Computationally Efficient Zero Coupon Swap Formulae

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In this short paper, we outline computationally efficient formulae for zero coupon swap pricing.

Zero coupon rates are defined as,

$$Zero\ Coupon\ Rate,\ Z_n = \prod_{i=1}^{n}(1 + r_i\,\tau_i\,) - 1 \tag{1}$$

We note that discount factors can be calculated using **simple compounding** as below,

$$DF(t_n) = \frac{1}{(1 + r_1\tau_1)(1 + r_2\tau_2)\ldots(1 + r_n\tau_n)} = \frac{1}{\prod_{i=1}^{n}(1 + r_i\,\tau_i\,)} \tag{2}$$

An alternative formula for the zero coupon rate that is **computationally efficient** can be derived by substituting (2) into (1),

$$Zero\ Coupon\ Rate,\ Z_n = \left(\frac{1}{DF_n}\right) - 1 \tag{3}$$

Therefore[1] the PV of a zero coupon swap leg can be calculated as follows,

$$\begin{aligned} PV(Zero\ Coupon\ Leg) &= N\,Z_n\,DF_n \\ &= N\left[\prod_{i=1}^{n}(1 + r_i\,\tau_i\,) - 1\right] DF_n \\ &= N\left[\left(\frac{1}{DF_n}\right) - 1\right] DF_n \\ &= N\,(\,1 - DF_n\,) \end{aligned} \tag{4}$$

where N denotes the trade notional, Z the zero coupon rate, r_i the fixing rate (or reset), τ_i the fixing year fraction and DF the discount factor.

[1] Typically for swaps PV(Coupon) = N r τ DF, but here the zero rate Z_n explicitly incorporates the fixing year fraction term, τ.

For zero coupon rates **with historical fixings** equation (1) becomes,

$$Zero\ Coupon\ Rate,\ Z_n = \underbrace{\prod_{\{i<t\}}(1 + r_i\,\tau_i\,)}_{\text{Accrued}}\underbrace{\prod_{\{i\geq t\}}(1 + r_i\,\tau_i\,)}_{\text{Projected}} - 1 \tag{5}$$

where t denotes today, {i<t} the set of known historical fixings and {i≥t} the projected fixings in the future.

Likewise the **computationally efficient** zero rate **with historical fixings** from equation (3) becomes,

$$Zero\ Coupon\ Rate,\ Z_n = \underbrace{\prod_{\{i<t\}}(1 + r_i\,\tau_i\,)}_{\text{Fixings}}\left(\frac{1}{DF_n}\right) - 1 \tag{6}$$

Finally the PV of a zero coupon swap leg **with historical fixings** can be calculated as,

$$\begin{aligned} PV(Zero\ Coupon\ Leg) &= N\,Z_n\,DF_n \\ &= N\left[\underbrace{\prod_{\{i<t\}}(1 + r_i\,\tau_i\,)}_{\text{Accrued}}\underbrace{\prod_{\{i\geq t\}}(1 + r_i\,\tau_i\,)}_{\text{Projected}} - 1\right] DF_n \\ &= N\left[\prod_{\{i<t\}}(1 + r_i\,\tau_i\,)\left(\frac{1}{DF_n}\right) - 1\right] DF_n \\ &= N\left(\underbrace{\prod_{\{i<t\}}(1 + r_i\,\tau_i\,)}_{\text{Fixings}} - DF_n\right) \end{aligned} \tag{7}$$