Libor Benchmark Reform: An Overview of Libor Changes and its Impact on Yield Curves, Pricing and Risk

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Abstract

Libor is arguably the world's most important number, with more than USD 350 trillion of derivatives, loans, securities and mortgages referencing this rate of which USD 200 trillion originates from US markets. The Libor benchmark rate is being replaced with alternative reference rates (ARRs) and there is no guarantee the rate will continue to be quoted beyond 2021.

In this paper Libor benchmark rate reform is discussed in detail and we assess the impact this has on yield curve construction, interest rate pricing and risk. We highlight why Libor is important, review its history and how it has evolved, which leads to a discussion as to what is wrong with Libor benchmarks. We outline market terminology with regards to both interest rates and yield curve construction, before proceeding to assess on the impact of Libor reform, reviewing the new benchmarks, fall-back rates and yield curve changes.

We review yield curve calibration and in doing so provide many charts and Excel workbook illustrations to demonstrate new features of ARR yield curves. We explain how to both bootstrap and globally calibrate curves to imply forward rates & discount factors. Moreover, we outline the interpolation, optimization and solving process, showing how to calibrate curves in such a way to capture the necessary risk metrics required to compute analytical risk and rebuild curves for ultra-fast performance.

It is hoped this paper will serve as a useful Libor benchmark rate reform and yield curve primer.

Keywords:
Libor, Benchmark Reform, IBOR, Alternative Reference Rate, Risk-Free Rate, SOFR, ESTR, Secured, Unsecured, Yield Curves, Calibration, Risk

Contents

1 Introduction

Interest rate benchmarks play an important role in global financial markets. These interest rate benchmarks are indexed by trillions of dollars of financial products worldwide, ranging from derivatives to residential mortgages. However excessive leverage and low liquidity in unsecured interbank funding markets, coupled with several high-profile rate manipulation scandals, have undermined confidence in the reliability and robustness of these interest rate benchmarks.

Consequently central banks and regulators alike are reforming the interest rates market and encouraging the migration to new alternative benchmark rates, which are compliant with newly proposed standards for financial benchmark rates by the International Organization of Securities Commission (IOSCO standards) [28]. The Financial Conduct Authority (FCA), the UK regulator overseeing the Libor benchmark has already indicated that it will not compel banks continue to publish Libor rates beyond 2021, [21], [22].

Libor benchmark rates are the international standard for interest rate markets and have been dubbed the world's most important number [32] and the risks facing households and the broad investment community worldwide are unprecedented. The financial risks associated with migrating to new benchmark, without a monetary transfer requirement, without changing the value of mortgages and financial contracts is considerable. Moreover, pricing models, financial systems and market infrastructure has been built around Libor benchmarks for several decades, which has made the size, scale, scope and migrating to new benchmark rates arguably one of the biggest challenges facing the financial industry.

It is hoped this paper will serve as both a Libor, benchmark rate reform and yield curve primer and address the model changes required when constructing yield curves needed to price and risk financial instruments.

We proceed as follows, firstly we give an overview of Libor rates, why they are important and why Libor has become a problem. Secondly we summarize market terminology used when discussing benchmark rate reform and yield curve modelling and construction. Thirdly we proceed to discuss the impact of Libor changes and fall-back rates, the contingency rate to be used in the event of Libor rates being unavailable. Fourthly we give an overview interest rate yield curve calibration and review the work required to remodel curves to work with alternative reference rates.

We provide a rich overview of curve calibration techniques, including bootstrapping, global curve calibration, interpolation and solving techniques. Moreover we also review new calibration instruments, curve term-structure dynamics and risk. Finally we supplement the above with many charts, tables, data and Excel curve construction workbooks to help the reader both visualize and experiment with the concepts discussed.

1.1 What is Libor?

In interest rate markets the London Interbank Offered Rate (Libor) is the underlying benchmark and reference rate used for pricing more than USD 370 trillion of rates transactions globally, ranging from mortgages, loans and derivatives. Libor interest rates measure the rate of interest major international panel banks in London were willing to pay to raise finance, on an unsecured[1] basis.

Originally designed to price floating rate notes in the 1960's Libor rates came into existence when Greek banker Minos Zombanakis syndicated loans bearing a fluctuating interest based on the six-month interbank rate in London, see [29], [32]. The rate became an increasingly popular transaction reference rate and was eventually adopted by the British Banker's Association as an official rate in 1986. Over time they grew to became an increasingly popular reference rate for many financial contracts, spanning 5 currencies namely USD, GBP, EUR, JPY and CHF for various loan maturities.

Libor rates have an international counterpart termed (IBOR), a term used indicate interbank lending rates set in major financial centres other than London e.g. EURIBOR (Europe/Brussels), HIBOR (Hong Kong), TIBOR (Tokyo) et al.

1.2 Why is Libor Important?

Libor rates are linked to transactions with notional amounts in the trillions of dollars making it arguably the world's most important number [32].

In the 2000s derivative products referencing Libor grew ten-fold from USD 10 trillion to more than USD 100 trillion of Notional outstanding in the USD markets alone, see [29].

The Association for Financial Markets in Europe (AFME) in [1] report market exposure based on total notional amount outstanding for IBOR products is greater than USD 370 trillion with USD Libor $150 tn, EURIBOR $150 tn, GBP Libor $30 tn and JPY Libor $30 tn as shown in figure (1).

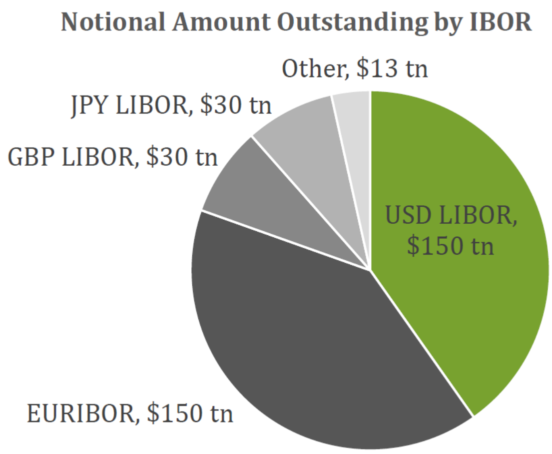

Figure 1: Source: AFME, IBOR Notional Outstanding

[1] or uncollateralised

1.3 Libor History & Evolution

The London Interbank Offered Rate (Libor) came into widespread use in the 1970s as a reference interest rate for transactions in offshore Eurodollar markets. In 1984, it became apparent that an increasing number of banks were trading actively in a variety of relatively new market instruments, notably forward rate agreements, interest rate swaps and foreign currency options. Over time as such instruments brought more business and greater depth to the London inter-bank market, bankers worried that future growth could be inhibited unless a measure of uniformity was introduced.

In October 1984, the British Bankers' Association (BBA) established the BBA standard for interest rate swaps or 'BBAIRS terms', which included the fixing of BBA interest settlement rates. In September 1985 the BBAIRS terms became standard market practice, which led to the introduction of official BBA Libor fixing rates. BBA Libor administration later transferred to the Intercontinental Exchange (ICE).

Libor is an interest rate average calculated from estimates submitted by the leading banks in London. Each bank estimates what it would be charged were it to borrow from other banks. Libor is the rate at which an individual contributor panel bank could borrow funds in the London inter-bank market at 11.00 GMT. We list the member panel banks contributing to the fixing of USD Libor below.

USD Libor Contributor Panel Banks

Bank of America	Bank of Tokyo-Mitsubishi UFJ	Barclays Bank
Citibank NA	Credit Agricole CIB	Credit Suisse
Deutsche Bank	HSBC	JP Morgan Chase
Lloyds Banking Group	Rabobank	Royal Bank of Canada
Société Générale	Sumitomo Mitsui Banking Corp.	Norinchukin Bank
Royal Bank of Scotland	UBS AG	

The London Inter-bank Offered Rate is the primary benchmark for short-term interest rates around the world. Libor helps set interest rates worldwide and affects the price of more than $300 trillion in mortgages, loans and derivatives.

Libor has been embedded in financial models and system infrastructure for more than 3 decades. Moreover, the size, scale and scope of usage make shifting from Libor to alternative reference rates one of the largest undertakings facing the financial industry.

1.4 What is Wrong with Libor?

The financial credit crisis of 2008 and the subsequent collapse of Lehman Brothers fundamentally changed the interbank market and how banks fund themselves. Banks and regulators alike recognized the risks inherent in the underlying unsecured borrowing and lending transactions. Post-crisis regulatory requirements on bank capital mandated that banks hold larger buffers of cash, debt and other securities to meet regulator stress tests and cushion losses. Banks became increasingly reluctant to lend to each another due to both the elevated credit risk and stricter regulator capital requirements. Consequently the interbank financing transaction volumes supporting Libor rates significantly diminished as banks altered how they fund themselves.

This was further exasperated by several high profile Libor rigging and rate manipulation scandals resulting in panel banks consequently being fined billions of USD by regulators, see [32]. From a systemic perspective IOSCO [28] argue that having such a large stock of contracts settle on a rate based on a relatively small market creates undesirable incentive problems, and adds to the risk of rate manipulation.

The low transaction volume has lead to Libor benchmark rate becoming increasingly dependent on expert panel bank judgment rather than actual transaction levels. It is a cause for concern as the rate becomes less measurable, transparent and robust, as illustrated in figure (2).

Specifically in European markets, as noted by [8], we observe low EONIA transaction volume of EUR 5 million notional supporting the EONIA rate with derived transactions referencing EONIA EUR 135 trillion (approx USD 150 trillion). As illustrated in figure (3) such low volume creates systemic problems in the market, whereby for example we observe large spikes in EONIA rates around regional holidays.

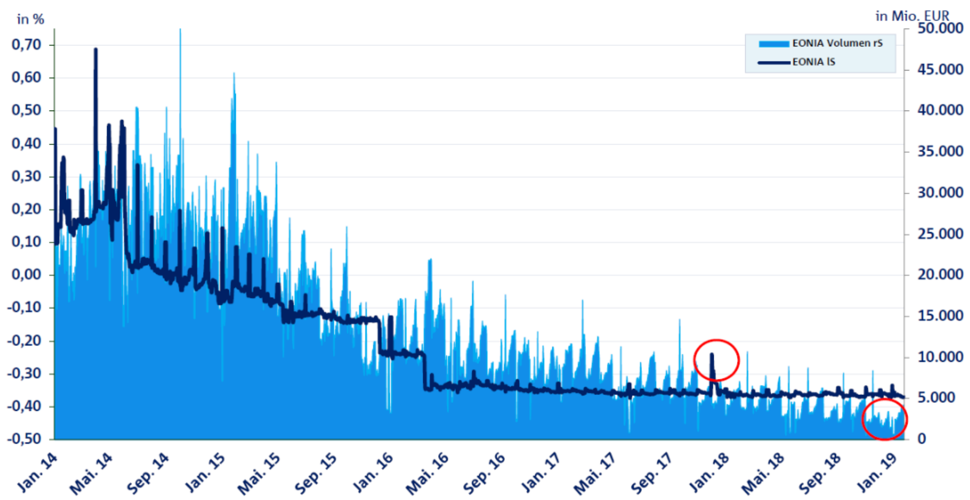

Figure 3: Source: Bayerische Landesbank, EONIA Rate & Volume

In USD interest rate markets we see that the Effective Fed Funds Rate (EFFR) trades at a similar level to SOFR rates as outlined in [17] and below in figure (4),

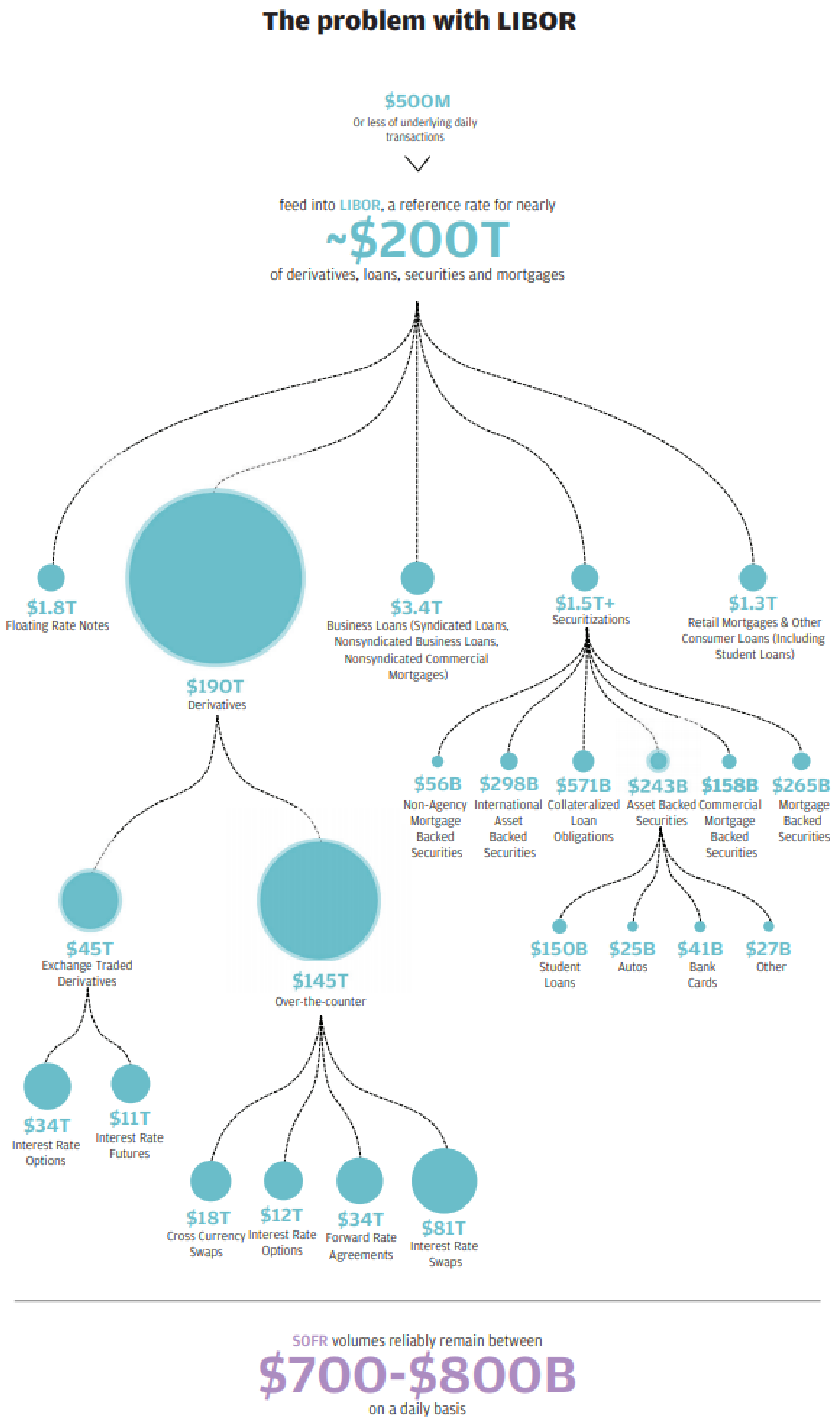

Figure 2: Source: J.P. Morgan, Transaction Volume: Libor vs contracts referencing Libor

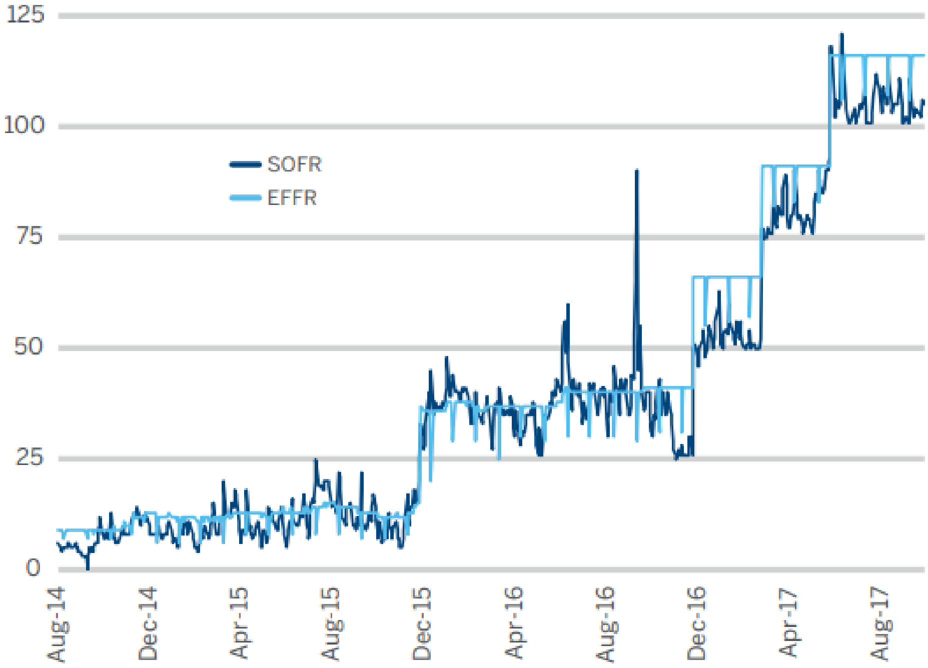

Figure 4: Source: Federal Reserve Bank New York - Daily EFFR and SOFR Values

However in terms of transaction volumes [29] confirm both Fed Fund and Libor volumes trade rather thinly, making these reference rates less robust relative to Libor replacement rates such as USD SOFR rates, as shown in figure (5).

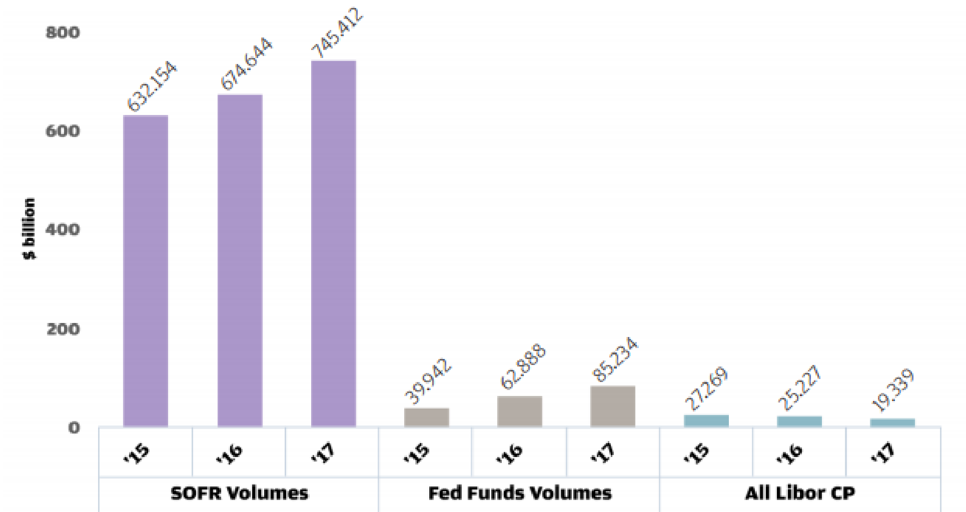

Figure 5: Source: J.P. Morgan, SOFR Volume vs Fed Funds

Libor has become increasingly based on the expert judgment of panel banks due to the declining amount of unsecured, wholesale borrowings by banks since the financial credit crisis of 2008. Consequently in 2013 the G20 nations tasked the Financial Stability Board (FSB) with the duty of reviewing and reforming major reference rate benchmarks. In July 2013, the International Organization of Securities Commissions (IOSCO) published a robust framework of principles for underlying benchmark rates in financial markets [28].

IOSCO Principles for Financial Benchmarks

The International Organization of Securities Commissions (IOSCO) recommend the following principles to promote the reliability of financial benchmarks [28].

- **Governance**, Administrators must have the appropriate government arrangements in place to protect the integrity of the benchmark determination process and address conflicts of interest.
- **Quality**, The design and data used to construct the benchmark rate should be such that the benchmark provides an accurate, reliable and transaction based representation of the underlying market.
- **Methodology**, The calculation methodology must be transparent with clear published controls in place to manage when material changes are planned. Furthermore administrators must establish credible contingency policies in case the benchmark becomes unavailable, ceases to exist or stakeholders need to transition to another benchmark.
- **Accountability**, The administrator must establish complaints processes, documentation standards and audit reviews and provide evidence of compliance to its quality standards.

This was followed by FSB proposals to adopt such standards as the starting point for robust reference rates [25]. Consequently the FCA has decided to no longer make the publication of Libor rates mandatory beyond 2021 and market participants are strongly encouraged to transition to other risk-free rates.

Libor Benchmark Rate Non-BMR Compliance

In summary Libor rates are generally non-BMR compliant with the following key flaws and issues.

- IBOR rates not representative of market levels
- Low volume and liquidity
- Geographical and sector bias
- Panel bank bias
- Non transparent and reliant on expert judgment

2 Market Terminology

In this section we outline key terms, phrases and acronyms to demystify some of the common market terminology used when referring to benchmark rate reform, interest rate markets and yield curve construction.

Alternative Reference Rates, ARRs
The alternative reference rate is the general phrase used to describe the Libor benchmark replacement rate.

Benchmark Rate, BMR
Another reference for the Alternative Reference Rate (ARR), used predominantly in the context of an interest rate being compliant and conforming to IOSCO standards.

Benchmark Rate Compliance (BMR Compliance)
The Financial Stability Board (FSB) was tasked by G20 nations with reforming interest rate benchmark (BMRs). The FSB has endorsed the use of IOSCO standards to evaluate benchmark rate robustness. Benchmarks that conform to such standards are deemed to be fit for purpose and categorized as benchmark rate compliant (BMR compliant).

Bootstrapping Curves
Curve bootstrapping is the process of implying forecast rates and discount factors needed to price financial instruments in a sequential manner. Specifically we infer forward rates and discount factors from liquid market instruments starting with the shortest tenor and progressively iterate sequentially over longer tenor instruments. This approach is considered inferior to Global Curve Calibration (see below), whereby we calibrate all instruments simultaneously, which has the benefit or incorporating all instrument cross effects into the implied Libor forecast rates and discount factors.

European Central Bank, ECB
The European Central Bank administers monetary policy for the Eurozone and Euro currency. They are also the administrators for the ARR European Short Term Rate, ESTR.

European Short Term Rate, ESTR
The EUR overnight European Short Term Rate. This is the alternative benchmark rate in Europe set to work side-by-side and potentially replace EURIBOR and EONIA. For an overview of the ESTR rate see [18].

Fall-back Rate, FR
This is the replacement rate to be used in the event that Libor rates are not published or cease to exist.

