Bond Option Pricing using the Vasicek Short Rate Model

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Abstract

An option is a financial instrument that allows the holder to buy or sell an underlying security in the future at an agreed strike or price set today. Many options are priced under the assumption of constant interest rates as seen in the Black-Scholes (1973) model. In interest rate markets however the underlying security is an interest rate, which cannot be assumed constant. Likewise bond markets have a similar requirement.

In what follows the assumption of a constant interest rate is relaxed. Bond option pricing using the Vasicek short rate model is examined in such a way that the methodology could be applied to any short rate model such as the classical Hull-White model (1990a)[1].

Firstly we discuss the preliminaries, namely numeraires and measures, where it can be seen that a careful choice of numeraire can simplify option calculations. Secondly we summarize the Vasicek short rate process and a change of numeraire to the terminal-forward measure is outlined, which simplifies bond option pricing calculations. Thirdly we review both pure discount and coupon bond pricing. Fourthly bond option pricing formulae are derived and 'Jamshidian's Trick' outlined.

Finally in conclusion practical implementation considerations and model extensions are discussed. The aim of this paper is to provide a general overview of option pricing using short rate models, using the Vasicek model as an important case study.

[1]Often referred to as the Extended-Vasicek model

1 Numeraires & Measures

Consider any option with a generic payoff denoted $\mathcal{H}_T$. The martingale representation theorem provides us with a framework to evaluate the price of an option using the below formula, whereby the price V_t at time t of such an option is evaluated with respect to a numeraire N and corresponding probability measure $\mathbb{Q}_N$.

$$\frac{V_t}{N_t} = \mathbb{E}^{\mathbb{Q}_N}\left[\frac{\mathcal{H}_T}{N_T}|\mathcal{F}_t\right] \tag{1}$$

which can also be written as

$$V_t = N_t.\mathbb{E}^{\mathbb{Q}_N}\left[\frac{\mathcal{H}_T}{N_T}|\mathcal{F}_t\right] \tag{2}$$

where
V_t is the option price evaluated at time t
N_t is the numeraire evaluated at time t
$\mathbb{E}^{\mathbb{Q}_N}[.]$ is an expectation with respect to the measure of numeraire N (discussed below)
$\mathcal{H}_T$ is a generic option payoff evaluated at time T

Under the 'Martingale Representation' approach an arbitrage free portfolio is formed to replicate the option using both the underlying and numeraire, which is a tradable asset. Together the underlying and numeraire form a perfect hedge. If the numeraire is to be part of a hedge portfolio it must be a positive tradable asset, which pays no dividends. The later condition ensures that we have a smooth continuous price process without jumps.

Each numeraire can be represented as a stochastic process[2] and therefore has a probability measure assigned to it. The probability measure corresponds to the probability density function governing the likelihood of price changes of the numeraire.

The numeraire also determines the denomination of the option price V_t or pricing units. If for example the option price is 100, the numeraire determines the units, e.g. pounds, euros, dollars. The numeraire is typically a cash instrument or bond, however it may be a completely different instrument such as, for example, a commodity or stock, provided it pays no dividends.

The most popular choice of numeraire would be a $savings\ account$, sometimes referred to as a cash bond, the associated equivalent probability measure is called the $risk$-$neutral$ measure and denoted $\mathbb{Q}$. A dollar savings account numeraire would denominate option prices V_t in dollars and would imply that the option replicating portfolio would comprise of the underlying and a dollar cash bond.

1.1 Savings Account Numeraire

Under the martingale representation theorem an option price is unique regardless of the choice of numeraire. Therefore option prices can be evaluated using any numeraire, subject to the

[2] One could also consider a numeraire as a random variable with a corresponding probability density function.

conditions above, namely that the numeraire is a positive tradable non-dividend paying asset. However in many cases it is not convenient and in some cases not possible to evaluate the expectation term in equations (1) and (2).

A savings account[3] is formed by holding cash in a risk-free account that accrues continuously compounded interest. Consider such a savings account process B_T[4] with dynamics

$$B_T = e^{\int_t^T r_u \, du} \tag{3}$$

or in difference form

$$dB_t = r_t B_t dt \tag{4}$$

Applying the martingale representation formula using the savings account numeraire gives

$$\frac{V_t}{B_t} = \mathbb{E}^Q \left[\frac{\mathcal{H}_T}{B_T} | \mathcal{F}_t \right] \tag{5}$$

Rerranging this gives

$$\begin{aligned} V_t &= B_t \mathbb{E}^Q \left[\frac{\mathcal{H}_T}{B_T} | \mathcal{F}_t \right] \\ V_t &= \mathbb{E}^Q \left[\frac{B_t}{B_T} \mathcal{H}_T | \mathcal{F}_t \right] \\ V_t &= \mathbb{E}^Q \left[e^{-\int_t^T r_u \, du} \mathcal{H}_T | \mathcal{F}_t \right] \end{aligned} \tag{6}$$

The rates process r_u in equation (6) under the savings account numeraire is stochastic and not trivial to evaluate, so at this point consider a change of numeraire.

The price of a derivative is invariant regardless of the choice of numeraire and therefore a numeraire can be chosen to simplify the calculation of the Expectation term within equation (5). Other considerations relating to the choice of measure include:

1. **Analytical Tractability**
   Can a closed form solution be reached?

2. **Implementation**
   Is the solution compatible with a recombining tree[5] and monte carlo pricing methods?

3. **Behaviour**
   Does the solution exhibit mean reversion[6]?

[3]A savings account can also be thought of as investing in a cash bond or depositing funds in a savings account, which is assumed risk-free and accumulating continuously compounded interest.

[4]Note that time t represents the filtration time or pricing date, T the maturity of the cash bond and $(T - t)$ the amount of time of the savings account funds are held on deposit.

[5]Trees allow us allow us to price American options.

[6]Mean Reversion is an empirical market observation whereby over time certain instruments, such as interest rates, revert back to a mean average level.

4. **Dynamics**
   Do the solution dynamics imply positive interest rates at all times?
   Can rates go negative[7]?

1.2 Choice of Measure

Firstly recall from section (1) that our replication portfolio consists of an underlying and a numeraire, and note that the underlying and numeraire could both be similiar instruments, both bond instruments perhaps. Secondly remembering that a measure or numeraire must be a tradable instrument, consider the tradable instruments available for selection.

In the interest rate market rates are derived from fixed income bonds. Typically these form the basis for our choice of numeraire. There are several bond instruments to choose from and the most popular choice is the savings account or cash bond as outlined above in section (1.1). Other choices of numeraire and measure are outlined below.

1.2.1 Risk-Neutral Measure, $\mathbb{Q}$

The tradable numeraire is a riskless cash bond or rolling savings account. The associated measure is called the risk-neutral measure $\mathbb{Q}$. This measure plays a key role in the Black-Scholes (1993) model.

1.2.2 Terminal-Forward Measure, $\mathbb{Q}_T$

The tradable numeraire is a zero coupon bond of maturity T, which is chosen to match the maturity of the underlying instrument to be priced. Hence the associated measure is called the terminal-forward measure $\mathbb{Q}_T$. This numeraire is used to price bonds, forwards and the like.

1.2.3 Forward Measure, $\mathbb{Q}_F$

The tradable numeraire is a zero coupon bond of maturity S, where $S > T$. That is to say the numeraire maturity S is greater than the maturity of the underlying instrument T. The associated measure is called the forward measure $\mathbb{Q}_F$.

1.2.4 Annuity Measure, $\mathbb{Q}_A$

The tradable numeraire is an annuity swap. The assocaited measure is callled the annuity measure $\mathbb{Q}_A$. This numeraire is used to price swaptions.

Finally for completeness and reference purposes we should mention the real-world measure $\mathbb{P}$, which is not typically used for derivatives pricing.

[7] Interest rates can certainly be negative, but not as frequent or for such prolonged periods as suggested by a normal distribution say.

1.2.5 Real-World Measure, $\mathbb{P}$

The real-world measure $\mathbb{P}$ gives the real-world probability of an event occurring. If an experiment were to be repeated many times this probability measure would be helpful in determining the long term average result.

Example: Real-World Measure
For example if we wanted to know the probability of rolling a fair die and landing on the number six. The real-world probability gives $P(die = 6) = \frac{1}{6}$, but this result would only be of use if we were to repeat the experiment rolling the die many times.

That is to say the number 6 should appear on average one time out of six. For a small number of die throws this often not the case. However as we increase the number of die throws we converge to the result and more so as the number of throws increases to infinity.

As far as derivatives pricing is concerned the real-world measure $\mathbb{P}$ provides an indication of the long term average price of a derivative but would not give an arbitrage free price.

1.3 Change of Measure

To change between measures the Radon-Nikodym Derivative is used, which is often encountered when changing from the real-world measure to the risk-neutral measure and denoted[8] $\frac{d\mathbb{Q}}{d\mathbb{P}}$.

Consider two numeraires N and M with associated equivalent martingale measures $\mathbb{Q}_N$ and $\mathbb{Q}_M$. Under the $\mathbb{Q}_N$ measure we have

$$V_t = N_t \mathbb{E}^{\mathbb{Q}_N}\left[\frac{\mathcal{H}_T}{N_T}|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}_N}\left[\frac{N_t}{N_T}\mathcal{H}_T|\mathcal{F}_t\right] \tag{7}$$

and under the $\mathbb{Q}_M$ measure we have

$$V_t = M_t \mathbb{E}^{\mathbb{Q}_M}\left[\frac{\mathcal{H}_T}{M_T}|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}_M}\left[\frac{M_t}{M_T}\mathcal{H}_T|\mathcal{F}_t\right] \tag{8}$$

equating equations (7) and (8) gives

$$\mathbb{E}^{\mathbb{Q}_N}\left[\mathcal{H}_T\frac{N_t}{N_T}|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}_M}\left[\mathcal{H}_T\frac{M_t}{M_T}|\mathcal{F}_t\right] \tag{9}$$

Stochastic Terms
The N_T and M_T terms in equations (7) to (9) above are stochastic and must remain within the expectation operator, however N_t and M_t are known values at the filtration time t and could be treated as constants.

The Radon-Nikodym derivative is a ratio of probability measures $\left(\frac{d\mathbb{Q}_{New}}{d\mathbb{Q}_{Old}}\right)$ such that we divide (and eliminate) the old measure and multiply (and introduce) the new measure.

