

Low Latency Interest Rate Markets

Theory, Pricing & Practice



Nicholas Burgess

PART ONE: Theory

IR Markets, Products & Models

- Introduction to IR Markets
- Interest Rate Swaps
- IR Products & CDS
- Yield Curves
- IR Risk
- Credit Models

PART TWO: Pricing & Practice

Case Studies

- IRS Pricing Formulae
- IRS Pricing Case Study
- Asset Swap Structuring
- Asset Swap Pricing Case Study
- Pricing Tricks & Rules of Thumb

Quant Research Papers

<https://ssrn.com/author=1728976>

Support Materials: Quant Research, C++ and Excel Examples

<https://github.com/nburgessx/SwapsBook>



PART ONE - THEORY

IR Markets, Products & Models

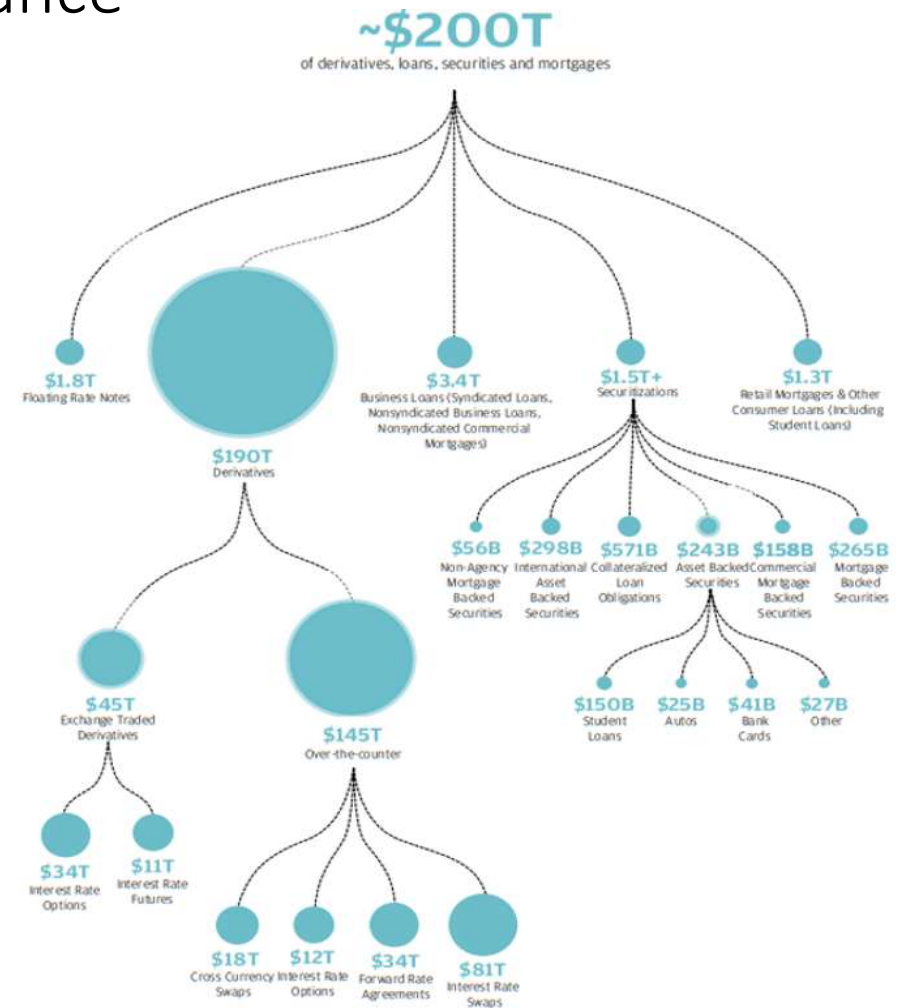
Interest Rate Markets - Project Finance

Purpose

- To Facilitate Government, Corporate & Project Finance
- Mortgages, Corporate Loans, Gov Projects & Infrastructure
- e.g. Hospitals, Transport (HS2), Energy & Defence Projects

Market Size

- Market Size by Notional: \$200T (US) + \$150T (EU)
- Derivatives, Loans & Securities
- All Referencing LIBOR, until Recently

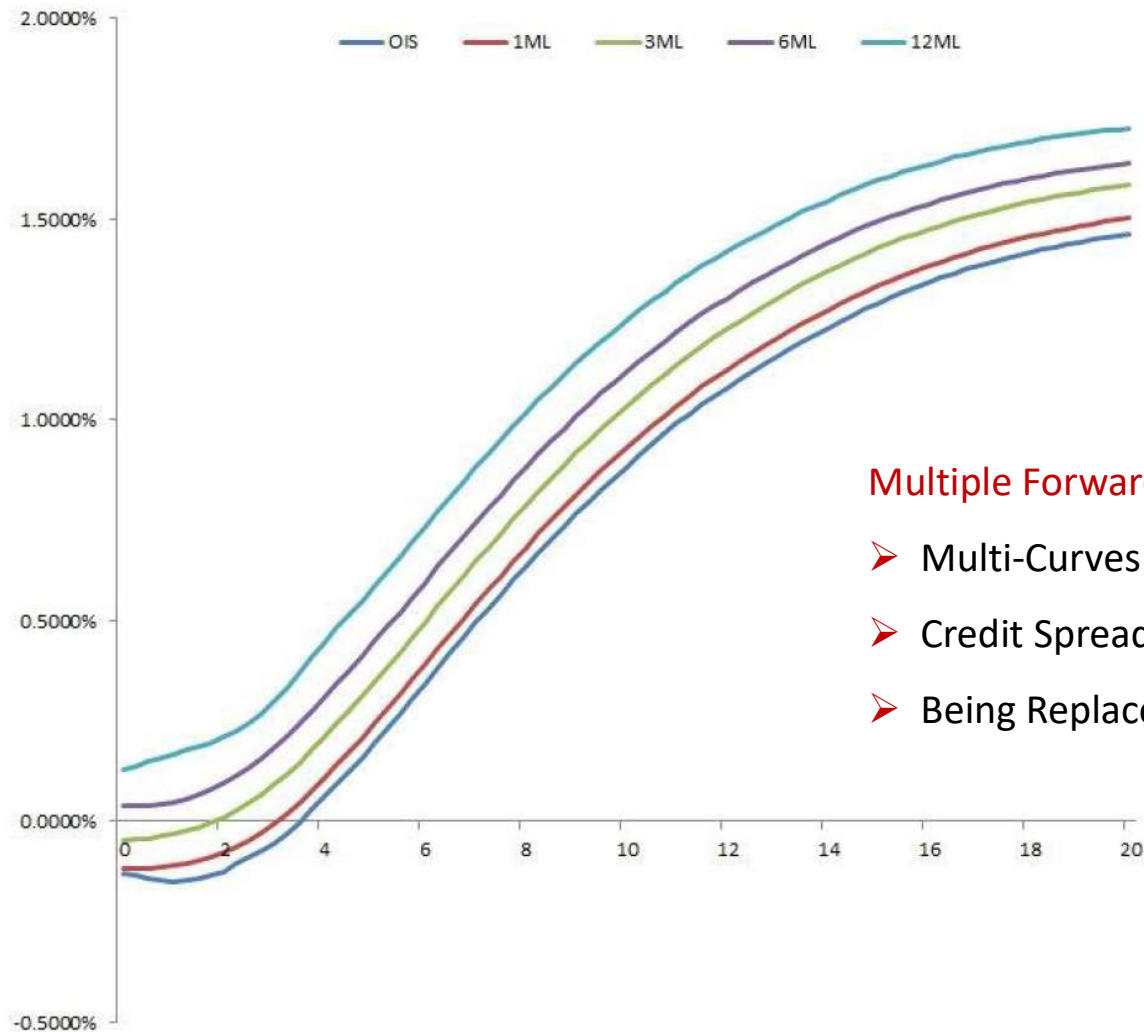


Interest Rate Markets – Why the need for Speed?

- Cleared **Electronic Trading** & Auto-Hedging
- Real-Time, Highly Liquid & High Precision (Bid-Offer 1/10th bps i.e. USD 10 per MM)
- Trading Horizon: **High Frequency Trading** (HFT) vs Long-Term Fund Performance

USD Semi vs 3M Libor					USD Spreads vs Treasuries				
31) 1 Year	0.750 / 0.754	+0.014	≡		71) 1 Year	4.282 / 5.295	+0.687		
32) 2 Year	1.045 / 1.049	+0.017	≡		72) 2 Year	10.248 / 10.806	-0.073	≡	
33) 3 Year	1.284 / 1.287	+0.018	≡		73) 3 Year	3.337 / 3.895	-0.029	≡	
34) 4 Year	1.467 / 1.471	+0.015	≡		74) 4 Year	1.350 / 1.900	+0.161		
35) 5 Year	1.617 / 1.621	+0.014	≡		75) 5 Year	-4.020 / -3.454	+0.138	≡	
36) 6 Year	1.750 / 1.754	+0.012	≡		76) 6 Year	-8.100 / -7.550	+0.157		
37) 7 Year	1.866 / 1.870	+0.011	≡		77) 7 Year	-13.577 / -13.036	+0.382	≡	
38) 8 Year	1.966 / 1.970	+0.011	≡		78) 8 Year	-11.100 / -10.550	+0.335		
39) 9 Year	2.052 / 2.056	+0.011	≡		79) 9 Year	-9.888 / -9.088	+0.492		
40) 10 Year	2.126 / 2.129	+0.011	≡		80) 10 Year	-9.775 / -9.275	+0.537	≡	
41) 12 Year	2.250 / 2.254	+0.007	≡		81) 12 Year	2.520 / 3.320	+0.204		
42) 15 Year	2.376 / 2.380	+0.006	≡		82) 15 Year	-3.599 / -2.799	+0.110		
43) 20 Year	2.497 / 2.501	+0.002	≡		83) 20 Year	-10.100 / -9.600	+0.150		
44) 25 Year	2.558 / 2.563	+0.003	≡		84) 25 Year	-22.800 / -22.250	+0.150		
45) 30 Year	2.592 / 2.597	+0.000	≡		85) 30 Year	-38.058 / -37.491	+0.351	≡	
46) 40 Year	2.612 / 2.621	+0.003	≡						
47) 50 Year	2.598 / 2.604	+0.004	≡						

Interest Rate Markets – Yield Curve Models



Required to Forecast Future Interest Rates

- Use **Liquid** Market Instruments
- To Imply Forward Rates & Disc. Factors

Multiple Forward Curves

- Multi-Curves Have In-Built **Credit Spread** (Tenor Homogenous)
- Credit Spread Determined by Loan Repayment Frequency
- Being Replaced by Single RFR Curves (Similar to OIS Curve)

Interest Rate Markets – The LIBOR Problem

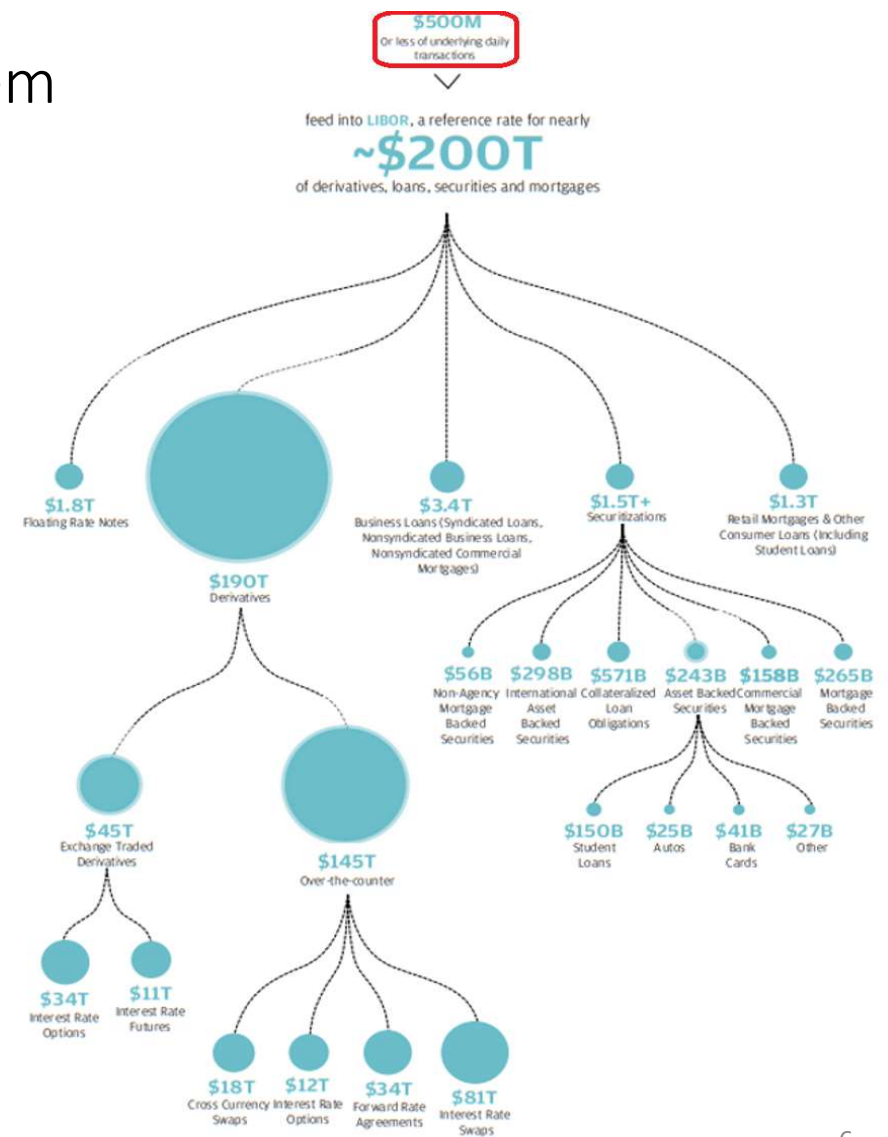
The Problem with LIBOR

- LIBOR Market Transactions < \$500M
- Rates Do Not Reflect Actual Borrowing Levels
- LIBOR Levels Increasingly Set by Panel/Expert Judgement

Market Size

- Market Size by Notional: \$200T (US) + \$150T (EU)

Large Market Driven by Small Number of LIBOR Transactions!!!