Federal Reserve Bank of New York, FED
The Federal Reserve Bank of New York is the principal bank within the network of 12 federal reserve banks of the United States. It is the cornerstone of the Federal Reserve System responsible for the setting of monetary policy and the administration of the SOFR benchmark rate.

Fixing in Advance/Arrears
Borrowing and Lending rates are typically set and fixed in advance of a loan being made. Fixing

in advance has been the norm for Libor benchmark rates, which is convenient since borrowing costs can be determined in advance. However, ARRs and Libor replacement benchmarks are to be transaction based and consequently will be fixed in arrears, meaning that we do not know our borrowing costs until the end of the loan period. This has led to much debate on the creation of term rates to determine borrowing costs in advance.

Global Curve Calibration
Global Curve Calibration is a technical process of implying forecast rates and discount factors needed to price financial instruments from liquid market instrument quotes. Specifically Global Calibration refers to the process of solving for all forward rates and discount factors simultaneously, which has the advantage of better incorporating instrument correlation and cross effects.

IBOR Rates
The Interbank Offered Rate. The rate at which a banks is willing to borrow and lend from each other in the interbank market. Whilst Libor rates are specifically set in London for 5 currencies namely USD, EUR, GBP, CHF and JPY, Ibor rates are the partner interbank benchmark rates published outside of the London financial centre, examples include Hong-Kong Hibor, Tokyo Tibor, Brussels Euro Euribor et al.

IOSCO Standards
The International Organization of Securities Commission (IOSCO) proposed standards for benchmark rates to be considered robust and fit for purpose, see IOSCO Principles for Financial Benchmarks [28]. Benchmark rates that comply with such principles are considered benchmark rate compliant (BMR Compliant).

ISDA Master Agreement
The ISDA Master Agreement, published by the International Swaps and Derivatives Association, is the most commonly used master service agreement for OTC derivative transactions internationally. It is part of a framework of documents, designed to enable OTC derivatives to be documented fully and flexibly.

Libor Rates
The London Interbank Offered Rate is the average interest rate which major global banks (panel banks) borrow from one another. There are 35 different rates spanning 5 currencies, namely USD, EUR, GBP, JPY, CHF, and various maturities typically Overnight O/N, SpotNext S/N, 1W, 1M, 3M, 6M and 12M. Libor rates are set or 'fixed' in-advance and implicitly incorporate panel bank credit risk into it's rate. The underlying transaction volume and liquidity supporting Libor rates has been diminishing. The rate has become increasingly less transparent and reliant on panel bank expert judgment to set and publish the official rate.

Risk-Free Rates, RFRs
Another name for the Alternative Reference Rate (ARR), this term is used to reflect that the alternative reference rate does not incorporate any interbank credit risk and is considered risk-free from this perspective. Furthermore much like OIS rates, IBOR replacement rates are daily rates and often secured potentially making close to risk-free from a credit risk perspective.

Secured Overnight Funding Rate, SOFR
The USD overnight Secured Overnight Financing Rate is the alternative reference rate set to replace USD Libor. Administered by the New York Federal Reserve Bank it broadly measures

the cost of borrowing cash overnight using U.S. Treasuries as collateral in the repo market. For more information on SOFR see [17] and [23].

Secured vs Unsecured
This is a reference as to whether collateral must be posted for a particular financial transaction to protect against potential counterparty credit defaults. Variable or floating interest rate transactions linked to benchmark rates are classified as secured if collateral posting is required and unsecured otherwise.

Single Curve Calibration
The process of building a yield curve for each curve index and currency separately in a particular order, for example firstly USDOIS, secondly USD3ML and finally USD6ML. The problem with this approach is that curve inter-dependencies are overlooked, which can lead to pricing and modelling errors when using basis instruments to calibrate outright curves before such basis curves have been constructed.

Spread Curves
A spread curve is a yield curve that trades as a spread to another. Tenor Basis curves are one example of a spread curve, whereby for example in the USD 3x6 Basis Curve USD3ML interest rates trade as a spread to USD6ML rates. Another more prominent example of this would be the ESTR risk-free curve, whereby ESTR interest rates (the alternative Eurozone benchmark) trades as a spread to EONIA overnight rates. Note that market quoted basis spreads are typically quoted as a positive number and therefore applied to the lower of the two rates being contrasted.

Sterling Overnight Index Average, SONIA
Sterling Overnight Index Average is the effective overnight interest rate paid by banks for unsecured transactions in the United Kingdom.

Term Rates
Benchmark rates that known in advance and lock-in the actual funding cost for a specified time horizon are referred to as Term rates. The majority of benchmark rates are backward locking in that they are based on actual transactions, typically from the previous working day. Currently the majority of alternative reference rates would determine daily rates in arrears and average or compound the daily rate over the specified time horizon to arrive at a effective term rate for the period. This is undesirable for market participants who want to know their marginal funding costs prior to entering a transaction.

Turn-of-Year Effect, TOY
The turn-of-year effect is a trading phenomena whereby rates spike on year-end as many funds and market participants all roll their interest rate transactions at the same time causing the rate to jump, sometimes drastically, creating a rate spike. There is also a similar quarter-end and month-end turn effect around central bank policy meeting dates that is less pronounced.

3 The Impact of Libor Changes

Traders and financial practitioners alike question the impact of Libor changes and don't necessarily understand the impact of the upcoming reform changes, specifically with regards to the impact on yield curve construction. ARR swap characteristics for example are not particularly well understood and more so the differences between OIS and ARR swaps. ARR instrument liquidity is still rather small relative to Libor and naturally makes understanding new market conventions and instruments somewhat opaque. There is also market reluctance to be the first to migrate to ARR rate based transactions, whilst there are opportunities there are also significant risks with limited liquidity and risk management mechanisms.

Many of the IT systems and analytics in place today within banks and financial service companies have been configured and specifically designed with Libor and OIS rates in mind. Numerical risk is a classical example of this. Consequently re-working and extending legacy systems is quite an undertaking and often they do not have the flexibility to evolve with the pending IBOR changes. In some cases such IT infrastructure is incompatible and cannot be used to evaluate products referencing the new alternative reference rates (ARRs) at all.

From a high-level benchmark rate perspective J.P. Morgan [29] conveniently summarize the differences between USD 3M Libor and replacement USD SOFR rates as shown below in figure (6),

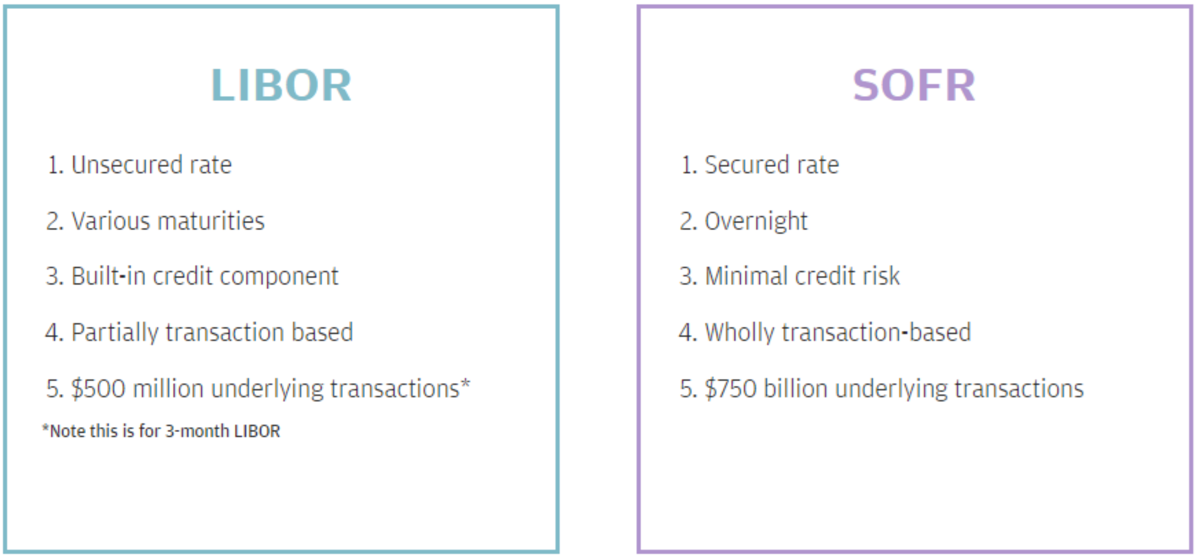

Figure 6: Source: J.P. Morgan, Key Differences between USD 3M Libor and SOFR

3.1 Structural Rate Changes

In structural terms the actual underlying rates themselves are different, which calls into question how to manage long dated Libor transactions beyond 2021, where there is a real and significant risk Libor benchmark rates may cease to be published.

With regards to the differences between Libor and ARR benchmarks and comparing the two, Libor rates have the following features,

Libor Rate Features

1. Forward Looking
2. Fixing in advance at 11am GMT
3. Cashflows based on Libor rates are known in advance
4. Therefore cashflows can be paid in any number of intermediate installments.
5. Has built-in bank credit component
6. Libor derivatives can very volatile especially near a fixing date.
7. Rates are less reflective of actual borrowing levels
8. Partially transaction based
9. Rates do not reflect broad demographics and geographies
10. Euribor for example jumps on regional European holidays

whereas Alternative Reference Rates on the other hand have the following behaviours and dynamics,

ARR Features

1. Backward Looking
2. Rates are published in arrears at 9am for the previous day
3. ARR term rates are daily averaged rates
4. Averaging of term rates makes them less volatile
5. Term rates not known in advance
6. ARR fixing tables are required
7. Term bank credit risk no longer incorporated in rate
8. Rate is subject to supply/demand fluctuations and jumps

The structural differences between Libor and ARR rates make it problematic to determine the variable or floating rate of interest to use should Libor no longer exist beyond 2021 as expected, which could lead to significant financial loss and legal litigation if not carefully managed. The replacement Libor rate in the event of its unavailability is referred to as the 'fall-back rate' which we discuss below.

3.2 Fall-back Rates

The UK Financial Conduct Authority (FCA) has decided that it would no longer compel Libor panel banks to publish Libor rates beyond 2021, see [21] and [1], [2], [6], [29] for more information. Furthermore, for trading contracts referencing Libor the FCA commented most recently that it expects Libor panels to dwindle and disappear after the end of 2021. Firms must be able to run their business without Libor from this date and reduce their stock of 'legacy' Libor contracts, see [22]

Effectively managing the trillions of dollars of existing Libor contracts without significant value transfer poses a major challenge. The legal mechanism in contracts to provide a back-up rate and contingency plan in the event of not knowing the Libor rate, the so-called 'fall-back', was written to manage temporary interruptions to Libor rate availability and publication, not a complete replacement of the rate.

This means that existing rates transactions that mature after a potential discontinuation of IBOR will require special treatment rather than rely on the existing provisions. Current callback provisions within the ISDA master agreements that govern the interest rate transaction logistics, as acknowledged by the ISDA Association, are not robust enough to support the discontinuation of IBOR rates. In some cases the fall-back language indicates that Libor rates would default to the most recently available value, in other words become a fixed rate!!!

The ARRC has released several public consultations including proposed fall-backs to Libor rates, fall-back trigger events and methods, see [2]. Market participants are required to both understand their contractual fall-back arrangements and ensure that those arrangements are robust enough to prevent potentially serious financial loss, liquidity and market disruption due to Libor cessation and uncertainties around large Libor related transactions, payments and receipts.

To better manage fall-back rates ISDA intend to update it's fall-back provisions, expected in December 2019. ISDA protocols provide an efficient way of implementing industry standards and contractual changes to the broad number of counterparties that have contractually signed-up to adhere to such protocols. The forthcoming protocols propose a fall-back methodology where Libor rates are replaced with an ARR rate plus a spread.

In some cases it is extremely challenging to select a fall-back rate or substitute calculation methodology. Simply substituting the replacement risk-free rate would not necessarily minimize the price impact and the subsequent value transfer required. From a Quant perspective as outlined by Henrard [27] it is important to understand the functional dynamics and differences between a Libor and risk-free rate to agree upon the best approach to rate substitution.

Libor rates are term rates, that is to say we know in-advance the interest rate to be used and transacted upon when borrowing or lending on a 3 month basis say, as shown in figure (7). Implicitly Libor rates and Libor forward curves are constructed such that instantaneous forward rates at every point on the forward curve have a corresponding accrual start, -end and fixing date.

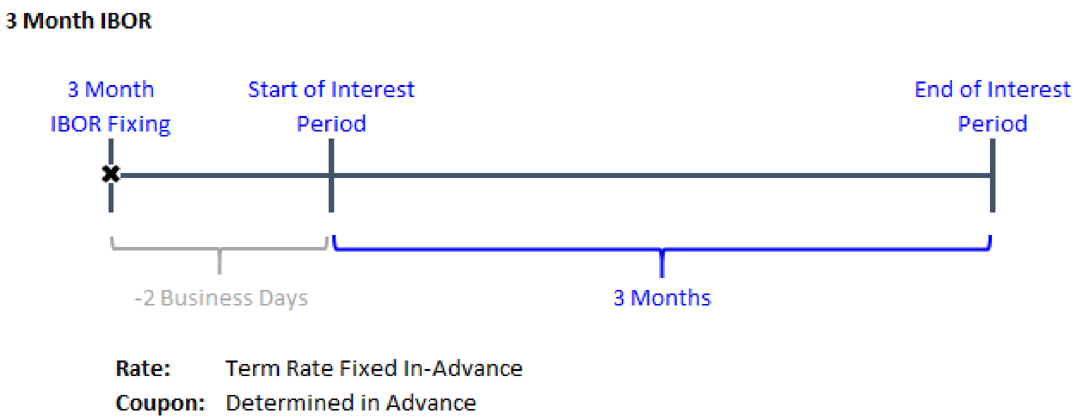

Figure 7: IBOR: Fixing In-Advance

Risk-free rates are overnight rates, not term rates, we do not know in advance the appropriate 3 month interest rate to use for borrowing and lending. Rather we fix and charge interest based on the average overnight interest rate over the 3 month period, as shown in figure (8).

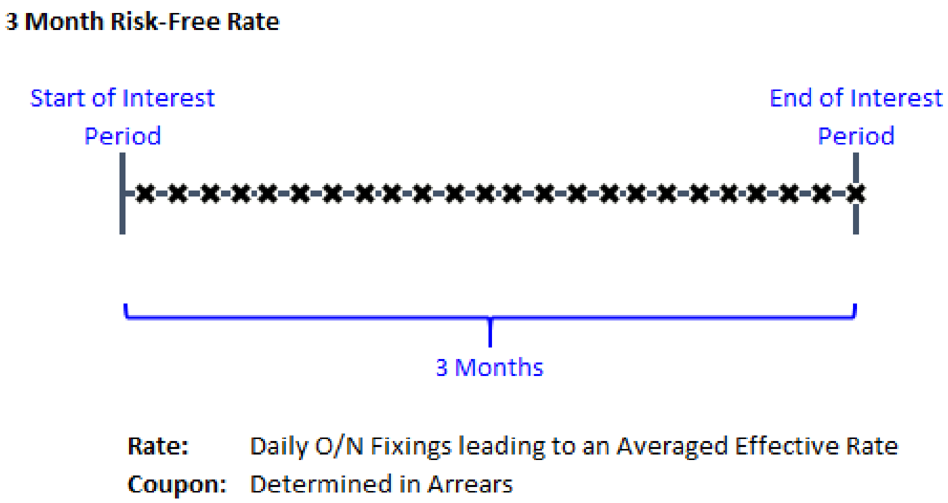

Figure 8: Risk-Free Rates: Daily Fixings with Effective Term Rate Fixing In-Arrears

An interest rate cap or floor referencing Libor is a European style contract, whereas the equivalent contract referencing a risk-free rate would be an Asian style contract with a significantly lower volatility[2]. Clearly this would lead to significant P&L shocks and monetary transfers using such a fall-back approach.

[2]Asian options are options on an average rate and clearly there significantly less variability in a moving average of rate compared to the rate itself.

3.3 Yield Curve Changes

We summarize the impact of yield curve changes here and elaborate in quite some detail in later chapters.

Loosely speaking yield curves imply Libor and ARR rates and discount factors from carefully selected liquid calibration instruments. These implied rates are then used to price and evaluate a much wider variety of instruments referencing the benchmark rates implied by the yield curve.

Yield curves are difficult to set-up in that it requires many calibration instruments be priced simultaneously at high speed with high precision. Calibration instruments are highly liquid, trading with exceptionally tight spreads and there is a vast amount of static and market data to manage. Moreover, risk sensitivities are also usually required and need to be computed and captured as part of curve calibration process, adding additional complexity.

A yield curve model can take a highly qualified professional several years to set-up, refine and test to such meet such high standards and requirements.

Now imagine the case whereby all curve instruments, conventions and risks fundamentally change. Updating yield curves to accommodate new ARR rates, new instruments and conventions, as confirmed by [31], is not a simple matter of tweaking an existing OIS curve, but rather a major undertaking to rewrite the yield curve framework, which we elaborate on in further detail in the subsequent chapters of this paper.

In section (6) we provide a summary of calibration, accuracy and performance requirements. One should also highlight that in order to price Cross Currency Swaps and instruments with CSA collateralization requirements the curve construction process requires multiple curves and currencies be calibrated simultaneously and with that the mathematical and computational effort increases considerably.

We refer the interested reader [9] and [10] for more information on trade CSAs and collateralization. Similarly for more information on Cross Currency Swaps see [12] and [14].

3.4 New Instruments & Interest Rate Markets

The introduction of alternative reference rates will naturally introduce new but similar interest rate instruments. Most notably with reference to yield curve calibration we have the following new futures and basis instruments,

- ARR 1M and 3M Futures
- ARR-OIS Tenor Basis Swaps
- ARR-Libor Tenor Basis Swaps
- Cross Currency Swaps: ARR Float vs Libor Float
- Cross Currency Swaps: ARR Float vs Fixed

It is understood that in markets such as the US, the OIS Fed-Fund and Libor instruments are transitory and will cease to exist beyond 2021. However in the Eurozone we expect OIS EONIA and ARR ESTR to co-exist for some time and the basis market to persist. For an extensive overview of ESTR see [18], [19] and [20].

As for Cross Currency Swaps we anticipate that there will be several transitory flavours of Cross Currency Swaps to support the several different combinations of trade legs with ARR forecast rates, ARR discount rates, Libor forecast rates, OIS discount rates at different stages of the market evolution cycle as ARR rates in different currencies go-live and develop sufficient liquidity for trading purposes. Flexible systems are paramount to managing the transition to different ARR roll-out schedules for each currency.

Market participants must be careful not to assume that the new instruments will behave in the same way as existing Libor referenced instruments and likewise not simply reuse existing Libor analytics for pricing and risk management.

There are several nuances to consider, the most notable of which we highlight below,

1. **Libor Versus ARR Rates**
   As outlined in section (3.2) Libor and Alternative Reference Rates are fundamentally different and do not exhibit the same behaviour, see [27] for an in-depth review and analysis.

2. **Libor Fixing In-Advance**
   Natural Libor rates fix in advance, unnatural Libor fixings require convexity adjustments, see [13].

3. **ARR Fixing In-Arrears**
   Natural Alternative Reference Rates fix in arrears, unnatural ARR fixings also require a similar, but not identical, convexity adjustment.

4. **Libor Futures Start-Date Roll**
   Libor futures contracts expire and roll over to the next contract on the futures start date, when the underlying rate is known.

5. **ARR Futures End-Date Roll**
   ARR futures contracts expire and roll over to the next contract on the futures end date, when all the underlying daily rates are known.

6. **Futures Convexity Calculations**
   Libor and ARR futures convexity calculations are not identical and convexity model volatility parameters are also different.

7. **European Libor Derivatives**
   Financial contracts referencing term Libor rates are typically European-Style (bullet payment).

8. **Asian ARR Derivatives**
   Financial contracts referencing term ARR rates are typically Asian-style (averaged).

9. **O/N ARR Rate Spikes**
   Individual overnight ARR rates can be more volatile than Libor rates due to repo market spikes and monetary policy open market operations.

10. **Term ARR Less Volatile**
    Term ARR rate volatility is typically lower compared to Libor as ARR rates are derived from an average of ARR overnight rates for the period.

The new ARR rates are prone to large jumps, as can be seen below in figure (9) daily Fed Fund rates can be spiky, however as shown in figure (11) the OIS effective rates are for the large part quite smooth.

Figure 9: Source: Bloomberg, Daily FED Funds Rate

In the Eurozone we also observe jumps in the daily EONIA rate,

Figure 10: Source: Bloomberg, Daily EONIA Rate

Interestingly [24] study the impact of liquidity spikes in the US markets and on the USD Libor replacement rate SOFR in particular. Despite daily ARR rates such as USD SOFR being prone to extremely large spikes in the daily rate the net average effect on the effective term rate, whist not immaterial is much smaller even for the largest of jumps as shown below in figure (11).

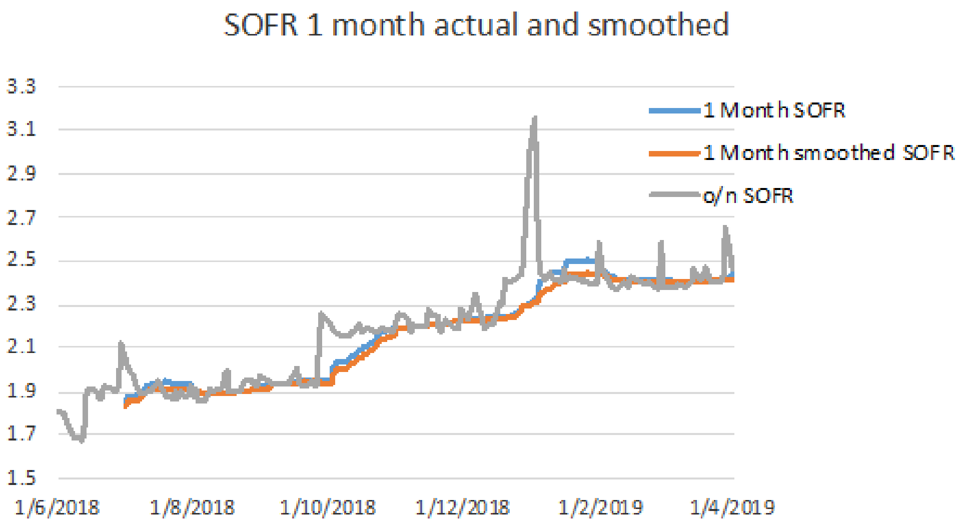

Figure 11: Source: Clarus Financial Technology, SOFR Overnight and Term Rates

3.5 New Risks to Manage

With regards to risk we make only a brief note here, whilst acknowledging risk framework set-up, configuration and testing is both important and quite involved. With the introduction of new benchmark rates and new instruments, both numerical and analytical risk systems will require significant modification.

