

MSDA 605 - Fundamentals of Computational Mathematics Final Exam

Ben Arancibia

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This final exam consists of three parts. The three parts are 1) Essential Concepts, 2) Coding, 3) Small Project.

Essential Concepts

1) What is the rank (number of linearly independent rows) of the following matrix:

```
##      [,1] [,2] [,3] [,4]
## [1,]   -1    1    3    5
## [2,]    2   -1    5    7
## [3,]    6  -10   -1    3
```

First step is multiply first row by -2, results in the following matrix:

```
##      [,1] [,2] [,3] [,4]
## [1,]    2   -2   -6  -10
## [2,]    2   -1    5    7
## [3,]    6  -10   -1    3
```

Second step is subtract first row from second row:

```
##      [,1] [,2] [,3] [,4]
## [1,]    1   -1   -3   -5
## [2,]    0    1   11   17
## [3,]    6  -10   -1    3
```

Third step multiply row 1 by 6 then subtract first row from third row.

```
##      [,1] [,2] [,3] [,4]
## [1,]    6   -6   -6  -30
## [2,]    0    1   11   17
## [3,]    0   -4   17   33
```

Fourth step multiply row 1 by -1 and then second row by -4.

```
##      [,1] [,2] [,3] [,4]
## [1,]   -1    1    3    5
## [2,]    0   -4  -44   68
## [3,]    0   -4   17   33
```

Last step subtract second row from third row.

```
##      [,1] [,2] [,3] [,4]
## [1,]  -1   1   3   5
## [2,]   0   1  11  17
## [3,]   0   0  61 101
```

The matrix rank is 3. All rows are linearly independent.

2) What is the determinant of the following matrix:

```
##      [,1] [,2] [,3] [,4]
## [1,]  -1   1   3   5
## [2,]   2  -1   5   7
## [3,]   6 -10  -1   3
```

It is not possible to calculate the determinant of the matrix because it is not a square matrix.

3) Define orthonormal basis vectors. Please write down at least one orthonormal basis for the 5-dimensional vector space \mathbb{R}^5 .

An Orthonormal Basis vector is when a orthogonal vector divided by its length = 1. An orthogonal vector is when

$$q_1, \dots, q_n$$

have dot products equal to zero (

$$q_i \cdot q_j$$

). Divide each vector by its length and the vectors become orthogonal unit vectors. The lengths are one.

A five dimensional orthonormal basis is the following:

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]   1   0   0   0   0
## [2,]   0   1   0   0   0
## [3,]   0   0   1   0   0
## [4,]   0   0   0   1   0
## [5,]   0   0   0   0   1
```

(this is just the standard basis)

4) Given the following matrix, what is its characteristic polynomial?

```
##      [,1] [,2] [,3]
## [1,]   2  -1   4
## [2,]  -1  -2   6
## [3,]   1   0  -3
```

Characteristic polynomial of a square matrix is a polynomial, which is invariant under matrix similarity and has the eigenvalues as roots.

$$\det(A - XI)$$

```
q4.1 = matrix(c(NA, -1, 1, -1, NA, 0, 4, 6, NA), nrow=3, ncol=3) # NA = X in this matrix
q4.1
```

```
##      [,1] [,2] [,3]
## [1,]   NA  -1    4
## [2,]  -1   NA    6
## [3,]   1    0   NA
```

Characteristic Polynomial =

$$-x^3 - 3x^2 + 9x + 17$$

5) What are its eigenvectors and eigenvalues of the following matrix?