[8] where $d\mathbb{Q}$ represents the risk-neutral measure we are changing to and $d\mathbb{P}$ is the real-world measure we are changing from.

We define the Radon-Nikodym derivative of $d\mathbb{Q}_M$ with respect to $d\mathbb{Q}_N$ as below

$$\frac{d\mathbb{Q}_M}{d\mathbb{Q}_N} = \frac{\left(\frac{M_t}{M_T}\right)}{\left(\frac{N_t}{N_T}\right)} = \left(\frac{M_t}{M_T}\frac{N_T}{N_t}\right) \tag{10}$$

multiplying the left-hand side LHS of equation (9) by the Radon-Nikodym derivative changes the LHS measure from $\mathbb{Q}_\mathbb{N}$ to $\mathbb{Q}_\mathbb{M}$ as demonstrated below

$$\mathbb{E}^{\mathbb{Q}_M}\left[\frac{N_t}{N_T}\frac{d\mathbb{Q}_M}{d\mathbb{Q}_N}\mathcal{H}_T|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}_M}\left[\mathcal{H}_T\frac{N_t}{N_T}\left(\frac{M_t}{M_T}\frac{N_T}{N_t}\right)|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}_M}\left[\mathcal{H}_T\frac{M_t}{M_T}|\mathcal{F}_t\right] \tag{11}$$

which leads to and implies

$$\mathbb{E}^{\mathbb{Q}_N}\left[\frac{N_t}{N_T}\mathcal{H}_T|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}_M}\left[\frac{N_t}{N_T}\frac{d\mathbb{Q}_M}{d\mathbb{Q}_N}\mathcal{H}_T|\mathcal{F}_t\right] \tag{12}$$

Equation (12) demonstrates how to move from one measure $\mathbb{Q}_N$ to another $\mathbb{Q}_M$, namely from equation (7) to (8).

2 Summary of the Vasicek Short Rate Model

In our last paper [4] 'An Overview of the Vasicek Short Rate Model' the Vasicek model was outlined and reviewed, below we summarize the the main points.

2.1 Short Rate Process

The Vasicek short rate model has an SDE with the following functional form

$$dr_t = (\theta - ar_t)dt + \sigma dB_t \tag{13}$$

which can also be represented as

$$dr_t = a(b - r_t)dt + \sigma dB_t \tag{14}$$

where
a = Speed of Mean Reversion, $0 \leq a \leq 1$
b = Mean Reversion Level
θ = ab
r_t = Short Rate at time, t
σ = Short Rate Volatility
B_t = Brownian Motion Process at time, t

Illustration: Vasicek Yield Curve
The yield curve for the Vasicek short rate model is illustrated below.

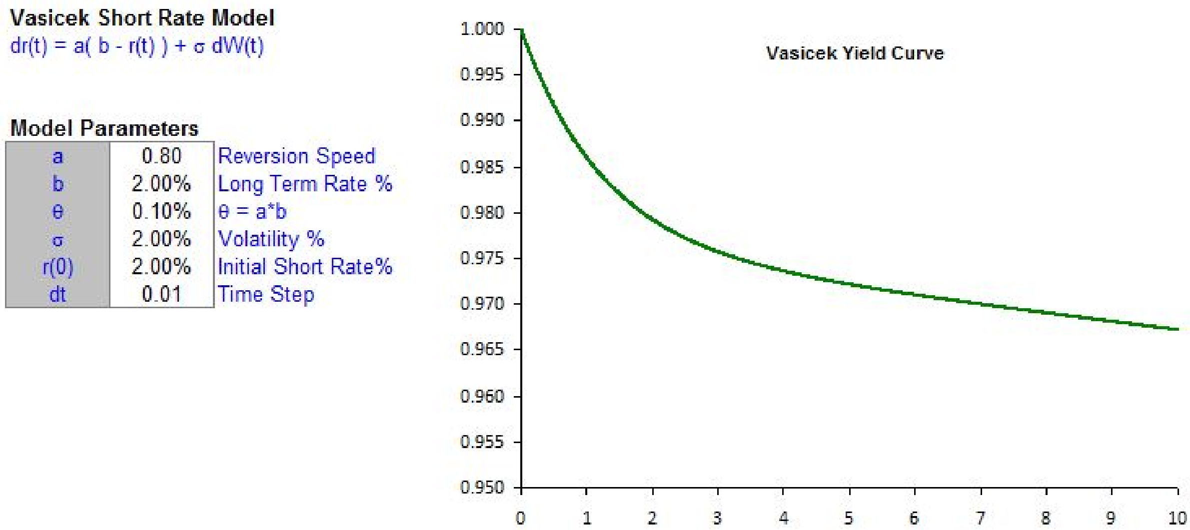

Figure 1: Vasicek Yield Curve

2.2 Short Rate Solution

The solution to the Vasicek SDE in equation (13) follows, whose derivation can be found in [4]. It is important to note that this solution is under the savings account numeraire with the corresponding risk-neutral measure $\mathbb{Q}$.

$$r_t = e^{-a(t-s)} r_s + \frac{\theta}{a}\left(1 - e^{-a(t-s)}\right) + \sigma \int_{u=s}^{t} e^{-a(t-u)}\, dB_u \tag{15}$$

2.3 Dynamics

The distribution of the short rate solution in equation (15) is primarily determined by the Brownian process, which is Gaussian having dynamics $B_t \sim \mathcal{N}(0, t)$.

$$r_t \sim N\left(\frac{\theta}{a}, \frac{\sigma^2}{2a}\right) \tag{16}$$

2.4 Why Change to the Terminal Forward Measure?

The price of an option at time t using the numeraire N with a corresponding risk-neutral measure $\mathbb{Q}_N$ is defined in equations (1) and (2) as

$$V_t = N_t \mathbb{E}^{\mathbb{Q}_N}\left[\frac{\mathcal{H}_T}{N_T} | \mathcal{F}_t\right] \tag{17}$$

Recalling that the option price is measure invariant[9], careful attention is paid to the $\left(\frac{\mathcal{H}_T}{N_T}\right)$ term within the expectation of (17) above and a measure is chosen to simplify the expectation as much as possible.

Using the cash account measure $\mathbb{Q}$ with a cash bond numeraire B_t as defined in equations (3) and (4) the option price is determined as

$$V_t = \underbrace{B_t}_{=1} \mathbb{E}^{\mathbb{Q}} \left[\frac{\mathcal{H}_T}{B_T} | \mathcal{F}_t \right] = \mathbb{E}^{\mathbb{Q}} \left[\frac{\mathcal{H}_T}{e^{\int_t^T r(u)\, du}} | \mathcal{F}_t \right] = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^T r(u)\, du} \mathcal{H}_T | \mathcal{F}_t \right] \tag{18}$$

In this scenario, when using the cash account measure, the bond term B_t outside the expectation in (18) simplifies to unity i.e. $B_t = 1$. However the bond term B_T inside the expectation remains and is stochastic. This stochastic term cannot be factored outside of the expectation operator. The cash measure does not simplify the calculation.

Under the terminal forward measure $\mathbb{Q}_T$ all bonds $P(t,T)$ both pure discount and coupon bearing mature at par[10]. Therefore at maturity $P(T,T) = 1$ and changing to this measure would conveniently lead to a simplified expression for the option price as follows.

$$V_t = P(t,T) \mathbb{E}^{\mathbb{Q}_T} \left[\frac{\mathcal{H}_T}{\underbrace{P(T,T)}_{=1}} | \mathcal{F}_t \right] = P(t,T) \mathbb{E}^{\mathbb{Q}_T} \left[\mathcal{H}_T | \mathcal{F}_t \right] \tag{19}$$

Specifically for a European style option the payoff $\mathcal{H}_T$ would be specified as

$$\mathcal{H}_T = max \left(\phi \left(P(t,T) - K \right), 0 \right) \tag{20}$$

or equivalently

$$\mathcal{H}_T = \phi \left(P(t,T) - K \right)^+ \tag{21}$$

where

$$\phi = \begin{cases} +1 & \text{for a call option} \\ -1 & \text{for a put option} \end{cases} \tag{22}$$

Using the terminal forward measure $\mathbb{Q}_T$ it follows that the European option price V_t at time t on an underlying bond $P(t,T)$ having maturity T with $t < T$ is

$$V_t = P(t,T) \mathbb{E}^{\mathbb{Q}_T} \left[\phi \left(P(t,T) - K \right)^+ | \mathcal{F}_t \right] \tag{23}$$

This expression is easier to evaluate than that under the cash measure.

[9] That is the price is constant regardless of the choice of numeraire. Furthermore since the choice of measure is discretionary, one typically selects a measure to simplify the calculation.

[10] This means that at maturity we receive back 100% of the bond's notional or face value.

2.5 How to Change to the Terminal Forward Measure

Having established that a switch of measure to the terminal forward measure is desirable, attention is drawn to the fact that at this point the solution to the Vasicek SDE has been determined under the cash account measure. Via the change of measure process we proceed to demonstrate how to change to the terminal measure and follow-up by determining the Vasicek solution under this new measure.

The risk-neutral measure $\mathbb{Q}$ is associated with the risk free cash account numeraire B_t. Consider another general numeraire N_t with the associated equivalent martingale measure $\mathbb{Q}_N$ and the below dynamics.