Interest Rate Markets – LIBOR Benchmark Replacement

LIBOR Rates

- Low Transaction Volume / Panel Based
- Forward Looking **Term Rate**, known **In-Advance**
- In Built Credit Risk Component

Risk-Free Rates (RFRs)

- Transaction Based
- Backward Looking Rate, Known **In-Arrears**
- No Credit Component i.e. Risk-Free

Market Changes

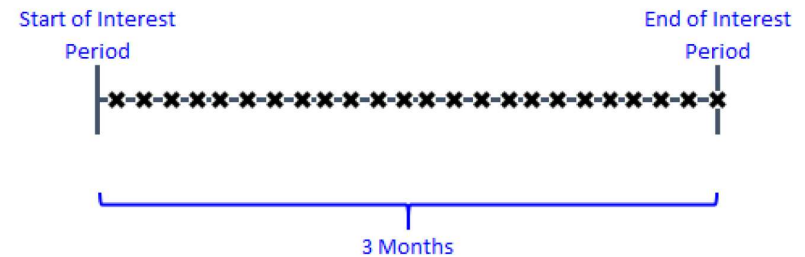
- Legacy LIBOR Contracts, Fall-Back Rates
- New RFR Products & Yield Curve Model Changes



Rate: Term Rate Fixed In-Advance

Coupon: Determined in Advance

3 Month Risk-Free Rate



Rate: Daily O/N Fixings leading to an Averaged Effective Rate

Coupon: Determined in Arrears

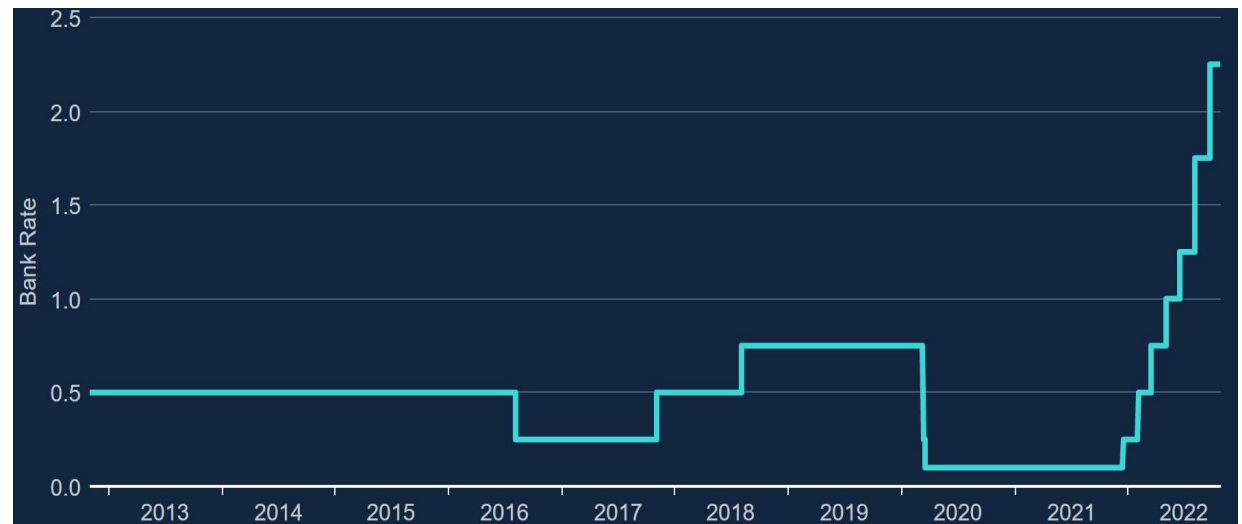
Interest Rate Markets – Project Finance Risks & Solutions

1. Interest Rate Risk

- Finance linked to variable interest rates
- Use IRS to Fix Borrowing Costs

2. Foreign Exchange / Currency Risk

- International Finance
- Use Cross Currency Swaps to Fix FX Rates



3. Credit Default Risk

- Bonds, Bi-Lateral and Non-Cleared Transactions
- Risk of Counterpart Default
- Credit Default Swaps, Collateral & CSA Agreements

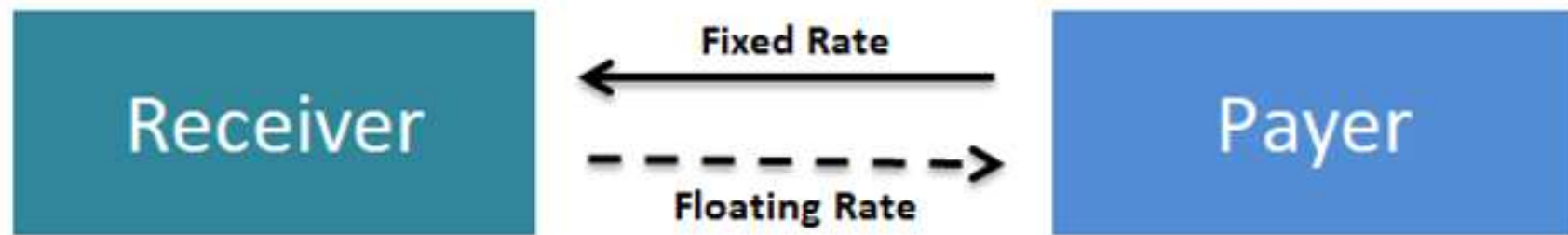
4. No money to invest?

- Use Asset Swaps to Borrow Funds to Invest in Bonds
- Pay LIBOR + Spread (Finance) to Receive Bond Coupons
- Floating Spread includes Funding + Credit Costs

Interest Rate Swaps – Fixed or Variable Borrowing Costs?

Project Finance

- Project Finance Naturally Incurs Variable Interest Costs (LIBOR + Spread)
- Exposed to Interest Rate Risk (Market may Move Against Us)



Hedging Interest Rate Risk

- Use IRS to Exchange Floating for Fixed Interest (or Vice Versa)
- We Can Choose to Fix Borrowing Costs
- We Also Trade IRS for Speculative Purposes

Interest Rate Swaps –Market Quotes & Pricing

USD Semi vs 3M Libor				USD Spreads vs Treasuries			
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- Standard Tenors: Spread Over US Treasury Yields
- New Swaps: Par Rate (%), since PV=0
- Existing Swaps: Present Value (USD)

Interest Rate Swaps – Present Value

The screenshot displays a financial software interface for Interest Rate Swaps. The main window is titled 'Swap Manager' and contains several tabs: 3) Main, 4) Details, 5) Curves, 6) Cashflow, 7) Resets, 9) Scenario, 10) Risk, 11) CVA, 12) Matrix, 20) Properties, 21) Calculators, and 23) More Greeks. The 'Main' tab is active, showing the 'Fixed Float Swap' deal details. The 'Deal' section includes 'CCP' (OTC), 'Counterparty' (SWAP CNTRPARTY), and 'Ticker' (SWAP). The 'Swap' section shows 'Leg 1: Fixed' (Receive) and 'Leg 2: Float' (Pay). The 'Valuation Settings' section includes 'Curve Date' (08/21/2015), 'Valuation' (08/25/2015), 'OIS DC Strip' (ON), and 'CSA Coll Ccy' (USD). The 'Market' section shows 'Dscnt' (42) and 'Fwd' (23) for 'USD Bloomberg Curv'. The 'Valuation Results' section shows 'Par Cpn' (1.548250), 'Principal' (167,892.11), 'Accrued' (0.00), and 'NPV' (167,892.11). The 'Premium' is 16.78921, and the 'BP Value' is 1678.92112. The 'PV01' is 486.40, 'DV01' is 532.42, and 'Gamma (1bp)' is 0.29.

Present Value is the Sum of Discounted Cash Flows

$$Swap\ PV = \underbrace{\sum_{i=1}^n N r \tau_i P(t_0, t_i)}_{\text{Fixed Cash Flows}} - \underbrace{\sum_{j=1}^m N (l_{j-1} + s) \tau_j P(t_0, t_j)}_{\text{Floating Cash Flows}}$$

Interest Rate Swaps – Par Rate

- New Swaps Trade at Par i.e. $PV = 0$
- Consequently such Swaps Quote as a Par Rate
- This is the fixed rate that makes both trade legs equal

$$Swap\ PV = \underbrace{r \sum_{i=1}^n N \tau_i P(t_0, t_i)}_{Fixed\ Cash\ Flows} - \underbrace{\sum_{j=1}^m N (l_{j-1} + s) \tau_j P(t_0, t_j)}_{Floating\ Cash\ Flows} = 0$$

Rearrange for the Fixed Rate r and call this the Par Rate, p

$$Par\ Rate, p = \frac{PV(Float\ Leg)}{\sum_{i=1}^n N \tau_i P(t_0, t_i)} = \frac{PV(Float\ Leg)}{Annuity(Fixed\ Leg)^1}$$

¹ Par Rates calculated in terms of Annuity or PV01

Interest Rate Swaps - Specification

- Majority of Swap Booking Schedule Related
- Trading **Templates**, Generators & Static Data

Swap Generator Template			
USD_SWAP_3M			
Dynamic Trade Info	LEG TYPE	LEG1:FIXED	LEG2:FLOAT
	PAY / RECEIVE	PAY	RECEIVE
	NOTIONAL	1,000,000	1,000,000
	FIXED RATE (%)	1.00%	-
	FLOAT SPREAD (BPS)	-	0.00
	EFFECTIVE DATE / LAG	2D	2D
	MATURITY DATE / TENOR	2Y	2Y
	LEG CURRENCY	USD	USD
	NOTIONAL EXCHANGE	NONE	NONE
	LEVERAGE	1.00	1.00
Static Data + Schedule Info	FRONT STUB INDEX	-	NATURAL
	BACK STUB INDEX	-	NATURAL
	VALUATION CURRENCY	USD	USD
	FORECAST INDEX	-	USD3M
	DISCOUNT INDEX	USDOIS	USDOIS
	INDEX COMPOUND METHOD	-	NONE
	SPREAD COMPOUND METHOD	-	NONE
	ROLL DAY	END	END
	STUB TYPE	SHORT START	SHORT START
	FIXING BUS DAY ADJUSTMENT	-	MODIFIED_FOLLOWING
	FIXING CALENDAR	-	NY+LDN
	FIXING LAG	-	2D
	FIXING IN-ADVANCE / IN-ARREARS	-	IN-ADVANCE
	ACCRUAL FREQUENCY	SEMI-ANNUAL	QUARTERLY
	ACCRUAL BUS DAY ADJUSTMENT	MODIFIED_FOLLOWING	MODIFIED_FOLLOWING
	ACCRUAL CALENDAR	NY	NY
	ACCRUAL DAYCOUNT	30/360	ACT/360
	PAYMENT FREQUENCY	SEMI-ANNUAL	QUARTERLY
	PAYMENT BUS DAY ADJUSTMENT	MODIFIED_FOLLOWING	MODIFIED_FOLLOWING
	PAYMENT CALENDAR	NY	NY
	PAYMENT LAG	2D	2D

TRADE PARAMETERS		LEG1	LEG2
TRADE ECONOMICS	LegType	FLOAT	FLOAT
	Currency	EUR	USD
	Notional	8,769,622	10,000,000
	NotionalExchange	ALL	ALL
	PayReceive	PAY	RECEIVE
	EffectiveDate	Fri, 26-Oct-18	Fri, 26-Oct-18
	MaturityDateOrTenor	1Y	1Y
	FixedRate (%)	-	-
	FloatSpread (Bps)	0.00	0.00
	IndexCompoundMethod	-	NONE
MTM SWAPS	SpreadCompoundMethod	-	NONE
	Leverage	1.00	1.00
	ForecastCurve	EUR3M	USD3M
	DiscountCurve	EURDF_USDCSA	USDDF
	IsMTMResetLeg	FALSE	TRUE
	ResetBaseFX	1.00000	1.14030
	ValuationCurrency	USD	USD
	CouponRollDay	NATURAL	NATURAL
	isEndOfMonth	TRUE	TRUE
	StubType	SHORT_START	SHORT_START
COUPON & STUB CONVENTIONS	FrontStubCurveIndex	NATURAL	NATURAL
	BackStubCurveIndex	NATURAL	NATURAL
	FrontStubDate	-	-
	BackStubDate	-	-
	AccrualFrequency	QUARTERLY	QUARTERLY
	AccrualCalendar	TGT+NY+LON	TGT+NY+LON
	AccrualBusDayConv	MOD_FOLLOWING	MOD_FOLLOWING
	AccrualDaycount	ACT/360	ACT/360
	IRFixingBusDayConv	MOD_FOLLOWING	MOD_FOLLOWING
	IRFixingCalendar	TGT+NY+LON	TGT+NY+LON
SCHEDULE INFORMATION	IRFixingLag	2D	2D
	IRFirstFixingLag	-	-
	PaymentFrequency	QUARTERLY	QUARTERLY
	PaymentBusDayConv	MOD_FOLLOWING	MOD_FOLLOWING
	PaymentCalendar	TGT+NY+LON	TGT+NY+LON
	PaymentLag	2D	2D
	IsNonDeliverable	FALSE	FALSE
	SettlementCurrency	-	-
	FXFixingLag	-	-
	FXFixingBusDayConv	-	-
NON-DELIVERABLES	FXFixingCalendar	-	-

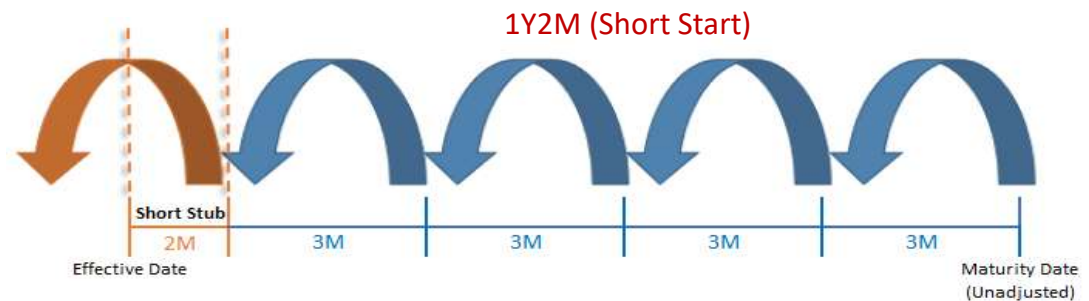
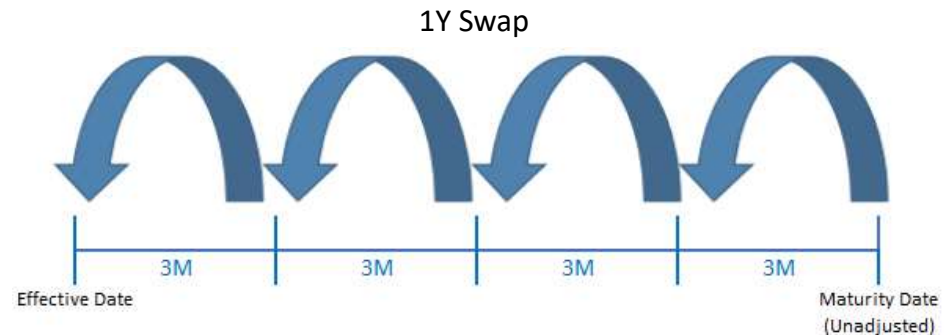
Interest Rate Swaps - Schedules & Stubs