```
##      [,1] [,2] [,3]
## [1,]    2  -1    4
## [2,]  -1  -2    6
## [3,]    1    0   -3
```

Characteristic Polynomial =

$$-x^3 - 3x^2 + 9x + 17$$

derived from question 4. Find the roots for the characterisitic polynomial.

Eigenvalue of x1 (2-x) = 2.691 Eigenvalue of x2 (-2-x) = -4.18 Eigenvalue of x3 (-3-x) = -1.511

With those values if you plug them in the following are the results eigenvectors: x1: (86.43 , 1 , 15.18) x2: (0.36 , 1 , -0.30) x3: (0.16 , 1 , 0.11)

6)When would a model be said to have a high bias and when would it be said to have a high variance?

Non-linear modules such as polynomial fits will have a low bias because they are able to go arbitrarily close to fi; however, they introduce a higher variance in the model as they may not fit any particular instance of the underlying data well. When there is a high variance but low bias, the model is referred to as overfitting, when there is a model of high bias but low variance, the model is known as underfitting. This is all part of the bias variance trade-off. Models with high bias tend to have low variance and models with high variance tend to have low bias.

7)Assuming that we are repeatedly sampling sets of numbers (each set is of size n) from an unknown probability density function. What can we say about the average value of each set?

Assuming that we are repeatedly sampling sets of numbers from an unknown probability density function dependson the type of random variable. Either for a continous or discrete random varibale the expect value of random variables states the average responses that should be obtained. The expected value of of PMF (discrete) can be calculated as the Sum of

$$P(x == i) * i$$

. If the variable is a continouse random variable you can calculte the entire PDF with Integral of

$$xp(x)dx$$

.

8)What is the derivative of $e^{2x\cos^2(x)}$

Take constant out so becomes:

$$e^2 d/dx(x\cos^2(x))$$

Product rule:

$$e^2((x)\cos^2(x) + (\cos^2(x))x)$$

Results in

$$e^2(1\cos^2(x) + (-2\cos(x)\sin(x))x)$$

Simplify:

$$e^2 \cos(x)(\cos(x) - 2x \sin(x))$$

9) What is the derivative of $\log(\sin(2x))$?

Apply log rule which is

$$\text{Log}(b) = -\ln(b)/\ln(a)$$

Result:

$$\ln(\sin(2x))/\ln(10)$$

10) What is the integral of

$$x^2 \sin(x) dx$$

?

Integration by parts: =

$$x^2(-\cos(x)) - \int 2x(-\cos(x))$$

$$= x^2(-\cos(x)) - \int 2x \cos(x)$$

$$\int -2x \cos(x) = -2(x \sin(x) + \cos(x))$$

$$= x^2(-\cos(x) - (-2(x \sin(x) + \cos(x))))$$

Simplified Integral:

$$2(x \sin(x) + \cos(x)) - x^2 \cos(x) + C$$

Compute with boundaries:

$$(0, \pi/2)$$

$$(0) : 2(x \sin(0) + \cos(0)) - x^2 \cos(0) = 2$$

$$(\pi/2) : 2(x \sin(\pi/2) + \cos(\pi/2)) - x^2 \cos(\pi/2) = \pi$$

Result:

$$\pi - 2$$

Mini Coding

2.1 Bayes Rule:

Working for a credit card company and 70% of customers have a good credit score. People with good credit have a loan default rate of 0.05. People with bad credit default on their loans at the rate of .10.

```
PRB <- (((.10*.30)+(.05*.30))/.7)
```

```
PRBA <- .10
```

```
PRA <- 0.5
```

```
PAB <- (PRBA*PRA)/PRB
```

```
PAB
```

```
## [1] 0.7777778
```

Suppose there are two full bowls of cookies. Bowl 1 has 5 chocolate chip and 35 oatmeal raisin cookies. Bowl 2 has 20 of each. Fred picks a bowl at random and then picks a cookie at random. Cookie is chocolate chip, how probable is it that Fred pick it out of bowl 2.

```
PRB2 <- (((0.125*20)+(0.5*20))/40)
PRBA2 <- .5
PRA2 <- .125
PAB2 <- (PRBA2*PRA2)/PRB2

PAB2 *.05 #multiplied by .5 because Fred has to select 1 of 2 bowls
```

```
## [1] 0.01
```

2.2 Central Limit Theorem

Assume a sample of size of 100 with mean of 50 and standard deviation 10. What is mean of sample and standard error?

```
x=rlnorm(100, log(50), log(10))

mean(x)
```

```
## [1] 312.6279
```

```
se <- sd(x)/sqrt(length(x))

se
```

```
## [1] 63.96241
```

Random sample of 100 observations with a mean of 40 and standard deviation of 25.

```
x=rnorm(100,mean=40,sd=25)

mean(x)
```

```
## [1] 42.29977
```

```
sd(x)
```

```
## [1] 26.65004
```

Random sample of 100 observations mean of 40 and standard deviation of 25. What is the probability that the mean of the sample will exceed 45?