Notation
To avoid confusion, between the cash bond and Brownian motion, W has been used in this section to denote the Brownian / Wiener process instead of the usual B.

$$dB_t = r_t B_t dt \tag{24}$$
$$dN_t = r_t N_t dt + \sigma_t^N N_t dW_t^{\mathbb{Q}} \tag{25}$$

Using Itô's Lemma to evaluate the dynamics for the ratio of B to N gives

$$d\left(\frac{B_t}{N_t}\right) = B_t d\left(\frac{1}{N_t}\right) + \left(\frac{1}{N_t}\right) dB_t \tag{26}$$

Let $X_t = d\left(\frac{1}{N_t}\right)$ and evaluate using Itô's Lemma

$$\begin{aligned} dX_t &= \frac{dX_t}{dN_t} dN_t + \frac{1}{2}\frac{d^2 X_t}{dN_t^2} dN_t^2 \\ &= -\frac{1}{N_t^2}\left(r_t N_t dt + \sigma_t^N N_t dW_t^{\mathbb{Q}}\right) + \frac{1}{2}\left(\frac{2}{N_t^3}\right)\left(\sigma_t^N\right)^2 N_t^2 dt \\ &= -\frac{r_t}{N_t} dr - \frac{\sigma_t^N}{N_t} dW_t^{\mathbb{Q}} + \frac{\left(\sigma_t^N\right)^2}{N_t} dt \\ &= \left(\frac{\left(\sigma_t^N\right)^2 - r_t}{N_t}\right) dt - \left(\frac{\sigma_t^N}{N_t}\right) dW_t^{\mathbb{Q}} \end{aligned} \tag{27}$$

substituting (27) into (26)

$$\begin{aligned} d\left(\frac{B_t}{N_t}\right) &= B_t dX_t + \left(\frac{r_t B_t}{N_t}\right) dt \\ &= B_t\left[\left(\frac{\left(\sigma_t^N\right)^2 - r_t}{N_t}\right) dt - \left(\frac{\sigma_t^N}{N_t}\right) dW_t^{\mathbb{Q}}\right] + \left(\frac{r_t B_t}{N_t}\right) dt \\ &= \frac{\left(\sigma_t^N\right)^2 B_t}{N_t} dt - \frac{r_t B_t}{N_t} dt - \frac{\sigma_t^N B_t}{N_t} dW_t^{\mathbb{Q}} + \frac{r_t B_t}{N_t} dt \\ &= \frac{\left(\sigma_t^N\right)^2 B_t}{N_t} dt - \frac{\sigma_t^N B_t}{N_t} dW_t^{\mathbb{Q}} \end{aligned} \tag{28}$$

we know that $d\left(\frac{B_t}{N_t}\right)$ in equation (28) is a martingale under $\mathbb{Q}_N$ therefore

$$d\left(\frac{B_t}{N_t}\right) = -\frac{\sigma_t^N B_t}{N_t} dW_t^{\mathbb{Q}_N} \tag{29}$$

comparing (28) and (29) leads to

$$\frac{\left(\sigma_t^N\right)^2 B_t}{N_t} dt - \frac{\sigma_t^N B_t}{N_t} dW_t^{\mathbb{Q}} = -\frac{\sigma_t^N B_t}{N_t} dW_t^{\mathbb{Q}_N} \tag{30}$$

simple factorization and rearrangement gives

$$dW_t^{\mathbb{Q}_N} = dW_t^{\mathbb{Q}} - \sigma_t^N dt \tag{31}$$

We should recognize (31) as the Girsanov result with σ_t^N as the market price of risk λ.

Cameron-Martin-Girsanov Theorem

The Cameron-Martin-Girsanov theorem states that if we have an existing $\mathbb{Q}$-Brownian motion and a new equivalent $\mathbb{Q}_N$-Brownian motion then there exists a previsible $\mathcal{F}_t$ measurable process λ that provides a mechanism to change from the existing measure to the new one, such that

$$dW_t^{\mathbb{Q}_N} = dW_t^{\mathbb{Q}} + \lambda dt \tag{32}$$

and the corresponding λ[11] to change from the old measure to the new one is quoted below

$$\begin{aligned} \lambda &= \left(\frac{\mu_{\text{Old}} - \mu_{\text{New}}}{\sigma}\right) \\ &= \left(\frac{\mu^{\mathbb{Q}} - \mu^{\mathbb{Q}_N}}{\sigma}\right) \end{aligned} \tag{33}$$

noting from (28) and (29) that $\mu^{\mathbb{Q}} = \left(\frac{\left(\sigma_t^N\right)^2 B_t}{N_t}\right)$, $\mu^{\mathbb{Q}_N} = 0$ and that $\sigma = -\left(\frac{\sigma_t^N B_t}{N_t}\right)$ gives

$$\lambda = \left(\frac{\left(\frac{\left(\sigma_t^N\right)^2 B_t}{N_t}\right) - 0}{-\left(\frac{\sigma_t^N B_t}{N_t}\right)}\right) = -\sigma_t^N \tag{34}$$

substituting (34) into (32) leads to (31) confirming that which was stated above.

$$dW_t^{\mathbb{Q}_N} = dW_t^{\mathbb{Q}} - \sigma_t^N dt \tag{35}$$

Negative Diffusion Terms

It is important to note that when the diffusion term $dW^{\mathbb{Q}}$ in the original stochastic process is negative we must use $-\lambda$ or equivalently $-\sigma$ in order to successfully change measure. Alternatively we could make a positive variable substitution for the negative drift or even absorb the

[11] It is important to note that we must use $-\lambda$ or equivalently $-\sigma$ when the original Brownian process has a negative diffusion term, as outlined below.

negative sign into the symmetric Brownian process, which all amount to the same course of action. Rearranging and substituting $dW_t^{\mathbb{Q}} = dW_t^{\mathbb{Q}_N} + \sigma_t^N dt$ from equation (35) into the original stochastic process (28) confirms this to be the correct course of action to change the drift from that under the original measure to that of the new measure whilst at the same time making no change to the diffusion term.

When the numeraire N_t is pure discount bond $Z(t,T)$ with associated equivalent terminal-forward martingale measure $\mathbb{Q}_T$ then (35) becomes

$$dW^{\mathbb{Q}_T} = dW^{\mathbb{Q}} - \sigma_t^Z dt \tag{36}$$

Recalling the definition of the pure discount bond from [4]

$$Z(t,T) = A(t,T)e^{-r_t B(t,T)} \tag{37}$$

where

$$A(t,T) = e^{\left((B(t,T)-(T-t))\left(\frac{\theta}{a}-\frac{\sigma^2}{2a^2}\right)-\left(\frac{\sigma^2 B(t,T)^2}{4a}\right)\right)}$$

and

$$B(t,T) = \left(\frac{1-e^{-a(T-t)}}{a}\right)$$

Since r_t is normally distributed as shown in (16) this implies that $Z(t,T)$ is lognormally distributed[12]. Consequently $\ln(Z(t,T))$ has normal dynamics giving

$$\begin{aligned}
Var(\ln(Z(t,T))) &= Var(ln(A(t,T)e^{-B(t,T)r_t})) \\
&= Var(\ln(A(t,T)) - B(t,T)r_t) \\
&= Var(\ln(A(t,T)) + Var(-B(t,T)r_t) \\
&= B(t,T)^2 Var(r_t) \\
&= B(t,T)^2\sigma^2
\end{aligned} \tag{38}$$

Defining $\sigma_Z^2 = Var(\ln(Z(t,T)))$ and taking the square root leads to an expression for the volatility

$$\sigma_Z = \pm\sqrt{B(t,T)^2\sigma^2} \tag{39}$$

from which we take the negative root[13] to get

$$\sigma_Z = -B(t,T)\sigma \tag{40}$$

it follows by substituting (40) into (36) that

$$dW^{\mathbb{Q}_T} = dW^{\mathbb{Q}} + B(t,T)\sigma dt \tag{41}$$

Notation

Attention is drawn to the fact that $B(t,T)$ above should not be confused with the cash account numeraire B_t.

Furthermore as discussed above can confirm the correctness of this measure change by substituting (41) into (28) to nullify the drift and make the process a martingale under the terminal forward measure. This will also confirm our intuition surrounding the use of the negative volatility parameter in (40).

[12] This comes from the fact that r_t is stochastic and normally distributed. The exponential of any normal process is lognormal. The A(t,T) and B(t,T) terms are deterministic.

[13] Since the diffusion term containing $dW^{\mathbb{Q}}$ in (28) is negative.

2.6 Short Rate Solution using the Terminal Forward Measure

Substituting the change of measure kernel for the Brownian process, namely equation (36) from section (2.4) the Vasicek short rate SDE in equation (13) becomes

$$\begin{aligned} dr_t &= (\theta - ar_t)\,dt + \sigma dB_t^{\mathbb{Q}} \\ &= (\theta - ar_t)\,dt + \sigma\left(dB_t^{\mathbb{Q}_T} - \sigma B(t,T)dt\right) \\ &= \left(\theta - ar_t - \sigma^2 B(t,T)\right)dt + \sigma dB_t^{\mathbb{Q}_T} \end{aligned} \tag{42}$$

This new SDE under the terminal measure can also be solved using the integrating factor short-hand from [4]. Rearranging (42) and multiplying by the Integrating Factor, $I_t = e^{at}$ gives

$$\begin{aligned} dr_t + ar_t dt &= \left(\theta - \sigma^2 B(t,T)\right)dt + \sigma dB_t^{\mathbb{Q}_T} \\ \underbrace{Idr_t + Iar_t dt}_{=d(Ir_t)} &= I\left(\theta - \sigma^2 B(t,T)\right)dt + \sigma I dB_t^{\mathbb{Q}_T} \\ d(Ir_t) &= I\left(\theta - \sigma^2 B(t,T)\right)dt + \sigma I dB_t^{\mathbb{Q}_T} \\ d(e^{at}r_t) &= e^{at}\left(\theta - \sigma^2 B(t,T)\right)dt + \sigma e^{at} dB_t^{\mathbb{Q}_T} \end{aligned} \tag{43}$$

substituting for $B(t,T)$ from equation (37) gives

$$\begin{aligned} d(e^{at}r_t) &= e^{at}\left(\theta - \sigma^2\left(\frac{1 - e^{-a(T-t)}}{a}\right)\right)dt + \sigma e^{at} dB_t^{\mathbb{Q}_\mathbb{T}} \\ &= \theta e^{at}dt - \sigma^2\left(\frac{e^{at} - e^{-a(T-2t)}}{a}\right)dt + \sigma e^{at} dB_t^{\mathbb{Q}_T} \\ &= \theta e^{at}dt - \left(\frac{\sigma^2}{a}\right)e^{at}dt + \left(\frac{\sigma^2}{a}\right)e^{-a(T-2t)}dt + \sigma e^{at} dB_t^{\mathbb{Q}_T} \end{aligned} \tag{44}$$

integrating over (s, t), where $0 < s < t$

$$\begin{aligned} e^{at}r_t - e^{as}r_s = {} & \frac{\theta}{a}\left(e^{at} - e^{as}\right) - \frac{\sigma^2}{a^2}\left(e^{at} - e^{as}\right) \\ & + \frac{\sigma^2}{2a^2}\left(e^{-a(T-2t)} - e^{-a(T-2s)}\right) + \sigma\int_s^t e^{au} dB_u^{\mathbb{Q}_T} \end{aligned} \tag{45}$$