Swap Schedules

- Backwards vs Forward Rolling Schedules
- Unadjusted to Preserve Roll Day
- Holiday Adjustments Ex-Ante
- Accrual Day Count Conventions

Broken-Dated Swaps

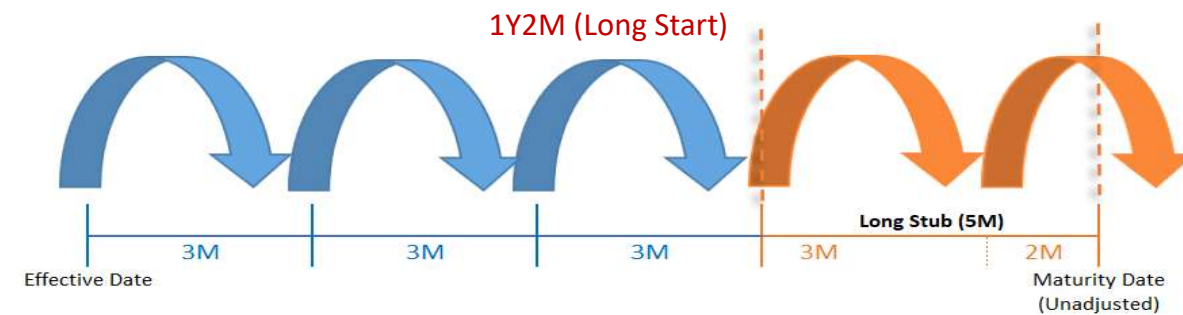
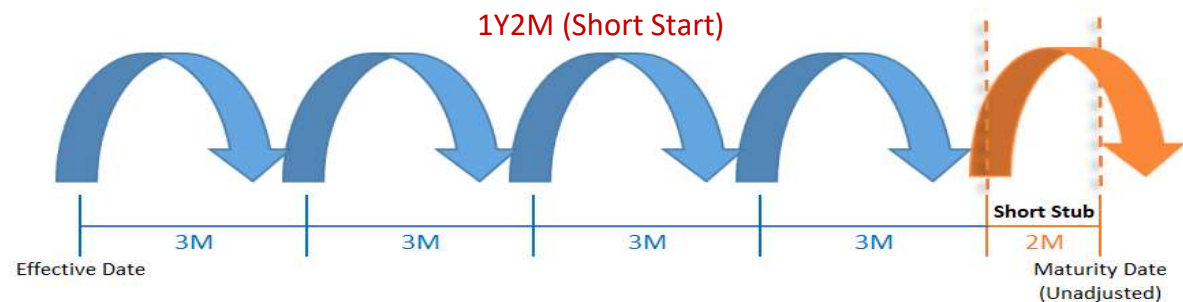
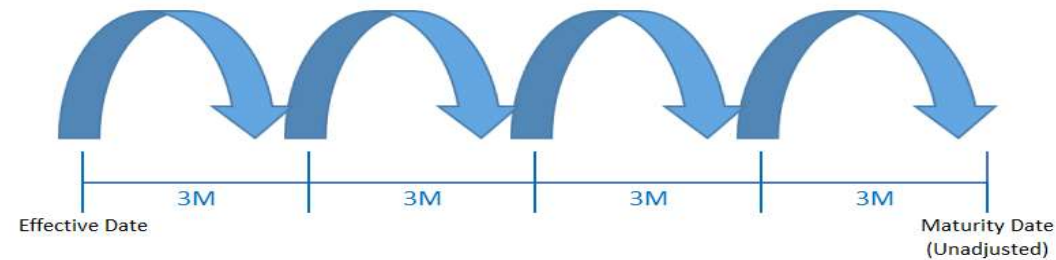
- Stubs & Stub Rates (Linear Interp)
- Short Start/End, Long Start/End
- Market Default: **Short Start**



Interest Rate Swaps - Forward Roll Schedules

Forward Roll Schedules

- End Stubs
- Regular, Short End or Long End
- Less Popular



IR Products – Tenor & Xccy Basis Swaps

Tenor Basis Swaps

- Float vs Float (Same Currency)
- Exchange USD3M for USD6M say
- Match Project Cash Flow Frequency

Tenor Basis Swap Formulae (December 30, 2015).

Available at SSRN: <https://ssrn.com/abstract=2959605>

Xccy Basis Swaps

- Float vs Float (Different Currencies)
- Exchange USD3M for EUR3M say
- Marked-to-Market / FX Notional Resets
- Reduces XVA Costs

An Illustrated Step-by-Step Guide of How to Price Cross Currency Swaps (November 11, 2018).

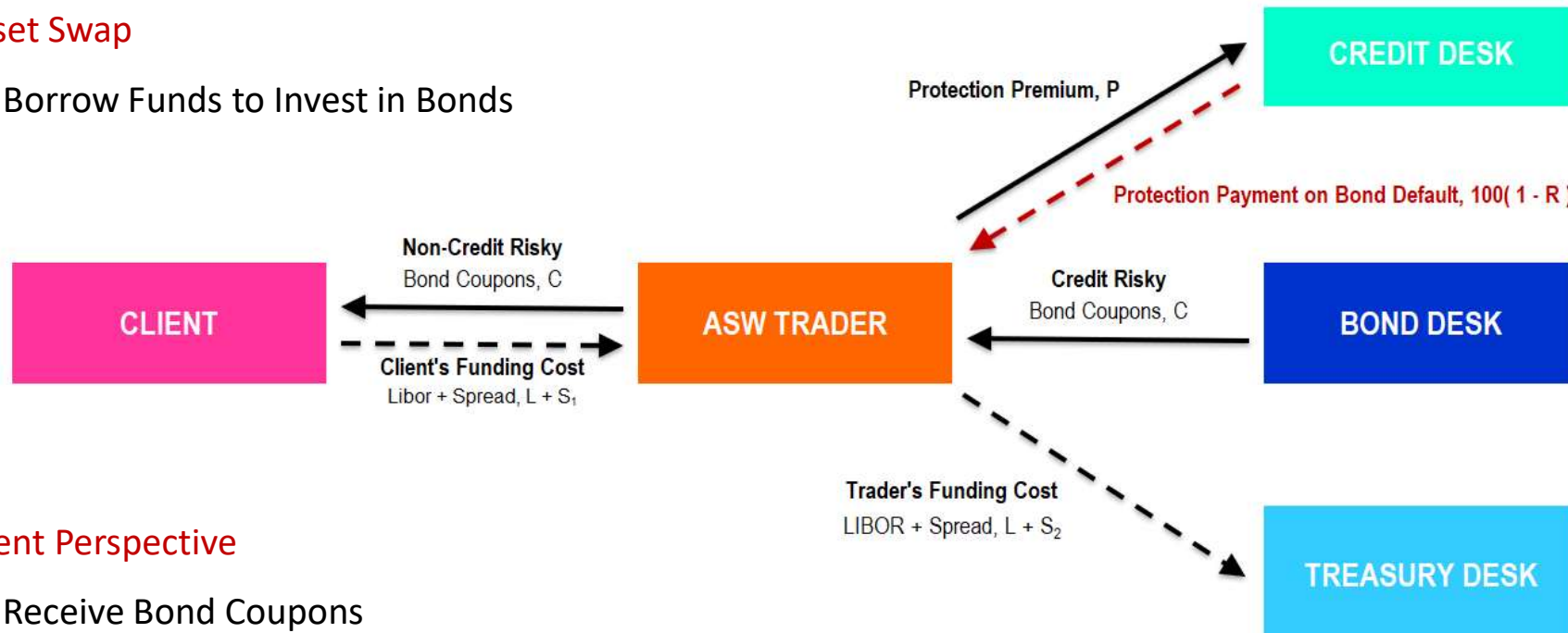
Available at SSRN: <https://ssrn.com/abstract=3278907>

91) Actions ▾	92) Products ▾	93) Views ▾	94) Info ▾	95) Settings ▾	Swap Manager
Solver (Premium) ▾	Load	Save	Trade ▾	CCP ▾	
3) Main	4) Details	5) Curves	6) Cashflow	7) Resets	9) Scenario
10) Risk	12) Matrix				
Deal	MTM XCCY Swap	Counterparty	SWAP CNTRPARTY ▾	Ticker / SWAP	20) Properties
Swap	*Notional Reset b...	3 Month Euribor	Valuation Settings		
Leg 1:Float	Receive ▾	Leg 2:Float	Pay ▾	Curve Date	03/22/2019
Notional	1MM	Notional	884,799.15	Valuation	03/26/2019
Currency	USD ▾	Currency	EUR ▾	CSA Coll Ccy	USD ▾
Effective	0D 03/26/2019	Effective	0D 03/26/2019	Valuation Ccy	USD ▾
Maturity	1Y 03/26/2020	Maturity	1Y 03/26/2020	FX Rate	1.130200
Index	3M US0003M	Index	3M EUR003M	<input checked="" type="checkbox"/> OIS DC Stripping	
Spread	0.000 bp	Spread	-12.625 bp		
Leverage	1.00000	Leverage	1.00000		
Latest Index	2.60988	Latest Index	-0.30900		
Reset Freq	Quarterly ▾	Reset Freq	Quarterly ▾		
Pay Freq	Quarterly ▾	Pay Freq	Quarterly ▾		
Day Count	ACT/360 ▾	Day Count	ACT/360 ▾		
Market					
Leg 1: NPV	1,002,566.12	Leg 2: NPV	-1,002,566.12		
Accrued	0.00	Accrued	0.00		
Premium	100.26	Premium	-100.26		
DV01	22.74	DV01	-22.74		
Valuation Results				22) Calculators ▾	
Principal	0.00	Premium	0.00000	BR01 92:EUR vs.	-102.10
Accrued	0.00	BP Value	0.00000	DV01	0.00
NPV	0.00			Gamma (1bp)	0.00

IR Products – Asset Swaps

Asset Swap

- Borrow Funds to Invest in Bonds



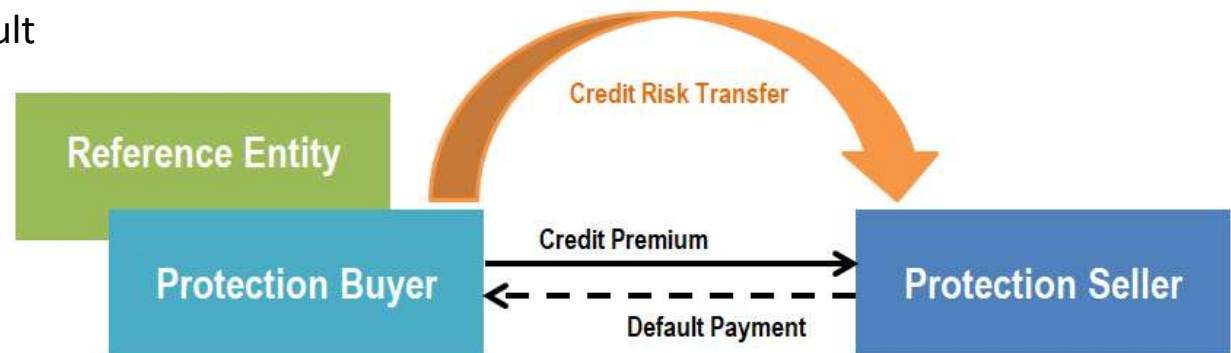
Client Perspective

- Receive Bond Coupons
- Pay LIBOR + Spread
- Spread Includes Finance + Credit Costs

IR Products – Credit Default Swaps (CDS)

Insurance Against Counterparty Default

- Insuring Bond Notional Invested
- Pay Fixed Insurance Premium
- Receive Protection Payment on Default



Credit Crisis & ISDA Big Bang (2008)

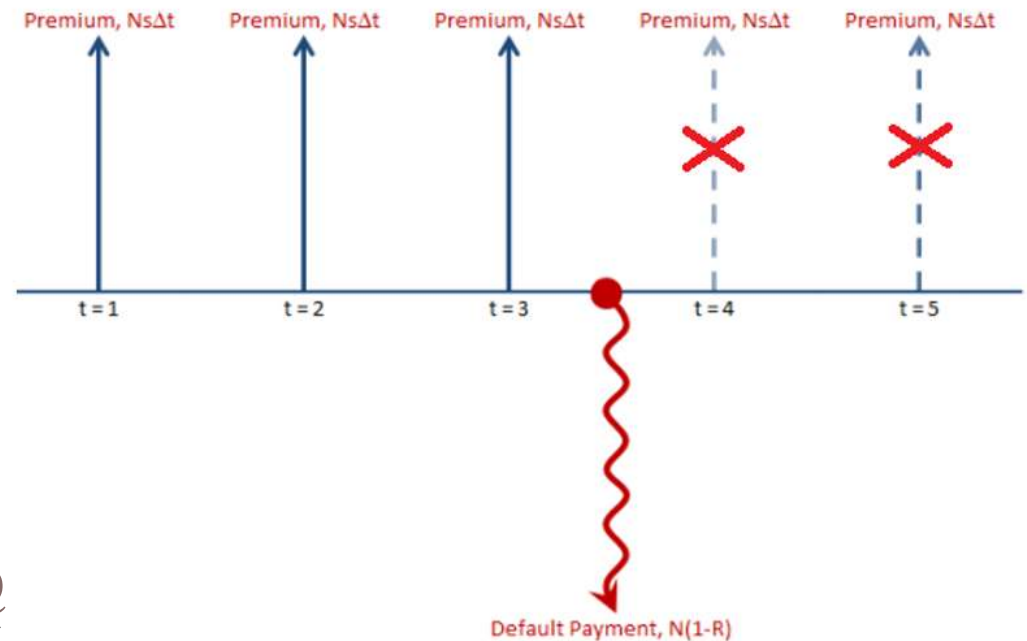
- Standardized & Cleared Contracts (IMM Dates¹)
- Increased Liquidity
- Accrued Interest, Clean & Dirty Prices

¹ Third Wednesday of Mar, June, Sep and Dec

IR Products – CDS Pricing

Pricing

- Similar to Interest Rate Swap Pricing
- With Additional Survival Probability Term, $Q(t,T)$
- $Q(t,T) = \exp\left(-\int_t^T \lambda(t,u)du\right)$
- λ is the 'Hazard Rate' (instantaneous prob of default)



Buying Credit Protection

$$PV = PV(\text{Protection Leg}) - PV(\text{Premium Leg})$$

$$PV(\text{Premium Leg}) = \sum_{i=1}^n \underbrace{N s \tau_i \Delta(t_{i-1}, t_i)}_{\text{Coupon}} \underbrace{\frac{Q(t_i)}{P(\text{Survive})}}_{P(\text{Survive})} \underbrace{\frac{P(t_0, t_i)}{\text{Discount Factor}}}_{\text{Discount Factor}}$$

$$PV(\text{Protection Leg}) = \sum_{i=1}^n \underbrace{N(1-R)}_{\text{Loss Given Default}} \underbrace{\frac{[Q(t_{i-1}) - Q(t_i)]}{\text{Default within Premium Period}}}_{\text{Default within Premium Period}} \underbrace{\frac{P(t_0, t_i)}{\text{Discount Factor}}}_{\text{Discount Factor}}$$

IR Risk

What are the main IR risks?