Furthermore scenario based risk management tools will also require enhancement. It should be noted that scenario based risk management tools requiring historic data, such as historical value-at-risk (VaR) will not have a rich history of data to work with. An alternative risk measure may be required in the absence of historical data. Alternatively one may need to resort to fall-back proxy and allow for proxy data differences.

4 Yield Curves

In this section we outline the impact of Libor rate reform on yield curves. We discuss what a yield curve actually is, why it is important and calibration or model changes required to accommodate migration to alternative reference rates.

For an excellent illustrative yield curve overview we recommend [4], for a more technical overview [5] and for an excellent academic review of yield curves see [3] and [26].

4.1 What is a Yield Curve?

Yield curves reflect the borrowing and lending rates over a range of maturities within a particular market and currency. There are different types of yield curves, reflecting the different markets, institutions and instruments investors can choose to secure financing. Government Bond curves reflect the rate of return or yield required for governments to secure financing. Likewise Swap Curves reflect the borrowing and lending rates available in interest rate swap markets for a variety of maturities and a given currency.

In this paper we focus on swap yield curves only. Swap yield curves are used to extract interest rate data from standardized interest rate products such as futures, FRAs and interest rate swaps including tenor- and cross currency basis swaps. This process is described as ‘curve calibration’ and the instruments used are called ‘calibration instruments’. We show in figure (12) the calibration instruments used the USD 3 Month Libor curve.

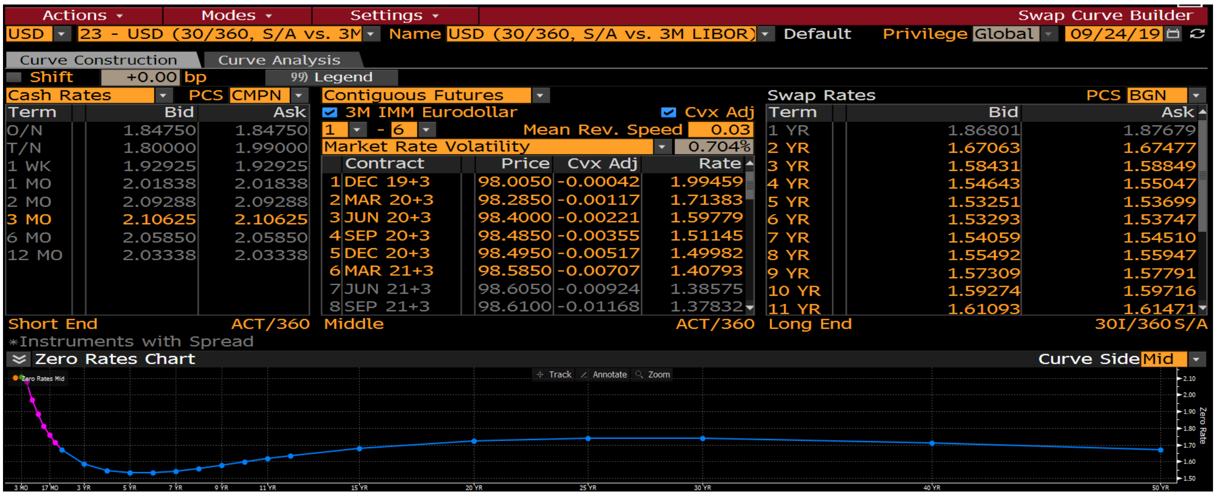

Figure 12: Source: Bloomberg, USD 3 Month Libor Curve

We calibrate curves to extract discount factors and forward rates from calibration instruments. Discount factors are used to calculate today’s price of a future cashflow, whereas forward rates imply future interest rates.

Remark: Discount Factor Example
Consider a cashflow of USD 100 in 1 year’s time. A 1 year USD discount factor of 0.9 implies that this cashflow has a value of USD 90. Conversely this means if we deposit USD 90 for 1 year our cash will earn interest and be worth USD 100 in 1 year’s time.

Remark: Forward Rate Example
Consider a 1 year loan of USD 1,000 being repaid quarterly. We know today’s interest rate is 2.0% and therefore we know we have to pay interest of USD 5 (1,000 x 0.25 x 2.0%) for the first quarter, but how do we calculate what our future payments will be? If we know from the yield curve that forward interest rates are 4.0% flat then we know that our future interest to be paid is USD 10 each quarter (1,000 x 0.25 x 4.0%).

For any given currency there exist multiple curves see [4] which are uniquely defined by the cashflow frequency of the curve calibration instruments e.g. 1D, 1M, 3M, 6M and 12M. We provide an illustration of this in figure (13) below.

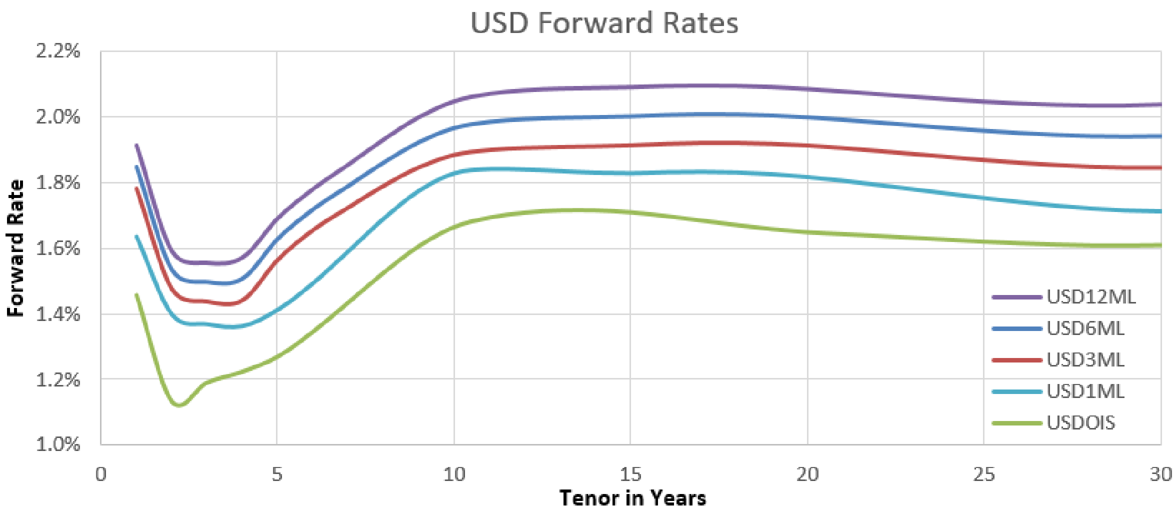

Figure 13: USD Curves & Forward Rates

Each curve captures the interest rate and corresponding interbank credit risk associated with banks lending for the given coupon frequency. Typically the credit spread is larger for longer coupon frequencies and hence corresponding forward rates are higher. The market implied interest rate for quarterly borrowing should be implied from the 3 month curve calibrated to instruments with quarterly coupons, likewise semi-annual interest rates would be implied from the 6 month curve etc.

4.2 Why are Yield Curves so Important?

Without yield curves it is impossible to evaluate the future value of money. Yield curves are extremely important, they provide the mechanism to price future cashflows and are required by **all** financial products. They also give an overview of the health of the economy [15].

Yield curves imply discount factors and forward rates from the financial products they are calibrated against. Discount factors are required to price future cashflows and forward rates are needed to evaluate the future floating or variable rates of interest embedded within mortgages, corporate loans, derivatives and other financial products.

Current estimates show we have assets with a notional value of USD 220 trillion referenced to USD Libor as shown in figure (2) and USD 350 trillion notional of assets referenced to IBOR worldwide as shown in figure (1).

4.3 Yield Curve Calibration

Yield curve calibration is the process of implying discount factors $P(t,T)$ and forward rates $f(t,T)$ from a chosen set of market calibration instruments see [3], [4], [5] and [26]. The calibration process involves repricing standard liquid market instruments (calibration instruments) and solving for the unknown discount factors and forward rates. We elaborate on discount factor and forward rate preliminaries in appendix (C) for the interested reader.

Curve Calibration Order
Discount factors are typically derived from OIS curves, so we calibrate the OIS curve first. Forward rates are derived from Libor curves and depend on the discount factor, so we calibrate these curves second.

The underlying calibration instruments depend on each other and drive the general curve build order, which we summarize below.

1. Outright OIS Curve
2. Outright Swap Curves
3. Tenor-Basis Curves
4. Xccy-Basis Curves
5. FX Forward CSA Collateral Adjustments

So for USD curves we would build USD curves in the following order: USDOIS, USD3ML, USD1ML, USD6ML then USD12ML as highlighted in figure (14).

Adjusting Discount Factors for CSA Collateral
CSA collateral adjustments to discount factors are required for collateralized trades, such as cross currency basis trades (Xccy-Basis). They adjust discount factors to account for collateral being posted and have no impact on forward rates. Discount factors are CSA collateral adjusted using the FX Forward Invariance relationship, outlined technically in [10] with examples in [14] and highlighted in appendix (D).

4.3.1 Bootstrapping

Calibration can be done in a sequential manner starting with the instruments having the shortest maturities to the longest this is called Bootstrapping. Bootstrapped curves ignore the effects that longer instruments have on shorter ones, which is undesirable. Furthermore, sequential bootstrapping methods imply that we can only build one curve at a time, meaning we also ignore the cross dependencies from instruments with different coupon frequencies, such as tenor-basis instruments. However on the positive side they give localized hedging and risk profiles, which reduce the distribution of risk, making risks more linear and risk management cost-effective in that we have less hedges to transact.

Single Libor curves would typically be built in the following order, see figure (14).

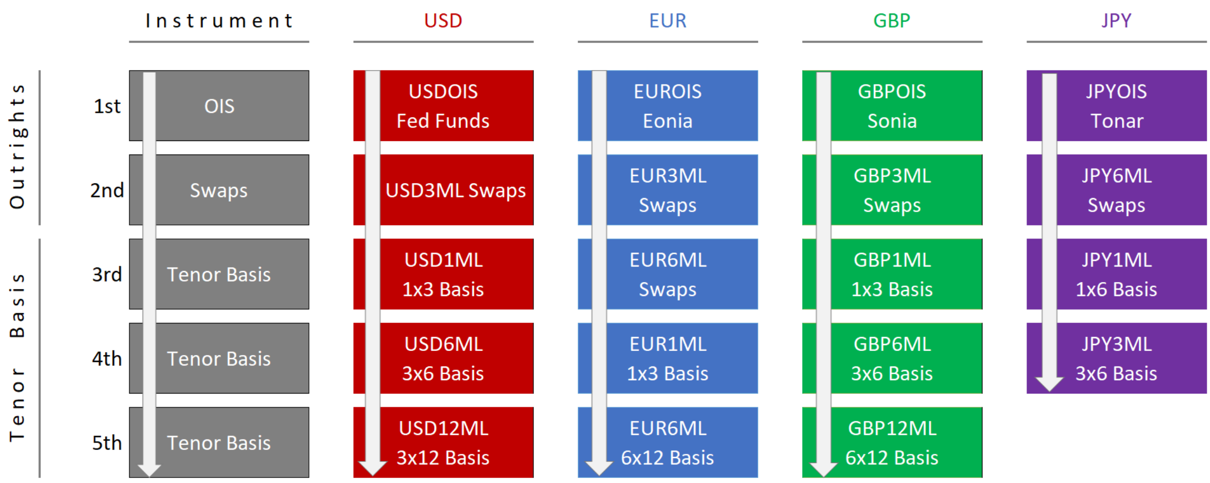

Figure 14: Single Curve Calibration Order

For demonstration purposes we provide a stylistic example to demonstrate how to bootstrap yield curves in Excel as shown below in figure (15), see appendix (G) for details.

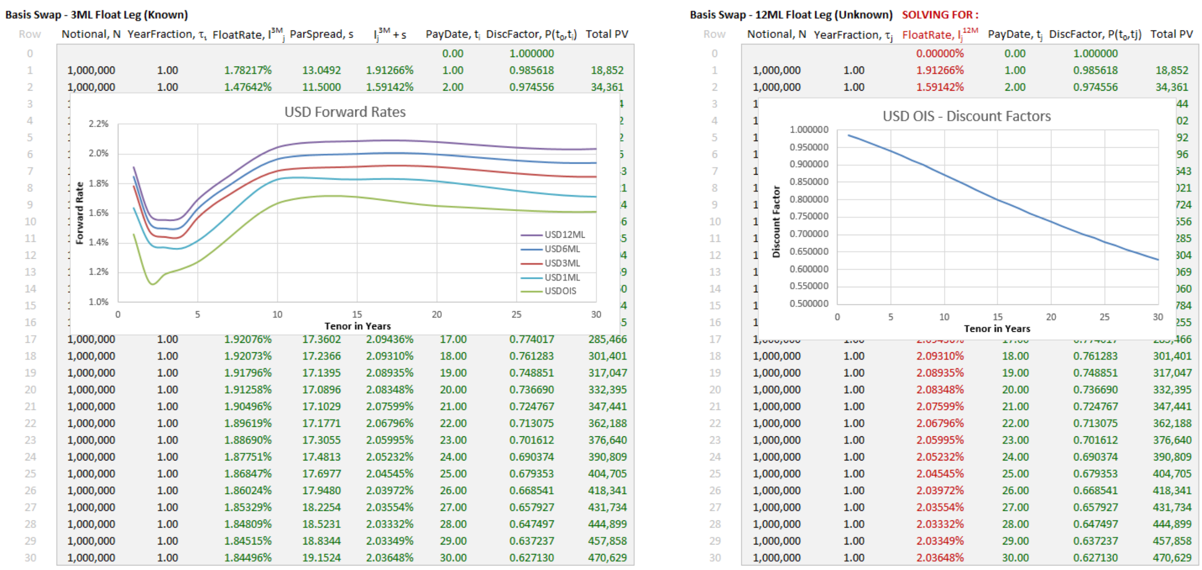

Figure 15: Bootstrapping Example

Important Remark: Common Bootstrapping Problem

A popular issue with Bootstrapping us that in the OIS curve we typically want to use Libor-OIS basis swaps from around 3 years onwards, but on the Libor swap curve we often prefer to use futures in the 2-5 year area, since these instruments are more liquid than Libor swaps.

The problem is two-fold, firstly we are using instruments in the OIS curve from a Swap curve that has yet to be built and secondly the Libor swaps referenced in the OIS curve are not part of *the swap curve. This leads to Libor-OIS swap mispricing in the OIS curve and false risks being generated from the OIS curve to instruments that are not part of the Libor swap curve.*

4.3.2 Global Calibration

As described in [31] global curve calibration is a sophisticated multi-curve construction method that simultaneously solves for implied discount factors and forward rates using all instruments on one or more curves. Global solving better captures information from market quotes, in particular from instruments that involve multiple curves such as basis swaps, incorporating cross effects and curve co-dependencies.

Multi-curve global curve calibration is the preferred calibration method for market practitioners, it is less biased, incorporates all instrument data evenly, improves risk and deals with issues outlined above with traditional bootstrapping approaches. Furthermore, global calibration allows us to use global interpolation schemes such as tension-splines, which cannot be employed in bootstrapping methods. Such interpolation methods give richer, smoother and more accurate forward rate results.

In contrast to single curve calibration as outlined in figure (14) the same curves can be simultaneously or globally calibrated as shown in figure (16).

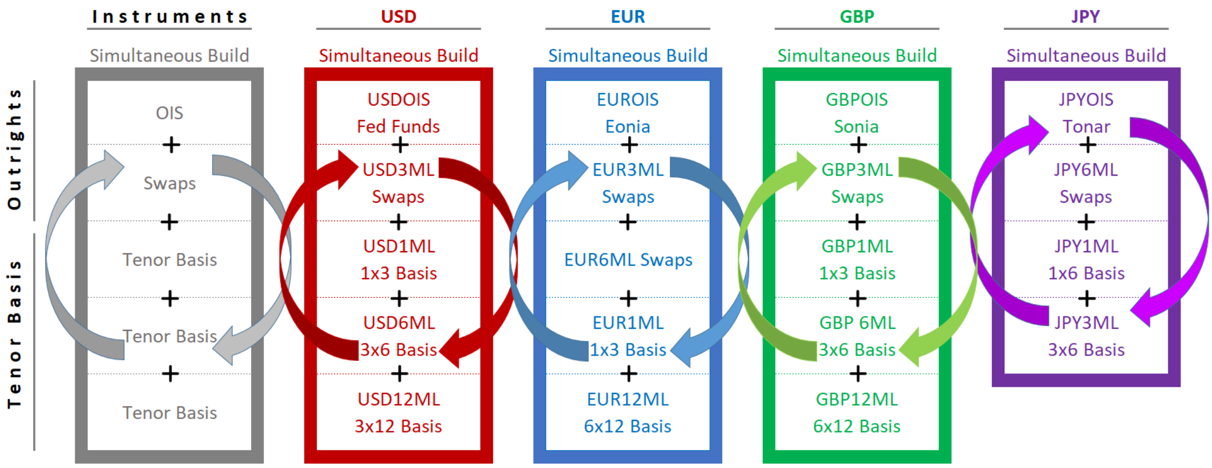

Figure 16: Global Curve Simultaneous Calibration

4.3.3 Implying Forwards & Discount Factors

Calibration Process

The goal of the calibration process is to select highly liquid market data instruments and use interpolation, optimization and solver routines to imply forward rates and discount factors required to price and evaluate a wide variety of financial instruments.

As shown in figure (12) we see that the principal calibration instrument is a vanilla interest rate swap, which quote as a par rate i.e. the rate that makes the swap price to zero or par, for more detailed information on swaps pricing and par rates, see [11].

To help understand the calibration process we briefly review how to price a vanilla interest rate swap, which is the principal calibration instrument. A vanilla swap involves the exchange of a fixed stream of cashflows with variable floating cashflows, with the later typically linked to Libor benchmark rates. Investors use interest rate swaps to immunize against adverse interest rate movements.

Investors can also swap a set of floating interest payments for another with a different coupon frequency. For example quarterly interest payments could be swapped for semi-annual interest payments, such a swap transaction is called a tenor basis swap.

In appendix (E) we present the main formulae required to price swaps and calculate the par rate. Swap pricing is extensively discussed in [11] with many pricing examples, we highlight the outright swap and tenor basis swap formulae below,

Curve Instrument Pricing Formulae

Outright Swap Formula
As defined in [11] the fixed rate r is quoted such that $PV = 0$

$$PV(\text{Swap}) = \phi \left(\sum_{i=1}^{n} N\ r\ \tau_i\ P(t_0, t_i) - \sum_{j=1}^{m} N\ (l_j + s)\ \tau_j\ P(t_0, t_j) \right) \tag{1}$$

Tenor Basis Swap Formula
As defined in [11] the float spread S is quoted on the shorter frequency trade leg such that $PV = 0$

$$PV(\text{3x6 Tenor-Basis}) = \phi \left(\sum_{i=1}^{n} N\ (l_i^{3M} + s)\ \tau_i\ P(t_0, t_i) - \sum_{j=1}^{m} N\ l_j^{6M}\ \tau_j\ P(t_0, t_j) \right) \tag{2}$$

We also note that for OIS Swaps daily forward rates are annualized and averaged / compounded into an effective rate, which is applied to each float leg coupon and replaces the l_j term in equation (1).

OIS Effective Rates

The OIS effective rate r_E when **geometric compounded** is calculated as follows,

$$r_E = \frac{\prod_{j=1}^{m}\left(1 + l_j^{OIS}\tau_j\right) - 1}{\tau_E} \tag{3}$$

and likewise when **arithmetic averaged** we have,

$$r_E = \frac{\sum_{j=1}^{m} l_j^{OIS}\tau_j}{\tau_E} \tag{4}$$

where τ_j is the jth daily year fraction and τ_E is year fraction for the entire effective rate period.

Outright vanilla interest rate swaps are quoted in the market as par rates. Tenor basis swaps on the other hand are quoted as par spreads. When calibrating yield curves we are required to solve for the all the individual forwards and discount factors that make our swap price to par. This is an under-determined or 'many-to-one' system, which we illustrate below.

Implying Discount Factors & Forward Rates from Swap Quotes

Interest Rate Swaps typically quote as par rates, i.e the fixed rate that makes the swap price to zero or par. We outline the swap pricing and par rate formulae in detail in [11] and appendix (E). A swap par rate can be calculated as follows.

$$\text{Par Rate} = \frac{\text{PV(Float Leg)}}{\text{Annuity(Fixed)}} = \left(\frac{\sum_{j=1}^{m} l_j \tau_j P(0,t_j)}{\sum_{i=1}^{n} \tau_i P(0,t_i)}\right) \tag{5}$$

where the jth Libor forward rate is denoted l_j and the discount factor at time t_i is denoted $P(0,t_i)$.

Illustration:
For a 2 year swap with annual cashflows assuming a day count of 30/360, which gives $\tau = 1$, we have,

$$\begin{aligned} \textit{Par Rate} &= \left(\frac{l_1\,\tau_1\,P(0,1) + l_2\,\tau_2\,P(0,2)}{\tau_1\,P(0,1) + \tau_2\,P(0,2)}\right) = \left(\frac{l_1\,P(0,1) + l_2\,P(0,2)}{P(0,1) + P(0,2)}\right) \\ &= l_1\left(\frac{P(0,1)}{P(0,1) + P(0,2)}\right) + l_2\left(\frac{P(0,1)}{P(0,1) + P(0,2)}\right) \end{aligned}$$

For each and every swap quote we are required to imply and solve for several Libor Forward rates and discount Factors. Interpolation enables us to do this as we outline in section (4.3.4).

Example: 5 Year Swap Quote
Given a 5 year EUR Interest Rate swap is trading with a par rate 0.3670% say, we can imply

discount factors and forward rates using equation (5) from [11] and the appropriate discount and forward rates from our yield curves, thus giving cashflows as shown below in figure (17).