rearranging terms

$$e^{at}r_t = e^{as}r_s + \left(\frac{\theta}{a} - \frac{\sigma^2}{a^2}\right)\left(e^{at} - e^{as}\right) + \frac{\sigma^2}{2a^2}\left(e^{-a(T-2t)} - e^{-a(T-2s)}\right) + \sigma\int_s^t e^{au} dB_u^{\mathbb{Q}_T} \tag{46}$$

leading to

$$r_t = e^{-a(t-s)}r_s + \left(\frac{\theta}{a} - \frac{\sigma^2}{a^2}\right)\left(1 - e^{-a(t-s)}\right) + \frac{\sigma^2}{2a^2}\left(e^{-a(T-t)} - e^{-a(T+t-2s)}\right) + \sigma\int_s^t e^{-a(t-u)} dB_u^{\mathbb{Q}_T} \tag{47}$$

for convenience we can factorize and express this as

$$r_t = e^{-a(t-s)}r_s + F^{\mathbb{Q}_T}(s,t) + \sigma\int_s^t e^{-a(t-u)} dB_u^{\mathbb{Q}_T} \tag{48}$$

where

$$F^{\mathbb{Q}_T}(s,t) = \left(\frac{\theta}{a} - \frac{\sigma^2}{a^2}\right)\left(1 - e^{-a(t-s)}\right) + \frac{\sigma^2}{2a^2}\left(e^{-a(T-t)} - e^{-a(T+t-2s)}\right) \tag{49}$$

This solution is identical to that under the cash measure, except that it contains an additional factor $F^{\mathbb{Q}_T}$.

2.7 Short Rate Dynamics under the Terminal Measure

The dynamics of the Vasicek solution under the terminal forward measure outlined in section (2.6) and equations (48) and (49) in particular are derived as follows.

Firstly observe that the short rate solution under the terminal measure is Gaussian, since the Brownian term is normally distributed by definition, whereby $B_t \sim \mathcal{N}(0,t)$.

The distribution mean μ can be found by taking the expectation of equation (48) and noting that the drift terms are deterministic and that both the expected value of the diffusion term and the stochastic integral are zero.

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}_T}(r_t|\mathcal{F}_s) &= \mathbb{E}^{\mathbb{Q}_T}\left[e^{-a(t-s)}r_s + F^{\mathbb{Q}_T}(s,t)|\mathcal{F}_s\right] + \underbrace{\mathbb{E}^{\mathbb{Q}_T}\left[\sigma\int_s^t e^{-a(t-u)}dB_u^{\mathbb{Q}_{\mathbb{T}}}|\mathcal{F}_s\right]}_{=0} \\
&= \mathbb{E}^{\mathbb{Q}_T}\left[e^{-a(t-s)}r_s + F^{\mathbb{Q}_T}(s,t)|\mathcal{F}_s\right] \\
&= e^{-a(t-s)}r_s + F^{\mathbb{Q}_{\mathbb{T}}}(s,t)
\end{aligned} \tag{50}$$

Similarly the variance σ^2 is

$$\begin{aligned}
Var^{\mathbb{Q}_T}(r_t|\mathcal{F}_s) &= \underbrace{Var^{\mathbb{Q}_T}\left(e^{-a(t-s)}r_s + F^{\mathbb{Q}_T}(s,t)|\mathcal{F}_s\right)}_{=0} + Var^{\mathbb{Q}_T}\left(\sigma\int_s^t e^{-a(t-u)}dB_u^{\mathbb{Q}_{\mathbb{T}}}|\mathcal{F}_s\right) \\
&= Var^{\mathbb{Q}_T}\left(\sigma\int_s^t e^{-a(t-u)}dB_u^{\mathbb{Q}_T}|\mathcal{F}_s\right) \\
&= \underbrace{\mathbb{E}^{\mathbb{Q}_T}\left[\left(\sigma\int_s^t e^{-a(t-u)}dB_u^{\mathbb{Q}_T}\right)^2|\mathcal{F}_s\right]}_{\text{Apply Itô Isometry}} - \underbrace{\mathbb{E}^{\mathbb{Q}_T}\left[\sigma\int_s^t e^{-a(t-u)}dB_u^{\mathbb{Q}_T}|\mathcal{F}_s\right]^2}_{=0}
\end{aligned} \tag{51}$$

The variance of the drift term in (51) was zero and likewise the term expectation of the stochastic integral is zero. What remains is to apply expand and solve the squared stochastic term using

Itô's isometry rule, which eliminates the randomness, since dB_t^2 becomes dt.

$$\begin{aligned} Var^{\mathbb{Q}_T}(r_t|\mathcal{F}_s) &= \mathbb{E}^{\mathbb{Q}_T}\left[\left(\sigma\int_s^t e^{-a(t-u)}dB_u^{\mathbb{Q}_T}\right)^2|\mathcal{F}_s\right] \\ &= \mathbb{E}^{\mathbb{Q}_T}\left[\sigma^2\int_s^t e^{-2a(t-u)}du|\mathcal{F}_s\right] \\ &= \sigma^2\int_s^t e^{-2a(t-u)}du \\ &= \frac{\sigma^2\left(1-e^{-2a(t-s)}\right)}{2a} \end{aligned} \tag{52}$$

Hence under the terminal forward measure the Vasicek short rate has dynamics

$$r_t^{\mathbb{Q}_T} \sim \mathcal{N}\left(e^{-a(t-s)}r_s + F^{\mathbb{Q}_T}(s,t), \frac{\sigma^2\left(1-e^{-2a(t-s)}\right)}{2a}\right) \tag{53}$$

where

$$F^{\mathbb{Q}_T}(s,t) = \left(\frac{\theta}{a}-\frac{\sigma^2}{a^2}\right)\left(1-e^{-a(t-s)}\right) + \frac{\sigma^2}{2a^2}\left(e^{-a(T-t)} - e^{-a(T+t-2s)}\right) \tag{54}$$

Finally the dynamics for the limiting case for t and T are considered. We examine the dynamics when the filtration time[14] $t \to \infty$ and bond maturity $T \to \infty$.

Limits under the Terminal Forward Measure

When using the savings account numeraire and the corrsponding cash measure $\mathbb{Q}$ the filtration time t had no upperbound and therefore we considered the limiting case to be $t \to \infty$. However when using a bond numeraire and the terminal forward measure $\mathbb{Q}_T$ the filtration time t cannot exceed the maturity of the underlying bond i.e. $t < T$. As a result under the terminal forward measure the limiting case for t becomes $t \to T$.

Knowing from (53) and (54) that

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}_T}[r_t|\mathcal{F}_s] &= e^{-a(t-s)}r_s + F^{\mathbb{Q}_T}(s,t) \\ &= e^{-a(t-s)}r_s + \left(\frac{\theta}{a}-\frac{\sigma^2}{a^2}\right)\left(1-e^{-a(t-s)}\right) + \left(\frac{\sigma^2}{2a^2}\right)\left(e^{-a(T-t)} - e^{-a(T+t-2s)}\right) \end{aligned} \tag{55}$$

the limit as $t \to \infty$ is given by

$$\begin{aligned} \lim_{t\to T}\left(\ \mathbb{E}^{\mathbb{Q}_T}[r_t|\mathcal{F}_s]\ \right) &= e^{-a(T-s)}r_s + \left(\frac{\theta}{a}-\frac{\sigma^2}{a^2}\right)\left(1-e^{-a(T-s)}\right) + \left(\frac{\sigma^2}{2a^2}\right)\left(\underbrace{e^{-a(T-T)}}_{=1} - e^{-a(T+T-2s)}\right) \\ &= e^{-a(T-s)}r_s + \left(\frac{\theta}{a}-\frac{\sigma^2}{a^2}\right)\left(1-e^{-a(T-s)}\right) + \left(\frac{\sigma^2}{2a^2}\right)\left(1-e^{-2a(T-s)}\right) \end{aligned} \tag{56}$$

[14] or pricing date

by further considering the limiting case where the bond maturity $T \to \infty$ we observe that the exponential terms in (56) tend to zero giving

$$\begin{aligned}\lim_{\substack{t \to T \\ T \to \infty}} \left(\mathbb{E}^{\mathbb{Q}_T}[r_t|\mathcal{F}_s] \right) &= \left(\frac{\theta}{a} - \frac{\sigma^2}{a^2}\right) + \left(\frac{\sigma^2}{2a^2}\right) \\ &= \left(\frac{\theta}{a} - \frac{\sigma^2}{2a^2}\right)\end{aligned} \tag{57}$$

and similarly the variance is given by (53)

$$Var^{\mathbb{Q}_T}(r_t|\mathcal{F}_s) = \frac{\sigma^2\left(1 - e^{-2a(t-s)}\right)}{2a} \tag{58}$$

letting $t \to T$ produces the below the limiting case

$$\lim_{t \to T} Var^{\mathbb{Q}_T}(r_t|\mathcal{F}_s) = \frac{\sigma^2\left(1 - e^{-2a(T-s)}\right)}{2a} \tag{59}$$

and by further taking the limit $T \to \infty$ this becomes

$$\lim_{\substack{t \to T \\ T \to \infty}} Var^{\mathbb{Q}_T}(r_t|\mathcal{F}_s) = \frac{\sigma^2}{2a} \tag{60}$$

Therefore in the limiting case where $t \to T$ and $T \to \infty$ under the terminal forward measure has the following dynamics

$$r_t^{\mathbb{Q}_T} \sim N\left(\left(\frac{\theta}{a} - \frac{\sigma^2}{2a^2}\right), \frac{\sigma^2}{2a}\right) \tag{61}$$

or equivalently

$$r_t^{\mathbb{Q}_T} \sim N\left(\left(b - \frac{\sigma^2}{2a^2}\right), \frac{\sigma^2}{2a}\right) \tag{62}$$

Model Dynamics under Different Measures
When changing measures only the mean of the model distribution is transformed and the variance term being measure invariant remains unchanged.

Comparing the dynamics under the cash and terminal measures, as expected, only the distribution mean is different and the variance is unchanged and measure invariant. It can be seen that the expected value tends to $\left(b - \frac{\sigma^2}{2a^2}\right)$. Hence r_t in the limit tends to the reversion level less some factor $\frac{\sigma^2}{2a^2}$. The variance tends to $\left(\frac{\sigma^2}{2a}\right)$, which is the model variance scaled by the speed of mean reversion.