- Discount Risk (DF01)
- Forward Risk (PV01)
- Discount + Forward Risk (DV01)

Risk Calculation Methods

- Analytical
- Numerical Risk (Benchmark)
- Using Yield Curve Jacobian
- Automatic Adjoint Differentiation (AAD)

USD SOFR YIELD CURVE - CALIBRATION INSTRUMENTS

Instrument	Term	Rate
USD SOFR Swap	ON	2.37000%
USD SOFR Swap	1W	2.36510%
USD SOFR Swap	2W	2.34960%
USD SOFR Swap	3W	2.35200%
USD SOFR Swap	1M	2.34550%
USD SOFR Swap	2M	2.30320%
USD SOFR Swap	3M	2.25590%
USD SOFR Swap	4M	2.19610%
USD SOFR Swap	5M	2.14750%
USD SOFR Swap	6M	2.10350%
USD SOFR Swap	1Y	1.89350%
USD SOFR Swap	2Y	1.68360%
USD SOFR Swap	3Y	1.62600%
USD SOFR Swap	4Y	1.61700%
USD SOFR Swap	5Y	1.64200%
USD SOFR Swap	6Y	1.67900%
USD SOFR Swap	7Y	1.71600%
USD SOFR Swap	8Y	1.75700%
USD SOFR Swap	9Y	1.79800%
USD SOFR Swap	10Y	1.83200%
USD SOFR Swap	15Y	1.96800%
USD SOFR Swap	20Y	2.03300%
USD SOFR Swap	25Y	2.04100%
USD SOFR Swap	30Y	2.04900%

Bucketed DV01, USD

Instrument	Tenor	DV01
USD SOFR Swap	ON	8
USD SOFR Swap	1W	0
USD SOFR Swap	2W	0
USD SOFR Swap	3W	0
USD SOFR Swap	1M	0
USD SOFR Swap	2M	0
USD SOFR Swap	3M	0
USD SOFR Swap	4M	0
USD SOFR Swap	5M	-1
USD SOFR Swap	6M	1
USD SOFR Swap	1Y	92
USD SOFR Swap	2Y	213
USD SOFR Swap	3Y	294
USD SOFR Swap	4Y	409
USD SOFR Swap	5Y	453
USD SOFR Swap	6Y	541
USD SOFR Swap	7Y	723
USD SOFR Swap	8Y	736
USD SOFR Swap	9Y	852
USD SOFR Swap	10Y	892
USD SOFR Swap	15Y	1,320
USD SOFR Swap	20Y	1,662
USD SOFR Swap	25Y	1,979
USD SOFR Swap	30Y	2,252
Total Risk		12,428

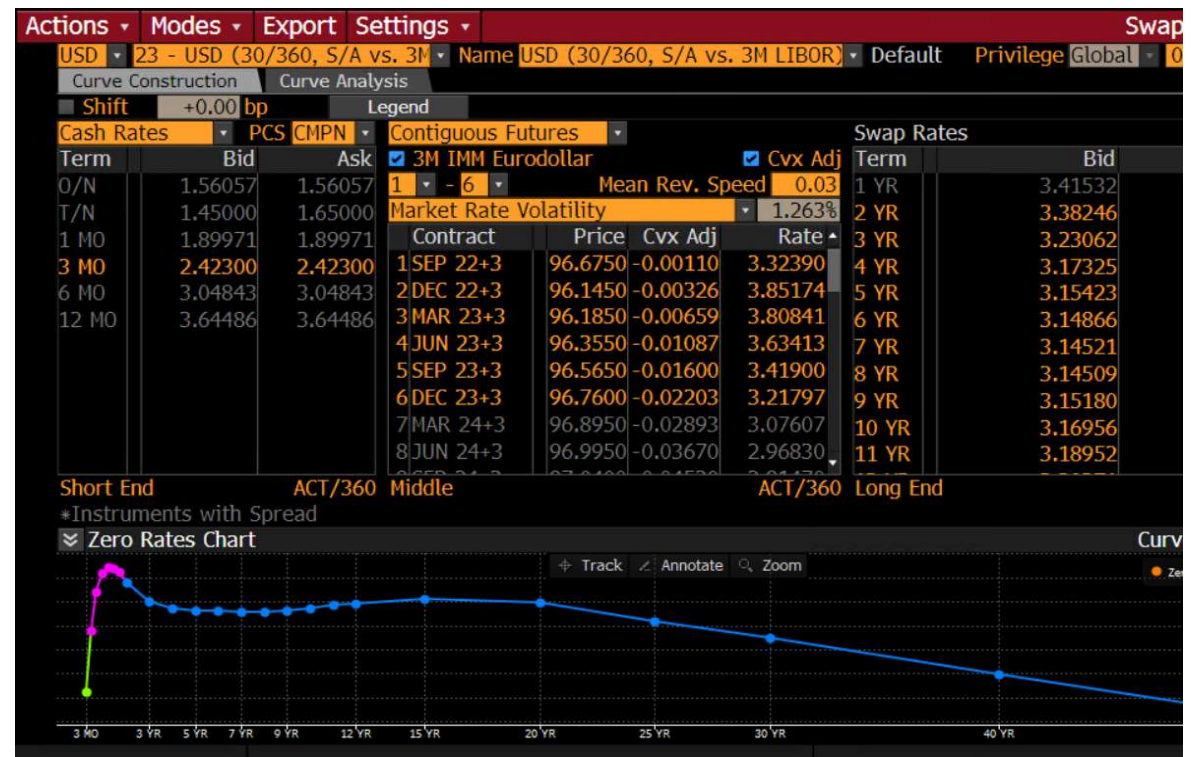
Yield Curves - Calibration

Model Inputs & Outputs

- Liquid Market Instrument Quotes [IN]
- Forward Rates [OUT]
- Discount Factors [OUT]

Calibration Process

- Choose State Variable¹
- Choose Interpolator (Functional Form)
- Solve and Imply Forwards & Disc Factors²



¹ Popular choices: forward rate, disc factor, logDF, zero rate etc.

² May need to differentiate and/or integrate state variable, $P(t, T) = \exp\left(-\int_t^T f(t, u) du\right)$

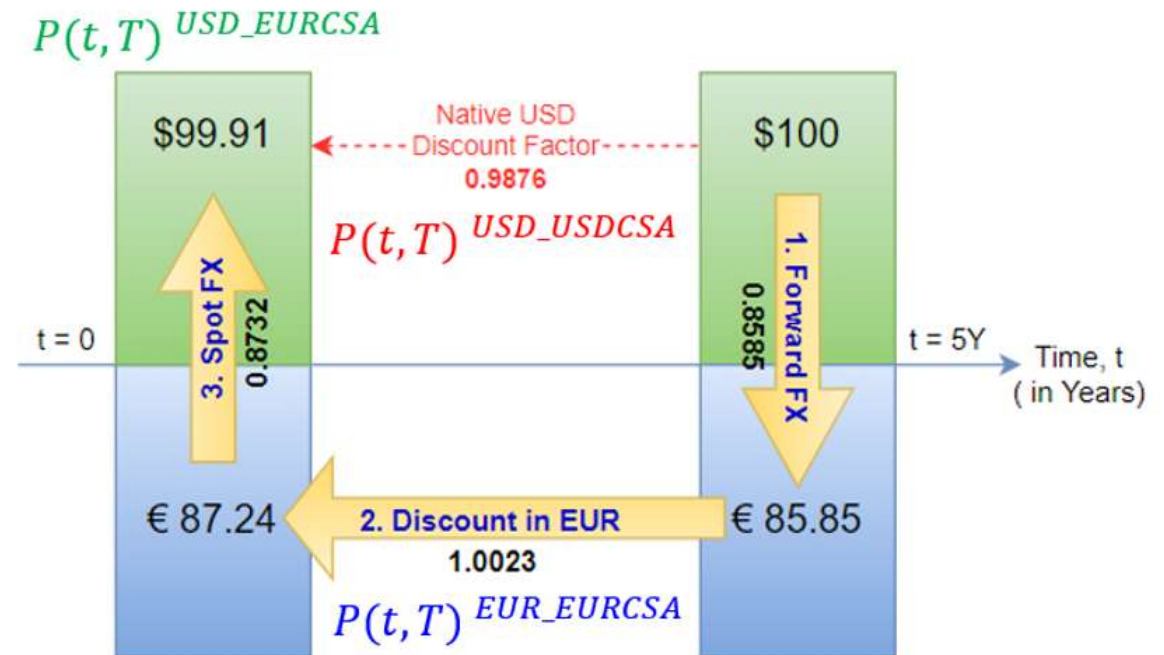
Yield Curves – Collateral & CSA Curves

Collateral & CSA Curves

- Calibrate to FX Forwards & Xccy Swaps
- **FX Forward Invariance** (FX Carry Trade)
- Impacts Discount Factors Only
- No Impact on Forward Rates

Advanced CSA Topics

- Cheapest to Deliver (Multiple CSAs)
- Collateral Switch Options



$$f(t, T)^{USD/EUR} = s(t)^{USD/EUR} \underbrace{\left(\frac{P(t, T)^{EUR_USD_CSA}}{P(t, T)^{USD_USD_CSA}} \right)}_{USD \text{ CSA}} = s(t)^{USD/EUR} \underbrace{\left(\frac{P(t, T)^{EUR_EUR_CSA}}{P(t, T)^{USD_EUR_CSA}} \right)}_{EUR \text{ CSA}}$$

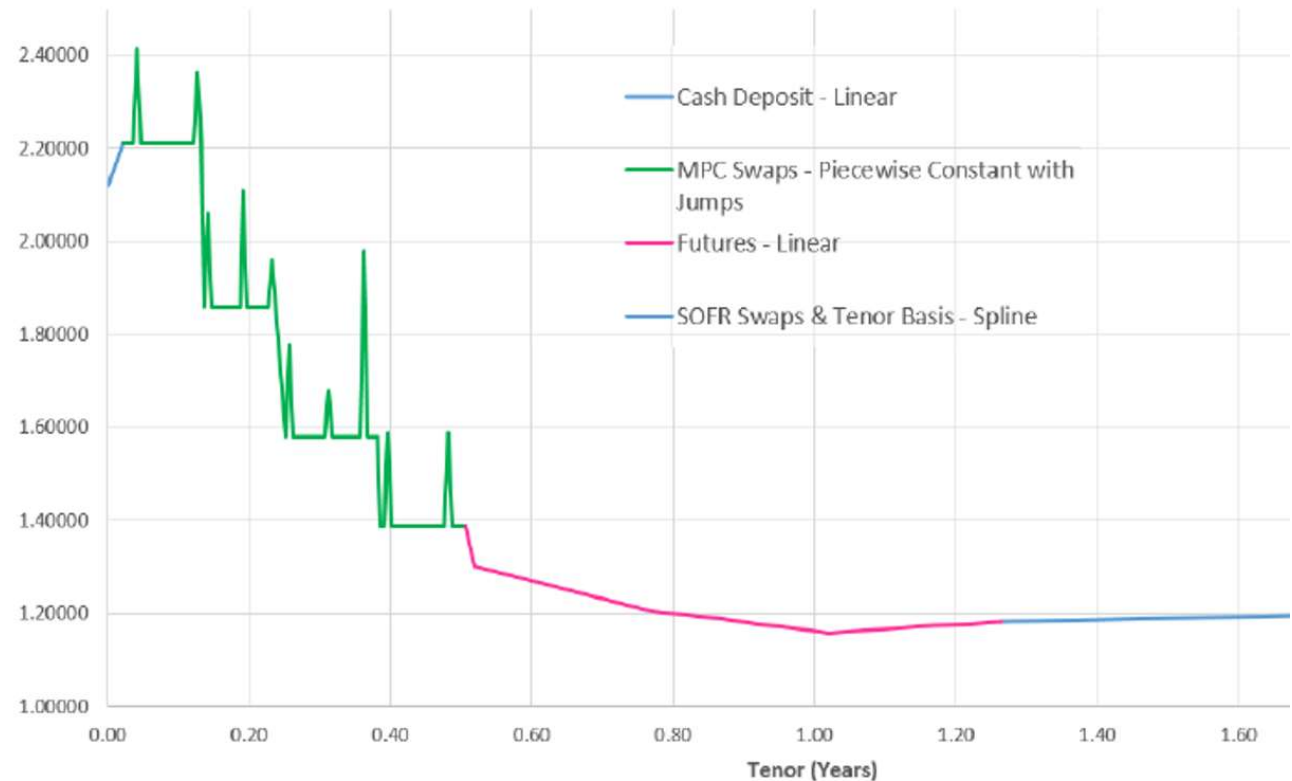
Yield Curves - Features

Curve Features & Considerations

- Underlying Instrument Behaviour
- Mixed Interpolation Schemes
- Turn-of-Year Effects (ToYs)

Advanced Features for Electronic Markets

- Curve Jacobian
- Ultra-Fast Curves & Analytical Risk
- Automatic Adjoint Differentiation (AAD)



Yield Curves – Curve Jacobian

Electronic HFT Usage

- Ultra-Fast Rebuilds
- Real-Time Risk
- Auto-Hedging

By-Product of Calibration Process

- Measures Changes in Market Instrument Quotes (P) on Forward Rates (L)
- First Order Derivative Matrix, dP/dL (Inverse Required)
- Controls Hedge and Risk Buckets (Same as Numerical Bumping)
- Use **Implicit Function Theorem** (IFT) to modify Risk Buckets (see Appendix)

