Interest Rate Markets &
The Forward Rate - Discount Factor Relationship Explained

Saïd Business School, Oxford University

Nicholas Burgess

nicholas.burgess@sbs.ox.ac.uk

nburgessx@gmail.com

October 2021

Abstract

In this paper we illustrate and explain the relationship between interest rates (or forward rates) and discount factors. We present the forward-discount factor relationship, which is popular and widely used in financial markets for yield curve construction, and derive the exact formulae using a replication argument approach. Furthermore we highlight the implicit assumption of simple compounded rates and consider the disadvantages, side-effects and potential hazards of this assumption. Finally to conclude we discuss how to relax this assumption and alternative approaches.

Forward Rates

A forward rate is the interest rate applied for borrowing funds in the future for a fixed period, three months say. It is a variable rate of interest that changes with market conditions. Forward rates are typically determined two days before the start of a loan period at which point they become known fixed rates and are no longer variable. Forward rates are required to price mortgages, corporate loans and a wide variety of financial instruments. In the London interbank markets LIBOR[1] is a popular forward rate used to reference over $350 trillion of transactions worldwide[2].

---

[1] London Interbank Offered Rate

[2] Burgess, Nicholas, Libor Benchmark Reform: An Overview of Libor Changes and Its Impact on Yield Curves, Pricing and Risk (September 6, 2019). Available at SSRN: https://ssrn.com/abstract=3479833 or http://dx.doi.org/10.2139/ssrn.3479833

Forward Rate Dynamics

In interest rate markets forward rates are instantaneous. At every point in the future we have a distinct forward rate representing an interest rate for borrowing funds on that fixing date; together the collection of forward rates forms a forward curve as illustrated in (figure 1).

Consider the USD 3M LIBOR curve below, at every point on the forward curve we have a distinct USD 3M LIBOR forward rate for borrowing USD funds for three months.

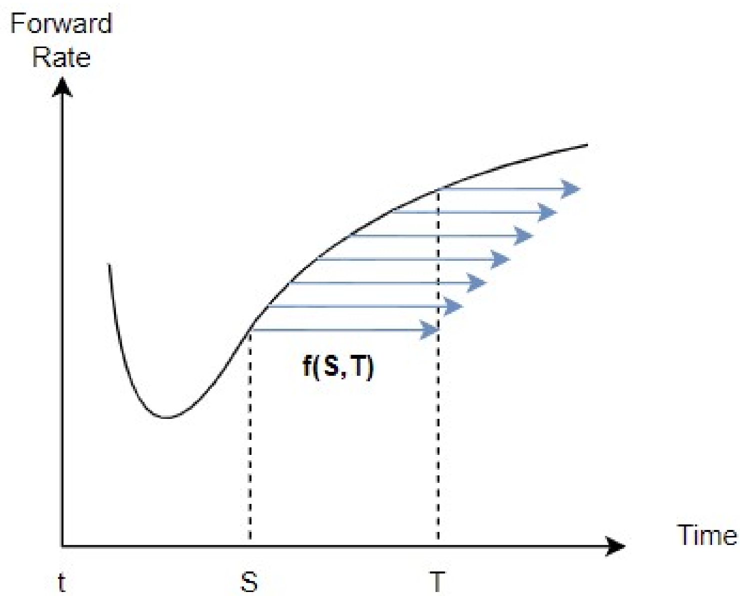

Figure 1: USD 3M LIBOR Forward Curve Illustration

The forward rate f(S,T) from time S to T is in this case a distance of 3 months apart. The blue arrows indicate other instantaneous USD 3M LIBOR forward rates all starting on the forward curve and ending 3 months later. The area under the forward curve corresponds to the discount factor, where for example P(S,T) is the discount factor from S to T and corresponds to the area under the curve from time S to T.

Forward – Discount Factor Relationship

Forward rates of interest and discount factors are related via the following formula, which is derived from a no arbitrage replication strategy and assumes we are applying simple compound interest,

$$f(S,T) = \left(\frac{P(t,S)}{P(t,T)} - 1\right) / (T - S) \tag{1}$$

The forward rate $f(S,T)$ which is the floating interest rate as at time t for borrowing funds from time S until time T with t < S < T.