3 Bond Pricing

In this section the analytical bond pricing formulae are reviewed firstly for pure discount bonds $Z(t,T)$ and secondly for coupon bearing bonds $P(t,T)$.

3.1 Pure Discount Bond Pricing

Zero coupon bonds accumulate interest over the life of the bond rather than paying regular coupon interest. Such bonds are issued at a discount to notional invested, at 90% say, but at maturity they redeem at par i.e. 100%. The discount compensates for the lack of interest. In effect interest is paid at maturity as part of the notional or face value redemption.

Discount Factors
A zero coupon bond price $Z(t,T)$ when quoted in percent is a discount factor. Discount factors are used to evaluate the price of a future cashflow. $Z(t,T)$ would give the price at time t of a unit cashflow paying at time T with $t < T$.

In the previous paper [4] an analytical solution for the zero coupon bond price $Z(t,T)$ was derived. The result of which we quote below, namely

$$Z(t,T) = A(t,T)e^{-r_t B(t,T)} \tag{63}$$

where

$$A(t,T) = e^{\left((B(t,T)-(T-t))\left(\frac{\theta}{a}-\frac{\sigma^2}{2a^2}\right)-\left(\frac{\sigma^2 B(t,T)^2}{4a}\right)\right)}$$

and

$$B(t,T) = \left(\frac{1-e^{-a(T-t)}}{a}\right)$$

Illustration: Pure Discount and Coupon Bond Prices
Pure discount and coupon bond prices for the Vasicek short rate model are illustrated below.

Vasicek Short Rate Model
dr(t) = a(b - r(t)) + σ dW(t)

Model Parameters

a	0.80	Reversion Speed
b	2.00%	Long Term Rate %
θ	0.10%	θ = a*b
σ	2.00%	Volatility %
r(0)	2.00%	Initial Short Rate%

Coupon Bond Specification

Notional	100.00
Coupon	2.00%
Frequency	1.00
Tenor	5.00

Pure Discount Bond Parameters

A(t,T)	0.99	Parameter A
B(t,T)	0.69	Parameter B
t	0.00	Intitial Time
r(t)	2.00%	Initial Rate
T	1.00	Tenor

Portfolio: Zero Coupon Bonds

Portfolio	1.9605%	1.9219%	1.8843%	1.8475%	92.3849%	
Coupon	2.00	2.00	2.00	2.00	102.00	
Z(t,T)	98.02%	96.10%	94.22%	92.38%	90.57%	Zero Price
A(t,T)	0.99	0.98	0.96	0.95	0.93	Parameter A
B(t,T)	0.69	1.00	1.14	1.20	1.23	Parameter B
t	0.00	0.00	0.00	0.00	0.00	Intitial Time
r(t)	2.00%	0.02	0.02	0.02	0.02	Initial Rate
T	1.00	2.00	3.00	4.00	5.00	Tenor

Pure Discount Bond Price

Z(t,T)	98.02%

Coupon Bond Price

P(t,T)	100.00%

Figure 2: Pure Discount and Coupon Bond Prices

3.2 Coupon Bond Pricing

Coupon bearing bonds, contrary to pure discount bonds, pay regular coupons and at maturity redeem at par i.e. 100% of notional invested.

A coupon bearing bond $P(t,T)$ could be priced as follows

$$P(t,T) = \left(\sum_{i=1}^{n} c_i Z(t,T_i) \right) + NZ(t,T) \tag{64}$$

where
n represents the number of bond coupons
c_i the coupon amount
T_i the payment date for ith coupon
T the bond maturity
N the bond notional.

All of the bond cashflows are adjusted by the appropriate discount factor to obtain their price or net present value (NPV) as at time t. As can be seen in equation (64) the coupons c_i are discounted by $Z(t,T_i)$ the discount factor at time T_i and the final notional exchange at maturity N is likewise discounted by $Z(t,T)$ the discount factor at maturity.

Illustration: Coupon Bond Decomposition into a Portfolio of Pure Discount Bonds
Consider a 5 year coupon bond with annual coupon payments of 2% as specified in figure (2). The bond cashflows are displayed in figure (3) whereby annual coupons (blue) are paid each year at time $t(i)$ where $1 \leq i \leq 5$ and the bond notional (orange) is redeemed at par or 100% at maturity $t(5)$.

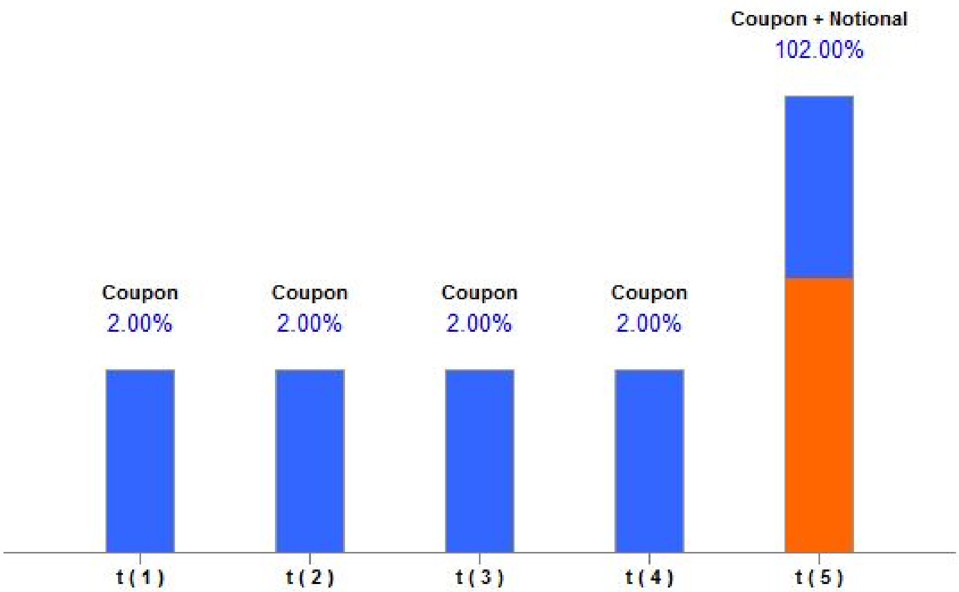

Figure 3: Coupon Bond Cashflows

Coupon bearing bonds can be decomposed and priced as a series of pure discount bonds. The above bond cashflows have been decomposed and priced as a portfolio of zero coupon bonds giving the coupon bond Net Present Value, NPV shown in figure (4). The portfolio is made up of 5 pure discount bonds with values 1.96%, 1.92%, 1.88%, 1.85% *and* 92.38% *giving a total NPV of* 100%[15].

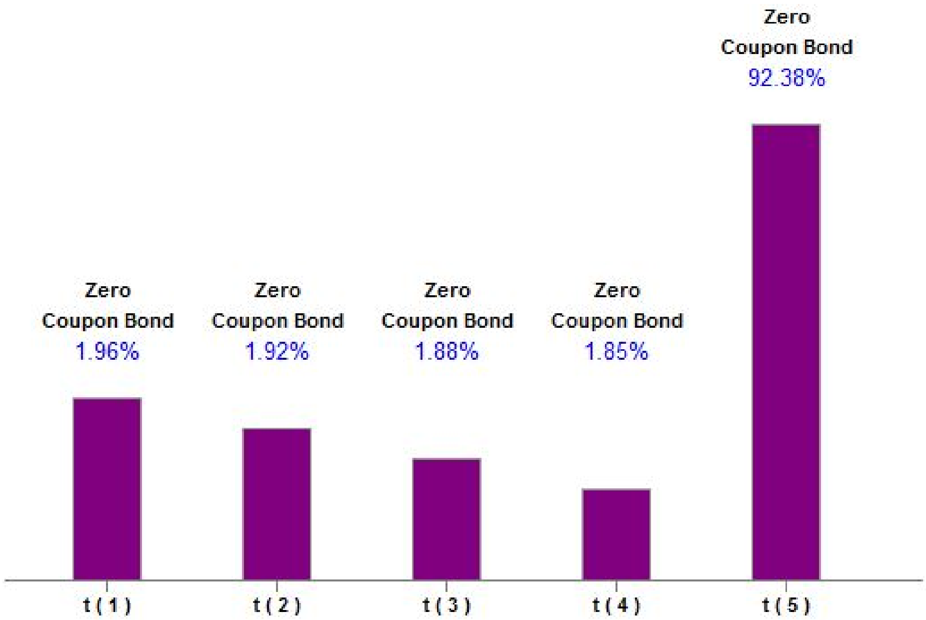

Figure 4: Pricing a Portfolio of Pure Discount Bonds

4 Bond Option Pricing

In this section European Bond option pricing is reviewed. Firstly options on pure discount bonds are considered and evaluated as the expected value of a log-normal process, which leads to a 'Black-Scholes' like expression. Secondly we outline how to price coupon bonds using 'Jamshidian's Trick'.

Option & Underlying Bond Maturities

Zero coupon bond options must be written on a bond that matures after the option expiry. It is not possible to have an option whereby the underlying bond already matured before the exercise date. Furthermore if the option expiry date and the underlying bond maturity date are identical then the option would be exercised into a bond that is maturing at the same time, such an option would generate a digital payoff[16]*, which is typically not desirable.*

[15] This is a par coupon bond having a price 100%. This should be fairly obvious, since the bond pays coupons of 2% and the model was calibrated to have both the initial and long term rate of 2% to match and a strong mean reversion, $a = 0.8$.

[16] At maturity we would have a 100 or 0 delta option position, equivalent to a cash or nothing position.

4.1 Pure Discount Bond Option Pricing

To price a European style pure discount bond option we break the problem down into smaller steps. Firstly we derive an expression for the expected payoff, which returns an expression quoted in terms of the mean and variance of the underlying risk process, r_T.

Secondly the mean and variance dynamics of the short rate under the terminal-forward measure calculated in equations (53) and (54) above are then applied to this expression.

Finally we evaluate the zero coupon bond option as the discounted expected value of the payoff and refactor the option price to quote the price in terms of the underlying bond.

4.1.1 Expected Value of a Log-Normal Process

To price our bond option we are required to evaluate it's expected payoff see equation (19). In what follows we outline a useful formula to do just that.