Inverse Curve Jacobian, dL/dP

Forward Pillars	Curve Calibration Instruments									
	dP_{1Y}^{OIS}	dP_{2Y}^{OIS}	dP_{3Y}^{OIS}	dP_{4Y}^{OIS}	dP_{5Y}^{OIS}	dP_{1Y}^{IRS}	dP_{2Y}^{IRS}	dP_{3Y}^{IRS}	dP_{4Y}^{IRS}	dP_{5Y}^{IRS}
dO_{1Y}	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
dO_{2Y}	-1.01	2.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
dO_{3Y}	0.00	-2.04	3.04	0.00	0.00	0.00	0.00	0.00	0.00	0.00
dO_{4Y}	0.00	0.00	-3.08	4.08	0.00	0.00	0.00	0.00	0.00	0.00
dO_{5Y}	0.00	0.00	0.00	-4.13	5.13	0.00	0.00	0.00	0.00	0.00
dL_{1Y}	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00
dL_{2Y}	0.00	0.00	0.00	0.00	0.00	-1.01	2.01	0.00	0.00	0.00
dL_{3Y}	0.00	0.00	0.00	0.00	0.00	0.00	-2.04	3.04	0.00	0.00
dL_{4Y}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-3.08	4.08	0.00
dL_{5Y}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-4.13	5.13

Yield Curves – Ultra-Fast Rebuilds

New Forwards

$$L_{New} = L_{Old} + dL$$

$$= L_{Old} + (dL/dP) \cdot dP$$

New Forwards		Original Forwards		Inverse Jacobian, dL/dP											Change in Mkt Data	
L _{NEW}		L _{OLD}													dP	
L _{1Y} ^{OIS}	1.44591%	L _{1Y} ^{OIS}	1.43591%	L _{1Y} ^{OIS}	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	L _{1Y} ^{OIS}	0.01%
L _{2Y} ^{OIS}	1.24323%	L _{2Y} ^{OIS}	1.23323%	L _{2Y} ^{OIS}	-1.01	2.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	L _{2Y} ^{OIS}	0.01%
L _{3Y} ^{OIS}	1.26107%	L _{3Y} ^{OIS}	1.25107%	L _{3Y} ^{OIS}	0.00	-2.04	3.04	0.00	0.00	0.00	0.00	0.00	0.00	0.00	L _{3Y} ^{OIS}	0.01%
L _{4Y} ^{OIS}	1.30130%	L _{4Y} ^{OIS}	1.29130%	L _{4Y} ^{OIS}	0.00	0.00	-3.08	4.08	0.00	0.00	0.00	0.00	0.00	0.00	L _{4Y} ^{OIS}	0.01%
L _{5Y} ^{OIS}	1.40782%	L _{5Y} ^{OIS}	1.39782%	L _{5Y} ^{OIS}	0.00	0.00	0.00	-4.13	5.13	0.00	0.00	0.00	0.00	0.00	L _{5Y} ^{OIS}	0.01%
L _{1Y} ^{IRS}	1.71896%	L _{1Y} ^{IRS}	1.70896%	L _{1Y} ^{IRS}	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00	L _{1Y} ^{IRS}	0.01%
L _{2Y} ^{IRS}	1.48359%	L _{2Y} ^{IRS}	1.47359%	L _{2Y} ^{IRS}	0.00	0.00	0.00	0.00	0.00	-1.01	2.01	0.00	0.00	0.00	L _{2Y} ^{IRS}	0.01%
L _{3Y} ^{IRS}	1.50531%	L _{3Y} ^{IRS}	1.49531%	L _{3Y} ^{IRS}	0.00	0.00	0.00	0.00	0.00	0.00	-2.04	3.04	0.00	0.00	L _{3Y} ^{IRS}	0.01%
L _{4Y} ^{IRS}	1.56934%	L _{4Y} ^{IRS}	1.55934%	L _{4Y} ^{IRS}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-3.08	4.08	0.00	L _{4Y} ^{IRS}	0.01%
L _{5Y} ^{IRS}	1.63999%	L _{5Y} ^{IRS}	1.62999%	L _{5Y} ^{IRS}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-4.13	5.13	L _{5Y} ^{IRS}	0.01%

Implementation

- Slow Curve (Full-Rebuild) Ticks in Background (ca. 10ms)
- Fast Curve (Jacobian Method) Used Between Refreshes (Real-Time)

Yield Curves – Real-Time Bucketed Risk

Requirements

- Curve Jacobian
- Trade or Portfolio Jacobian

$$DV01(Analytical) = \underbrace{1bps \times \frac{dPV}{dL}}_{\text{Pricing Jacobian}} \times \underbrace{\frac{dL}{dP}}_{\text{Curve Jacobian}}$$

Risk as a Matrix Operation

- Can be Parallelized / Vectorized
- Matrix Dimensions Must Agree
- Interpolation & Forward Mapping
- Barycentric Weights, $w_j(t)$

$$p(t) = \sum_{j=0}^n w_j(t) f(t_j), \quad w_j(t) = \frac{\prod_{k=0, k \neq j}^n (t - t_k)}{\prod_{k=0, k \neq j}^n (t_j - t_k)}$$

Inverse Curve Jacobian, dL/dP

Forward Pillars	Curve Calibration Instruments									
	dP_{1Y}^{OIS}	dP_{2Y}^{OIS}	dP_{3Y}^{OIS}	dP_{4Y}^{OIS}	dP_{5Y}^{OIS}	dP_{1Y}^{IRS}	dP_{2Y}^{IRS}	dP_{3Y}^{IRS}	dP_{4Y}^{IRS}	dP_{5Y}^{IRS}
dO_{1Y}	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
dO_{2Y}	-1.01	2.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
dO_{3Y}	0.00	-2.04	3.04	0.00	0.00	0.00	0.00	0.00	0.00	0.00
dO_{4Y}	0.00	0.00	-3.08	4.08	0.00	0.00	0.00	0.00	0.00	0.00
dO_{5Y}	0.00	0.00	0.00	-4.13	5.13	0.00	0.00	0.00	0.00	0.00
dL_{1Y}	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00
dL_{2Y}	0.00	0.00	0.00	0.00	0.00	-1.01	2.01	0.00	0.00	0.00
dL_{3Y}	0.00	0.00	0.00	0.00	0.00	0.00	-2.04	3.04	0.00	0.00
dL_{4Y}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-3.08	4.08	0.00
dL_{5Y}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-4.13	5.13

=

Trade	
Risk Bucket	IRS 3Y
OIS 1Y	0
OIS 2Y	0
OIS 3Y	0
OIS 4Y	-1
OIS 5Y	1
IRS 1Y	0
IRS 2Y	0
IRS 3Y	291
IRS 4Y	0
IRS 5Y	0

Trade Jacobian, dPV/dL
3Y Par Swap

Trade	OIS Curve (Discount Risk)					Swap Curve (Forward Risk)				
	dO_{1Y}	dO_{2Y}	dO_{3Y}	dO_{4Y}	dO_{5Y}	dL_{1Y}	dL_{2Y}	dL_{3Y}	dL_{4Y}	dL_{5Y}
dS_{3Y}^{IRS}	0	0	0	0	0	98	97	96	0	0

Total Trade DV01

IRS 3Y
291

Yield Curves – Automatic Adjoint Differentiation (AAD)

Trade Jacobian

- AAD Can Compute Instrument Price & Risk Simultaneously
- Direct Differentiation of Code + Implicit Function Theorem (IFT)
- Exact & Fast (X4 Pricing Time)

Tangent & Adjoint Modes

- Tangent Mode (dot) : **Forward** Mode - **One Risk at a Time**
- Adjoint Mode (bar) : **Backward** Mode - **All Risks Simultaneously**
- Activation Inputs Control Risk Outputs

Implementation Methods

- By Hand (See Appendix for Swap DV01 Risk Example)
- Derivative Code by Overloading, DCO/C++
- Professional Tools: Adept, NAG

Pricing Calculations

$$x \rightarrow f(x) \rightarrow g(f) \rightarrow h(g) \rightarrow y$$

Chain Rule: Forwards

$$\frac{df}{dx} \cdot \frac{dg}{df} \cdot \frac{dh}{dg} \cdot \frac{dy}{dh} = \frac{dy}{dx}$$

Chain Rule: Backwards

$$\frac{dy}{dh} \cdot \frac{dh}{dg} \cdot \frac{dg}{df} \cdot \frac{df}{dx} = \frac{dy}{dx}$$

Yield Curves – AD Tangent Mode Example

Tangent Mode

- Differentiate Forwards using 'Dot' Notation
- One Risk at a Time, Controlled by Dot **Input Activation Variables** 1 or 0
- For $\frac{df}{dx_1}$ and $\frac{df}{dx_2}$ must call tangent method twice

```

01 double function( double x1, double x2 )
02 {
03     double a = x1*x1;           // Step 1:    a = x12
04     double b = 2*a;             // Step 2:    b = 2x12
05     double c = x2;              // Step 3:    c = x2
06     double d = 3*c;             // Step 4:    d = 3x2
07     double f = b + d;           // Step 5:    f = 2x12 + 3x2
08     return f;
09 }

```

Simple Function: $f(x_1, x_2) = 2x_1^2 + 3x_2$

Source Code: <https://onlinegdb.com/kKqaS6hJT>

```

01 tangent(2.0, 3.0, 1.0, 0.0); // Input: x1 = 2, x2 = 3, x1_d = 1, x2_d = 0   Output: 8
02 tangent(2.0, 3.0, 0.0, 1.0); // Input: x1 = 2, x2 = 3, x1_d = 0, x2_d = 1   Output: 3

```

Function Derivatives using Tangent Mode

```

01 double tangent( double x1, double x2, double x1_dot, double x2_dot )
02 {
03     double a = x1*x1;           // Step 1:    a = x12
04     double a_dot = 2*x1*x1_dot; // Tangent:   $\dot{a} = 2x_1 \cdot \dot{x}_1$        $\dot{a} = 2x_1$ 
05     double b = 2*a;             // Step 2:    b = a
06     double b_dot = 2*a_dot;     // Tangent:   $\dot{b} = 2 \cdot \dot{a}$            $\dot{b} = 4x_1$ 
07     double c = x2;              // Step 3:    c = x2
08     double c_dot = x2_dot;      // Tangent:   $\dot{c} = \dot{x}_2$            $\dot{c} = 1$ 
09     double d = 3*c;             // Step 4:    d = 3c
10     double d_dot = 3*c_dot;     // Tangent:   $\dot{d} = 3 \cdot \dot{c}$          $\dot{d} = 3$ 
11     double f = b + d;           // Step 5:    f = 2x12 + 3x2
12     double f_dot = b_dot + d_dot; // Tangent:   $\dot{f} = \dot{b} + \dot{d}$ 
13     return f_dot;              // Result:     $\dot{f} = 4x_1 + 3$ 
14 }

```

Simple Function $f(x_1, x_2) = 2x_1^2 + 3x_2$ with Tangent Derivatives

Yield Curves – AD Adjoint Mode Example

Adjoint Mode (Reverse Mode)

- Backwards Differentiation with 'Bar' Notation
- Forward Sweep then Back Propagate Risk
- Computes All Risks at Same Time
- Risk Controlled By Bar Input Activation Variable 1 or 0
- Adjoint Method Calculates Both $\frac{df}{dx_1}$ and $\frac{df}{dx_2}$

```

01 double function( double x1, double x2 )
02 {
03     double a = x1*x1;           // Step 1:  a = x12
04     double b = 2*a;             // Step 2:  b = 2x12
05     double c = x2;             // Step 3:  c = x2
06     double d = 3*c;            // Step 4:  d = 3x2
07     double f = b + d;          // Step 5:  f = 2x12 + 3x2
08     return f;
09 }

```

Simple Function: $f(x_1, x_2) = 2x_1^2 + 3x_2$

```

01 adjoint(2.0, 3.0, 1.0); // Input: x1 = 3, x2 = 2, f_bar Output: df/dx1 = 8 and df/dx2 = 3

```

Function Derivatives using Adjoint Mode

```

01 void adjoint( double x1, double x2, double f_bar )
02 {
03     // Forward Sweep
04     double a = x1*x1;           // Step 1:  a = x12
05     double b = 2*a;             // Step 2:  b = 2x12
06     double c = x2;             // Step 3:  c = x2
07     double d = 3*c;            // Step 4:  d = 3x2
10     double f = b + d;          // Step 5:  f = 2x12 + 3x2
08
09     // Back Propagation
10     double b_bar = f_bar;       // Step 5:  b_bar = 1    from input variable
11     double d_bar = f_bar;       // Step 5:  d_bar = 1    from input variable
12     double c_bar = 3*d_bar;     // Step 4:  c_bar = 3
13     double x2_bar = c_bar;      // Step 3:  x2_bar = 3    df/dx2 = 3
14     double a_bar = 2*b_bar;     // Step 2:  a_bar = 2
15     double x1_bar = 2*x1*a_bar; // Step 1:  x1_bar = 4x1  df/dx1 = 4x1
16
17     // Display Results
18     std::cout << "df/dx1: " << x1_bar << std::endl; // x̄1 = df/dx1 = 4x1
19     std::cout << "df/dx2: " << x2_bar << std::endl; // x̄2 = df/dx2 = 3
20 }

```

Simple Function $f(x_1, x_2) = 2x_1^2 + 3x_2$ with Adjoint Derivatives

Credit Models – Hazard Rates & Survival Probabilities

Calibration Summary

- Yield Curve is an Input
- Calibrate to Bonds or CDS
- Imply Hazard Rates, λ
- Used for Survival Prob, $Q(t,T)$

Common Assumptions

- Piecewise Constant¹
- Deterministic Hazard Rates

Rule of Thumb

$$\lambda = \frac{S}{(1 - R)}$$

¹ As often there is only a single calibration instrument



$$Q(t,T) = \exp\left(-\int_t^T \lambda(t,u)du\right)$$

$$P(t,T) = \exp\left(-\int_t^T f(t,u)du\right)$$

PART TWO – PRICING & PRACTICE

Case Studies Interest Rate Swaps & Asset Swaps

Interest Rate Swap – Annuity is the Key Pricing & Risk Factor

It's All About Annuity

- Pricing & Risk Expressed in Terms of Annuity
- Similarly Float Legs Expressed in Annuity Terms
- Can Be Used to Convert a Float Leg to Fixed Leg
- Useful for Low Latency Pricing

Key Formulae:

- $PV = (r - p) \text{Annuity(Fixed)}$
- $\text{Par Rate} = PV(\text{Float}) / \text{Annuity(Fixed)}$
- $PV01 = \text{Annuity(Fixed)} \times 0.01\%$
- $DV01 = PV01 + DF01 = PV01$ for Par Swaps

$$A_N = N \sum_{i=1}^n \tau_i P(t_0, t_i)$$

Low Latency Interest Rate Swap Pricing

Electronic Rates Markets & Low Latency Interest Rate Swap Calculations (May 31, 2022).