```
x=rnorm(100,mean=40,sd=25)

y=mean(x)

1-pnorm(y, 40, 25)
```

```
## [1] 0.5440023
```

2.3 Sample from function.

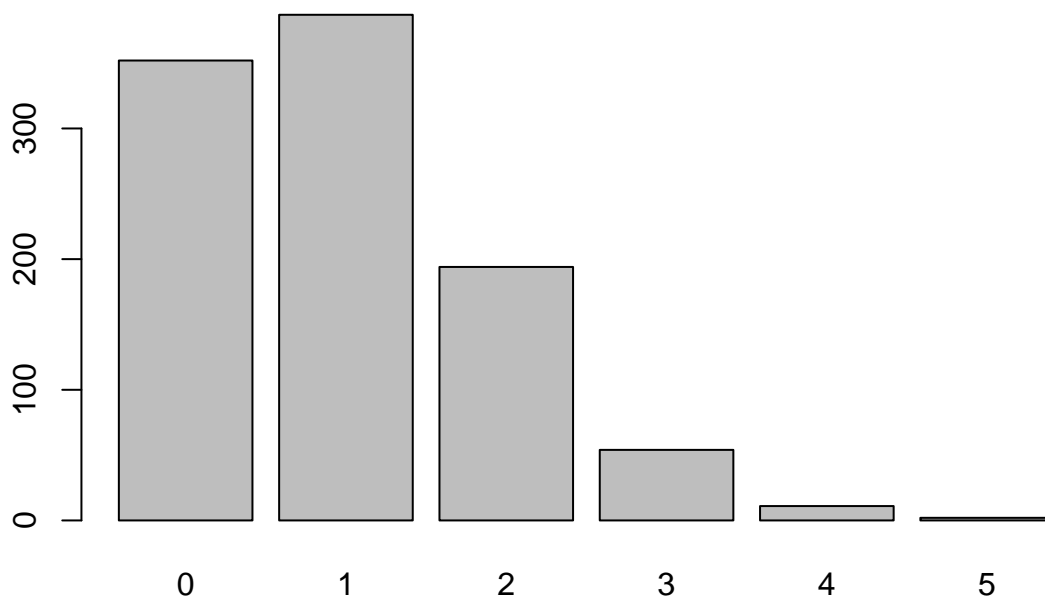
First create a binominal distribution to samepl $p = 0.1$, $q = 1-p$ and $x[0,10]$

```
rbinom(1, 10, .1)
```

```
## [1] 3
```

Generate 1000 samples using this and plot a histogram.

```
barplot(table(rbinom(1000,10,.1)))
```

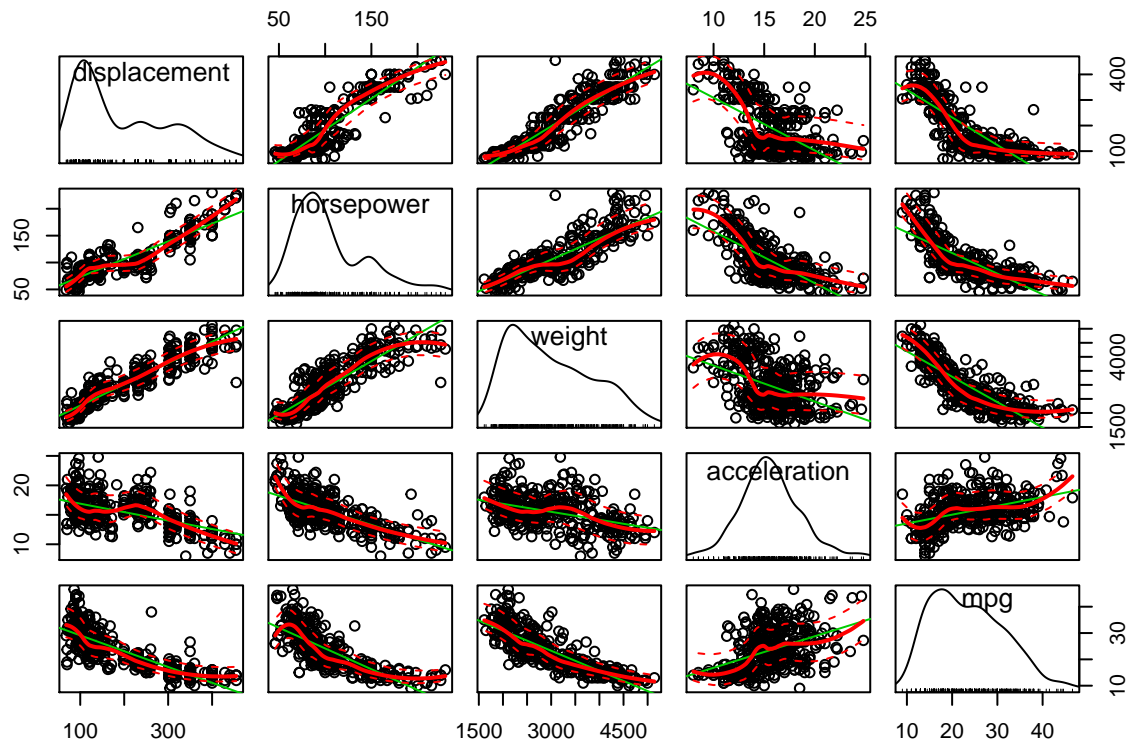


2.4 Principal Components Analysis

Perform a principal components analysis by performing an SVD on 4 independent variables (mpg is dependent variable) and select top 2 directions. Create a scatter plot of the data and print out the two orthogonal vectors.