Replication Argument

To give some intuition as to where (equation 1) comes from, consider the following replication argument. If we deposit funds today at time t making a long deposit until time T then the interest received on this deposit should be the same as when making a short deposit to time S and rolling this deposit forward to time T as shown in (figure 2).

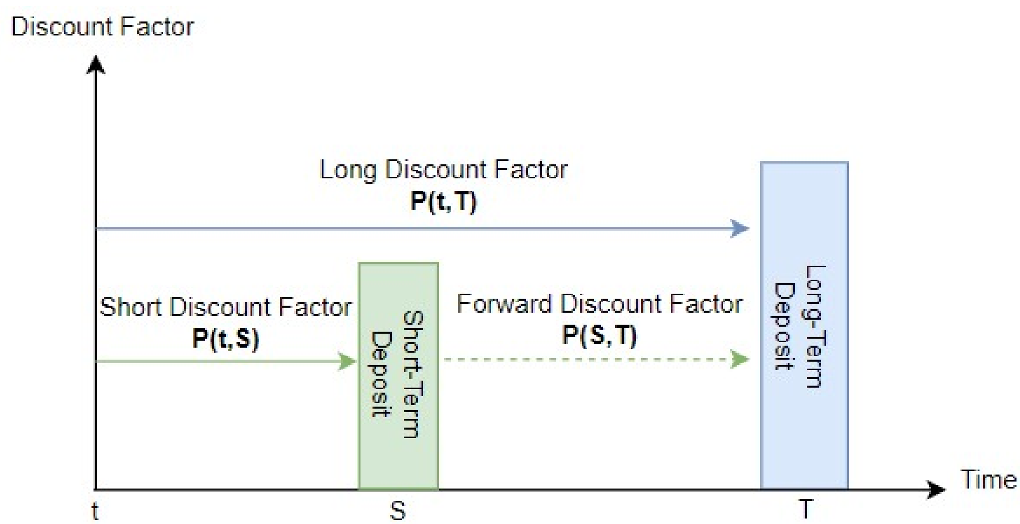

Figure 2: Forward Rate Replication Argument

The replication argument is equivalent to stating that the long discount factor is equal to the short discount factor multiplied by the forward discount factor. Mathematically we can represent this as follows,

$$P(t,S).P(S,T) = P(t,T) \tag{2}$$

Rearranging this for the forward discount factor we have,

$$P(S,T) \stackrel{\text{def}}{=} \frac{P(t,T)}{P(t,S)} \tag{3}$$

Simple Compounding

If we assume a constant interest rate r over the period $[S,T]$ with simple compounding then forward discount factors can be represented as,

$$P(S,T) \stackrel{\text{def}}{=} \frac{1}{(1+r(T-S))} \tag{4}$$

Relabelling r as $f(S,T)$ to indicate it is a forward rate and substituting (equation 4) into (equation 3) leads to,

$$\frac{1}{(1+f(S,T)(T-S))} = \frac{P(t,T)}{P(t,S)} \tag{5}$$

Rearranging gives,

$$\big(1+f(S,T)(T-S)\big) = \frac{P(t,S)}{P(t,T)} \tag{6}$$

Giving the forward rate result assuming interest rates with simple compounding,

$$f(S,T) = \left(\frac{P(t,S)}{P(t,T)} - 1\right) / (T-S) \tag{7}$$

Ghost Features

A disadvantage of this approach is that we require two discount factors for each forward rate with the forward rate dependent on the discount factor on the forward rate's fixing start and end date e.g. for USD 3M LIBOR the fixing date and the fixing date + 3M.

Modelling forward rates under this assumption, forward rates are dependent on the discount factor on the fixing start date which is natural, as this is the market fixing date and also on the fixing end date which is un-natural. Ideally interest rates should not be dependent on discount factors or any market data after their fixing date.