Available at SSRN: <https://ssrn.com/abstract=4125565>

$$\text{Swap PV} = PV^{\text{Fixed Leg}} - PV^{\text{Float Leg}}$$

$$= r \sum_{i=1}^n N_i \tau_i P(t_0, t_i) - \sum_{j=1}^m N_j l_{j-1} \tau_j P(t_0, t_j)$$

$$= (r - p) A_{\text{Fixed}}$$

Interest Rate Swap – Pricing & Risk Example

Compute Annuity A_N

= USD 4,863,971.74

$$A_N = N \sum_{i=1}^n \tau_i P(t_0, t_i)$$

$PV = (r - p) A_N$

= (5.00% - 1.59%) A_N

= USD 167,892.11

The screenshot displays a financial software interface for swap pricing and risk management. The main window is titled 'Swap Manager' and contains several tabs: 3) Main, 4) Details, 5) Curves, 6) Cashflow, 7) Resets, 9) Scenario, 10) Risk, 11) CVA, 12) Matrix, and 20) Properties. The 'Main' tab is active, showing a 'Fixed Float Swap' with a 'Counterparty' of 'SWAP CNTRPARTY'. The swap details include: Leg 1: Fixed (Receive), Notional 1MM, Currency USD, Effective 08/25/2015, Maturity 08/25/2020, Coupon 5.000000%, Pay Freq SemiAnnual, Day Count 30I/360, Calc Basis Money Mkt. Leg 2: Float (Pay), Notional 1MM, Currency USD, Effective 08/25/2015, Maturity 08/25/2020, Index 3M US0003M, Spread 0.000 bp, Latest Index 0.32910, Day Count ACT/360, Reset Freq Quarterly, Pay Freq Quarterly. The 'Valuation Settings' panel shows Curve Date 08/21/2015, Valuation 08/25/2015, OIS DC Strip ON, and CSA Coll Ccy USD. The 'Valuation Results' panel shows Par Cpn 1.548250, Premium 16.78921, Principal 167,892.11, BP Value 1678.92112, Accrued 0.00, and NPV 167,892.11. The 'More Greeks' panel shows PV01 486.40, DV01 532.42, and Gamma (1bp) 0.29.

Par Rate = $PV(\text{Float}) / A_N$

= 75,306 / A_N

= 1.5482%

PV01

= $A_N \times 0.01\%$

= USD 486.40

Credit Default Swap – Pricing & Risk Example

Compute Risky Annuity \tilde{A}_N

= USD 49,512,369.11

$$\tilde{A}_N = N \sum_{i=1}^n \tau_i Q(t_i) P(t_0, t_i)$$

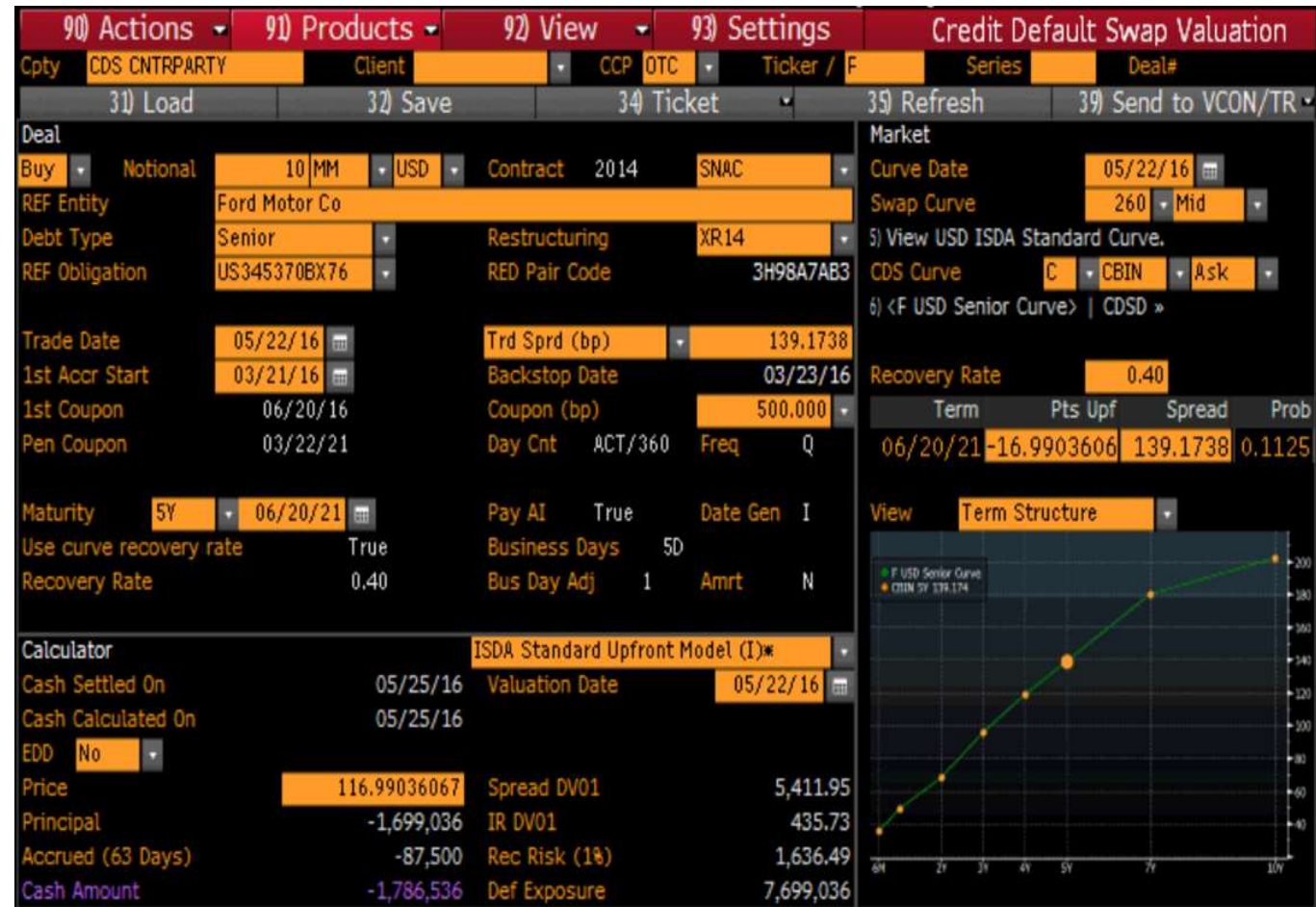
$PV = (r - p) \tilde{A}_N$

= (5.00% - 1.39%) \tilde{A}_N

= USD 1,786,536

$CS01 = \tilde{A}_N \times 0.01\%$

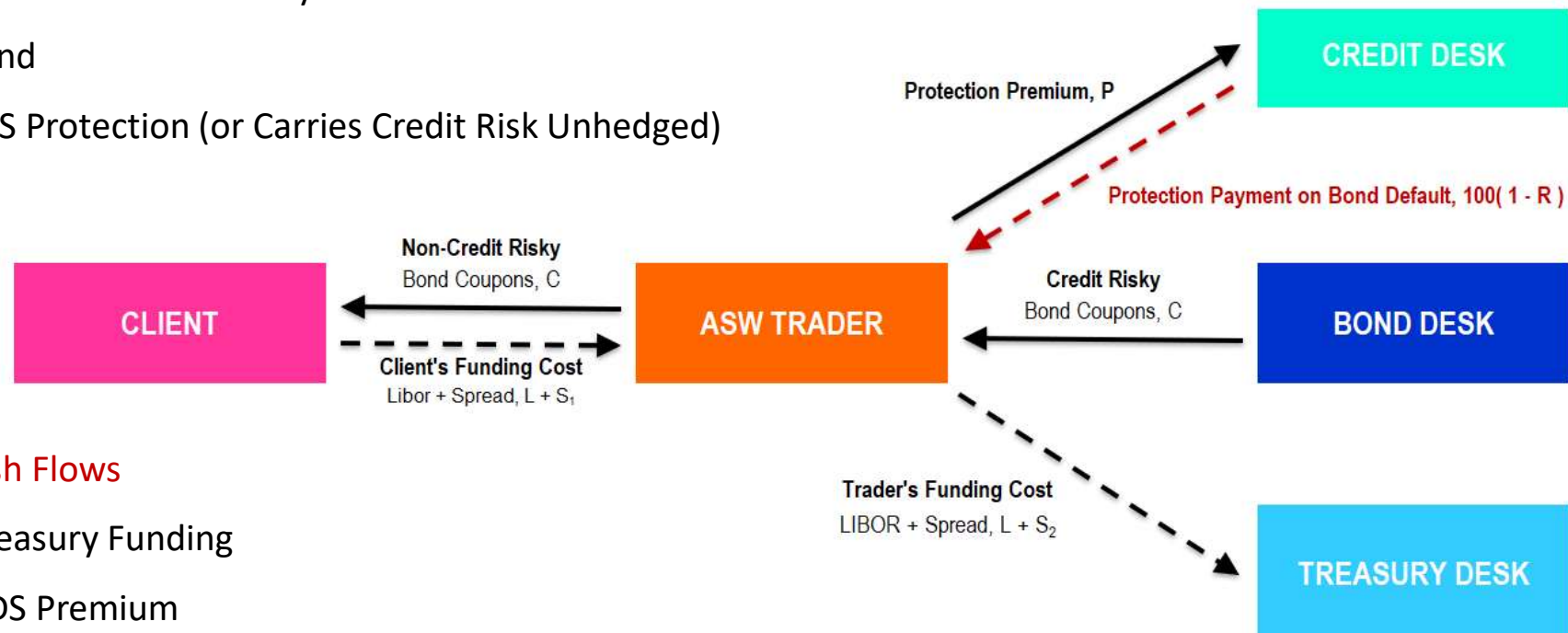
= USD 486.40



Asset Swap – Structuring the Asset Swap Spread

Trader Creates Synthetic Asset Swap

- Borrow Cash from Treasury to Purchase Bond
- Buy Bond
- Buy CDS Protection (or Carries Credit Risk Unhedged)



Trader Cash Flows

- Pays Treasury Funding
- Pays CDS Premium
- Receives Bond Coupons and Passes on to Client
- Client Pays All Costs + Commission as a **Spread over LIBOR** (or RFR)

Asset Swap – Pricing as a Spread Over LIBOR (or RFR)

DBR 0 1/2 02/15/26		5 Actions		6 Settings		Asset Swap Calculator	
1 Pricing		2 Cashflow		3 Relative Value		4 Deal Summary	
Asset Swap Analysis				Price	104.5800		
Calculate				Z-Spread	-40.9	ASW Spread	-40.6
Price → ASW Spread				Yield(%)	0.02595	MMS Spread	-41.2
Bond JV503423		Swap		Par-Par <input checked="" type="radio"/> Matched Maturity		B) Swap Detail SWPM	
Par Amount	1MM	Leg 1: Fixed	Pay	Leg 2: Float	Receive		
Workout	02/15/2026	Notional	1MM	Notional	1MM		
Workout Price	100.0000	Currency	EUR	Currency	EUR		
Pay Freq	Annual	Effective Date	01/15/2016	Effective Date	01/15/2016		
Day Count	ACT/ACT	Maturity Date	02/15/2026	Maturity Date	02/15/2026		
		Coupon	0.5	Latest Index	-0.112		
		Pay/Reset Freq	Annual	Index	EUR006M		
		Day Count	ACT/ACT	Pay/Reset Freq	SemiAnnual		
				Day Count	ACT/360		
Implied Value	100.5736	<input checked="" type="checkbox"/> Include Accrued		<input checked="" type="checkbox"/> Include Accrued			
Market		Discount Curve		Discount Curve			
Curve Date	06/09/2016	133 Mid		133 Mid			
Settle Date	06/13/2016			45 Mid			
Swapped Spread Detail							
Clean Price	104.5800			Money		Spread(bp)	
Swap Price	100.0000	Cash Out	4.5800	-45,800.0		-46.0	
Swap Rate(%)	0.44104	Bond Cpn(%)	0.5000	5,736.5		5.8	
Redemption(%)	0.0000			0.0		0.0	
Funding	Spread(bp)	0.0		0.0		0.0	
Swapped Spread				-40,063.5		-40.0	

- **ASW Spread** - Par-Par Spread
- **MMS Spread** - Yield-Yield Spread¹

¹ Y/Y Spread Between Swap Rate and Benchmark Gov't Bond Yield

Asset Swap – Pricing using Par-Par Method

Pricing as a PV

- Valuation Method for Existing Swaps, Unwinds and Novations (trade transfers)
- Again Present Value is Simply the Sum of Incoming and Outgoing Cash Flows
- An Upfront Par-Adjustment is Made if the Underlying Bond not Trading at Par, i.e., 100

$$PV^{Asset\ Swap} = \underbrace{\phi r^{Fixed} \sum_{i=1}^n N_i \tau_i P(t_0, t_i)}_{Fixed\ Leg} - \underbrace{\phi \sum_{j=1}^m N_j (l_{j-1} + s) \tau_j P(t_0, t_j)}_{Float\ Leg} + \underbrace{\phi N_1 (100 - B)\%}_{Par\ Adjustment}$$

Pricing as a Par Spread

- New Asset Swaps Price to Par i.e., zero
- Instead Quote as a Par Spread s
- Rearrangement of PV formula with $PV=0$

$$s = \left(\frac{(r^{Fixed} - p^{Market}) A^{Fixed} + (100 - B)\%}{A^{Float}} \right)$$

Fast Pricing & Risk – Using Annuity Factors

Multiples Pricing and Risk

Pricing Tricks - Identifying Annuity Factors

Pricing Tricks & Rules of Thumb

- Identify the Annuity Factor
- Key Pricing and Risk Factor
- Assume $A_N^{Fixed} = A_N^{Float}$

Annuity

$$A_N = N \sum_{i=1}^n \tau_i P(t_0, t_i)$$

Credit Risky Annuity

$$\widetilde{A}_N = N \sum_{i=1}^n \tau_i Q(t_i) P(t_0, t_i)$$

Interest Rate Swaps

$$PV^{Swap} = \phi(r - p) A_N$$

$$DV01^{Swap} = \phi A_N \times 0.01\%$$

Credit Default Swaps

$$PV^{CDS} = \phi(s - p) \widetilde{A}_N$$

$$CS01^{CDS} = \phi \widetilde{A}_N \times 0.01\%$$

Asset Swap Spreads

$$S^{ASW} = \left(\frac{(r-p)\% A + (100-B)\%}{A} \right)$$

Pricing Tricks – Fast Annuity Factors

Pricing Tricks & Rules of Thumb

- Assume Annual Coupons & Discount Factors = 1.0
- This gives $A_N = NT$ and $A = T$
- Resulting Prices are an upper-bound