Interest Rate Swap (Par Swap)
EUR 1MM Fixed 0.3670% vs EUR6ML 5 Year

Fixed Leg — **PV Fixed: 18,475**

	Accrual Start	Accrual End	Pay Date	t_i	N	r^{Fixed}	τ_i	$P(t_E, t_i)$	PV^{Fixed}
1	25-Sep-19	24-Sep-20	24-Sep-20	1.00	1,000,000	0.3670%	1.00	**1.004132**	3,685
2	24-Sep-20	24-Sep-21	24-Sep-21	2.00	1,000,000	0.3670%	1.00	**1.006928**	3,695
3	24-Sep-21	24-Sep-22	24-Sep-22	3.00	1,000,000	0.3670%	1.00	**1.008377**	3,700
4	24-Sep-22	25-Sep-23	25-Sep-23	4.00	1,000,000	0.3670%	1.00	**1.008377**	3,700
5	25-Sep-23	24-Sep-24	24-Sep-24	5.00	1,000,000	0.3670%	1.00	**1.006928**	3,695

Float Leg — **PV Float: 18,475**

	Fixing Date	Accrual Start	Accrual End	Pay Date	t_j	N	l_{j-1}	s	$l_{j-1}+s$	τ_j	$P(t_E, t_j)$	PV^{Float}
1	25-Sep-19	25-Sep-19	25-Mar-20	25-Mar-20	0.50	1,000,000	**0.0487%**	0.00	0.0487%	0.50	**1.001933**	244
2	25-Mar-20	25-Mar-20	24-Sep-20	24-Sep-20	1.00	1,000,000	**0.1087%**	0.00	0.1087%	0.50	**1.004132**	546
3	24-Sep-20	24-Sep-20	25-Mar-21	25-Mar-21	1.50	1,000,000	**0.1887%**	0.00	0.1887%	0.50	**1.005494**	949
4	25-Mar-21	25-Mar-21	24-Sep-21	24-Sep-21	2.00	1,000,000	**0.2487%**	0.00	0.2487%	0.50	**1.006928**	1,252
5	24-Sep-21	24-Sep-21	26-Mar-22	26-Mar-22	2.50	1,000,000	**0.3287%**	0.00	0.3287%	0.50	**1.007922**	1,657
6	26-Mar-22	26-Mar-22	24-Sep-22	24-Sep-22	3.00	1,000,000	**0.4087%**	0.00	0.4087%	0.50	**1.008377**	2,061
7	24-Sep-22	24-Sep-22	26-Mar-23	26-Mar-23	3.50	1,000,000	**0.4687%**	0.00	0.4687%	0.50	**1.008598**	2,364
8	26-Mar-23	26-Mar-23	25-Sep-23	25-Sep-23	4.00	1,000,000	**0.5487%**	0.00	0.5487%	0.50	**1.008377**	2,767
9	25-Sep-23	25-Sep-23	25-Mar-24	25-Mar-24	4.50	1,000,000	**0.6287%**	0.00	0.6287%	0.50	**1.007922**	3,169
10	25-Mar-24	25-Mar-24	24-Sep-24	24-Sep-24	5.00	1,000,000	**0.6887%**	0.00	0.6887%	0.50	**1.006928**	3,468

Figure 17: Interest Rate Swap Pricing Example - Fixed vs Float 6ML

The purpose of the calibration process is to solve and imply the discount factors and forward rates from a yield curve such as that in figure (18), so that we can price swaps. Prior to calibration the discount factors and forwards are not known.

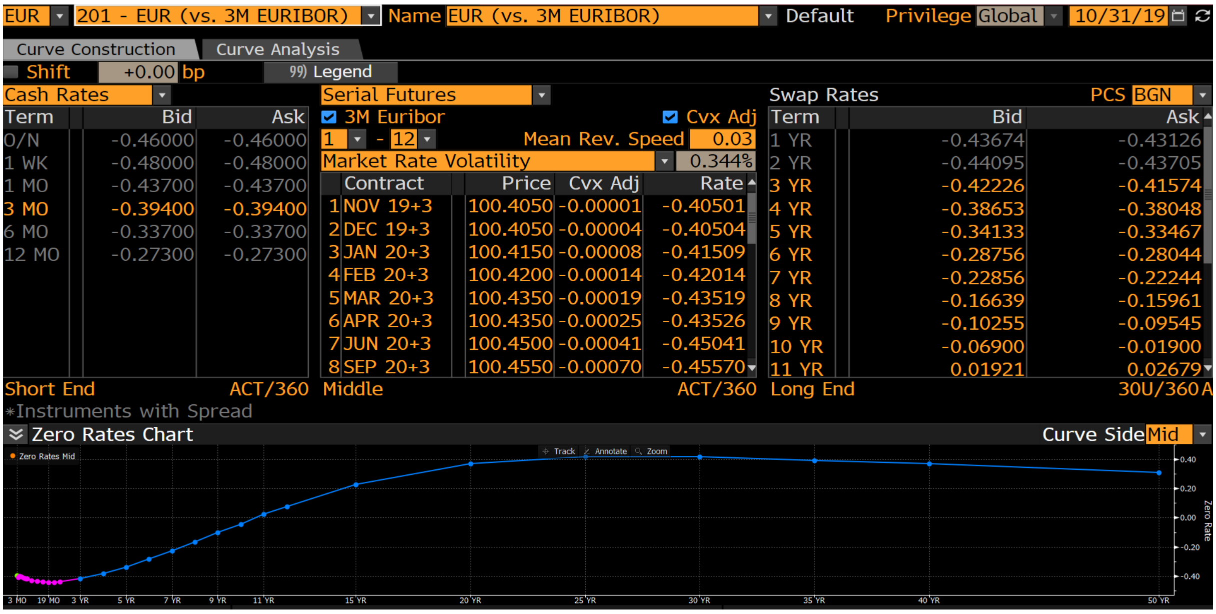

Figure 18: Source: Bloomberg, EUR 3M Libor Curve

In the case of the single swap above in figure (17) the calibration process is required to solve and imply 10 Libor forward rates L highlighted in blue on the floating leg given a single par rate for the trade. Furthermore we would also have to solve for the 15 discount factors P on both the fixed and float legs, highlighted in yellow.

The above example in figure (17) is for a single swap, whilst in reality we would have to calibrate many swaps simultaneously. The question then becomes how do we solve for so many unknown variables given only a handful of par rates / market quotes. The problem of finding many Libor forward rates and discount factors is solved by making assumptions about their functional form.

Market participants assume that discount factors and forwards can be described using a well defined linear function or a smooth spline say. This is where interpolation becomes key to solving for so many unknowns. Interpolation cannot be separated from the calibration process and is a vital component, which we discuss next.

We give a stylized example of how to solve for multiple Libor forecast rates simultaneously in appendix (H), which we highlight below in figure (19).

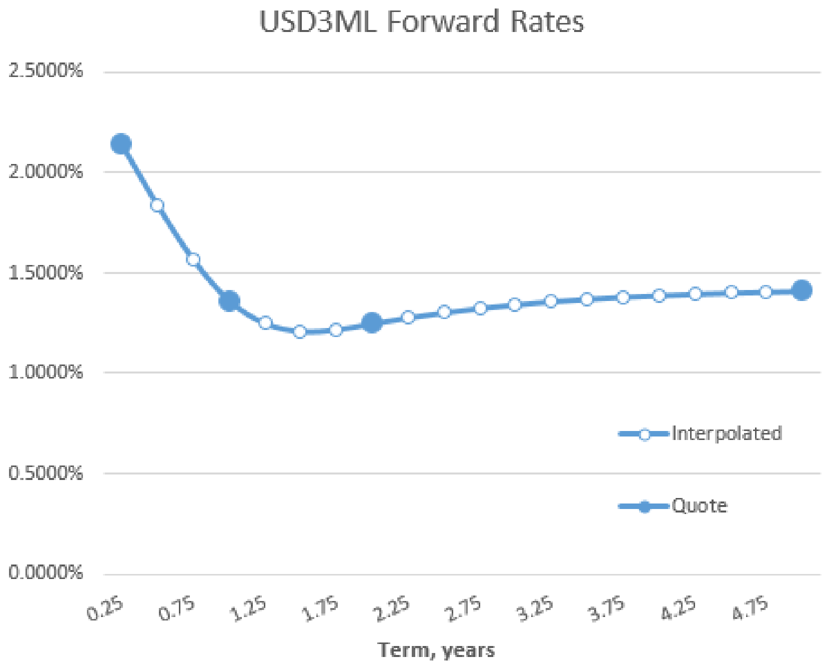

Figure 20: Interpolation & Calibration

Multi-Dimensional Newton-Raphson Algorithm

$X_{n+1} = X_n - J^{-1} f(X_n)$

Tolerance: 1.00E-08

RMSE: 8.72E-12

Iteration: 4

Curve	Term	Time, t	X_{n+1}	X_n	Initial Guess X_0	$f(X_n)$	Epsilon
USDOIS	1Y	1.00	**1.43591%**	1.43591%	2.00000%	0.00000%	0.00E+00
USDOIS	2Y	2.00	**1.23323%**	1.23323%	2.00000%	0.00000%	2.69E-12
USDOIS	3Y	3.00	**1.25107%**	1.25107%	2.00000%	0.00000%	3.86E-12
USDOIS	4Y	4.00	**1.29130%**	1.29130%	2.00000%	0.00000%	1.00E-12
USDOIS	5Y	5.00	**1.39782%**	1.39782%	2.00000%	0.00000%	-3.89E-12
USD3ML	1Y	1.00	**1.70896%**	1.70896%	2.00000%	0.00000%	0.00E+00
USD3ML	2Y	2.00	**1.47359%**	1.47359%	2.00000%	0.00000%	3.13E-12
USD3ML	3Y	3.00	**1.49531%**	1.49531%	2.00000%	0.00000%	4.44E-12
USD3ML	4Y	4.00	**1.55934%**	1.55934%	2.00000%	0.00000%	5.28E-14
USD3ML	5Y	5.00	**1.62999%**	1.62999%	2.00000%	0.00000%	-2.89E-12

USDOIS Discount Factors

Integrate USDOIS Forward Polynomial

Time, t	DiscFactor	Integrand
1.00	**0.982281**	1.78781%
2.00	**0.969579**	3.08936%
3.00	**0.957671**	4.32509%
4.00	**0.945574**	5.59628%
5.00	**0.933074**	6.92710%

Update Solver

Figure 19: Global Solving for USD Discount Factors & Forwards

4.3.4 Interpolation

Yield curve calibration typically involves solving for many forward rates for each and every swap or calibration instrument as illustrated in figure (20).

We have to solve an under-determined system of forward rates or discount factors i.e. we have more unknowns than known data points. This problem is simplified by assuming that the underlying state variable follows a particular parametric functional form. This is typically achieved by assuming forward rates lie exactly on an interpolation method of choice, often a cubic spline of some form. The choice of interpolation method is implicitly part of the curve calibration process and not a separable feature.

In interest rate markets we observe flat forwards between monetary policy meeting dates (since central banks do not modify interest rates between meeting dates), linear futures (due to daily exchange margining effectively removing convexity) and swaps including basis swaps to be smooth.

Interpolation Schemes

Forward rates for different segments of the yield curve behave and should be modelled or interpolated differently. We generally select an interpolation scheme to generate forward rates with a shape that broadly matches market behaviour.

- **Right-Continuous** (Piecewise-Constant)
  Forward rates between monetary policy committee (MPC) meeting dates are assumed constant.
- **Linear**
  Forward rates derived from futures instruments are assumed to be linear
- **Tension-Spline** (Smooth)
  Forward rates from swaps and basis instruments are assumed to be smooth.
- **Mixed Interpolation** (Linear/Spline)
  For curves with a mixture of futures and swaps we require a mixed linear and smooth interpolator.

We also have to choose an interpolation state variable, ideally we want to interpolate on forward rates since the functional form and shape of the forward curve is generally well understood and even traded via curve spreads (slope) and swap butterfly trades (curvature), see [16].

Often practitioners interpolate based on the following discount factor formula, which assumes continuous compounding,

Discount Factors: Continuous Compounding

$$P(0,t) = \exp(-z\,t) \tag{6}$$

where $P(0,t)$ is the spot discount factor and z is the zero rate or interest rate from time 0 to t with $t > 0$.

or when using simple compounding,

Discount Factors: Simple Compounding

$$P(0,t) = \frac{1}{(1 + z\,t)} \tag{7}$$

where $P(0,t)$ is the spot discount factor and z is the zero rate or interest rate from time 0 to t with $t > 0$.

Interpolation works best on smooth, linear or affine functions, so practitioners often choose to work on the log-DF, DF, zero-rate, zero-rate times Time and transform the results back into forward rates by rearranging equation (6) or (7), see appendix (C) for details.

Interpolation State Variables

- Discount Factors
- Log of Discount Factors (Minus Zero Rate times Time)
- Zero Rate
- Zero Rate times Time (Minus Log of Discount Factors)
- Forward Rates

Ametrano [5] gives an illustration of the interpolation impact on curve shape as shown in figure (21),

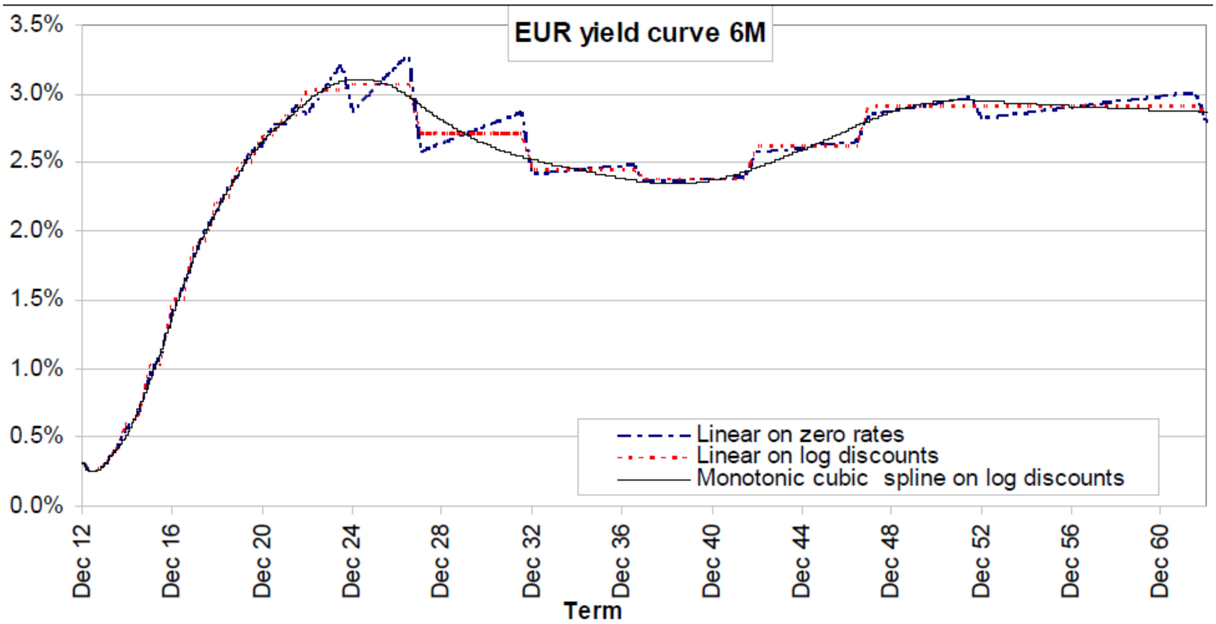

Figure 21: Source: Ametrano (2013), The impact of interpolation on curve shape

The choice of interpolation scheme this can lead to a combination of spiky, disjointed, overfitted, extreme and unusual forward rate results. Therefore interpolation on continuously compounded instantaneous forward rates is the desired choice of state variable using equation (4.3.4),

Discount Factors from Instantaneous Forward Rates

$$P(0,t) = \exp\left(-\int_0^t f(u)\,du\right) \tag{8}$$

where $P(0,t)$ is the spot discount factor and z is the zero rate or interest rate from time 0 to t with $t > 0$.

However rearranging (4.3.4) to imply the forward rate involves numerical integration within an already computationally intensive calibration and optimization routine. High accuracy is required and numerical integration can result in performance and accuracy issues.

As part of the interpolation process we must also carefully manage jumps and turn-of-year (TOY) effects. Banca IMI provide a good overview of the turn effect, see [4]. Conveniently [3] suggests we manage jumps and turns using an overlay curve, whereby we model and interpolate the forward rate as follows,

Forward Rate Jumps & Turns

$$f'(t) = f(t) + \mathbb{1}_{(t=\tau_i)}\epsilon_i \tag{9}$$

where $f'(t)$ denotes the adjusted forward rate, $f(t)$ the unadjusted forward rate, τ_i the ith jump date, $\mathbb{1}$ is an indicator function and ϵ_i the ith corresponding jump spread.

Important Remark: Discount Factors & Forward Rates
On the same curve forward rates and discount factors only match on interpolation pillar points, we should not expect that these quantities match at any other point. This is because when we interpolate on different values and transform the state variable, the transformation process stretches and distorts the state variable function.

4.3.5 Solving & Optimization

As discussed above each and every swap or calibration instrument quote is made up of several forward rates and discount factors. We would typically use a multi-dimensional Newton-Raphson solver or similar optimization mechanism to solve for the yield curve forward rates and discount factors required to reprice our chosen calibration instruments.

Newton-Raphson Formula

The Newton-Raphson formula is a useful iterative technique for solving a given target function $f(x)$ dependent on an unknown variable x given an initial guess x_0 of our choice,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{10}$$

Alternatively written to emphasize that the usefulness of the gradient as a risk Jacobian, which can be used to calculate analytical risk,

$$x_{n+1} = x_n - \mathcal{J}_n^{-1} f(x_n) \tag{11}$$

where $\mathcal{J}_n = \partial f / \partial x_n$ is the risk Jacobian.

Remark: Using the Jacobian for Fast Model Re-Calibration & Delta Risk
When optimising a model with a gradient-checking solver, such as the Newton-Raphson approach, the Jacobian matrix holds valuable delta risk information that should not be discarded. It captures the delta risk, which is useful in itself. Moreover, it gives the change in results for a change in input data. With the Jacobian we can quickly update the results given a change in model inputs without having to recalibrate. We give an example of this in appendix (H).

Curve calibration looks to imply the set of forward rates Libor forward Rates and discount factors required to price our calibration instruments, where we represent the vector of forward rates as,

$$\bar{L} = (l_1, l_2, ...l_n) \tag{12}$$

and discount factors,

$$\overline{DF} = (P(0,t_1), P(0,t_2), ..., P(0,t_n)) \tag{13}$$

For a single swap with par rate p_1 dependent on $\bar{L}$ and $\bar{P}$ we have,

$$p_1(\bar{L}, \overline{DF}) = p_1^{Market} \tag{14}$$

Therefore for a single swap we are looking to discover the set of Libor forward rates and discount factors which makes our par rate reprice back to the market quote, namely we solve for the below target function $f(x_n)$,

$$f_1(\bar{L}, \overline{DF}) = \left(p_1(\bar{L}, \overline{DF}) - p_1^{Market}\right) = 0 \tag{15}$$

If we define $\theta := (\bar{L}, \overline{DF})^T$ then we have,

$$f_1(\theta) = \left(p_1(\theta) - p_1^{Market}\right) = 0 \tag{16}$$

Likewise in the general case, for other instruments, such as Futures or Basis swaps we denote kth instrument model price $p_k(\theta)$ given $\theta = (\bar{L}, \overline{DF})$ and the market quoted price p_k^{Market}.

Therefore for a single instrument we can solve for and imply forwards and discount factors θ using the Newton-Raphson formula as follows,

$$\theta_{n+1} = \theta_n - \mathcal{J}_n^{-1} f_1(\theta_n) \tag{17}$$

where $\mathcal{J}_n = (\partial f_1 / \partial \theta_n)$

Extending the above for k calibration instruments we have,

$$\begin{aligned}
\theta_{n+1} &= \theta_n - \bar{\mathcal{J}}_n^{-1} f_1(\theta_n) \\
\theta_{n+1} &= \theta_n - \bar{\mathcal{J}}_n^{-1} f_2(\theta_n) \\
\theta_{n+1} &= \theta_n - \bar{\mathcal{J}}_n^{-1} f_3(\theta_n) \\
&\dots \\
\theta_{n+1} &= \theta_n - \bar{\mathcal{J}}_n^{-1} f_k(\theta_n)
\end{aligned} \tag{18}$$

where $\bar{\mathcal{J}}_n = \left(\partial \bar{f} / \partial \theta_n\right) = \nabla \left(\partial f / \partial \theta_n\right)$ with gradient vector or del operator $\nabla(.)$

denoting the calibration instrument set $\bar{f}(\theta) = (f_1(\theta), f_2(\theta), ..., f_k(\theta))$ we can succinctly write this as,

$$\theta_{n+1} = \theta_n - \mathcal{J}_n^{-1} \bar{f}(\theta_n) \tag{19}$$

This is the system we are required to solve.

Example calibration System:
For a small system consisting 2 calibration instruments requiring 3 Libor Rates and 3 discount factors have,

$$\begin{aligned}
\theta &= (\bar{L}, \overline{DF})^T \\
\bar{f}(\theta) &= (f_1(\bar{L}, \overline{DF}), f_2(\bar{L}, \overline{DF}))^T
\end{aligned}$$

with

$$\begin{aligned}
\bar{L} &= (l_1, l_2, l_3) \\
\overline{DF} &= (P(0, t_1), P(0, t_2), P(0, t_3))
\end{aligned}$$

which substituting into (19) and expanding gives,

$$\begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ P(0,t_1) \\ P(0,t_2) \\ P(0,t_3) \end{bmatrix}_{n+1} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ P(0,t_1) \\ P(0,t_2) \\ P(0,t_3) \end{bmatrix}_{n} - \bar{\mathcal{J}}_n^{-1} \begin{bmatrix} f_1(\bar{L}, \overline{DF}) \\ f_2(\bar{L}, \overline{DF}) \end{bmatrix}_n \tag{20}$$

where

$$\bar{\mathcal{J}}_n = \begin{bmatrix} \partial f_1/\partial l_1 & \partial f_2/\partial l_1 \\ \partial f_1/\partial l_2 & \partial f_2/\partial l_2 \\ \partial f_1/\partial l_3 & \partial f_2/\partial l_3 \\ \partial f_1/\partial P(0,t_1) & \partial f_2/\partial P(0,t_1) \\ \partial f_1/\partial P(0,t_2) & \partial f_2/\partial P(0,t_2) \\ \partial f_1/\partial P(0,t_3) & \partial f_2/\partial P(0,t_3) \end{bmatrix}_n$$

which we more conveniently write as,

$$\theta_{n+1} = \theta_n - \bar{\mathcal{J}}_n^{-1} \bar{f}(\theta_n)$$

where $\theta_n = (\bar{L}, \overline{DF})_n$ and $\bar{\mathcal{J}}_n = \nabla(\partial f / \partial \theta_n)$

Summary: Solving & Optimization

We can imply a set of forward rates $\bar{L} = (l_1, l_2, ...l_n)$ and a set of discount factors $\overline{DF} = (P(0,t_1), P(0,t_2), ..., P(0,t_n))$ for a given interpolation scheme $\mathcal{I}$ using the Newton-Raphson formula,

$$\theta_{n+1} = \theta_n - \bar{\mathcal{J}}_n^{-1} \bar{f}(\theta_n) \tag{21}$$

where $\theta_n = (\bar{L}, \overline{DF})_n$ and $\bar{\mathcal{J}}_n = \nabla(\partial f / \partial \theta_n)$

For a given set of k target calibration instruments $\bar{f}(\theta) = (f_1(\theta), f_2(\theta), ..., f_k(\theta))$ where $f_k(\theta) = p_k(\theta) - p_k^{Market}$ with the kth instrument having model price $p_k(\theta)$ and market quoted price p_k^{Market}.