If X is a random variable that is lognormally distributed then let us define $Y := ln(X)$ with mean μ and variance σ^2 with $Y \sim N(\mu, \sigma^2)$. Knowing that the expectation of a random variable, X is defined as $\mathbb{E}(X) = \int_{-\infty}^{+\infty} X f(x)\, dx$, where f(x) denotes the probability density function of X we deduce that

$$\mathbb{E}^{\mathbb{Q}_T}\left[\phi\,(X-K)^+\right] = \int_{-\infty}^{\infty} \left[\phi\,(X-K)\right]^+ \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{\ln x-\mu}{\sigma}\right)^2}\, d\log x \tag{65}$$

Since $Y := ln(X)$ this can be written in terms of Y as

$$\mathbb{E}^{\mathbb{Q}_T}\left[\phi\,(X-K)^+\right] = \int_{-\infty}^{\infty} \left[\phi\,(e^y-K)\right]^+ \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}\, dy \tag{66}$$

The max operator can be eliminated by carefully selecting the integration bounds such that the expression within the maximum operator is always non-negative and the maximum operator is no longer required. In this case such integration limits must satisfy the below condition

$$\begin{aligned} &\phi\,(e^y-K) \geq 0 \\ &\phi\,(y-\ln(K)) \geq 0 \end{aligned} \tag{67}$$

that is

$$\begin{aligned} &y \geq \ln(K) \quad \text{when } \phi = 1 \\ &y \leq \ln(K) \quad \text{when } \phi = -1 \end{aligned} \tag{68}$$

giving for some function $f(y)$ integration bounds of

$$\int_{\ln(K)}^{\infty} f(y)\, dy \qquad \text{when } \phi = 1 \tag{69}$$

and

$$\int_{-\infty}^{\ln(K)} f(y)\, dy \qquad \text{when } \phi = -1 \tag{70}$$

or more generically

$$\int_{\ln(K)}^{\phi\infty} f(y)\, dy \tag{71}$$

Therefore the maximum operator on the right-hand side of (66) can be omitted, including the ϕ term, provided the integrand bounds are respected. As a result the equation becomes

$$\mathbb{E}^{\mathbb{Q}_T}\left[\phi\left(X-K\right)^{+}\right] = \int_{\ln(K)}^{\phi\infty} \left(e^{y}-K\right)\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}\, dy \tag{72}$$

Next we simplify the exponential term making the below variable substitution with the aim of transforming the integrand into the cumulative standard normal density function.

$$z = \left(\frac{y-\mu}{\sigma}\right) \tag{73}$$

or equivalently

$$y = \mu + \sigma z \tag{74}$$

making the substitution for y in equation (72) leads to following. We remind the reader to transform both the integration limits from y to z and the integration variable from dy to σdz [17] using equations (73) and (74) respectively.

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}_T}\left[\phi\left(X-K\right)^{+}\right] &= \int_{\frac{\ln(K)-\mu}{\sigma}}^{\phi\infty} \left(e^{\mu+\sigma z}-K\right)\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}z^2}\sigma\, dz \\
&= \int_{\frac{\ln(K)-\mu}{\sigma}}^{\phi\infty} \left(e^{\mu+\sigma z}-K\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\, dz \\
&= \int_{\frac{\ln(K)-\mu}{\sigma}}^{\phi\infty} \frac{1}{\sqrt{2\pi}}\underbrace{e^{\mu+\sigma z}e^{-\frac{1}{2}z^2}}_{factorize}\, dz - K\int_{\frac{\ln(K)-\mu}{\sigma}}^{\phi\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\, dz
\end{aligned} \tag{75}$$

factorizing of the exponential term indicated and then completing the square allows the exponential term to be expressed to allow us to transform the integrand into the standard cumulative normal density function.

Using the completing square method and observing that

$$\left(-\frac{1}{2}z^2+\sigma z\right) = -\left(\frac{1}{2}z^2-\sigma z\right) = -\frac{1}{2}\left(z+\sigma\right)^2+\frac{1}{2}\sigma^2 \tag{76}$$

we can refactorize the exponential term as below

$$\begin{aligned}
e^{\mu+\sigma z}e^{-\frac{1}{2}z^2} &= e^{\mu}e^{-\frac{1}{2}z^2+\sigma z} \\
&= e^{\mu}e^{-\frac{1}{2}(z-\sigma)^2}e^{\frac{1}{2}\sigma^2} \\
&= e^{\left(\mu+\frac{1}{2}\sigma^2\right)}e^{-\frac{1}{2}(z-\sigma)^2}
\end{aligned} \tag{77}$$

[17] This comes from equation (74) we differentiate the expression and note that μ is constant.

leading to

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}_T}\left[\phi\left(X-K\right)^+\right] &= \int_{\frac{\ln(K)-\mu}{\sigma}}^{\phi\infty} \frac{1}{\sqrt{2\pi}} e^{\left(\mu+\frac{1}{2}\sigma^2\right)} e^{-\frac{1}{2}(z-\sigma)^2}\, dz - K\int_{\frac{\ln(K)-\mu}{\sigma}}^{\phi\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz \\
&= e^{\left(\mu+\frac{1}{2}\sigma^2\right)} \int_{\frac{\ln(K)-\mu}{\sigma}}^{\phi\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sigma)^2}\, dz - K\int_{\frac{\ln(K)-\mu}{\sigma}}^{\phi\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz
\end{aligned} \tag{78}$$

The left integrand can be simplified further by making the following substitution $x = z - \sigma$ as follows

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}_T}\left[\phi\left(X-K\right)^+\right] &= e^{\left(\mu+\frac{1}{2}\sigma^2\right)} \int_{\frac{\ln(K)-\mu-\sigma^2}{\sigma}}^{\phi\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx - K\int_{\frac{\ln(K)-\mu}{\sigma}}^{\phi\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz \\
&= e^{\left(\mu+\frac{1}{2}\sigma^2\right)} \left(\Phi\left(\phi\infty\right) - \Phi\left(\frac{\ln(K)-\mu-\sigma^2}{\sigma}\right)\right) \\
&- K\left(\Phi\left(\phi\infty\right) - \Phi\left(\frac{\ln(K)-\mu}{\sigma}\right)\right)
\end{aligned} \tag{79}$$

where $\Phi(.)$ represents the standard normal cumulative density function. Now since $\Phi(+\infty) = 1$ and $\Phi(-\infty) = 0$ we know that

$$\Phi(\phi\infty) - \Phi(x) = \begin{cases} 1 - \Phi(x) & \text{when } \phi = 1 \\ -\Phi(x) & \text{when } \phi = \text{-1} \end{cases} \tag{80}$$

and using the symmetry property[18] of the standard normal distribution, shown in figure (5) below, leads to the following

$$\Phi(\phi\infty) - \Phi(x) = \begin{cases} \Phi(-x) & \text{when } \phi = 1 \\ -\Phi(x) & \text{when } \phi = \text{-1} \end{cases} \tag{81}$$

which could be stated in generic terms as

$$\Phi(\phi\infty) - \Phi(x) = \phi\Phi(-\phi x) \tag{82}$$

[18] That is $1 - \Phi(z) = \Phi(-z)$

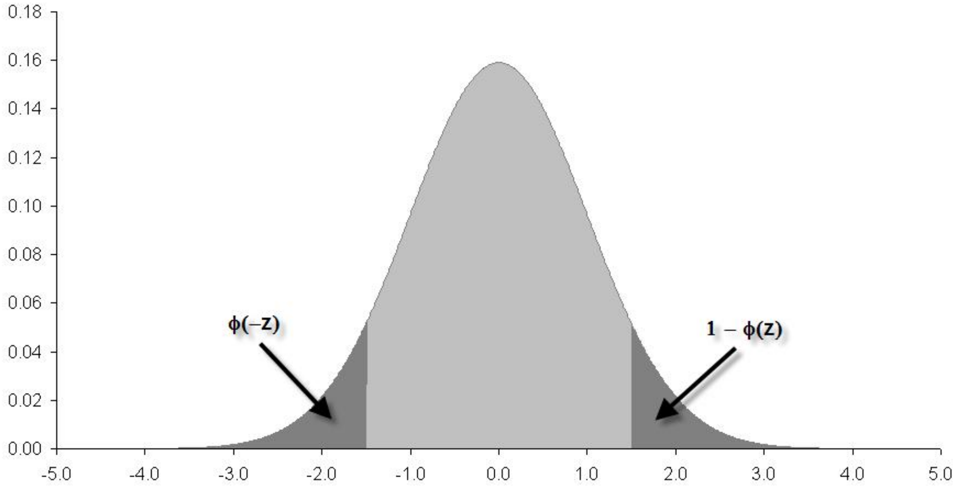

Figure 5: Symmerty Property of Standard Normal Cumuative Density Function

Applying (82) to (79) gives the below solution

$$\mathbb{E}^{\mathbb{Q}_T}\left[\phi\left(X-K\right)^{+}\right]=\phi e^{\mu+\frac{1}{2}\sigma^2}\Phi\left(\phi\left(\frac{\mu+\sigma^2-\ln(K)}{\sigma}\right)\right)-\phi K\Phi\left(\phi\left(\frac{\mu-\ln(K)}{\sigma}\right)\right) \quad (83)$$

4.1.2 Zero Coupon Bond Option Formula

As determined above in section (2.4) and equation (19) the value of a European option on a zero coupon bond $Z(t,T)$ under the terminal-forward measure can be represented as

$$V_t = Z(t,T)\mathbb{E}^{\mathbb{Q}_T}\left[\mathcal{H}_T|\mathcal{F}_t\right] \quad (84)$$

Option and Underlying Bond Maturities

As outlined above, careful attention is drawn to the fact that bond options are quoted on bonds, where the bond maturity, S that is strictly greater than the option maturity, T i.e. $T<S$. *This is a practical restriction, since it is not possible to have an option on an underlying that has already matured. Also whilst it is possible to have the option and bond maturities coincide this is generally not desirable.*

In light of the above the payoff $\mathcal{H}_T = Z(T,S)$[19] where T is the option expiry, S is the underling bond maturity and $T<S$. Setting the payoff $\mathcal{H}_T=\phi\left(Z(T,S)-K\right)^+$ gives

$$V_t = Z(t,T)\mathbb{E}^{\mathbb{Q}_T}\left[\phi\left(Z(t,T)-K\right)^{+}|\mathcal{F}_t\right] \quad (85)$$

and applying equation (83) leads to

$$V_t = Z(t,T)\left[\phi e^{\mu_z+\frac{1}{2}\sigma_z^2}\Phi\left(\phi\left(\frac{\mu_z-\ln(K)+\sigma_z^2}{\sigma_z}\right)\right)-\phi K\Phi\left(\phi\left(\frac{\mu_z-\ln(K)}{\sigma_z}\right)\right)\right] \quad (86)$$

[19]That is the bond matures after the option expiry or exercise date. A payoff of $\mathcal{H}_T=Z(T,T)$ is not desirable.