Annuity

$$A_N = N \sum_{i=1}^n \tau_i P(t_0, t_i) = NT$$

Credit Risky Annuity

$$\widetilde{A}_N = N \sum_{i=1}^n \tau_i Q(t_i) P(t_0, t_i) = NT$$

- We could also assume survival probabilities $Q(t) = 1.0$

Interest Rate Swaps

$$PV^{Swap} = \phi(r - p)NT$$

$$DV01^{Swap} = \phi NT \times 0.01\%$$

Credit Default Swaps

$$PV^{CDS} = \phi(s - p) NT$$

$$CS01^{CDS} = \phi NT \times 0.01\%$$

Asset Swap Spreads

$$s^{ASW} = \underbrace{(r - p)\%}_{Cpn\ Factor} + \underbrace{(100 - B)\% / T}_{Price\ Factor}$$

Pricing Tricks – Multiples Pricing & Risk

Swap Multiples Pricing

- Knowledge of Liquid Market Par Rates Required
- Precompute a Base Case / Reference Price
- Determine all Prices as a **Multiple of a Base Case**
- Prices computed this way are an **Upper-Bound**

Swap Reference Prices

- Price Base Case Units: **Per Million, Per Bps, Per Year**
- $N = \text{USD } 1,000,000$, $(r - p) = 1\text{bps}$ and $T = 1.0$
- $PV(\text{Base Swap}) = \text{USD } 100$

Swap Reference Risk

- Price Base Case Units: **Per Million, Per Year**
- $DV01(\text{Base Swap}) = \text{USD } 100$

Swap PV Multiples

$$PV(1mm, 1bps, 1y) = \text{USD } 100$$

$$PV(5mm, 1bps, 1y) = \text{USD } 500$$

$$PV(1mm, 5bps, 1y) = \text{USD } 500$$

$$PV(1mm, 1bps, 5y) = \text{USD } 500$$

$$PV(5mm, 5bps, 5y) = \text{USD } 12,500$$

Swap DV01 Multiples

$$DV01(1mm, 1y) = \text{USD } 100$$

$$DV01(1mm, 5y) = \text{USD } 500$$

$$DV01(5mm, 5y) = \text{USD } 2,500$$

Pricing Tricks – CDS Multiples & ASW Spread Factors

CDS Multiples

- Similar to IRS Multiples
- $PV(\text{Base CDS}) = \text{USD } 100$
- $CS01(\text{Base CDS}) = \text{USD } 100$

Asset Swap Spread Factors

- Simple addition of Bond Coupon and Bond Price Factors
- Coupon Factor, $C_F = (r - p)$
- Price Factor, $P_F = (100 - B)\% / T$
- Note bond price factor can be negative if B below Par

CDS Multiples

$$PV(1\text{mm}, 1\text{bps}, 1\text{y}) = \text{USD } 100$$

$$PV(1\text{mm}, 1\text{bps}, 5\text{y}) = \text{USD } 500$$

$$PV(1\text{mm}, 5\text{bps}, 5\text{y}) = \text{USD } 2,500$$

$$PV(5\text{mm}, 5\text{bps}, 5\text{y}) = \text{USD } 12,500$$

ASW Spread Factors

$$s^{\text{ASW}}(C_F=10\text{bps}, P_F=\{100\text{bps}, 10\text{Y}\}) \\ = 10 + 100/10 = 20 \text{ bps}$$

$$s^{\text{ASW}}(C_F=10\text{bps}, P_F=\{0\text{bps}, 10\text{Y}\}) \\ = 10 + 0/10 = 10 \text{ bps}$$

$$s^{\text{ASW}}(C_F=10\text{bps}, P_F=\{-100\text{bps}, 10\text{Y}\}) \\ = 10 - 100/10 = 0 \text{ bps}$$

Pricing Tricks – Interest Rate Swap Multiples

IRS Base Cases

- $PV(\text{Base Case}) = 100$
- $DV01(\text{Base Case}) = 100$

Market Par Rate

- 5Y Par Rate = 150 bps
- $\Delta r = (r - p) = (500 - 150) = 350 \text{ bps}$

IRS Multiples

- Here $\Delta N = 1$, $\Delta r = 350$, $\Delta T = 5$
- $PV = 100 \times 1 \times 350 \times 5 = \text{USD } 175K$
- $DV01 = 100 \times 1 \times 5 = \text{USD } 500$

Reference Price USD 100 per Million per Year per Δr in bps

The screenshot displays the Swap Manager interface with the following details:

- Deal:** Fixed Float Swap, Counterparty: SWAP CNTRPARTY, CCP: OTC.
- Swap Details:**
 - Leg 1: Fixed:** Receive, Notional: 1MM, Currency: USD, Effective: 0D (08/25/2015), Maturity: 5Y (08/25/2020), Coupon: 5.000000%, Pay Freq: SemiAnnual, Day Count: 30I/360, Calc Basis: Money Mkt.
 - Leg 2: Float:** Pay, Notional: 1MM, Currency: USD, Effective: 0D (08/25/2015), Maturity: 5Y (08/25/2020), Index: 3M US0003M, Spread: 0.000 bp, Latest Index: 0.32910, Day Count: ACT/360, Reset Freq: Quarterly, Pay Freq: Quarterly.
- Valuation Settings:** Curve Date: 08/21/2015, Valuation: 08/25/2015, OIS DC Strip: ON, CSA Coll Ccy: USD.
- Market:** Dscnt: 42 M USD Bloomberg Curv, Fwd: 23 M USD Bloomberg Curv.
- Valuation Results:**

Par Cpn	1.548250	Premium	16.78921	PV01	486.40
Principal	167,892.11	BP Value	1678.92112	DV01	532.42
Accrued	0.00			Gamma (1bp)	0.29
NPV	167,892.11				

Pricing Tricks – Credit Default Swap Multiples

CDS Base Cases

- $PV(\text{Base Case}) = 100$
- $CS01(\text{Base Case}) = 100$

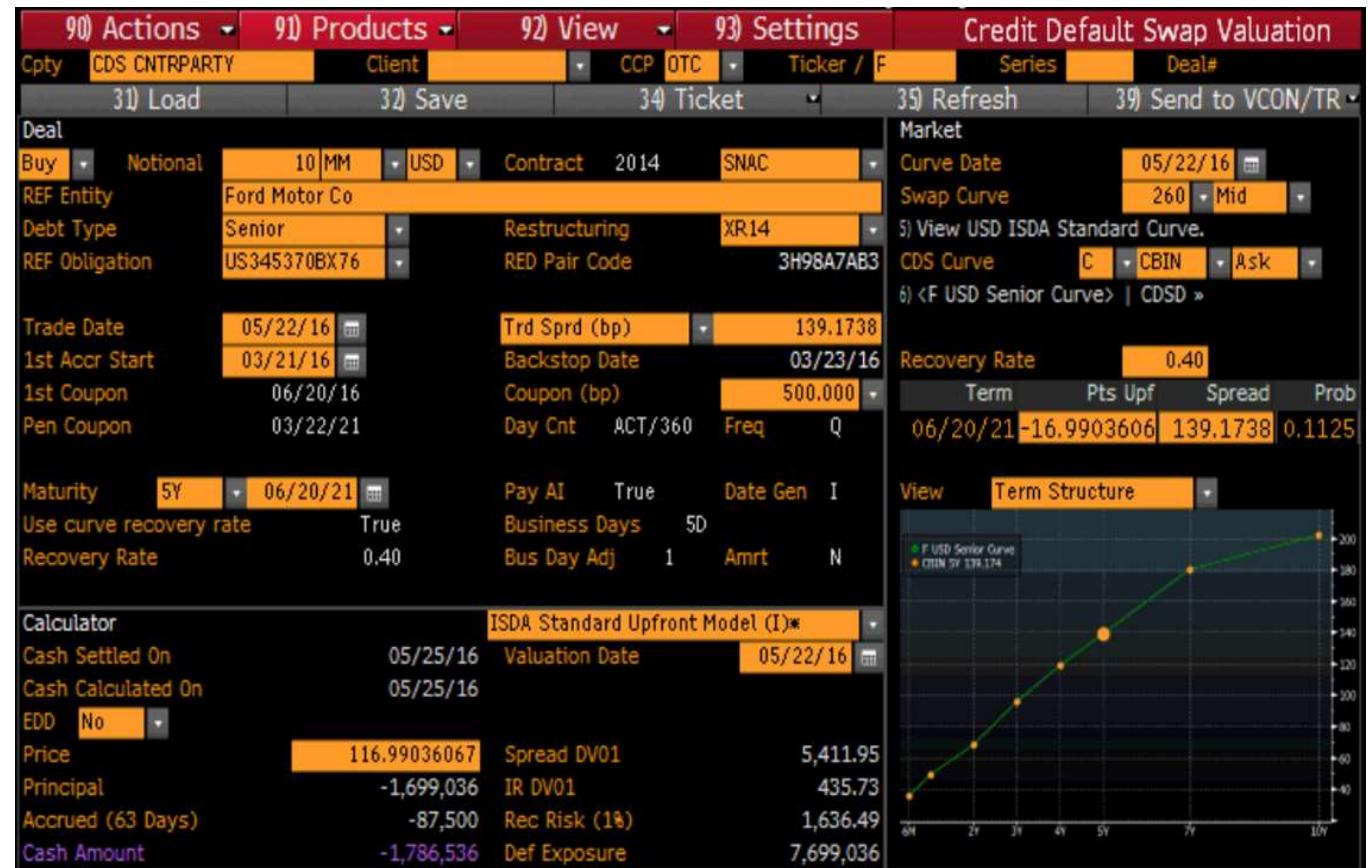
Market CDS Par Rate

- 5Y CDS Par Rate ≈ 140 bps
- $\Delta r = (r - p) = (500 - 140) = 360$ bps

CDS Multiples

- Here $\Delta N = 10$, $\Delta r = 360$, $\Delta T = 5$
- $PV = 100 \times 10 \times 360 \times 5 = \text{USD } 1.8\text{mm}$
- $CS01 = 100 \times 10 \times 5 = \text{USD } 5,000$

Reference Price USD 100 per Million per Year per Δr in bps



Pricing Tricks – Asset Swap Spread Factors

Par-Par Spread Factors

$$s = (r - p) + (100 - B)\% / T$$

= Coupon Factor C_F + Price Factor P_F

We compute $C_F = (r - p)$ in bps

and $P_F = (B - 100)\% / T$ in bps

Par-Par Spread

For this German Bund we have,

$$C_F = 0.50\% - 0.44\% = 6 \text{ bps}$$

$$P_F = (100 - 104.580)\% / 10$$

$$= -458/10 \approx -46 \text{ bps}$$

$$S = 6 - 46 = -40 \text{ bps}$$

DBR 0 1/2 02/15/26		Actions		Settings		Asset Swap Calculator		
<div> <div>1 Pricing</div> <div>2 Cashflow</div> <div>3 Relative Value</div> <div>4 Deal Summary</div> </div>								
<div>Asset Swap Analysis</div> <div>Calculate</div> <div>Price -> ASW Spread</div>				<div>Price</div> <div>Z-Spread</div> <div>Yield(%)</div>				<div>104.5800</div> <div>-40.9</div> <div>0.02595</div>
				<div>ASW Spread</div> <div>MMS Spread</div>		<div>-40.6</div> <div>-41.2</div>		
<div>Bond</div> <div>JV503423</div>		<div>Swap</div> <div>Par-Par</div> <div>Matched Maturity</div>		<div>8) Swap Detail SWPM »</div>				
<div>Par Amount</div> <div>1MM</div>		<div>Leg 1: Fixed</div> <div>Pay</div>		<div>Leg 2: Float</div> <div>Receive</div>				
<div>Workout</div> <div>02/15/2026</div>		<div>Notional</div> <div>1MM</div>		<div>Notional</div> <div>1MM</div>				
<div>Workout Price</div> <div>100.0000</div>		<div>Currency</div> <div>EUR</div>		<div>Currency</div> <div>EUR</div>				
<div>Pay Freq</div> <div>Annual</div>		<div>Effective Date</div> <div>01/15/2016</div>		<div>Effective Date</div> <div>01/15/2016</div>				
<div>Day Count</div> <div>ACT/ACT</div>		<div>Maturity Date</div> <div>02/15/2026</div>		<div>Maturity Date</div> <div>02/15/2026</div>				
		<div>Coupon</div> <div>0.5</div>		<div>Latest Index</div> <div>-0.112</div>				
		<div>Pay/Reset Freq</div> <div>Annual</div>		<div>Index</div> <div>EUR006M</div>				
		<div>Day Count</div> <div>ACT/ACT</div>		<div>Pay/Reset Freq</div> <div>SemiAnnual</div>				
				<div>Day Count</div> <div>ACT/360</div>				
<div>Implied Value</div> <div>100.5736</div>		<div>Include Accrued</div> <div>Include Accrued</div>						
<div>Market</div> <div>Curve Date</div> <div>06/09/2016</div>		<div>Discount Curve</div> <div>133</div> <div>Mid</div>		<div>Discount Curve</div> <div>133</div> <div>Mid</div>				
<div>Settle Date</div> <div>06/13/2016</div>				<div>Forward Curve</div> <div>45</div> <div>Mid</div>				
<div>Swapped Spread Detail</div>								
<div>Clean Price</div> <div>104.5800</div>				<div>Money</div> <div>Spread(bp)</div>				
<div>Swap Price</div> <div>100.0000</div>		<div>Cash Out</div> <div>4.5800</div>		<div>-45,800.0</div> <div>-46.4</div>				
<div>Swap Rate(%)</div> <div>0.44104</div>		<div>Bond Cpn(%)</div> <div>0.5000</div>		<div>5,736.5</div> <div>5.8</div>				
<div>Redemption(%)</div> <div>0.0000</div>				<div>0.0</div> <div>0.0</div>				
<div>Funding</div> <div>Spread(bp)</div>		<div>0.0</div>		<div>0.0</div> <div>0.0</div>				
<div>Swapped Spread</div>				<div>-40,063.5</div> <div>-40.6</div>				

Swap Risk – Curve Calibration & Risk Hedges

Swap Curve

- Calibrated using 1Y, 2Y, 3Y, 4Y and 5Y swaps
- Bucketed DV01 risk profile shown
- Calibration instruments are the risk hedge instruments

Risk Hedge Instruments

- Consider a portfolio of calibration instruments
- Each with USD 1mm Notional
- Risk from each calibration instrument fits perfectly into ... calibration risk buckets