We give a stylized example of how to solve for multiple USD 3M Libor forecast rates denoted X_{n+1} and in blue within figure (22) below, see appendix (H) for more information.

Multi-Dimensional Newton-Raphson Algorithm

$X_{n+1} = X_n - J^{-1} f(X_n)$

Tolerance: 1.00E-08

RMSE: 8.72E-12

Iteration: 4

Curve	Term	Time, t	X_{n+1}	X_n	X_0 (Initial Guess)	$f(X_n)$	Epsilon
USDOIS	1Y	1.00	1.43591%	1.43591%	2.00000%	0.00000%	0.00E+00
USDOIS	2Y	2.00	1.23323%	1.23323%	2.00000%	0.00000%	2.69E-12
USDOIS	3Y	3.00	1.25107%	1.25107%	2.00000%	0.00000%	3.86E-12
USDOIS	4Y	4.00	1.29130%	1.29130%	2.00000%	0.00000%	1.00E-12
USDOIS	5Y	5.00	1.39782%	1.39782%	2.00000%	0.00000%	-3.89E-12
USD3ML	1Y	1.00	1.70896%	1.70896%	2.00000%	0.00000%	0.00E+00
USD3ML	2Y	2.00	1.47359%	1.47359%	2.00000%	0.00000%	3.13E-12
USD3ML	3Y	3.00	1.49531%	1.49531%	2.00000%	0.00000%	4.44E-12
USD3ML	4Y	4.00	1.55934%	1.55934%	2.00000%	0.00000%	5.28E-14
USD3ML	5Y	5.00	1.62999%	1.62999%	2.00000%	0.00000%	-2.89E-12

USDOIS Discount Factors

Integrate USDOIS Forward Polynomial

Time, t	DiscFactor	Integrand
1.00	0.982281	1.78781%
2.00	0.969579	3.08936%
3.00	0.957671	4.32509%
4.00	0.945574	5.59628%
5.00	0.933074	6.92710%

Update Solver

Figure 22: Implying USD3ML Forwards using Multi-Dimensional Newton-Raphson

4.4 Calibration Instrument Selection

Not all instruments can be calibration instruments, in particular if we have two instruments in the same time region of the curve, futures and swaps say, we have to make a priority call and make a choice. Typically instruments are chosen based on practitioner liquidity preferences.

For any model to work well we need to provide it with good data. Stale and illiquid data points give unreliable and off-market model outputs. Calibrating yield curves to illiquid instruments can result in swap pricing discrepancies of several basis points, which when compared with market bid-offer spreads of around 1/10th of a basis point is quite large.

In what follows we describe standard instruments selected for calibration with reference to the short-, medium- and long-sections of the curve, which we loosely define as,

Region	**Years**
Short-End	$< 2Y$
Medium-End	$2Y - 5Y$
Long-End	$> 5Y$

Table 1: Curve Regions

4.4.1 Libor Instruments

For an OIS curve it is common select calibration instruments as shown below in figure (23), using cash deposits on the short-end, outright swaps in the mid-section, followed by Libor-OIS tenor basis swaps on the long end of the daily curve. Note here that the floating leg of OIS swaps are priced with an effective floating rate comprising of daily averaged / compounded forward rates.

OIS Curve (1D)

Instrument	Tenor	Quote	Interpolation Style
Cash Deposit	1D	2.02480	Linear
OIS Swap	6M	7.73450	Spline
OIS Swap	1Y	1.59890	Spline
OIS Swap	18M	1.52050	Spline
OIS Swap	2Y	1.46050	Spline
OIS Swap	5Y	1.36900	Spline
LIBOR-OIS Basis Swap	7Y	0.26563	Spline
LIBOR-OIS Basis Swap	10Y	0.26063	Spline
LIBOR-OIS Basis Swap	15Y	0.25500	Spline
LIBOR-OIS Basis Swap	20Y	0.25375	Spline
LIBOR-OIS Basis Swap	30Y	0.25375	Spline
LIBOR-OIS Basis Swap	40Y	0.25375	Spline
LIBOR-OIS Basis Swap	50Y	0.25375	Spline

Figure 23: OIS Curve Calibration Instruments, USDOIS

For swap curves referencing predominantly outright interest rate swaps, such as USD3ML, a popular choice of calibration instruments would be to use cash deposits and futures on the short end of the curve, swaps in the middle and tenor basis swaps on the long-end (5Y onwards), since these instruments are most liquid in their respective sections of the curve.

Swap Curve (USD3ML)

Instrument	Tenor	Quote	Interpolation Style
Cash Deposit	3M	2.13940	Linear
Future1	SEP-19	97.85500	Linear
Future2	DEC-19	97.97500	Linear
Future3	MAR-20	98.23500	Linear
Future4	JUN-20	98.33500	Linear
Future5	SEP-20	98.39500	Linear
Future6	DEC-20	98.38000	Linear
LIBOR Swap	3Y	1.69450	Spline
LIBOR Swap	5Y	1.65880	Spline
LIBOR Swap	7Y	1.67880	Spline
LIBOR Swap	10Y	1.74720	Spline
LIBOR Swap	15Y	1.84090	Spline
LIBOR Swap	20Y	1.89680	Spline
LIBOR Swap	30Y	1.92460	Spline
LIBOR Swap	40Y	1.92460	Spline
LIBOR Swap	50Y	1.92460	Spline

Figure 24: Swap Curve Calibration Instruments, USD3ML

For tenor basis curves, such as USD6ML, we calibrate to cash deposits and tenor basis swaps,

Tenor-Basis Curve (USD6ML)

Instrument	Term (Years)	Quote	Interpolation Style
Cash Deposit	6M	2.70300	Linear
LIBOR Basis Swap (3X6)	1Y	0.01750	Spline
LIBOR Basis Swap (3X6)	2Y	0.04125	Spline
LIBOR Basis Swap (3X6)	3Y	0.05125	Spline
LIBOR Basis Swap (3X6)	4Y	0.05625	Spline
LIBOR Basis Swap (3X6)	5Y	0.06125	Spline
LIBOR Basis Swap (3X6)	7Y	0.07375	Spline
LIBOR Basis Swap (3X6)	10Y	0.08500	Spline
LIBOR Basis Swap (3X6)	15Y	0.09625	Spline
LIBOR Basis Swap (3X6)	20Y	0.10375	Spline
LIBOR Basis Swap (3X6)	30Y	0.11000	Spline
LIBOR Basis Swap (3X6)	40Y	0.11000	Spline
LIBOR Basis Swap (3X6)	50Y	0.11000	Spline

Figure 25: Tenor-Basis Curve Calibration Instruments, USD6ML

4.4.2 ARR Instruments

Curves built from Alternative Reference Rate (ARR) Instruments have much more choice of calibration instrument, when compared to Libor curves. At this point in time it is difficult to judge which instruments to use on the long-end of ARR curves, due to the lack of liquidity on the long-end of the curve in any instrument. Numerix nicely illustrate the different basis relationships in figure 26 below,

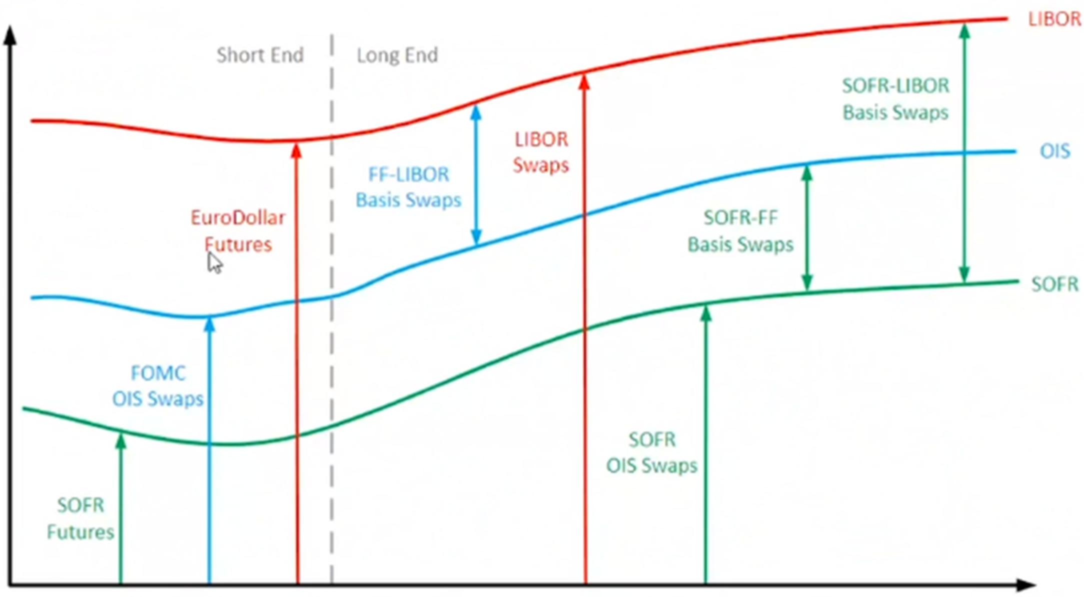

Figure 26: USD SOFR Basis Instruments, Source: Numerix

USD cleared swaps with LCH and CME clearing houses are expected to migrate to ARR / Risk-Free discounting in October 2020. The clearing houses will migrate existing Libor rates, cash compensate investors for any price difference and issue LIBOR-SOFR basis swaps to generate equivalent client ARR risk portfolio positions. The issuing of basis instruments will generate and build basis instrument liquidity.

Furthermore implied forwards from the different instruments have new behaviours. In particular we have a stronger jump and turn-of-year effects to manage on the short end of the curve; central bank open market operations in the repo markets can cause the underlying ARR rates to spike sharply.

Forward rates from ARR instruments require segmented interpolation treatment, with forward rates behaving spiky, constant, linear and smooth in different areas of the curve as referenced in figure (27) and illustrated in (28).

SOFR Curve

Instrument	Term (Years)	Quote	Interpolation Style
Cash Deposit	0.00	2.12000	Linear
Monetary Policy SOFR Swap	0.02	2.21266	Piecewise-Constant with Jumps
Monetary Policy SOFR Swap	0.14	1.85987	Piecewise-Constant with Jumps
Monetary Policy SOFR Swap	0.25	1.57939	Piecewise-Constant with Jumps
Monetary Policy SOFR Swap	0.39	1.38860	Piecewise-Constant with Jumps
Future 5	0.52	98.69748	Linear
Future 6	0.77	98.79385	Linear
Future 7	1.02	98.84050	Linear
Future 8	1.27	98.81677	Linear
SOFR Swap	3	1.22559	Spline
SOFR Swap	5	1.20502	Spline
SOFR Swap	7	1.23028	Spline
SOFR-OIS Basis Swap	10	0.01000	Spline
SOFR-OIS Basis Swap	15	0.02500	Spline
SOFR-OIS Basis Swap	20	0.05000	Spline
SOFR-LIBOR Basis Swap	30	0.07500	Spline
SOFR-LIBOR Basis Swap	40	0.08000	Spline
SOFR-LIBOR Basis Swap	50	0.10000	Spline

Figure 27: SOFR Curve Instruments and Interpolation Style

In the USD markets the ARR SOFR curve exhibits the following shape and behaviours,

Figure 28: ARR Curve Shape and Behaviour - USD SOFR

Once again we note the spikes and jumps observed in the underlying daily SOFR rates can be rather pronounced. This is often due to market events such as monetary policy open market operations and treasuries turning special.

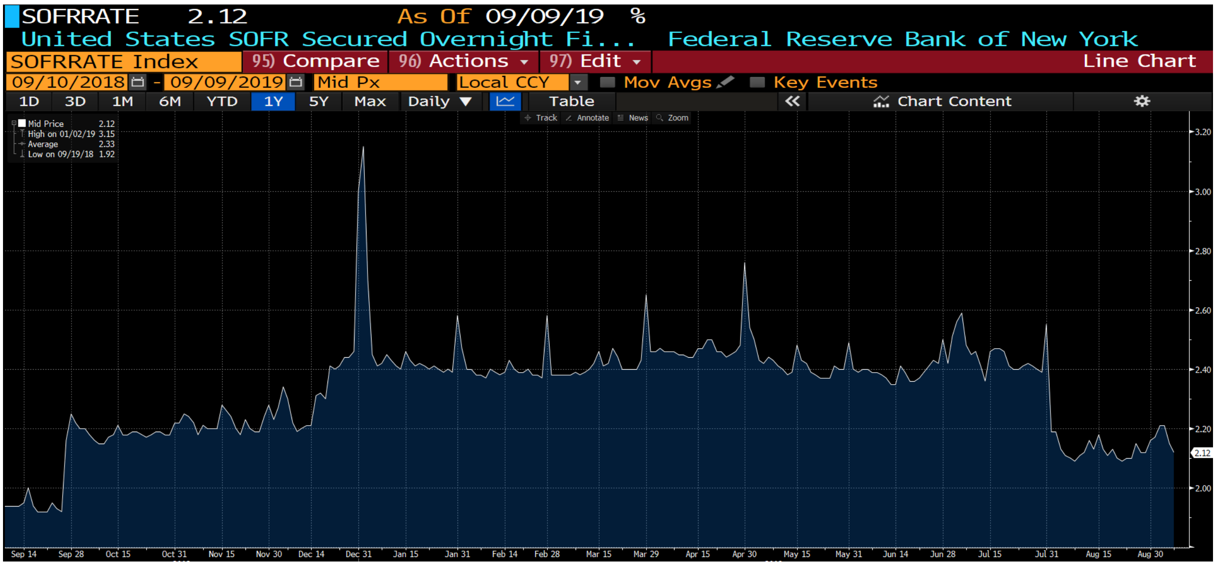

Figure 29: Source: Bloomberg, Daily USD SOFR Rate

5 Yield Curve Calibration Steps: Putting Everything Together

In summary we firstly need to select our calibration instruments and we typically do so to include the most liquid instruments possible. The inclusion of stale data points can distort forward rates significantly. Market quotes for liquid calibration instruments such as futures and swaps can be extremely tight. It is not unusual for swap par rates to quote with a bid-offer spread of 1/10th of a basis point[3] i.e. 0.00001. Using illiquid calibration instruments will not give a good forward curve that one could rely on for trading and risk management purposes.

Secondly we have to select a suitable interpolation method and state variable bearing in mind we generally know the term-structure of forward rates expected. Typically we want forward rates between monetary policy meeting dates to be flat or right-continuous, since we expect rates to be stable between meeting dates, we expect futures to be linear (due to daily exchange margining removing convexity) and swaps including basis swaps to be smooth.

Thirdly we need to employ a suitable optimization or solver routine to imply forwards and discount factors from the chosen calibration instruments simultaneously. Furthermore any risk metrics generated in the solving process should be captured since this information is valuable and most useful for analytical risk calculations and speedy curve rebuilding.

Typical calibration steps we need to follow and consider are outlined in detail below and explained in detail by [3] and [33].

Curve Calibration Steps:

[3] A basis point (bps) is 1/100th of a percent or 0.0001.

1. **Instrument Selection**
   Select liquid calibration instruments, taking into account new futures and new tenor basis instruments.

2. **Fixing Tables**
   Maintain and update fixing tables for daily ARR rates, required to price and calibrate futures and swap instruments.

3. **Convexity Adjustments**
   Model and apply new futures convexity adjustments

4. **State Variable**
   Select the interpolation state variable. Ideally we wish to interpolate on forwards or an appropriately smooth variable.

5. **Interpolation Scheme**
   Select the interpolation scheme(s). Often a mixed interpolation scheme is required that can correctly explain market dynamics for each of the curve instruments and tenor segments.

6. **Jumps & Turns**
   Empirically model, maintain and update a jump and turn-of-year table.

7. **Solver**
   Use a solver such as Multi-Variate Newton-Raphson to imply discount factors and / or forward rates.

8. **Global Calibration**
   Solve for all forwards and discount factors simultaneously and all curves as required.

9. **Risk Metrics**
   Compute and capture the Risk Jacobian during calibration.

10. **Instrument Repricing Accuracy**
    Ensure all calibration instruments reprice exactly.

11. **Performance**
    Ensure the curve calibration process is fast, ideally less than 10 milliseconds for moderate performance.

6 Yield Curve Requirements: Why is Calibration so Hard?

Curve calibration is complicated because we have to re-price all calibration instruments to high precision simultaneously, we are typically required to achieve 8 decimal places of accuracy to achieve 5 decimal places of accuracy in the forward rate. Forwards are typically quoted to 5 decimal places with a bid-offer spread of 1/10th basis point, making the requirement for high precision curve calibration paramount; there is no room for error when calibrating yield curves. We summarize the curve calibration requirements below,

Yield Curve Requirements, Accuracy & Performance

To help provide insight on the complexity of yield curve calibration, an average USD curve model with moderate performance requires,

- Over **100** instruments must be calibrated **simultaneously**
- Over **10,000** forecast rates and **10,000** discount factors must be implied
- Instruments must reprice with at least **1/100th** basis point accuracy (i.e. 0.000001)
- Calibration speed of **5-10 milliseconds** is required for moderate performance
- **Risk sensitivities** may also need to be calculated and captured

A yield curve model can take a highly qualified professional several years to set-up and test to meet such high standards and requirements. Now imagine the case whereby the curve instruments, conventions and risks fundamentally change. Clearly there is much work to be done in order to reconfigure yield curves to satisfy such high expectations.

We expect the new ARR curves to look as follows for USD, for the short-end of the curve,

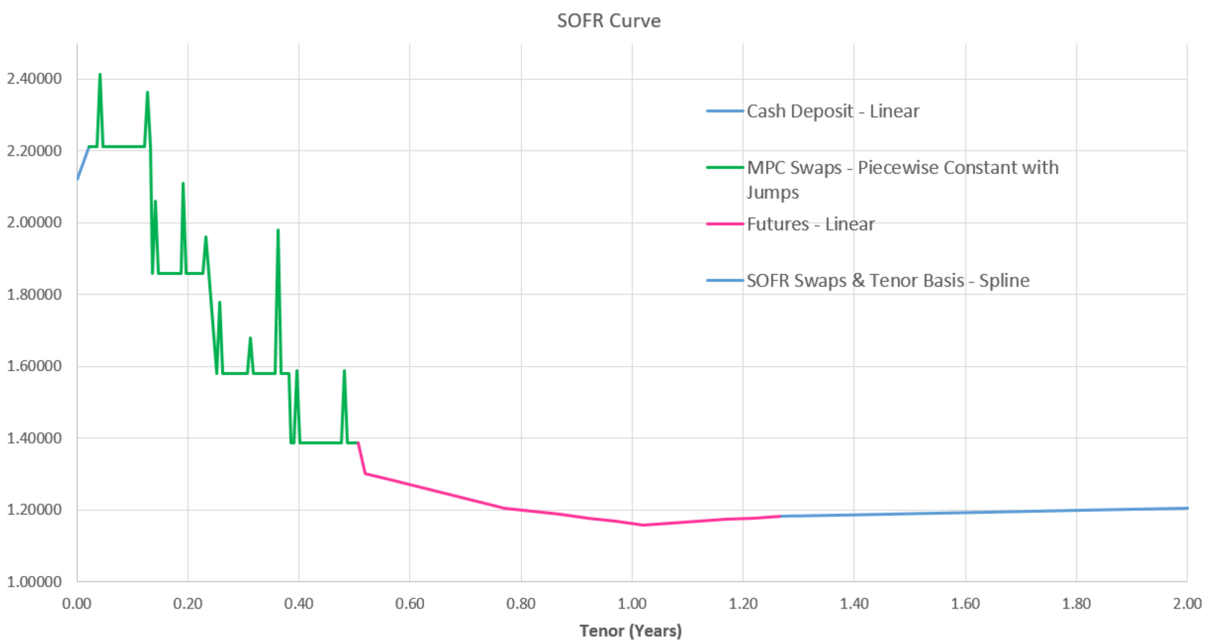

Figure 30: SOFR Curve Interpolation - 2 Year View

and likewise on the long-end,

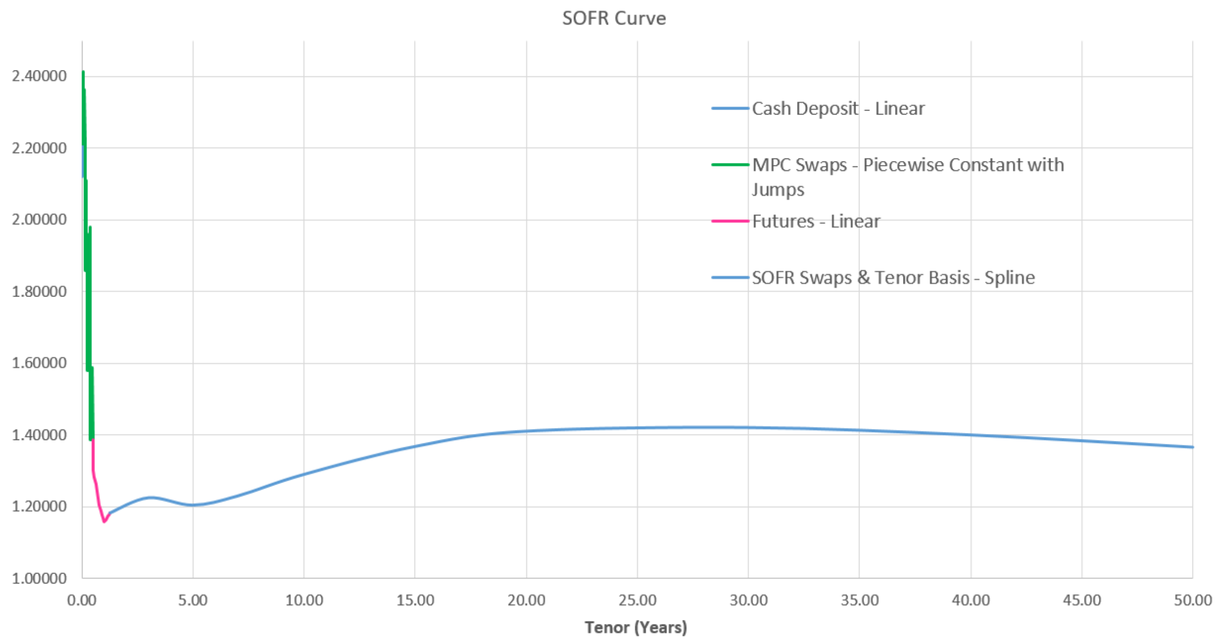

Figure 31: SOFR Curve Interpolation - 50 Year View

Updating yield curves to accommodate the new ARRs rates, instruments and conventions, as confirmed by [31], is not a simple matter tweaking an existing OIS curve, but rather a case of a major reworking or rewriting of the yield curve framework. Numerix [31] accurately and elegantly highlight the main changes as below,

New ARR curves must:

- Solve Globally Multiple Curves Simultaneously.
- Accommodate central bank meeting dates and capturing jumps and drops (spikes) in interest rates.
- Be capable of handling the turn effect, which happens at month-, quarter-, and year-end based on supply demand.
- Allow curves to be defined as spread curves i.e. as a spread to another curve.
- Should allow different interpolation methods to be used for different segments of the same curve.