Note that μ_z and σ_z^2 above are defined in accordance with section (4.1.1), whereby the random variable X is defined as $X = Z(T,S)$ and Y is defined as $Y = \ln(Z(T,S)) \sim N(\mu_z, \sigma_z^2)$, which can be written in terms of the terminal-forward measure dynamics.

$$\begin{aligned} X = Z(T,S) &= A(T,S)e^{-B(T,S)r_T} \\ &\sim \mathcal{LN}(\mu_z, \sigma_z^2) \end{aligned} \tag{87}$$

and

$$\begin{aligned} Y = \ln(X) &= \ln(A(T,S)) - B(T,S)r_T \\ &\sim \mathcal{N}(\mu_z, \sigma_z^2) \end{aligned} \tag{88}$$

giving

$$\begin{aligned} \mu_z &= \mathbb{E}^{\mathbb{Q}_T}[Y] \\ &= \mathbb{E}^{\mathbb{Q}_T}[\ln(A(T,S)) - B(T,S)r_T] \\ &= \mathbb{E}^{\mathbb{Q}_T}[\ln(A(T,S))] - \mathbb{E}^{\mathbb{Q}_T}[-B(T,S)r_T] \\ &= \ln(A(T,S)) - B(T,S)\mathbb{E}^{\mathbb{Q}_T}[r_T] \end{aligned} \tag{89}$$

and likewise

$$\begin{aligned} \sigma_z^2 &= Var^{\mathbb{Q}_T}(\ln(A(T,S)) - B(T,S)r_T) \\ &= Var^{\mathbb{Q}_T}(\ln(A(T,S))) + B(T,S)^2 Var^{\mathbb{Q}_T}(r_T) \\ &= Var^{\mathbb{Q}_T}(r_T)\, B(T,S)^2 \end{aligned} \tag{90}$$

substituting the terminal-forward mean and variance dynamics $\mathbb{E}^{\mathbb{Q}_T}[r_T]$ and $Var^{\mathbb{Q}_T}(r_T)$ as defined above in equations (53) and (54) leads to

$$\mu_z = \ln(A(T,S)) - B(T,S)\left(e^{-a(S-T)}r_T + F^{\mathbb{Q}_T}(T,S)\right) \tag{91}$$

where

$$\begin{aligned} F^{\mathbb{Q}_T}(T,S) &= \left(\frac{\theta}{a} - \frac{\sigma^2}{a^2}\right)\left(1 - e^{-a(S-T)}\right) + \frac{\sigma^2}{2a^2}\left(e^{-a(S-t)} - e^{-a(S+S-2T)}\right) \\ &= \left(\frac{\theta}{a} - \frac{\sigma^2}{a^2}\right)\left(1 - e^{-a(S-T)}\right) + \frac{\sigma^2}{2a^2}\left(e^{-a(S-t)} - e^{-2a(S-T)}\right) \end{aligned} \tag{92}$$

and

$$\sigma_p = \sigma\sqrt{\frac{(1 - e^{-2a(T-t)})}{2a}}B(T,S) \tag{93}$$

Next using the notation from section (4.1.1) above and applying the normal moment generating function with the mean and variance terms as defined in (89) and (90) gives

$$\begin{aligned} Z(T,S) = \mathbb{E}[X] &= \mathbb{E}\left[e^Y\right] \\ &= e^{\mathbb{E}[Y] + \frac{1}{2}Var(Y)} \\ &= e^{\mu_z + \frac{1}{2}\sigma_z^2} \end{aligned} \tag{94}$$

Therefore substituting (94) equation (86) becomes

$$V_t = Z(t,T)\left(\phi Z(T,S)\Phi\left(\phi\left(\frac{\mu_z - \ln(K) + \sigma_z^2}{\sigma_z}\right)\right) - \phi K\Phi\left(\phi\left(\frac{\mu_z - \ln(K)}{\sigma_z}\right) - \sigma_z^2\right)\right) \tag{95}$$

leading to

$$V_t = \phi Z(t,S)\Phi\left(\phi\left(\underbrace{\frac{\mu_z - \ln(K) + \sigma_z^2}{\sigma_z}}_{\text{Term A = h}}\right)\right) - \phi Z(t,T)K\Phi\left(\phi\left(\underbrace{\frac{\mu_z - \ln(K)}{\sigma_z}}_{\text{Term B = h } -\sigma_z^2}\right)\right) \tag{96}$$

This equation and specifically terms A and B can be further factorized in terms of Z(T,S) once again by using the normal moment generating function expression from (94), albeit with some careful thought and trickery perhaps.

Firstly in equation (96) observe that Term B= $h - \sigma_z$, where h is defined as Term A. Secondly consider and factorize term A as follows

$$\begin{aligned} h =& \text{ Term A} \\ =& \left(\frac{\mu_z - ln(K) + \sigma_z^2}{\sigma_z}\right) \\ =& \left(\frac{\mu_z + \frac{1}{2}\sigma_z^2 - ln(K) + \frac{1}{2}\sigma_z^2}{\sigma_z}\right) \end{aligned} \tag{97}$$

Here we take the log of the exponential, which cancel each other. This allows us to once again apply the moment generating function from (94).

$$\begin{aligned} h =& \left(\frac{\ln\left(e^{\mu_z + \frac{1}{2}\sigma_z^2}\right) - ln(K) + \frac{1}{2}\sigma_z^2}{\sigma_z}\right) \\ =& \left(\frac{\ln\left(Z(T,S)\right) - ln(K) + \frac{1}{2}\sigma_z^2}{\sigma_z}\right) \\ =& \left(\frac{\ln\left(\frac{Z(t,S)}{Z(t,T)}\right) - ln(K) + \frac{1}{2}\sigma_z^2}{\sigma_z}\right) \\ =& \left(\frac{\ln\left(\frac{Z(t,S)}{Z(t,T)K}\right) + \frac{1}{2}\sigma_z^2}{\sigma_z}\right) \\ =& \frac{1}{\sigma_z}\left(\ln\left(\frac{Z(t,S)}{Z(t,T)K}\right)\right) + \frac{1}{2}\sigma_z \end{aligned} \tag{98}$$

Substituting h from (98) above into the option pricing formula (96) leads to the below expression to evaluate the value V_t at time t of a European Call Option ($\phi = 1$) or Put ($\phi = -1$), which expires at time T on an underlying Pure Discount Bond maturing at time S, where $t \leq T \leq S$.

$$V_t = \phi Z(t,S)\Phi\left(\phi h\right) - \phi Z(t,T)K\Phi\left(\phi\left(h - \sigma_z\right)\right) \tag{99}$$

where

$$h = \frac{1}{\sigma_z}\left(\ln\left(\frac{Z(t,S)}{Z(t,T)K}\right)\right) + \frac{1}{2}\sigma_z \tag{100}$$

and

$$\sigma_z = \sigma\sqrt{\frac{(1 - e^{-2a(T-t)})}{2a}}B(T,S) \tag{101}$$

Illustration: Bond Option Prices

Bond option pricing under the Vasicek model are demonstrated below, where the underlying bond is a pure discount bond $Z(t,T)$. In this particular case we evaluate the price of both a European ATMF[20] Call and Put option where the option expires in 1 year and the underlying bond expires in 2 years.

Model Parameters

a	0.80	Reversion Speed
b	2.00%	Long Term Rate %
θ	0.10%	θ = a*b
σ	2.00%	Volatility %
r(0)	2.00%	Initial Short Rate%

Option Parameters

ϕ	1	Call (ϕ = 1) / Put (ϕ = -1)
T	1.00	Option Maturity
S	2.00	Bond Maturity (S > T)
K	98.02%	Option Strike

Option Calculations

t	0.00	Initial Time
r(t)	2.00%	Initial Rate
Z(t,T)	98.02%	Zero Price, Z(t,T)
A(t,T)	0.99	Parameter A
B(t,T)	0.69	Parameter B
Z(t,S)	96.10%	Zero Price, Z(t,S)
A(t,S)	0.98	Parameter A
B(t,S)	1.00	Parameter B
h	0.0164	Parameter h
σ(p)	0.97%	Option Volatility

Option Result

Call	0.003782

Model Parameters

a	0.10	Reversion Speed
b	1.00%	Long Term Rate %
θ	1.60%	θ = a*b
σ	10.00%	Volatility %
r(0)	1.00%	Initial Short Rate%

Option Parameters

ϕ	-1	Call (ϕ = 1) / Put (ϕ = -1)
T	1.00	Option Maturity
S	2.00	Bond Maturity (S > T)
K	98.02%	Option Strike

Option Calculations

t	0.00	Initial Time
r(t)	2.00%	Initial Rate
Z(t,T)	98.02%	Zero Price, Z(t,T)
A(t,T)	0.99	Parameter A
B(t,T)	0.69	Parameter B
Z(t,S)	96.10%	Zero Price, Z(t,S)
A(t,S)	0.98	Parameter A
B(t,S)	1.00	Parameter B
h	0.0164	Parameter h
σ(p)	0.97%	Option Volatility

Option Result

Put	0.003674

Figure 6: Bond Option Prices

In the appendix, see figures (7) and (8), we provide a further illustration of a range of bond option prices for various price levels of underlying bond.

4.2 Coupon Bond Option Pricing & Jamshidian's Trick

In this section we outline how to price European options on coupon bonds $P(t,T)$. Such bonds make a series of coupon payments in addition to a single redemption of notional payment at

[20] At-the-Money Forward options are denoted ATMF, meaning that the strike of the option is set such that it matches the forward value of the underlying.

maturity. The approach taken here from Jamshidian (1989) is commonly referred to as Jamshidian's Trick and is outlined below.

