation.

Another important property of quadratics concerns their sum and product of roots. **Vieta's Formulae** state that the roots r_1, r_2 of ax^2+bx+c satisfy $r_1+r_2 = -\frac{b}{a}$ and $r_1r_2 = \frac{c}{a}$. Whenever you see roots mentioned in a problem, consider using Vieta's Formulae.

The following examples illustrate these ideas in action.

Example 3.1

Solve the equation

$$\frac{(2x-1)^2}{2} + \frac{(3x-1)^2}{3} + \frac{(6x-1)^2}{6} = 1.$$

Solution. Expanding and collecting terms, we find $11x^2-6x=0$, which has solutions $x = \boxed{0, \frac{6}{11}}$. $\square$

Example 3.2

Simplify

$$\frac{1}{\sqrt{x+2\sqrt{x-1}}} + \frac{1}{\sqrt{x-2\sqrt{x-1}}}$$

for $x \in [1,2)$.

Solution. We complete the squares in the denominators as $x \pm 2\sqrt{x-1} = x - 1 \pm 2\sqrt{x-1} + 1 = (\sqrt{x-1} \pm 1)^2$. As $\sqrt{a^2} = |a|$ and $\sqrt{x-1} - 1 < 0$ for $x \in [1, 2)$, our expression equals

$$\frac{1}{1+\sqrt{x-1}} + \frac{1}{1-\sqrt{x-1}} = \boxed{\frac{2}{2-x}}.$$

□

Example 3.3

Solve in real numbers the equation

$$(x+1)(x+2)(x+3)(x+4) = 360.$$

Solution. We pair the factors as $(x+1)(x+4) = x^2+5x+4$ and $(x+2)(x+3) = x^2+5x+6$. We notice the common expression $x^2 + 5x$ and substitute $y = x^2 + 5x$. Our equation becomes $(y + 4)(y + 6) = 360 \implies y^2 + 10y - 336 = 0 \implies y = 14, -24$. These correspond to the equations $x^2 + 5x = 14$ and $x^2 + 5x = -24$. The former has real solutions $x = \boxed{-7, 2}$ while the latter has no real solutions. □

§4 Problem Set

Easy Problems

1. Simplify the expression $\frac{1}{ab} + \frac{1}{a^2-ab} + \frac{1}{b^2-ba}$.
2. If a is a real number such that $a - \frac{1}{a} = 1$, find $a^4 - \frac{1}{a^4}$.
3. Solve the equation
$$\frac{1}{3x-1} + \frac{1}{4x-1} + \frac{1}{7x-1} = 1.$$
4. Find all pairs (x, y) of real numbers such that
$$4x^2 + 9y^2 + 1 = 12(x + y - 1).$$
5. Using Example 2.2, prove the Sophie-Germain identity.

Medium Problems

1. Factor the expression $a^{1024} - b^{1024}$. Do you see a pattern?
2. Solve the system of equations $\begin{cases} x + \frac{1}{y} = -1 \\ y + \frac{1}{z} = \frac{1}{2} \\ z + \frac{1}{x} = 2. \end{cases}$
3. Solve the system of equations $\begin{cases} x - y = 3 \\ x^2 + (x+1)^2 = y^2 + (y+1)^2 + (y+2)^2. \end{cases}$

4. Find the greatest integer n for which the equation

$$\frac{1}{x-1} - \frac{1}{nx} + \frac{1}{x+1} = 0$$

has real solutions.

5. Positive real numbers a, b satisfy $\begin{cases} a + \frac{1}{b} = 7 \\ b + \frac{1}{a} = 5 \end{cases}$. Find $ab + \frac{1}{ab}$.

6. What is the least possible value of $(x+1)(x+2)(x+3)(x+4) + 2019$ where x is a real number?

Hard Problems

1. Solve in real numbers the system of equations $\begin{cases} x + y = 2z \\ x^3 + y^3 = 2z^3 \end{cases}$.

2. Solve in real numbers the system of equations $\begin{cases} x + y = xy - 5 \\ y + z = yz - 7 \\ z + x = zx - 11 \end{cases}$.

3. The equation $x^4 - 4x = 1$ as two real roots. Find their product.

4. Let x and y be real numbers between $-\frac{1}{2}$ and $\frac{1}{2}$ inclusive. What is the minimum possible value of

$$x^2 - 3x + y^2 - 3y + \frac{9}{2} + 2xy?$$

5. If a, b, c are real numbers such that

$$\begin{cases} a + b + c = 1 \\ a^2 + b^2 + c^2 = 17 \\ a^3 + b^3 + c^3 = 11, \end{cases}$$

find abc.

6. (a) Factor the expression $x^5 + x + 1$.

   (b) Find the prime factorization of 100011.

7. Let $f(x)$ and $g(x)$ be monic quadratic polynomials with positive real roots such that the roots of $g(x)$ are the reciprocals of the roots of $f(x)$. What is the minimum value of $f(-1) \cdot g(-1)$?

8. If $-3 \le x < \frac{3}{2}$ and $x \ne 1$, define $C(x) = \frac{x^3}{1-x}$. The real root of the cubic $2x^3 + 3x - 7$ is of the form $pC^{-1}(q)$, where p and q are rational numbers. What is the ordered pair (p, q)?