Challenges:

- Curves, Instruments and Risk must change and keep-up with evolving market
- Different conventions and practices for each currency and market
- Need to add and configure new calibration instruments as the market evolves

- Numerical Risk must be updated to incorporate new additional instruments
- Management of Backward vs Forward looking rates
- Convexity adjustments required for unnatural ARR rates, see [13]

7 Basic & Advanced ARR Curves

One approach to modelling ARR Yield Curves would be to take existing Libor analytics for the OIS Curve and simply reconfigure the instrument conventions and use ARR market data. Such an approach could be used to create a 'Basic' ARR curve. We do not think such a curve could be used for live trading without adjustment, but perhaps this is acceptable for indicative purposes. We show potential differences between a '**Basic**' and an '**Advanced**' curve with all the ARR features[4] implemented below. Firstly for the short-end of the curve we observe the following,

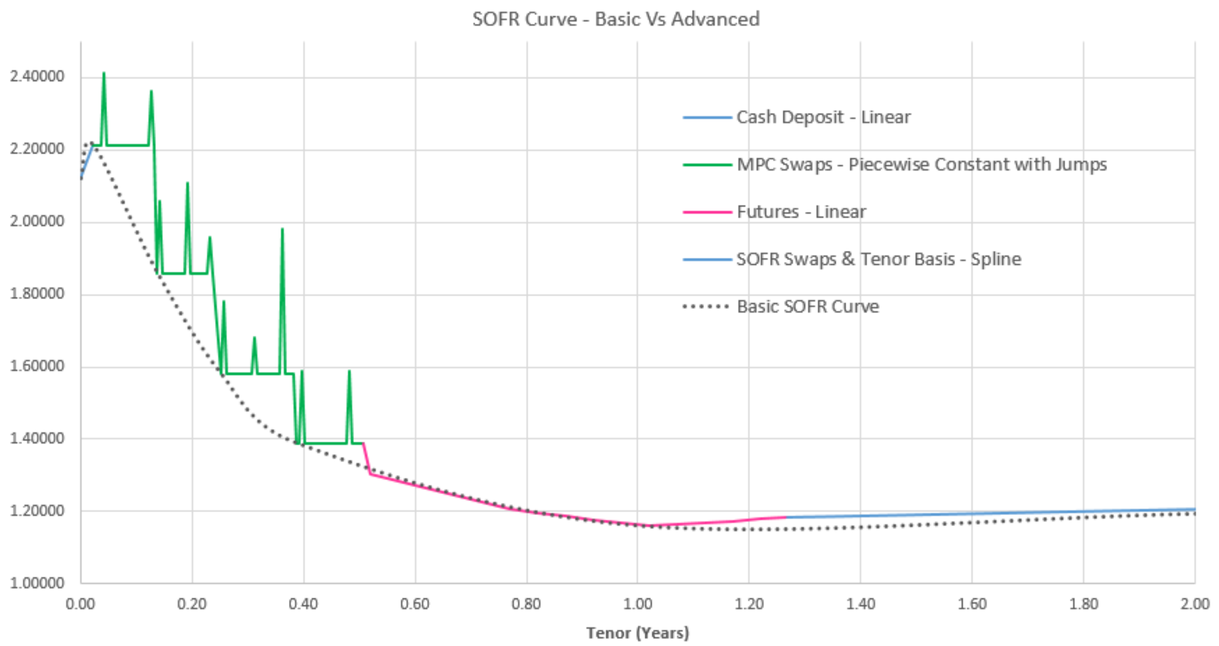

Figure 32: SOFR Curve 2 Year View - Basic vs Advanced

and likewise for the long-end of the curve,

[4] ARR features as listed in section (5).

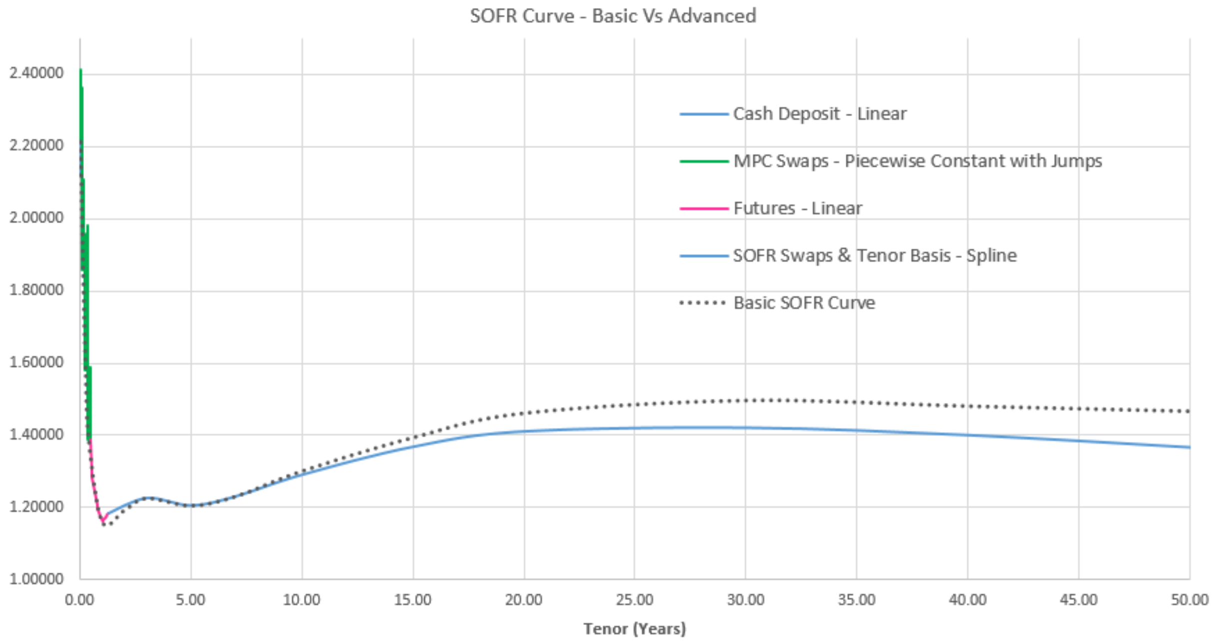

Figure 33: SOFR Curve 50 Year View- Basic vs Advanced

Summary of ARR Curve Requirements

- Simultaneous Global Solving of All Calibration Instruments for Multiple Curves
- Allow different interpolation methods on different segments of the same curve
- Ability to incorporate Jumps and Drops in rates around central bank meeting dates
- Handling of the supply/demand 'Turn-of-Year' effect (TOY) around month / quarter and year-end
- Ability to define spread curves such as ESTR as a spread to EONIA
- New ARR Tenor Basis Instruments e.g. ARR-OIS and ARR-Libor Basis
- New 1M and 3M Futures and Corresponding Convexity Adjustments
- New standard of Fixing In-Arrears rather than Fixing In-Advance
- Ability to handle and incorporate daily ARR fixing tables

Conclusion

In conclusion we have reviewed what the Libor rate is, what this benchmark is used for and why the rate is so important. It is used to price over USD 200 trillion of assets in the US markets alone and is arguably the world's most important number. We looked at the evolution of Libor and discussed what is wrong with Libor benchmarks, namely there is no transaction volume to support the rate, it no longer represents a fair view of the market and is no longer a reliable benchmark.

We discussed Libor benchmark rate reform and the alternative reference rates (ARRs), which are also known as risk-free rates, since the new benchmarks don't incorporate interbank credit spreads. This naturally led to a discussion on the impact of Libor changes. Structural differences between Libor and ARRs were also looked at and this lead to a discussion on fall-back rates and what to do in the event Libor rates become unavailable. It is anticipated by central banks and regulators that Libor benchmarks will cease to be published beyond 2021.

The impact of benchmark reform on interest rate yield curves was discussed in detail, including bootstrapping, global calibration, interpolation, optimization. We looked at how to imply forwards and discount factors in such a way to capture risk metrics that can be used for analytical risk and for ultra-fast yield curve calibration.

Libor rates and ARRs are fundamentally different and simply configuring Libor analytics for ARR curve construction is inadequate. Yield curve calibration is a complex process that requires many instruments to be repriced simultaneously to high precision with high performance in such a way to capture risk metrics. These requirements of a yield curve calibration framework make it a tough task that requires significant effort.

It is hoped this paper will serve as a useful primer on both Libor reform and yield curve calibration. To this end we have provided many charts, examples to illustrate concepts, formulae and also Excel workbooks to allow readers to play and experiment with the same, kindly email the author to receive a copy.

To conclude we present a useful Libor reform timeline, produced by RBC Capital Markets, see figure (34).

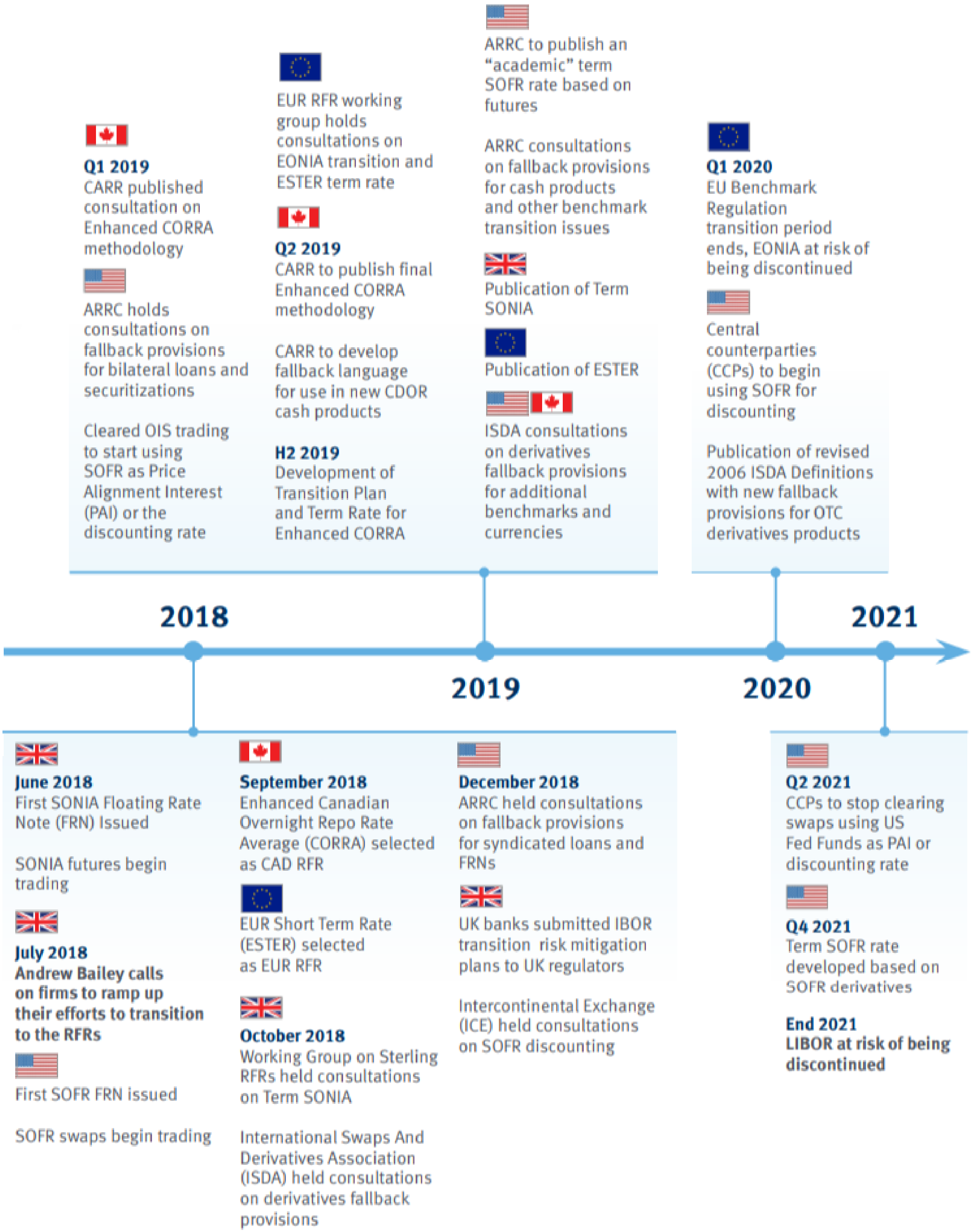

Figure 34: Libor Reform Timeline, Source: RBC Capital Markets

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A Overview Alternative Reference Rates

Overview of identified alternative RFRs in selected currency areas

	United States	United Kingdom	Euro area	Switzerland	Japan
Alternative rate	**SOFR** (secured overnight financing rate)	**SONIA** (sterling overnight index average)	**ESTER** (euro short-term rate)	**SARON** (Swiss average overnight rate)	**TONA** (Tokyo overnight average rate)
Administrator	Federal Reserve Bank of New York	Bank of England	ECB	SIX Swiss Exchange	Bank of Japan
Data source	Triparty repo, FICC GCF, FICC bilateral	Form SMMD (BoE data collection)	MMSR	CHF interbank repo	Money market brokers
Wholesale non-bank counterparties	**Yes**	**Yes**	**Yes**	**No**	**Yes**
Secured	**Yes**	**No**	**No**	**Yes**	**No**
Overnight rate	**Yes**	**Yes**	**Yes**	**Yes**	**Yes**
Available now?	Yes	Yes	Oct 2019	Yes	Yes

FICC = Fixed Income Clearing Corporation; GCF = general collateral financing; MMSR = money market statistical reporting; SMMD = sterling money market data collection reporting.

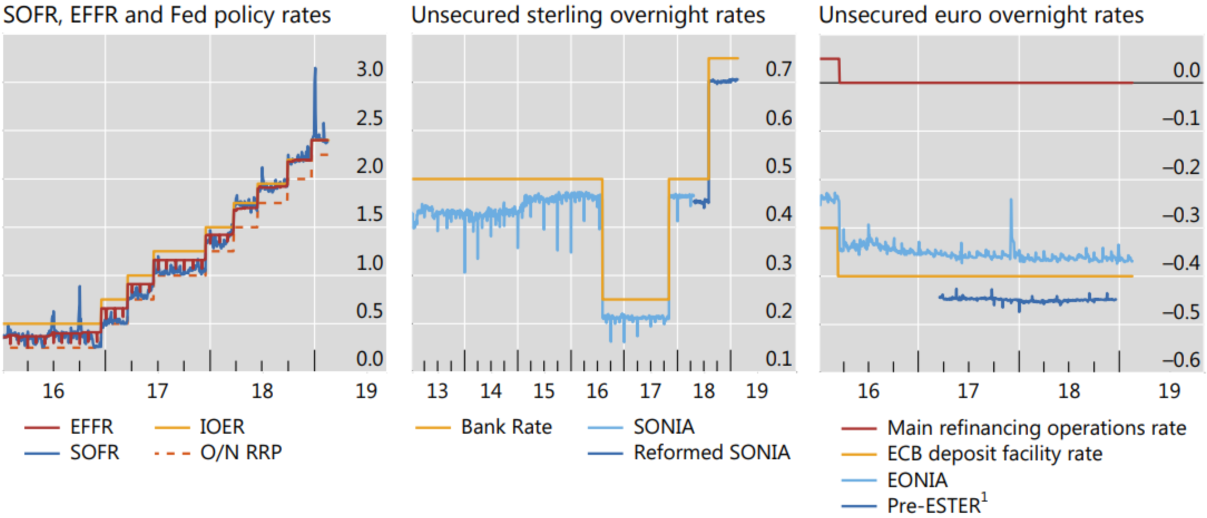
Sources: ECB; Bank of Japan; Bank of England; Federal Reserve Bank of New York; Financial Stability Board; Bank of America Merrill Lynch; International Swaps and Derivatives Association.

Figure 35: Source: BIS, Overview of Alternative Reference Rates

B Overnight Rate Comparison

BIS outline and compare the different overnight rates for US Dollar, Sterling and Euro in [7], sourcing the data from the Federal Reserve bank of New York and Bloomberg.

US dollar, sterling and euro overnight rates



EFFR = effective federal funds rate; EONIA = euro overnight index average; ESTER = euro short-term rate; IOER = interest rate on excess reserves; O/N RRP = interest rate on the overnight reverse repo facility; SOFR = secured overnight financing rate; SONIA = sterling overnight index average.

[1] Until ESTER becomes available in October 2019, the ECB is publishing figures referred to as pre-ESTER, which market participants can use to assess the suitability of the new rate.

Figure 36: Source: BIS, Overnight Rate Comparison

C Useful Discount Factor and Forward Rate Formulae

As shown by [30] we can imply forwards from discount factors on the same curve, by assuming the following relationship i.e. a short deposit to time T rolled over to time S should value the same as a long deposit to time S, where $t < T < S$

$$P(t,T)P(S,T) = P(t,S) \tag{22}$$

for a given curve and tenor discount factors and forward rates are related by the following formula, in the case of simple compounded rates we have,

$$P(t,T) = \frac{1}{1 + f(t,T)(T-t)} \tag{23}$$

substituting (23) into (22) gives,

$$P(t,T)\left(\frac{1}{1 + f(T,S)(S-T)}\right) = P(t,S) \tag{24}$$

which we can rearrange giving an expression for the forward rate, $f(S,T)$ for the period $[S,T]$ in terms of discount factors $P(t,S)$ and $P(t,T)$,

Forward Rate Formula - Simple Compounded Rates

$$f(T,S) = \left(\frac{P(t,S)}{P(t,T)} - 1\right)/(S-T) \tag{25}$$

where $t < T < S$.

Likewise for a continuously compounded rate we have,

$$P(t,T) = \exp(-f(t,T)(T-t)) \tag{26}$$

again substituting (26) into (22) leads to,

$$P(t,T)\exp(-f(T,S)(S-T)) = P(t,S) \tag{27}$$

which we can rearrange giving,

$$f(T,S) = -\ln\left(\frac{P(t,S)}{P(t,T)} - 1\right)/(S-T) \tag{28}$$

or equivalently,

Forward Rate Formula - Continuously Compounded Rates

$$f(T,S) = \ln\left(\frac{P(t,T)}{P(t,S)} - 1\right)/(S-T) \tag{29}$$

where $t < T < S$.

D Discount Factors incorporating CSA Collateral

The below FX Forward Invariance relationship to imply discount factors adjusted to account for CSA collateral posting, which is derived from non-arbitrage relationships and outlined in [14].

FX Forward Invariance Example for EUR_JPYCSA

$$FwdFX(t,T)^{EUR/JPY} = \underbrace{S\frac{P(t,T)^{EUR\_JPYCSA}}{P(t,T)^{JPY\_JPYCSA}}}_{JPY\_JPYCSA \text{ from OIS Curve}} = \underbrace{S\frac{P(t,T)^{EUR\_USDCSA}}{P(t,T)^{JPY\_USDCSA}}}_{\text{From Xccy Curve}}$$

where t denotes the valuation date, S the FX spot rate, P(t,T) the discount factor at time t for tenor T with $0 \leq t \leq T$

Figure 37: FX Forward Invariance Formula for EUR Discount Factors adjusted for JPY CSA Collateral

E Useful Swap Pricing Formulae

A vanilla swap involves the exchange of a fixed stream of cashflows with variable floating cashflows, with the later typically linked to Libor benchmark rates. Investors use interest rate swaps to immunize against adverse interest rate movements. As highlighted in [14] to price a fixed vs float semi-Annual Libor swap we can use the swap pricing formula from figure (38).

Swap Specification & Pricing

To specify a swap many parameters are required to generate the swap cashflow schedules accurately. To price a swap we require Libor forecast rates, OIS discount rates and a Swap pricing formula.

$$PV^{Swap} = N\sum_{\forall i} r^{Fixed}\tau_i P(t_0, t_i) - N\sum_{\forall j}(L_j + s)\,\tau_j P(t_0, t_j)$$

Figure 38: Interest Rate Swap Pricing Formula - Fixed vs Float 6ML

where in this example we are referencing 6 month Libor and where N denotes the trade notional, r the fixed rate, τ the accrual or coupon period in years, L the forward or Libor rate, P the discount factor and the index i and j represent the ith fixed and jth float coupon respectively.

In the calibration process we take liquid instruments, often swaps, and given the quoted par rate[5] observed in the market and we solve for the discount factors and Libor forward rates that make the swap price back to zero or par. The par rate formula is shown below in figure (39) as outlined in [11].

Swap Price

$$PV(Swap) = \phi\,(PV(Fixed) - PV(Float))$$

Fixed Leg

$$PV(Fixed) = N \times r^{Fixed} \underbrace{\sum_{\forall i} \tau_i P(t_0, t_i)}_{Annuity}$$

Float Leg

$$PV(Float) = N \sum_{\forall j} (L_j + s)\,\tau_j P(t_0, t_j)$$

Swap Rate

$$ParRate = \frac{PV(Float)}{N \times Annuity}$$

Figure 39: Useful Swap Pricing Formulae

[5] The par rate or swap rate is the fixed rate, r that makes the swap price to zero or par.

