Advanced Algebraic Manipulations

with a Focus on Trigonometric Substitutions and Symmetric Polynomials

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Introduction

In algebra, a clever manipulation can severely simplify a problem. This lecture will cover two types of algebraic manipulations – symmetric polynomials manipulations and trigonometric manipulations – that will help you in the AMC series and beyond!

§1 Manipulations Regarding $x + y$ and xy

A common type of manipulation is where we express everything in terms of $x + y$ and xy. This type of manipulation works best on symmetric polynomials, polynomials in which the value remains unchanged when their variables are swapped, as symmetric polynomials can be written in terms of simpler symmetric polynomials.

For example,

$$x^4 + 3x^3y + 3xy^3 + y^4 = (x + y)^4 - xy(x + y)^2 - 4(xy)^2$$

The following examples illustrate this sort of manipulation in action!

Example 1.1
What are all ordered pairs of numbers (x, y) which satisfy $x^2 - xy + y^2 = 7$ and $x - xy + y = 1$?

Solution. Let $a = x + y$ and $b = xy$. We express both equations in terms of a and b. We get:

$$a^2 - 3b = 7$$
$$a - b = 1$$

The second equation yields $a = b + 1$. Substitute this in for a, we get $(b + 1)^2 - 3b = 7 \implies b^2 - b - 6 = 0 \implies (b - 3)(b + 2) = 0$. Ergo, $b = 3$ or $b = -2$. This yield solution pairs $(4, 3)$ and $(-1, -2)$. However, note that we are not done - we must find solution pairs (x, y)!!! Using Vieta's we create the quadratics:

$$x^2 - 4x + 3 = 0$$
$$x^2 + x - 2 = 0$$

This yields solutions $(3, 1)$, $(1, 3)$, $(1, -2)$, $(-2, 1)$. $\square$

Example 1.2

Find x, y, and z if $x+y+z=8$, $x^2+y^2+z^2=62$, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=90$, and $x \geq y \geq z$.

§2 Manipulations Regarding Trigonometry

Another common type of manipulation is where one substitutes trig functions for the variables in the expression. Signs to do so can be found if there are lots of squares, square roots, or trigonometric identities in terms of other variables.
Keep on the lookout for these identities:

1. $x^2+y^2=1 \implies$ substitute $x=\cos\theta$ and $y=\sin\theta$
2. $\frac{x+y}{1-xy} \implies$ substitute $x=\tan\alpha$ and $y=\tan\beta$ to obtain tan addition formula!
3. $1+x^2 \implies$ substitute $x=\tan\theta$ to get $\sec^2\theta$
4. $\sqrt{1-x^2} \implies$ substitute $x=\sin(\theta)$ to get $\cos(\theta)$

Let's look at some examples below.

Example 2.1

(Original Problem) What is the maximum value for $x\sqrt{25-4x^2}$?

Example 2.2

(Original Problem) Let x, y be real numbers. Given that $x^2+y^2=1$ what is maximum possible value of x^2-y^2+2xy.

Solution. Substitute $x=\cos\theta$ and $y=\sin\theta$; this is motivated by the fact that $\cos^2\theta+\sin^2\theta=1$. Thus, we are trying to maximize $\cos^2\theta-\sin^2\theta+2\cos\theta\sin\theta$. By double-angle identity, the quantity we are trying to maximize is $\cos(2\theta)+\sin(2\theta)$. Squaring this quantity yields $1+2\cos(2\theta)\sin(2\theta)=1+\sin(4\theta)$, which is maximized $\sin(4\theta)=1$. Ergo, $[\cos(2\theta)+\sin(2\theta)]^2$ has maximum 2, so $\cos(2\theta)+\sin(2\theta)$ has maximum $\boxed{\sqrt{2}}$. □

Example 2.3

(Modified HMMT Feburary 2008 General Round 2 #3) Suppose that a, b , c, d are real numbers satisfying $a \geq b \geq c \geq d \geq 0$, $a^2=(1-d)(1+d)$, $b^2=(1-c)(1+c)$, and $ac+bd=\frac{1}{3}$. Find the value of $ab-cd$.

Substitute $a=\cos(\alpha)$, $b=\cos(\beta)$, $c=\sin(\beta)$ and $d=\sin(\alpha)$.

Example 2.4

(Original Problem) There are 6 real numbers such that none of them are the negative reciprocal of the other, what is the largest possible non-zero minimum of the set $|\frac{x-y}{1+xy}|$ where x, y permutes every combination of the 6 numbers.

Substitute $x = \tan(\theta)$ and $y = \tan(\phi)$

Example 2.5

(PUMaC 2018 A Division Algebra #6) Let a, b, c be non-zero real numbers that satisfy $\frac{1}{abc} + \frac{1}{a} + \frac{1}{c} = \frac{1}{b}$. The expression $\frac{4}{1+a^2} + \frac{4}{1+c^2} + \frac{7}{1+b^2}$ has a maximum value M. Find the sum of the numerator and denominator of the reduced form of M.