1. Recall that a coupon bond $P(t,T)$ can be represented as a portfolio of pure discount bonds
$$P(t,T) = \left(\sum_{i=1}^{n} c_i Z(t,T_i)\right) + NZ(t,T)$$

2. Find the fixed interest rate r^* that makes the coupon bond price $P(t,T)$ match the option strike K by solving the below equation
$$K = \left(\sum_{i=1}^{n} c_i Z(t,T_i,r^*)\right) + NZ(t,T,r^*)$$

3. Price each zero coupon bond using the Vasicek Bond formula from equation (63) but with r^* as the interest rate. The resulting price will give the effective local strike K_i.
$$K_i = Z(t,T_i,r^*)$$

4. The option payoff $\mathcal{H}_T = \phi(P(t,T) - K)^+$ can be decomposed as follows
$$\begin{aligned}
\mathcal{H}_T &= \phi\left(P(t,T) - K\right)^+ \\
&= \phi\left[\left(\sum_{i=1}^{n} c_i Z(t,T_i)\right) + NZ(t,T) - \left(\sum_{i=1}^{n} c_i Z(t,T_i,r^*)\right) - NZ(t,T,r^*)\right]^+ \\
&= \phi\left[\sum_{i=1}^{n} c_i\left(Z(t,T_i) - Z(t,T_i,r^*)\right) + N\left(Z(t,T) - Z(t,T,r^*)\right)\right]^+ \\
&= \underbrace{\phi\left[\sum_{i=1}^{n} c_i\left(Z(t,T_i) - K_i\right)\right]^+}_{\text{Option Payoff for Coupons}} + \underbrace{\phi\left[N\left(Z(t,T) - K_n\right)\right]^+}_{\text{Option Payoff for Notional}} \\
&= \left(\sum_{i=1}^{n} \mathcal{H}_i\right) + \mathcal{H}_n
\end{aligned}$$

5. As can be seen from the previous point an option on a coupon bond can be decomposed into a portfolio of zero coupon bonds on the individual coupons and the notional exchange. It is important to note the that each option has it's own unique local strike K_i as defined above.

6. Therefore we can decompose the option into a portfolio π of n options on zero coupon bonds, where $V(t_i, K_i)$ is as defined in equations (96) to (100) with the strike set to K_i.
$$\pi = \left(\sum_{i=1}^{n} c_i V(t_i,K_i)\right) + NV(t_n,K_n)$$

We price each option within the portfolio with the new effective local strike K_i. The linear aggregate sum of the options in the portfolio is the price of the coupon bond option.

As outlined in Jamshidian (1989) this only works for 1 factor models, since we can only form an average rate r^* when we assume perfect correction of interest rates. A monotonically increasing (or decreasing) function is required in order to obtain a unique value for r^*.

5 Model Extensions

Here we consider some of the practical implementation issues surrounding bond option pricing using short rate models and discuss possible model extensions.

5.1 Credit Risk

So far we have not discussed credit risk i.e. the risk of not receiving payment from the counterparty. The pricing formulae above intrinsically assume that our counterparty is risk-free and does not default. As such one should consider extending the model and to incorporate counterparty default risk, perhaps by adding an additional hazard rate factor λ, which could be made stochastic.

5.2 Cash vs Physical Settlement

There is an implicit assumption in this paper is that the bond options are cash settled. In the cash settled case one receives cash at maturity which carries no default risk, but in the physical settlement case, one receives the underlying bond which may default. Physical settlement generates additional credit risk for which investors will require a risk premium.

5.3 European and American Style Options

Attention is drawn to the fact that only European options have been discussed, whereby the option holder can only exercise the option on the option expiry date at the end of the option's life. American options allow the holder to exercise at any time. Extra consideration is required to determine at each exercise date whether it is best to exercise the option or hold the option in order to exercise later under better conditions. Optimal exercise could be modeled as part of a Tree or PDE implementation or by using a Longstaff-Schwartz Monte Carlo[21] type process.

[21] A simple Monte Carlo is not sufficient, since on an individual Monte Carlo path one cannot tell if it is best to hold or exercice the option.

5.4 Volatility Smile

The ’volatility smile’ is an important feature observed in option markets. This arises from the fact that the distribution of an underlying asset cannot be modeled exactly with a Gaussian distribution, since extreme or ’fat-tail’ events are typically more likely than predicted, and particularly so for short dated options. Option volatility parameters are adjusted to account for this and are typically adjusted upwards the further the option strike is from the at-the-money price (ATM) of the underlying asset. When plotting volatility versus option strikes for a given and fixed maturity the resulting volatility chart often looks like a smile.

The Vasicek model calibrates to a single volatility parameter, which cannot facilitate smile dynamics. To incorporate smile one possibility would be to replace the model volatility parameter with a polynomial expression with an appropriate functional form and recalibrate the model accordingly.

5.5 Convexity Adjustments

In the Fixed Income market options on Bond Futures are liquid instruments. The underlying bond future and the option are both traded on exchange with daily margining. Daily profits and losses resulting from movements in the underlying are credited and debited to the investor. This has the effect of removing convexity from the underlying instrument. Depending on whether the investor has a long or short position one can benefit from this, whereby daily profits can be reinvested at a higher interest rate and likewise losses funded at lower interest rates. The market factors this benefit into the instrument’s price. This benefit is called a ’Convexity Adjustment’. Convexity adjustments occur when the natural payment frequency of the interest rate or the currency of the interest payment is adjusted. The later is sometimes called a ’Quanto Adjustment’.

5.6 Two Factor Models

The Vasicek model discussed in this paper is a one factor model. Implicitly one factor models assume that interest rates are 100% correlated. Bonds converge to par at maturity and at different speeds, so clearly bonds of different maturities and their underlying interest rates do not demonstrate perfect correlation. One factor models can price instruments with a single bond underlying well and replicate parallel shifts in the yield curve well, however they cannot incorporate twists and slope changes in the yield curve. Hence they cannot adequately model the behavior of instruments such as Constant Maturity Swaps, which trade the steepness of the yield curve. Two Factor models address this problem by introducing an additional stochastic model parameter, which once calibrated to typically provide a richer description of interest rates.

6 Conclusion

In conclusion we have discussed numeraires, measures and how and why sometimes we apply a change of measure. We reviewed the Vasicek short rate model and it's dynamics and looked at pure discount and coupon bond pricing using the Vasicek short rate process. We derived bond option pricing formulae whereby changing from a risk neutral measure to a terminal-forward measure made the calculations easier and looked at how to evaulate options on coupon bearing options and considered Jamshidian's trick. Finally we discussed some of the practical implementation issues around bond option pricing.

Appendix

Illustration: Bond Option Price Scenarios
Bond option prices under the Vasicek model are tabulated below for various price levels of the underlying pure discount bond.

Option Results						**Spot**				**Step Size**	**0.5%**
Z(t,S)	93.60%	94.10%	94.60%	95.10%	95.60%	**96.10%**	96.60%	97.10%	97.60%	98.10%	98.60%
Call	0.000010	0.000052	0.000212	0.000684	0.001772	0.003782	0.006833	0.010783	0.015325	0.020164	0.025120
Put	0.024902	0.019944	0.015104	0.010576	0.006664	0.003674	0.001725	0.000675	0.000216	0.000056	0.000012

Option Calculations											
Z(t,T)	98.02%	98.02%	98.02%	98.02%	98.02%	98.02%	98.02%	98.02%	98.02%	98.02%	98.02%
A(t,T)	0.9938	0.9938	0.9938	0.9938	0.9938	0.9938	0.9938	0.9938	0.9938	0.9938	0.9938
B(t,T)	0.6883	0.6883	0.6883	0.6883	0.6883	0.6883	0.6883	0.6883	0.6883	0.6883	0.6883
t	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
r(t)	2.00%	2.00%	2.00%	2.00%	2.00%	2.00%	2.00%	2.00%	2.00%	2.00%	2.00%
h	-2.6946	-2.1467	-1.6016	-1.0594	-0.5201	0.0164	0.5502	1.0812	1.6094	2.1350	2.6579
σ(p)	0.97%	0.97%	0.97%	0.97%	0.97%	0.97%	0.97%	0.97%	0.97%	0.97%	0.97%

Figure 7: Bond Option Prices for Various Price Levels of the Underlying Pure Discount Bond

Illustration: Bond Option Price Chart
Bond option prices under the Vasicek model are charted below for various price levels of the underlying pure discount bond.

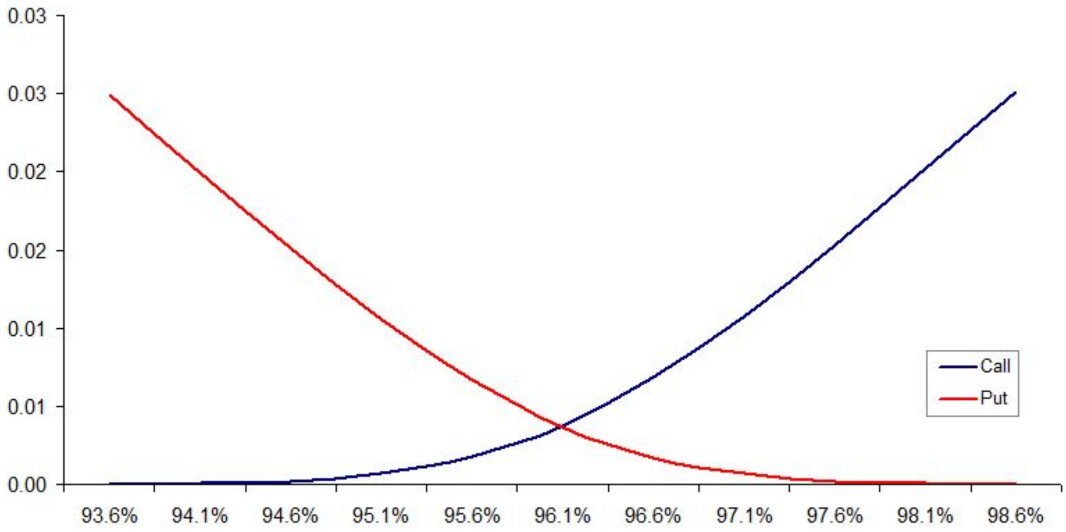

Figure 8: Bond Option Price Chart for Various Price Levels of Underlying Pure Discount Bond

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