F USD Curve Instruments & Statistics

In this appendix we highlight typical USD Curve Calibration instruments that would be used to calibrate a USD curve in a Live trading environment. We also give some statistics as to how many calibration instruments are used and also indicate the number of forward rates and discount factors required, which would need to be implied by the yield curve model. These are required to price the calibration instrument set for the USD curve and each curve index component.

USD curve calibration statistics both totals and a breakdown by curve index are presented below in figures (40) and (41),

USD Curve Calibration Instruments & Statistics

USD Curve Total Statistics

No. USD OIS Instruments	30
No. USD 1ML Instruments	23
No. USD 3ML Instruments	18
No. USD 6ML Instruments	18
No. USD 12ML Instruments	15
Total No. Instruments Required	104
No. Libor Rates Required for Pricing	13,550
No. Discount Factors Required for Pricing	13,550

Figure 40: Total USD Curve Calibration Statistics

Outright Instrument Curves

USD OIS Calibration Instruments & Statistics

OIS Curve

No. Instruments	30
Max Tenor in Years	50
No. Libor Rates Req'd Per Year	252
No. Libor Rates Required for Pricing	12,600

USD 3ML Calibration Instruments & Statistics

Swap Curve

No. Instruments	23
Max Tenor in Years	50
No. Libor Rates Req'd Per Year	4
No. Libor Rates Required for Pricing	200

Basis Instrument Curves

USD 6ML Calibration Instruments & Statistics

Tenor Basis Curve

No. Instruments	18
Max Tenor in Years	50
No. Libor Rates Req'd Per Year	2
No. Libor Rates Required for Pricing	100

USD 1ML Calibration Instruments & Statistics

Tenor Basis Curve

No. Instruments	18
Max Tenor in Years	50
No. Libor Rates Req'd Per Year	12
No. Libor Rates Required for Pricing	600

USD 12ML Calibration Instruments & Statistics

Tenor Basis Curve

No. Instruments	15
Max Tenor in Years	50
No. Libor Rates Req'd Per Year	1
No. Libor Rates Required for Pricing	50

Figure 41: USD Curve Calibration Statistics by Curve Index

Furthermore we present a typical live trading calibration instrument set below in figures (42) and (43) below,

USD OIS

OIS SWAPS Tenor	OIS SWAPS Rate	LIBOR-OIS BASIS Tenor	LIBOR-OIS BASIS Rate
ON	1.90000%	1Y	0.27813%
1W	1.83100%	2Y	0.25750%
2W	1.77400%	3Y	0.24500%
3W	1.72700%	4Y	0.24250%
1M	1.69570%	5Y	0.24000%
2M	1.66020%	7Y	0.23500%
3M	1.62770%	10Y	0.23500%
4M	1.59440%	12Y	0.23500%
5M	1.57410%	15Y	0.23125%
6M	1.55310%	20Y	0.22875%
9M	1.50530%	25Y	0.22625%
1Y	1.46850%	30Y	0.22500%
2Y	1.36450%	40Y	0.22500%
3Y	1.32950%	50Y	0.22500%
4Y	1.31600%		
5Y	1.31800%		

USD 3ML

LIBOR FIXING Tenor	LIBOR FIXING Rate	FUTURES Contract	StartDate	EndDate	Rate	Volatility	SWAPS Tenor	SWAPS Rate
3M	1.96588%	ED1	18-Dec-19	18-Mar-20	98.100	1.45000%	3Y	1.59430%
		ED2	18-Mar-20	17-Jun-20	98.325	1.45000%	4Y	1.57510%
		ED3	17-Jun-20	16-Sep-20	98.410	1.45000%	5Y	1.57510%
		ED4	16-Sep-20	16-Dec-20	98.460	1.45000%	7Y	1.60800%
		ED5	16-Dec-20	17-Mar-21	98.455	1.45000%	8Y	1.63280%
		ED6	17-Mar-21	16-Jun-21	98.520	1.45000%	9Y	1.66070%
		ED7	16-Jun-21	15-Sep-21	98.525	1.45000%	10Y	1.68780%
		ED8	15-Sep-21	15-Dec-21	98.525	1.45000%	12Y	1.73690%
							15Y	1.79120%
							20Y	1.84720%
							25Y	1.86790%
							30Y	1.87410%
							40Y	1.87410%
							50Y	1.87410%

Figure 42: USD OIS and Swap Curve Calibration Instruments

USD 6ML

LIBOR FIXING Tenor	LIBOR FIXING Rate	BASIS SWAPS Term	BASIS SWAPS Rate
6M	1.97450%	1Y	8.00
		2Y	7.13
		3Y	7.00
		4Y	7.25
		5Y	7.63
		6Y	7.88
		7Y	8.00
		8Y	8.38
		9Y	8.63
		10Y	8.75
		12Y	9.13
		15Y	9.50
		20Y	10.00
		25Y	10.25
		30Y	10.38
		40Y	10.38
		50Y	10.38

USD 1ML

LIBOR FIXING Tenor	LIBOR FIXING Rate	BASIS SWAPS Term	BASIS SWAPS Rate
1M	1.84638%	1Y	14.88
		2Y	12.88
		3Y	11.88
		4Y	11.38
		5Y	11.25
		6Y	11.13
		7Y	11.00
		8Y	10.88
		9Y	10.88
		10Y	10.88
		12Y	10.75
		15Y	10.75
		20Y	10.63
		25Y	10.63
		30Y	10.63
		40Y	10.63
		50Y	10.63

USD 12ML

LIBOR FIXING Tenor	LIBOR FIXING Rate	BASIS SWAPS Term	BASIS SWAPS Rate
12M	1.99313%	1Y	25.13
		2Y	22.75
		3Y	22.25
		4Y	20.88
		5Y	20.88
		7Y	20.13
		10Y	19.75
		12Y	19.75
		15Y	19.75
		20Y	20.63
		25Y	21.25
		30Y	22.13
		40Y	22.13
		50Y	22.13

Figure 43: USD Tenor Basis Curve Calibration Instruments

G Curve Bootstrapping

With this paper we provide an Excel workbook to demonstrate curve bootstrapping, kindly email the author to receive a copy. We follow the bootstrapping build order for the USD curve as outlined in figure (14).

The demonstration workbook includes a toy stylistic model, where we made several simplifying examples for brevity and demonstration purposes, which we list in the workbook. This conveniently allows us to demonstrate the curve bootstrapping process without over-emphasizing conventional nuances, albeit in a stylistic setting.

To calibrate the USD OIS curve we need to imply both the OIS forecast rates and the OIS discount factors simultaneously. We defined and simplified the OIS par swap pricing formula as outlined in the model assumptions as follows,

$$\begin{aligned} \text{PV(Fixed Leg)} &= \text{PV(OIS Float Leg)} \\ \sum_{i=1}^{n} N\tau_i p P(0,t_i) &= \sum_{j=1}^{m} N\tau_j (l_j + s) P(0,t_j) \end{aligned} \tag{30}$$

Now using forward rate formula (25) for the Libor forward rate l_j from appendix (C) we have,

$$\sum_{i=1}^{n} N\tau_i p P(0,t_i) = \sum_{j=1}^{m} N\tau_j \left(\left(\frac{P(0,t_{j-1})}{P(0,t_j)} - 1 \right) / \tau_j + s \right) P(0,t_j) \tag{31}$$

We know that our calibration instruments have a zero spread so we set $s = 0$, we also set $N = 1$ since N terms cancel. Furthermore setting $\tau = 1$, $i = j$ and $n = m$ as per our assumptions and rearranging for $P(t_0, t_n)$ leads to the following recursive relationship for the OIS discount factor.

Simplified OIS Discount Factor Bootstrapping Formula

$$P(0,t_n) = \frac{\text{PV(Fixed excluding last cpn)} - \text{PV(Float excluding last cpn)} + P(0,t_{n-1})}{(p+1)} \tag{32}$$

where we assume for simplicity that fixed and float leg coupons are paid and accrue with annual coupons with unit year fractions.

Following a similar approach for interest rate swaps we can derive a recursive formula to imply bootstrapped forward rates for outright swap curves as follows,

Simplified Swap Curve Forward Rate Bootstrapping Formula

$$l_n = \frac{\text{PV(Fixed)} - \text{PV(Float excluding last cpn)}}{P(t_0,t_n)} \tag{33}$$

where again we assume for simplicity that fixed and float leg coupons are paid and accrue with annual coupons with unit year fractions.

Likewise for tenor-basis curves and using the pricing formulae from [11] we can derive the recursive formula as follows,

Simplified Tenor-Basis Curve Forward Rate Bootstrapping Formula

The tenor basis spread is always applied to the shorter frequency leg, when implying forward rates on the longer frequency leg without the basis spread we have,

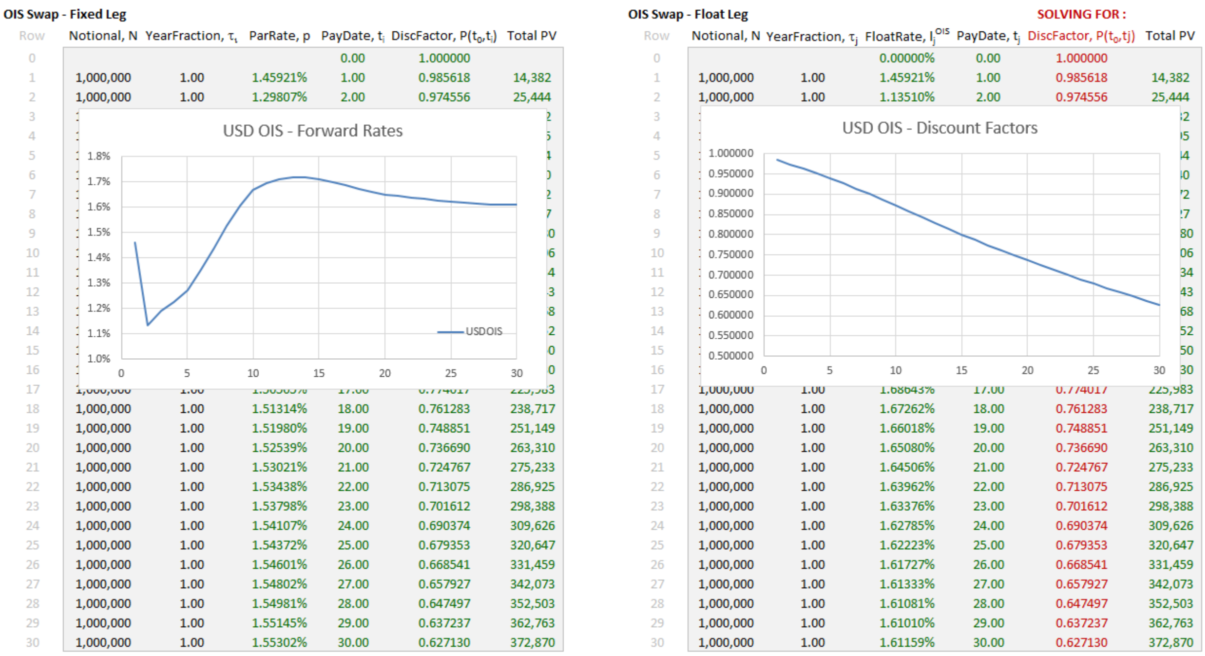
$$l_n = \frac{\text{PV}(\text{Float}[S]) - \text{PV}(\text{Float}[L] \text{ excluding last cpn})}{P(t_0, t_n)} \tag{34}$$

and when implying forward rates on the shorter frequency leg with the basis spread we have,

$$l_n = \frac{\text{PV}(\text{Float}[L]) - \text{PV}(\text{Float}[S] \text{ no spread excluding last cpn}) - \text{PV}(\text{Float}[S] \text{ spread only})}{P(t_0, t_n)} \tag{35}$$

where PV(Float$[L]$) denotes the PV of the longer frequency leg, PV(Float$[S]$) the PV of the shorter frequency leg and again we assume for simplicity that fixed and float leg coupons are paid and accrue with annual coupons with unit year fractions.

We bootstrap the OIS Curve using the recursive relationship from (32) with $P(t_0, 0) = 1$ to imply OIS discount factors and equation (25) to imply the OIS forward rates,



OIS Swap - Fixed Leg

Row	Notional, N	YearFraction, τ_i	ParRate, p	PayDate, t_i	DiscFactor, $P(t_0,t_i)$	Total PV
0				0.00	1.000000	
1	1,000,000	1.00	1.45921%	1.00	0.985618	14,382
2	1,000,000	1.00	1.29807%	2.00	0.974556	25,444
18	1,000,000	1.00	1.51314%	18.00	0.761283	238,717
19	1,000,000	1.00	1.51980%	19.00	0.748851	251,149
20	1,000,000	1.00	1.52539%	20.00	0.736690	263,310
21	1,000,000	1.00	1.53021%	21.00	0.724767	275,233
22	1,000,000	1.00	1.53438%	22.00	0.713075	286,925
23	1,000,000	1.00	1.53798%	23.00	0.701612	298,388
24	1,000,000	1.00	1.54107%	24.00	0.690374	309,626
25	1,000,000	1.00	1.54372%	25.00	0.679353	320,647
26	1,000,000	1.00	1.54601%	26.00	0.668541	331,459
27	1,000,000	1.00	1.54802%	27.00	0.657927	342,073
28	1,000,000	1.00	1.54981%	28.00	0.647497	352,503
29	1,000,000	1.00	1.55145%	29.00	0.637237	362,763
30	1,000,000	1.00	1.55302%	30.00	0.627130	372,870

OIS Swap - Float Leg

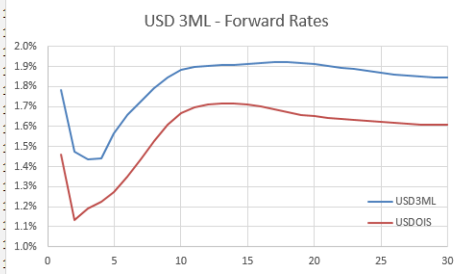
Row	Notional, N	YearFraction, τ_j	FloatRate, l_j^{OIS}	PayDate, t_j	DiscFactor, $P(t_0,tj)$	Total PV
0			0.00000%	0.00	1.000000	
1	1,000,000	1.00	1.45921%	1.00	0.985618	14,382
2	1,000,000	1.00	1.13510%	2.00	0.974556	25,444
18	1,000,000	1.00	1.67262%	18.00	0.761283	238,717
19	1,000,000	1.00	1.66018%	19.00	0.748851	251,149
20	1,000,000	1.00	1.65080%	20.00	0.736690	263,310
21	1,000,000	1.00	1.64506%	21.00	0.724767	275,233
22	1,000,000	1.00	1.63962%	22.00	0.713075	286,925
23	1,000,000	1.00	1.63376%	23.00	0.701612	298,388
24	1,000,000	1.00	1.62785%	24.00	0.690374	309,626
25	1,000,000	1.00	1.62223%	25.00	0.679353	320,647
26	1,000,000	1.00	1.61727%	26.00	0.668541	331,459
27	1,000,000	1.00	1.61333%	27.00	0.657927	342,073
28	1,000,000	1.00	1.61081%	28.00	0.647497	352,503
29	1,000,000	1.00	1.61010%	29.00	0.637237	362,763
30	1,000,000	1.00	1.61159%	30.00	0.627130	372,870

Figure 44: USD OIS Curve Bootstrapping for USD OIS Discount Factors

Next we calibrate the USD 3M Swap curve, again following the order outlined in figure (14),

IRS Swap - Fixed Leg

Row	Notional, N	YearFraction, τ_i	ParRate, p	PayDate, t_i	DiscFactor, $P(t_0,t_i)$	Total PV
0				0.00	1.000000	
1	1,000,000	1.00	1.78217%	1.00	0.985618	17,565
2	1,000,000	1.00	1.63016%	2.00	0.974556	31,954
17	1,000,000	1.00	1.75285%	17.00	0.774017	263,190
18	1,000,000	1.00	1.76095%	18.00	0.761283	277,812
19	1,000,000	1.00	1.76806%	19.00	0.748851	292,175
20	1,000,000	1.00	1.77423%	20.00	0.736690	306,265
21	1,000,000	1.00	1.77950%	21.00	0.724767	320,071
22	1,000,000	1.00	1.78395%	22.00	0.713075	333,593
23	1,000,000	1.00	1.78767%	23.00	0.701612	346,831
24	1,000,000	1.00	1.79076%	24.00	0.690374	359,793
25	1,000,000	1.00	1.79330%	25.00	0.679353	372,487
26	1,000,000	1.00	1.79539%	26.00	0.668541	384,923
27	1,000,000	1.00	1.79711%	27.00	0.657927	397,116
28	1,000,000	1.00	1.79856%	28.00	0.647497	409,083
29	1,000,000	1.00	1.79983%	29.00	0.637237	420,841
30	1,000,000	1.00	1.80101%	30.00	0.627130	432,411



IRS Swap - Float Leg SOLVING FOR :

Row	Notional, N	YearFraction, τ_j	FloatRate, l_j^{3M}	PayDate, t_j	DiscFactor, $P(t_0,tj)$	Total PV
0			0.00000%	0.00	1.000000	
1	1,000,000	1.00	1.78217%	1.00	0.985618	17,565
2	1,000,000	1.00	1.47642%	2.00	0.974556	31,954
17	1,000,000	1.00	1.92076%	17.00	0.774017	263,190
18	1,000,000	1.00	1.92073%	18.00	0.761283	277,812
19	1,000,000	1.00	1.91796%	19.00	0.748851	292,175
20	1,000,000	1.00	1.91258%	20.00	0.736690	306,265
21	1,000,000	1.00	1.90496%	21.00	0.724767	320,071
22	1,000,000	1.00	1.89619%	22.00	0.713075	333,593
23	1,000,000	1.00	1.88690%	23.00	0.701612	346,831
24	1,000,000	1.00	1.87751%	24.00	0.690374	359,793
25	1,000,000	1.00	1.86847%	25.00	0.679353	372,487
26	1,000,000	1.00	1.86024%	26.00	0.668541	384,923
27	1,000,000	1.00	1.85329%	27.00	0.657927	397,116
28	1,000,000	1.00	1.84809%	28.00	0.647497	409,083
29	1,000,000	1.00	1.84515%	29.00	0.637237	420,841
30	1,000,000	1.00	1.84496%	30.00	0.627130	432,411

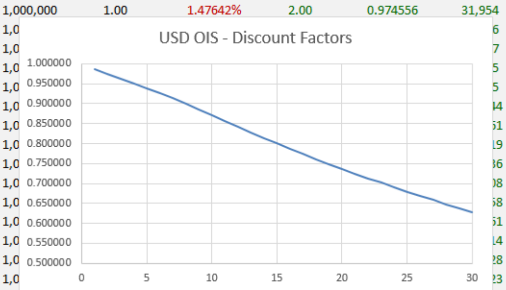
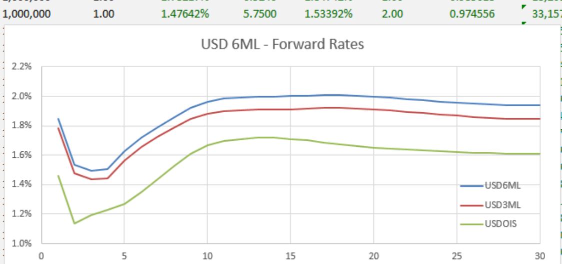


Figure 45: USD 3ML Swap Curve Bootstrapping for USD3ML Forecast Rates

Next the USD 6M Basis curve,

Basis Swap - 3ML Float Leg (Known)

Row	Notional, N	YearFraction, τ_i	FloatRate, l_j^{3M}	ParSpread, s	$l_j^{3M}+s$	PayDate, t_i	DiscFactor, $P(t_0,t_i)$	Total PV
0						0.00	1.000000	
1	1,000,000	1.00	1.78217%	6.5246	1.84742%	1.00	0.985618	18,208
2	1,000,000	1.00	1.47642%	5.7500	1.53392%	2.00	0.974556	33,157
18	1,000,000	1.00	1.92073%	8.6185	2.00692%	18.00	0.761283	289,607
19	1,000,000	1.00	1.91796%	8.5699	2.00366%	19.00	0.748851	304,612
20	1,000,000	1.00	1.91258%	8.5448	1.99803%	20.00	0.736690	319,331
21	1,000,000	1.00	1.90496%	8.5513	1.99047%	21.00	0.724767	333,757
22	1,000,000	1.00	1.89619%	8.5884	1.98208%	22.00	0.713075	347,891
23	1,000,000	1.00	1.88690%	8.6525	1.97342%	23.00	0.701612	361,737
24	1,000,000	1.00	1.87751%	8.7404	1.96491%	24.00	0.690374	375,302
25	1,000,000	1.00	1.86847%	8.8487	1.95696%	25.00	0.679353	388,596
26	1,000,000	1.00	1.86024%	8.9738	1.94998%	26.00	0.668541	401,633
27	1,000,000	1.00	1.85329%	9.1126	1.94441%	27.00	0.657927	414,426
28	1,000,000	1.00	1.84809%	9.2615	1.94071%	28.00	0.647497	426,992
29	1,000,000	1.00	1.84515%	9.4171	1.93932%	29.00	0.637237	439,350
30	1,000,000	1.00	1.84496%	9.5762	1.94072%	30.00	0.627130	451,521



Basis Swap - 6ML Float Leg (Unknown) SOLVING FOR :

Row	Notional, N	YearFraction, τ_j	FloatRate, l_j^{6M}	PayDate, t_j	DiscFactor, $P(t_0,tj)$	Total PV
0			0.00000%	0.00	1.000000	
1	1,000,000	1.00	1.84742%	1.00	0.985618	18,208
2	1,000,000	1.00	1.53392%	2.00	0.974556	33,157
18	1,000,000	1.00	2.00692%	18.00	0.761283	289,607
19	1,000,000	1.00	2.00366%	19.00	0.748851	304,612
20	1,000,000	1.00	1.99803%	20.00	0.736690	319,331
21	1,000,000	1.00	1.99047%	21.00	0.724767	333,757
22	1,000,000	1.00	1.98208%	22.00	0.713075	347,891
23	1,000,000	1.00	1.97342%	23.00	0.701612	361,737
24	1,000,000	1.00	1.96491%	24.00	0.690374	375,302
25	1,000,000	1.00	1.95696%	25.00	0.679353	388,596
26	1,000,000	1.00	1.94998%	26.00	0.668541	401,633
27	1,000,000	1.00	1.94441%	27.00	0.657927	414,426
28	1,000,000	1.00	1.94071%	28.00	0.647497	426,992
29	1,000,000	1.00	1.93932%	29.00	0.637237	439,350
30	1,000,000	1.00	1.94072%	30.00	0.627130	451,521