Substitute $a = \tan(\alpha)$, $b = \tan(\beta)$, and $c = \tan(\gamma)$.

§3 Problem Set

Easy Problems

1. (Original Problem) What is xyz if $x^2+y^2+z^2 = 10$, $x+y+z = 4$, and $x^3+y^3+z^3 = 34$?

2. (HMMT 2013 Algebra #1) Let x and y be real numbers with $x > y$ such that $x^2y^2 + x^2 + y^2 + 2xy = 40$ and $xy + x + y = 8$. Find the value of x.

3. (HMMT 2019 Algebra #3) Let x and y be positive real numbers. Define $a = 1 + x$ and $b = 1 + y$. If $a^2 + b^2 = 15$, compute $a^3 + b^3$.

4. (Math League HS 1990-1991) What is the ordered pair of numbers (x, y), with $x > y$, for hich $x^2 + xy + y^2 = 84$ and $x + \sqrt{xy} + y = 14$?

5. (Advanced Manipulations Alcumus) If $abc = 13$ and
$$\left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{a}\right) = \left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right),$$
find $a + b + c$.

Intermediate Problems

1. (HMMT 2004 Algebra #8) Let x be a real number such that $x^3+4x = 8$. Determine the value of $x^7 + 64x^2$.

2. (HMMT 2015 Algebra #5) Let a, b, c be positive real numbers such that $a+b+c = 10$ and $ab + bc + ca = 25$. Let $m = \min(ab, bc, ca)$. Find the largest possible value of m.

3. (Advanced Manipulations Alcumus) Let a, b, c be nonzero real numbers such that $a + b + c = 2$ and $a^2 + b^2 + c^2 = 4$. Find
$$\frac{ab}{c} + \frac{ac}{b} + \frac{bc}{a}.$$

4. (Advanced Manipulations Alcumus) Let x, y, and z be distinct, nonzero real numbers such that $x + y + z = 0$. Simplify
$$\left(\frac{x-y}{z} + \frac{y-z}{x} + \frac{z-x}{y}\right)\left(\frac{z}{x-y} + \frac{x}{y-z} + \frac{y}{z-x}\right).$$

5. (2019 AIME I #8) Let x be a real number such that $\sin^{10} x + \cos^{10} x = \frac{11}{36}$. Then $\sin^{12} x + \cos^{12} x = \frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Hard Problems

1. (AIME 1990 #15) Find $ax^5 + by^5$ if the real numbers a, b, x, and y satisfy the equations

$$ax + by = 3,$$
$$ax^2 + by^2 = 7,$$
$$ax^3 + by^3 = 16,$$
$$ax^4 + by^4 = 42.$$

2. (2012 Math Prize For Girls) Evaluate the expression

$$\frac{121\left(\frac{1}{13} - \frac{1}{17}\right) + 169\left(\frac{1}{17} - \frac{1}{11}\right) + 289\left(\frac{1}{11} - \frac{1}{13}\right)}{11\left(\frac{1}{13} - \frac{1}{17}\right) + 13\left(\frac{1}{17} - \frac{1}{11}\right) + 17\left(\frac{1}{11} - \frac{1}{13}\right)}.$$

3. (2013 Math Prize For Girls) If $-3 \le x < \frac{3}{2}$ and $x \ne 1$, define $C(x) = \frac{x^3}{1-x}$. The real root of the cubic $2x^3 + 3x - 7$ is of the form $pC^{-1}(q)$, where p and q are rational numbers. What is the ordered pair (p, q)?

4. (HMMT 2017 November General Round #9) Find the minimum possible value of $\sqrt{58 - 42x} + \sqrt{149 - 140\sqrt{1 - x^2}}$ where $-1 \le x \le 1$

5. (HMMT 2015 Algebra #10) Find all ordered 4-tuples of integers (a, b, c, d) (not necessarily distinct) satisfying the following system of equations:

$$\begin{aligned} a^2 - b^2 - c^2 - d^2 &= c - b - 2 \\ 2ab &= a - d - 32 \\ 2ac &= 28 - a - d \\ 2ad &= b + c + 31. \end{aligned}$$

6. (HMMT 2013 Algebra #8) Let x, y be complex numbers such that $\frac{x^2+y^2}{x+y} = 4$ and $\frac{x^4+y^4}{x^3+y^3} = 2$. Find all possible values of $x^6 + y^6$.

Please use this link for solutions to the problems: http://bit.ly/SimonSolutions. For HMMT/PUMaC/MPFG problems, you can simply go to http://bit.ly/ProblemLevelP#. The difficulty for the sections are Easy, Interm and Hard. For example, if I wanted to find the solution to P1 of the Intermediate section I would go to http://bit.ly/IntermP1. Exceptions:

- Problems from other contests, in this case please use the SimonSolutions bit.ly link.
- Hard #3 is http://bit.ly/HardPro3
- Hard #5 is http://bit.ly/IntermP2
- Hard #6 is http://bit.ly/EasyP2