When forward rates depend on pairs of discount factors forward rate volatility and market jumps in value results in pairs of jumps in discount factors, namely a jump on the forward fixing date and an artificial ghost jump on the forward end date. This can cause a ripple effect on neighbouring forward rates in the forward curve that are dependent on discount factors at the ghost point. Consequently this approach is problematic for curve construction, risk-management and can give rise to unrealistic / unstable forward curves.

Furthermore one must note that this assumption acts as an approximation and introduces errors when the underlying yield curve from which we imply forwards has not been calibrated assuming simple compounding of forward rates.

Continuous Compounding

We can relax the assumption that interest rates compound with simple interest by modifying our discount expression in (equation 4). If instead we assume a constant interest rate r over the period $[S, T]$ that is continuously compounded we then define the discount factor as,

$$P(S,T) \stackrel{\text{def}}{=} \exp(-r(T-S)) \tag{8}$$

Once again relabelling r as $f(S,T)$ to indicate it is a forward rate and substituting (equation 8) into (equation 3) leads to,

$$\exp\big(-f(S,T)(T-S)\big) = \frac{P(t,T)}{P(t,S)} \tag{9}$$

Rearranging gives,

$$f(S,T)(T-S) = -ln\left(\frac{P(t,T)}{P(t,S)}\right) = ln\left(\frac{P(t,S)}{P(t,T)}\right) \tag{10}$$

Giving the forward rate result assuming interest rates with continuous compounding,

$$\boxed{f(S,T) = ln\left(\frac{P(t,S)}{P(t,T)}\right)/(T-S) \tag{11}}$$

Instantaneous Forward Rates

Whilst the interest rate for a given forward interval is indeed a single rate that is reset or fixed on the given fixing date as shown in (figure 1); modelling interest rates to be piecewise constant over forward rate intervals restricts the choice of yield curve calibration instruments and can introduce curve inconsistencies where intervals overlap. Ideally we want to discount factors to use instantaneous discount factors and incorporate the full term-structure of forward rates.

To achieve this we can relax assumptions further to incorporate the term-structure of forward rates by further modifying our discount expression in (equation 8) and define the discount factor as follows,

$$P(S,T) \stackrel{\text{def}}{=} \exp\left(-\int_S^T f(t,u)du\right) \tag{12}$$

Once again relabelling r as $f(S,T)$ to indicate it is a forward rate and substituting (equation 12) into the forward rate replication formula from (equation 3) leads to,

$$\exp\left(-\int_S^T f(t,u)du\right) = \frac{P(t,T)}{P(t,S)} \tag{13}$$

Taking the natural log of both the LHS and RHS and simple rearrangement gives,

$$\int_S^T f(t,u)du = -ln\left(\frac{P(t,T)}{P(t,S)}\right) = ln\left(\frac{P(t,S)}{P(t,T)}\right) \tag{14}$$

Differentiating with respect to T,

$$f(t,T) = \frac{\partial}{\partial T}\left(ln\left(\frac{P(t,S)}{P(t,T)}\right)\right) \tag{15}$$

Noting that t is a free variable and setting $t = S$ gives,

$$f(S,T) = \frac{\partial}{\partial T}\left(ln\left(\frac{P(S,S)}{P(S,T)}\right)\right) = \frac{\partial}{\partial T}\left(ln\left(\frac{1}{P(S,T)}\right)\right) \tag{16}$$

Using the definition from (equation 3) we have,

$$\boxed{f(S,T) = \frac{\partial}{\partial T}\left(ln\left(\frac{P(t,S)}{P(t,T)}\right)\right)} \tag{17}$$

This result from (equation 17) requires the forward curve interpolation function to be continuous and sufficiently smooth. This is often not the case for forwards derived from daily or overnight composite indices (OIS indices) such as US Fed Funds and US SOFR, where forward market data is very spiky.

To avoid model complexity some market participants prefer to use the analytically tractable solutions offered by (equation 7), (equation 11) however this might not be the most suitable choice if one is to achieve the best forward curve fit or to model the advanced features outlined in (Burgess 2021)[3].

[3] Burgess, N. (2021) - Advanced Yield Curve Calibration, Mixed Interpolation Schemes & How to Incorporate Jumps and the Turn-of-Year Effect. Available at: https://ssrn.com/abstract=3898069