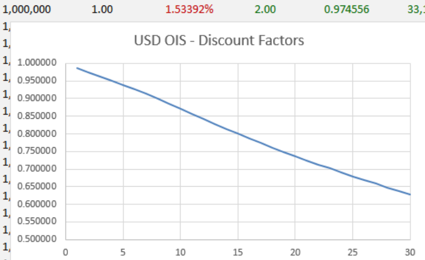
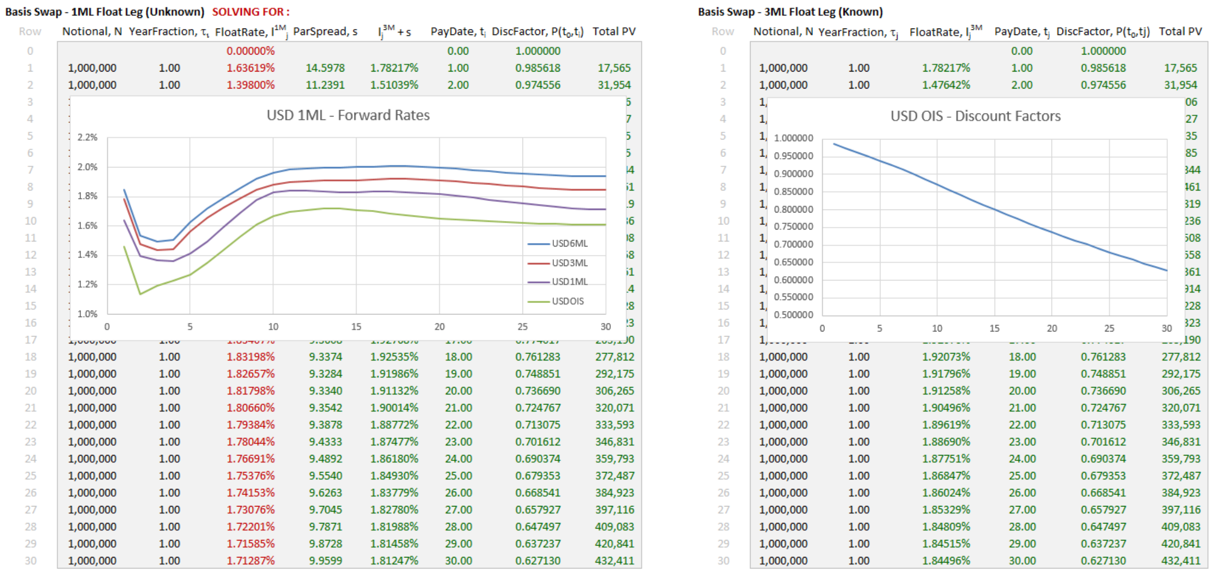


Figure 46: USD 6ML Basis Curve Bootstrapping for USD 6ML Forecast Rates

Followed by the USD 1M Basis curve,



Basis Swap - 1ML Float Leg (Unknown) SOLVING FOR :

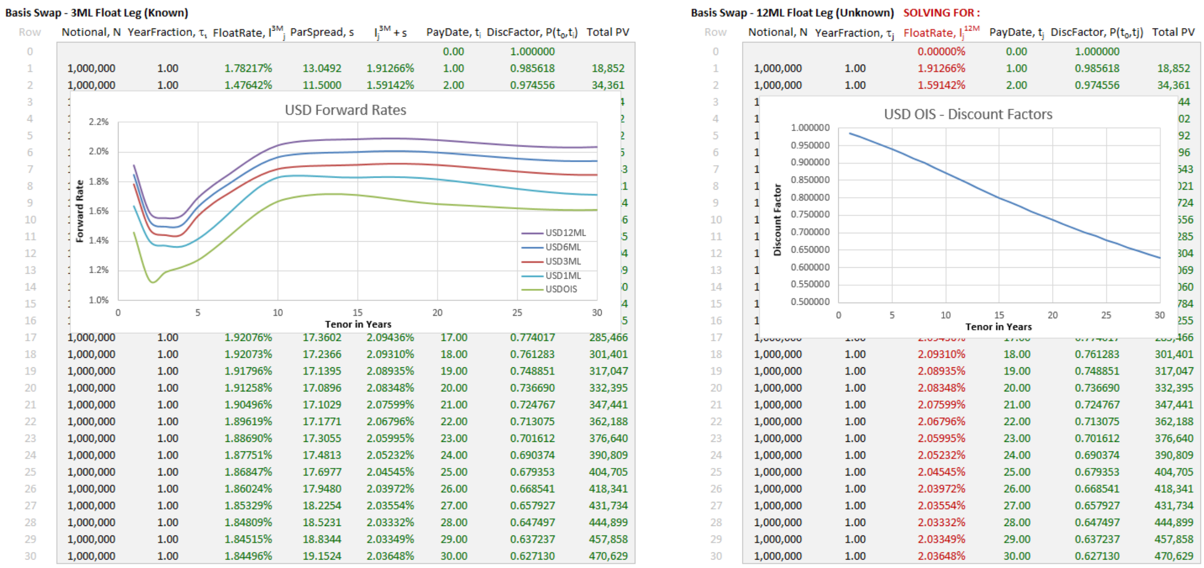
Row	Notional, N	YearFraction, τ_i	FloatRate, l^{1M}_j	ParSpread, s	l_j^{3M} + s	PayDate, t_i	DiscFactor, $P(t_0,t_i)$	Total PV
0			0.00000%			0.00	1.000000	
1	1,000,000	1.00	1.63619%	14.5978	1.78217%	1.00	0.985618	17,565
2	1,000,000	1.00	1.39800%	11.2391	1.51039%	2.00	0.974556	31,954
18	1,000,000	1.00	1.83198%	9.3374	1.92535%	18.00	0.761283	277,812
19	1,000,000	1.00	1.82657%	9.3284	1.91986%	19.00	0.748851	292,175
20	1,000,000	1.00	1.81798%	9.3340	1.91132%	20.00	0.736690	306,265
21	1,000,000	1.00	1.80660%	9.3542	1.90014%	21.00	0.724767	320,071
22	1,000,000	1.00	1.79384%	9.3878	1.88772%	22.00	0.713075	333,593
23	1,000,000	1.00	1.78044%	9.4333	1.87477%	23.00	0.701612	346,831
24	1,000,000	1.00	1.76691%	9.4892	1.86180%	24.00	0.690374	359,793
25	1,000,000	1.00	1.75376%	9.5540	1.84930%	25.00	0.679353	372,487
26	1,000,000	1.00	1.74153%	9.6263	1.83779%	26.00	0.668541	384,923
27	1,000,000	1.00	1.73076%	9.7045	1.82780%	27.00	0.657927	397,116
28	1,000,000	1.00	1.72201%	9.7871	1.81988%	28.00	0.647497	409,083
29	1,000,000	1.00	1.71585%	9.8728	1.81458%	29.00	0.637237	420,841
30	1,000,000	1.00	1.71287%	9.9599	1.81247%	30.00	0.627130	432,411

Basis Swap - 3ML Float Leg (Known)

Row	Notional, N	YearFraction, τ_j	FloatRate, l_j^{3M}	PayDate, t_j	DiscFactor, $P(t_0,t_j)$	Total PV
0				0.00	1.000000	
1	1,000,000	1.00	1.78217%	1.00	0.985618	17,565
2	1,000,000	1.00	1.47642%	2.00	0.974556	31,954
18	1,000,000	1.00	1.92073%	18.00	0.761283	277,812
19	1,000,000	1.00	1.91796%	19.00	0.748851	292,175
20	1,000,000	1.00	1.91258%	20.00	0.736690	306,265
21	1,000,000	1.00	1.90496%	21.00	0.724767	320,071
22	1,000,000	1.00	1.89619%	22.00	0.713075	333,593
23	1,000,000	1.00	1.88690%	23.00	0.701612	346,831
24	1,000,000	1.00	1.87751%	24.00	0.690374	359,793
25	1,000,000	1.00	1.86847%	25.00	0.679353	372,487
26	1,000,000	1.00	1.86024%	26.00	0.668541	384,923
27	1,000,000	1.00	1.85329%	27.00	0.657927	397,116
28	1,000,000	1.00	1.84809%	28.00	0.647497	409,083
29	1,000,000	1.00	1.84515%	29.00	0.637237	420,841
30	1,000,000	1.00	1.84496%	30.00	0.627130	432,411

Figure 47: USD 1ML Basis Curve Bootstrapping for USD 1ML Forecast Rates

and finally the USD 12M Basis curve as follows,



Basis Swap - 3ML Float Leg (Known)

Row	Notional, N	YearFraction, τ_i	FloatRate, l^{3M}_j	ParSpread, s	l_j^{3M} + s	PayDate, t_i	DiscFactor, $P(t_0,t_i)$	Total PV
0						0.00	1.000000	
1	1,000,000	1.00	1.78217%	13.0492	1.91266%	1.00	0.985618	18,852
2	1,000,000	1.00	1.47642%	11.5000	1.59142%	2.00	0.974556	34,361
17	1,000,000	1.00	1.92076%	17.3602	2.09436%	17.00	0.774017	285,466
18	1,000,000	1.00	1.92073%	17.2366	2.09310%	18.00	0.761283	301,401
19	1,000,000	1.00	1.91796%	17.1395	2.08935%	19.00	0.748851	317,047
20	1,000,000	1.00	1.91258%	17.0896	2.08348%	20.00	0.736690	332,395
21	1,000,000	1.00	1.90496%	17.1029	2.07599%	21.00	0.724767	347,441
22	1,000,000	1.00	1.89619%	17.1771	2.06796%	22.00	0.713075	362,188
23	1,000,000	1.00	1.88690%	17.3055	2.05995%	23.00	0.701612	376,640
24	1,000,000	1.00	1.87751%	17.4813	2.05232%	24.00	0.690374	390,809
25	1,000,000	1.00	1.86847%	17.6977	2.04545%	25.00	0.679353	404,705
26	1,000,000	1.00	1.86024%	17.9480	2.03972%	26.00	0.668541	418,341
27	1,000,000	1.00	1.85329%	18.2254	2.03554%	27.00	0.657927	431,734
28	1,000,000	1.00	1.84809%	18.5231	2.03332%	28.00	0.647497	444,899
29	1,000,000	1.00	1.84515%	18.8344	2.03349%	29.00	0.637237	457,858
30	1,000,000	1.00	1.84496%	19.1524	2.03648%	30.00	0.627130	470,629

Basis Swap - 12ML Float Leg (Unknown) SOLVING FOR :

Row	Notional, N	YearFraction, τ_j	FloatRate, l_j^{12M}	PayDate, t_j	DiscFactor, $P(t_0,t_j)$	Total PV
0			0.00000%	0.00	1.000000	
1	1,000,000	1.00	1.91266%	1.00	0.985618	18,852
2	1,000,000	1.00	1.59142%	2.00	0.974556	34,361
18	1,000,000	1.00	2.09310%	18.00	0.761283	301,401
19	1,000,000	1.00	2.08935%	19.00	0.748851	317,047
20	1,000,000	1.00	2.08348%	20.00	0.736690	332,395
21	1,000,000	1.00	2.07599%	21.00	0.724767	347,441
22	1,000,000	1.00	2.06796%	22.00	0.713075	362,188
23	1,000,000	1.00	2.05995%	23.00	0.701612	376,640
24	1,000,000	1.00	2.05232%	24.00	0.690374	390,809
25	1,000,000	1.00	2.04545%	25.00	0.679353	404,705
26	1,000,000	1.00	2.03972%	26.00	0.668541	418,341
27	1,000,000	1.00	2.03554%	27.00	0.657927	431,734
28	1,000,000	1.00	2.03332%	28.00	0.647497	444,899
29	1,000,000	1.00	2.03349%	29.00	0.637237	457,858
30	1,000,000	1.00	2.03648%	30.00	0.627130	470,629

Figure 48: USD 12ML Basis Curve Bootstrapping for USD 12ML Forecast Rates

H Curve Calibration, Solving for Forward Rates

Here we present a stylistic example of how to calibrate a USD swap curve. Given USD market par rate quotes we can simultaneously imply OIS discount factors, OIS- and 3M Libor forward rates.

Multi-Dimensional Newton-Raphson Algorithm

$X_{n+1} = X_n - J^{-1} f(X_n)$

Tolerance	RMSE
1.00E-08	8.72E-12

Iteration: 4

Curve	Term	Time, t	X_{n+1}	X_n	Initial Guess X_0	$f(X_n)$	Epsilon
USDOIS	1Y	1.00	1.43591%	1.43591%	2.00000%	0.00000%	0.00E+00
USDOIS	2Y	2.00	1.23323%	1.23323%	2.00000%	0.00000%	2.69E-12
USDOIS	3Y	3.00	1.25107%	1.25107%	2.00000%	0.00000%	3.86E-12
USDOIS	4Y	4.00	1.29130%	1.29130%	2.00000%	0.00000%	1.00E-12
USDOIS	5Y	5.00	1.39782%	1.39782%	2.00000%	0.00000%	-3.89E-12
USD3ML	1Y	1.00	1.70896%	1.70896%	2.00000%	0.00000%	0.00E+00
USD3ML	2Y	2.00	1.47359%	1.47359%	2.00000%	0.00000%	3.13E-12
USD3ML	3Y	3.00	1.49531%	1.49531%	2.00000%	0.00000%	4.44E-12
USD3ML	4Y	4.00	1.55934%	1.55934%	2.00000%	0.00000%	5.28E-14
USD3ML	5Y	5.00	1.62999%	1.62999%	2.00000%	0.00000%	-2.89E-12

USDOIS Discount Factors

Integrate USDOIS Forward Polynomial

Time, t	DiscFactor	Integrand
1.00	0.982281	1.78781%
2.00	0.969579	3.08936%
3.00	0.957671	4.32509%
4.00	0.945574	5.59628%
5.00	0.933074	6.92710%

Update Solver

Figure 49: USD Curve Calibration using a Multi-Variate Newton-Raphson Algorithm

Calibration is achieved using a multi-variate Newton-Raphson technique, similar to that outlined in section (4.3.5). Using the recursive multi-dimensional Newton-Raphson algorithm given an initial guess X_0 we have,

$$X_{n+1} = X_n - \mathcal{J}^{-1} f(X_n) \tag{36}$$

where X_n denotes the Libor forward rate for the nth iteration, $f(X_n)$ is the swap par rate as a function of the Libor forward rate and $\mathcal{J}^{-1}$ denotes the inverse of the Jacobian matrix $\nabla \partial f / \partial X_n$.

In this particular case each element of the Jacobian matrix corresponds to $\partial p_i / \partial L_j$, where i represents the ith instrument par rate and j the jth Libor rate constituent required to price the instrument.

Jacobian, dParRate/dLibor (dp/dL)

p = PV(Float Leg)/Annuity(Fixed Leg)

$dp_j/dL_j = N\tau_j DF_j \,/\, A_i(\text{Fixed}) = DF_j \,/\, A_i(\text{Fixed})$, since N=1 and τ=1

$$J = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{bmatrix}$$

	dL_{1Y}^{OIS}	dL_{2Y}^{OIS}	dL_{3Y}^{OIS}	dL_{4Y}^{OIS}	dL_{5Y}^{OIS}	dL_{1Y}^{IRS}	dL_{2Y}^{IRS}	dL_{3Y}^{IRS}	dL_{4Y}^{IRS}	dL_{5Y}^{IRS}
$dP(OIS_{1Y})$	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$dP(OIS_{2Y})$	0.50	0.50	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$dP(OIS_{3Y})$	0.34	0.33	0.33	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$dP(OIS_{4Y})$	0.25	0.25	0.25	0.25	0.00	0.00	0.00	0.00	0.00	0.00
$dP(OIS_{5Y})$	0.21	0.20	0.20	0.20	0.19	0.00	0.00	0.00	0.00	0.00
$dP(IRS_{1Y})$	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00
$dP(IRS_{2Y})$	0.00	0.00	0.00	0.00	0.00	0.50	0.50	0.00	0.00	0.00
$dP(IRS_{3Y})$	0.00	0.00	0.00	0.00	0.00	0.34	0.33	0.33	0.00	0.00
$dP(IRS_{4Y})$	0.00	0.00	0.00	0.00	0.00	0.25	0.25	0.25	0.25	0.00
$dP(IRS_{5Y})$	0.00	0.00	0.00	0.00	0.00	0.21	0.20	0.20	0.20	0.19

Figure 50: USD Curve Jacobian Matrix

Using this approach we are able to calibrate the USD curve quickly in 4 iterations with precision 1.0e-8, which is quite typical.

Iteration Results

Curve	Term	X_0	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
USDOIS	1Y	2.00000%	1.43591%	1.43591%	1.43591%	1.43591%						
USDOIS	2Y	2.00000%	1.23252%	1.23323%	1.23323%	1.23323%						
USDOIS	3Y	2.00000%	1.25044%	1.25107%	1.25107%	1.25107%						
USDOIS	4Y	2.00000%	1.29112%	1.29130%	1.29130%	1.29130%						
USDOIS	5Y	2.00000%	1.39919%	1.39782%	1.39782%	1.39782%						
USD3ML	1Y	2.00000%	1.70896%	1.70896%	1.70896%	1.70896%						
USD3ML	2Y	2.00000%	1.47276%	1.47359%	1.47359%	1.47359%						
USD3ML	3Y	2.00000%	1.49458%	1.49531%	1.49531%	1.49531%						
USD3ML	4Y	2.00000%	1.55933%	1.55934%	1.55934%	1.55934%						
USD3ML	5Y	2.00000%	1.63100%	1.62999%	1.62999%	1.62999%						

Figure 51: USD Curve Calibration, Iteration Results

We integrate the forward rates to imply discount factors at the same time using,

$$P(t,T) = \exp\left(-\int_t^T f(u)\,du\right) \tag{37}$$

where $P(t,T)$ denotes the discount factor observed at time t for a cashflow payment at time T with $t < T$ and $f(t)$ denotes the instantaneous forward rate. This gives the following results,

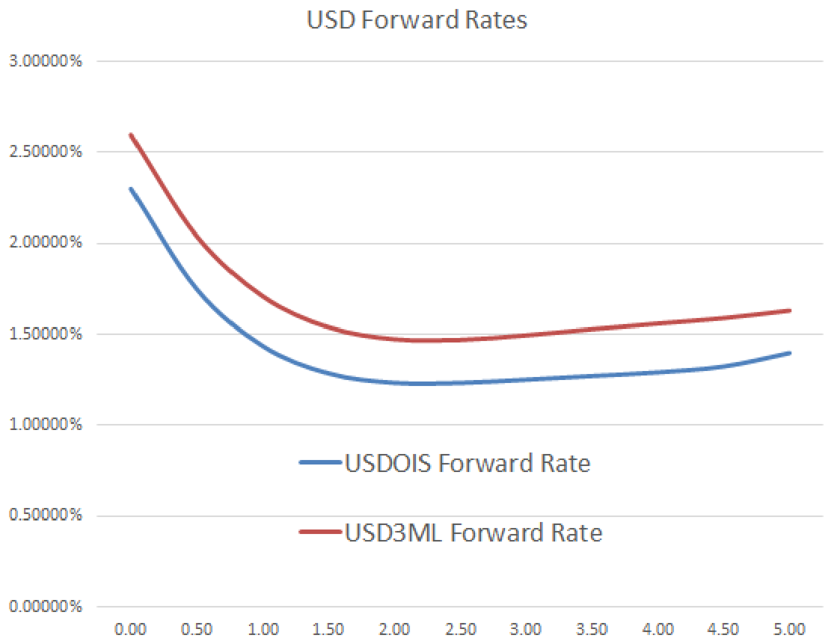


Figure 52: USD Curve Forward Rates

Numerical integration generally requires many abscissae to achieve a reasonable forward rate precision and can consequently be quite slow. Therefore market practitioners like to work with a well-defined interpolation function such as a polynomial to mitigate this issue and apply analytical integration.

Finally it should be noted that when using slope or gradient based solvers retaining the gradient matrix or **Jacobian**, $\mathcal{J}$ is most helpful, since the Jacobian provides a means to calculate analytical risk and perform ultra-fast curve calibration as outlined below. We provide an Excel demo for this, kindly email the author for a copy.

Usefulness of Jacobian, J

1. Analytical Risk

PV01 = Shift Size . J . Annuity

Tenor	ShiftSize	dP	Annuity	PV01 *
1Y	0.01%	1.00E-04	985,846	**98.58**
2Y	0.01%	9.94E-05	1,959,657	**194.76**
3Y	0.01%	9.96E-05	2,921,402	**290.94**
4Y	0.01%	9.97E-05	3,870,746	**385.87**
5Y	0.01%	9.97E-05	4,807,437	**479.54**

* PV01 = PV Change for 1 Basis Point Libor Rate Change

Derivative Matrices

$J = f'(X_n) = dp/dL$

	L_1	L_2	L_3	L_4	L_5
f_1	1.00	0.00	0.00	0.00	0.00
f_2	0.50	0.50	0.00	0.00	0.00
f_3	0.33	0.33	0.33	0.00	0.00
f_4	0.25	0.25	0.25	0.25	0.00
f_5	0.20	0.20	0.20	0.20	0.19

2. Ultra-Fast Yield Curves

$L_{NEW} = L_{OLD} + Mkt\ Data\ Chg\ .\ J_{OLD}$

Tenor	Original Libor Rate	Change in Mkt Data	New Libor Rate	+/-
1Y	1.73357%	1.00000%	**2.73357%**	1.00000%
2Y	1.48088%	0.75000%	**2.35050%**	0.86963%
3Y	1.48139%	0.50000%	**2.22934%**	0.74794%
4Y	1.51764%	0.25000%	**2.14345%**	0.62582%
5Y	1.58209%	0.00000%	**2.08598%**	0.50388%

Inverse J = Inverse $f'(X_n)$

	L_1	L_2	L_3	L_4	L_5
f_1	1.00	0.00	0.00	0.00	0.00
f_2	-1.00	2.01	0.00	0.00	0.00
f_3	0.00	-2.04	3.04	0.00	0.00
f_4	0.00	0.00	-3.08	4.08	0.00
f_5	0.00	0.00	0.00	-4.13	5.13

Figure 53: Usefulness of the Curve Jacobian