Actual Risk

Hedge Trade Risk

Risk Bucket	Hedge Trades				
	IRS 1Y	IRS 2Y	IRS 3Y	IRS 4Y	IRS 5Y
OIS 1Y	0	0	0	0	0
OIS 2Y	0	0	0	0	0
OIS 3Y	0	0	0	0	0
OIS 4Y	0	0	0	0	0
OIS 5Y	0	0	0	0	0
IRS 1Y	98	0	0	0	0
IRS 2Y	0	195	0	0	0
IRS 3Y	0	0	291	0	0
IRS 4Y	0	0	0	386	0
IRS 5Y	0	0	0	0	479

Total Trade DV01

IRS 1Y	IRS 2Y	IRS 3Y	IRS 4Y	IRS 5Y
98	195	291	386	479

Swap Risk - Trade Positions & DV01 Risk

Trade Positions

- **IRS 1Y:** USD 1mm Spot Starting 1Y IRS
- **IRS(4Y, 5Y):** USD 1mm Forward Starting IRS
Starts in 4Y and Ends in 5Y
- **IRS(4.5Y):** USD 1mm Spot Starting 4.5Y IRS

Risk Profiles

- **IRS 1Y:** Same Risk as Calibration Instrument
- **IRS(4Y, 5Y):** Equivalent to Long 5Y IRS and Short 4Y IRS
- **IRS(4.5Y):** Equivalent to 50% 4Y IRS and 50% 5Y IRS

Actual Risk

Portfolio Risk - Trade Level

Risk Bucket	IRS 1Y	IRS(4Y, 5Y)	IRS(4.5Y)
OIS 1Y	0	0	0
OIS 2Y	0	0	0
OIS 3Y	0	0	0
OIS 4Y	0	0	0
OIS 5Y	0	0	0
IRS 1Y	98	0	0
IRS 2Y	0	0	0
IRS 3Y	0	0	0
IRS 4Y	0	-386	193
IRS 5Y	0	479	239

Actual Risk

Portfolio Risk - Total

Risk Bucket	Risk Total
OIS 1Y	0
OIS 2Y	0
OIS 3Y	0
OIS 4Y	0
OIS 5Y	0
IRS 1Y	98
IRS 2Y	0
IRS 3Y	0
IRS 4Y	-193
IRS 5Y	718

Actual Risk

Portfolio Hedges

Hedge	Qty
OIS 1Y	0
OIS 2Y	0
OIS 3Y	0
OIS 4Y	0
OIS 5Y	0
IRS 1Y	-1
IRS 2Y	0
IRS 3Y	0
IRS 4Y	0.50
IRS 5Y	-1.50

Total Trade DV01

IRS 1Y	IRS(4Y, 5Y)	IRS(4.5Y)
98	93	432

Total DV01

624

Fast Swap Risk – Curve Calibration Instruments

Fast Swap Risk

- Use **Multiples Approach** for intuition
- Gives a quick risk overview
- A close approximation & upper-bound
- **DV01(Base Case) = 100 per Million per Year**

Quick Risk

Hedge Trade Risk

Hedge Trades

Risk Bucket	IRS 1Y	IRS 2Y	IRS 3Y	IRS 4Y	IRS 5Y
OIS 1Y	0	0	0	0	0
OIS 2Y	0	0	0	0	0
OIS 3Y	0	0	0	0	0
OIS 4Y	0	0	0	0	0
OIS 5Y	0	0	0	0	0
IRS 1Y	100	0	0	0	0
IRS 2Y	0	200	0	0	0
IRS 3Y	0	0	300	0	0
IRS 4Y	0	0	0	400	0
IRS 5Y	0	0	0	0	500

Total Trade DV01

IRS 1Y	IRS 2Y	IRS 3Y	IRS 4Y	IRS 5Y
100	200	300	400	500

Fast Swap Risk – Trade Positions & DV01 Risk

Risk Profiles

- **IRS 1Y:** Same Risk as Calibration Instrument
- **IRS(4Y, 5Y):** Long 5Y IRS and Short 4Y IRS
- **IRS(4.5Y):** 50% of 4Y IRS and 50% of 5Y IRS

DV01 Calculations

- **IRS 1Y:** DV01(Base Case)=100
- **IRS(4Y, 5Y):** DV01(1mm, 5Y) – DV01(1mm, 4Y)
- **IRS(4.5Y):** 0.5 x DV01(1mm 4Y) + 0.5 x DV01(1mm 5Y)

Quick Risk

Portfolio Risk - Trade Level

Risk Bucket	IRS 1Y	IRS(4Y, 5Y)	IRS(4.5Y)
OIS 1Y	0	0	0
OIS 2Y	0	0	0
OIS 3Y	0	0	0
OIS 4Y	0	0	0
OIS 5Y	0	0	0
IRS 1Y	100	0	0
IRS 2Y	0	0	0
IRS 3Y	0	0	0
IRS 4Y	0	-400	200
IRS 5Y	0	500	250

Quick Risk

Portfolio Risk - Total

Risk Bucket	Risk Total
OIS 1Y	0
OIS 2Y	0
OIS 3Y	0
OIS 4Y	0
OIS 5Y	0
IRS 1Y	100
IRS 2Y	0
IRS 3Y	0
IRS 4Y	-200
IRS 5Y	750

Quick Risk

Portfolio Hedges

Hedge	Qty
OIS 1Y	0
OIS 2Y	0
OIS 3Y	0
OIS 4Y	0
OIS 5Y	0
IRS 1Y	-1
IRS 2Y	0
IRS 3Y	0
IRS 4Y	0.50
IRS 5Y	-1.50

Total Trade DV01

IRS 1Y	IRS(4Y, 5Y)	IRS(4.5Y)
100	100	450

Total DV01

650

References

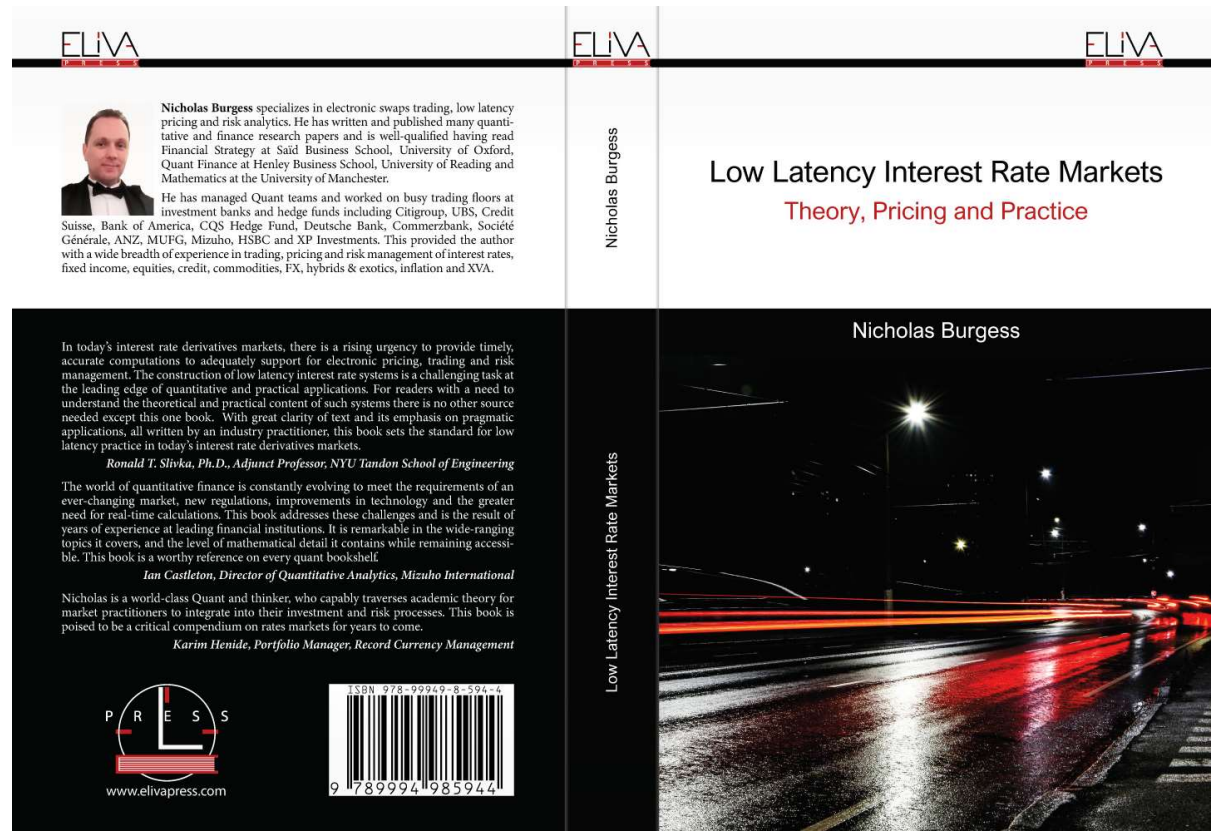


Quant Research Papers

<https://ssrn.com/author=1728976>

Support Materials, C++ & Excel Examples

<https://github.com/nburgessx/SwapsBook>



Available from Amazon

UK: <https://www.amazon.co.uk/dp/9994985949>

USA: <https://www.amazon.com/dp/9994985949>

Appendix – Implicit Function Theorem (IFT)

IFT Theorem

To gain some intuition consider the following function $f(x, y) = 0$ for which we have a solution (a, b) . Near the solution we can express y as function of x namely $f(x, y(x)) = 0$. Using this expression, we can compute the derivative in terms of x only by differentiating with respect to x as follows,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial x} = 0$$

which gives,

$$\frac{\partial y}{\partial x} = - \left(\frac{\partial f / \partial x}{\partial f / \partial y} \right)$$

We have a solution under the condition, $\partial f / \partial y \neq 0$, since we cannot divide by zero.

Yield Curve Application

In the context of a yield curve calibration, we solve for the solution of a helper target function, $H(L, P) = 0$, where L is the LIBOR forward rate state variable (model output) and P the yield curve par rate (model input). The helper target function computes the difference between model par rates as a function of the forward state variable L and a market instrument par rate quote,

$$H(L, P) = \text{Model Par Rate}(L) - \text{Market Par Rate}$$

How does this Help with Sensitivity Calculations?

The IFT theorem says that having found a solution to the continuously differentiable function $H(L, P) = 0$ in two variables we can express the solution solely in terms of the model output L namely $H(L, P(L)) = 0$ and that the Jacobian derivative can be computed independent of model inputs i.e., the yield curve instruments and par rates as,

$$\frac{\partial P}{\partial L} = - \left(\frac{\partial H / \partial L}{\partial H / \partial P} \right)$$

Now, from the definition of the function $H(L, P)$ we can easily determine $dH/dP = -1$ which leads to,

$$\begin{aligned} \frac{\partial P}{\partial L} &= - \left(\frac{\partial H / \partial L}{\partial H / \partial P} \right) = \frac{\partial H}{\partial L} \\ &= \frac{d}{dL} (\text{Model Par Rate}) \end{aligned}$$

For an Interest Rate Swap

$$\text{Par Rate}, p = \frac{PV(\text{Float Leg})}{\sum_{i=1}^n N \tau_i P(t_0, t_i)} = \frac{\sum_{j=1}^m N (l_{j-1} + s) \tau_j P(t_0, t_j)}{\text{Annuity}(\text{Fixed})}$$

- The derivative with respect to L is trivial to calculate
- We can calculate for any set of calibration instruments
- This allows us to modify and select any risk & hedge buckets

Appendix – Swap DV01 Risk Example using AAD (Part I)

IRS Present Value Code

- Swap Price Implementation
- Simplified for Demo Purposes
- For Full Example See

<https://bit.ly/SwapCodeAAD>

```
01 // Swap Inputs
02 // phi    Pay or Receive Fixed: Pay = 1, Receive = -1
03 // n      Swap Notional
04 // r      Fixed rate
05 // tau    Accrual year fraction
06 // t      Coupon Payment Time
07 // f      Floating Forward Rate
08 // s      Floating Spread
09 // z      Discounting Zero Rate for Discount Factor, where df = exp(-z*t)
10
11 double swap_pv(double phi, double n, double r, double tau, double t, double f, double s,
12 double z)
13 {
14     double df = exp(-z*t);           // Step 1. Discount Factor using zero rate, z
15     double pv_fixed = phi*n*r*tau*df; // Step 2. Fixed PV =  $\varphi N r \tau_1 P(0, t_1)$ 
16     double pv_float = -phi*n*(f+s)*tau*df; // Step 3. Float PV =  $\varphi N (l_1 + s) \tau_1 P(0, t_1)$ 
17     double pv_swap = pv_fixed+pv_float; // Step 4. Swap PV = Fixed PV + Float PV
18     return pv_swap;
19 }
```

Swap Price

Appendix – Swap DV01 Risk Example using AAD (Part II)

Analytical DV01 Risk

- Using Adjoint Mode (AAD)
- Forward Sweep for Price
- Back Propagation for Risk
- Simultaneous Forward and Discount Risk

```

01 double adjoint(double phi, double n, double r, double tau, double t, double f, double s, double z,
02 double pv_bar)
03 {
04     // Forward Sweep
05     double df = exp(-z*t); // Step 1. Discount Factor using zero rate, z
06     double pv_fixed = phi*n*r*tau*df; // Step 2. Fixed PV =  $\phi N r \tau_1 P(0, t_1)$ 
07     double pv_float = -phi*n*(f+s)*tau*df; // Step 3. Float PV =  $\phi N (l_1 + s) \tau_1 P(0, t_1)$ 
08     double pv_swap = pv_fixed + pv_float; // Step 4. Swap PV = Fixed PV + Float PV
09     // Backward Propagation
10     double pv_fixed_bar = pv_bar; // Step 4.
11     double pv_float_bar = pv_bar; // Step 4.
12     double f_bar = -phi*n*tau*df*pv_float_bar*shift_size_f; // Step 3. *
13     double df_bar = -phi*n*f*tau*pv_float_bar*shift_size_df; // Step 3. *
14     df_bar += phi*n*r*tau*pv_fixed_bar*shift_size_df; // Step 2. *
15     double z_bar = -t*exp(-z*t)*df_bar; // Step 1.
16
17     // DV01 Result
18     return f_bar + df_bar; // Sensitivity to 1 bps change in forwards and discount factors
19 }

```

Swap DV01 using AD in Adjoint Mode

Source Code: <https://onlinedb.com/5U3lChYiD>

```

01 // inputs( phi, n, r, tau, t, f, s, z, pv_bar )
02 adjoint( 1, 1000000, 0.02, 1, 1, 0.01, 0, 0.02, 1 ); // Output DV01 Risk

```

Swap DV01 Risk using Adjoint Mode

Have questions or want further info?

Contact

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