```
library(car)
data <- read.table("/Users/bcarancia/CUNY_IS_605/FinalExam/auto-mpg.data")
names(data) <- c("displacement", "horsepower", "weight", "acceleration", "mpg")

scatterplotMatrix(data) #take a look at the data first
```



```
standard <- as.data.frame(scale(data))
car.pca <- prcomp(standard)
summary(car.pca)
```

```
## Importance of components:
```

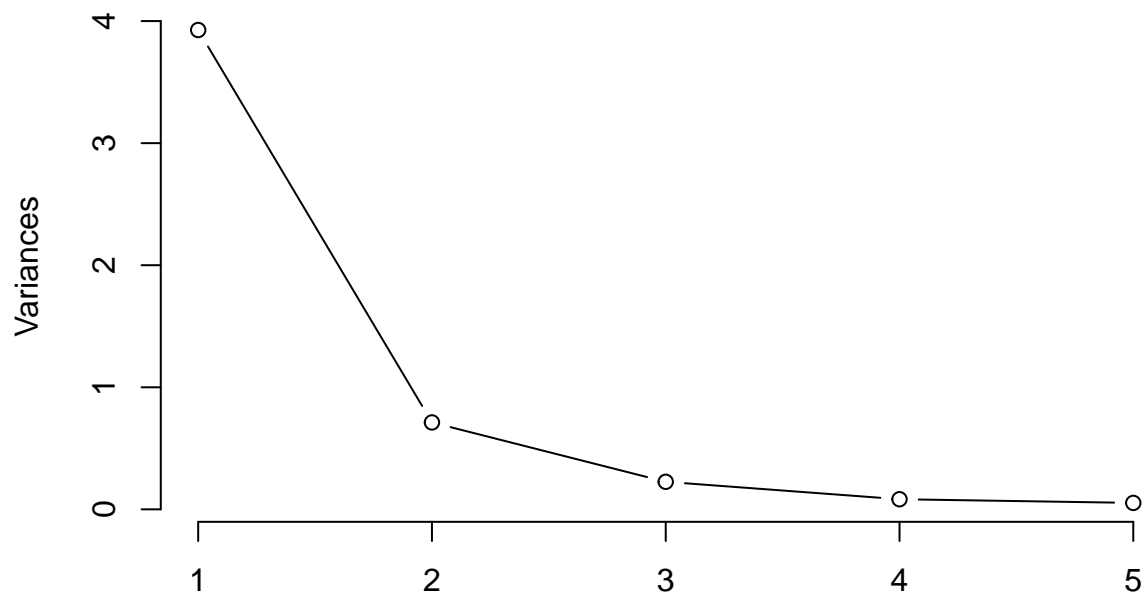
```
##              PC1    PC2    PC3    PC4    PC5
## Standard deviation  1.9816 0.8438 0.47500 0.28788 0.22966
## Proportion of Variance 0.7853 0.1424 0.04512 0.01658 0.01055
## Cumulative Proportion 0.7853 0.9277 0.97288 0.98945 1.00000
```

```
car.pca$sdev
```

```
## [1] 1.9816040 0.8438043 0.4749966 0.2878810 0.2296579
```

```
screeplot(car.pca, type = "lines")
```

car.pca



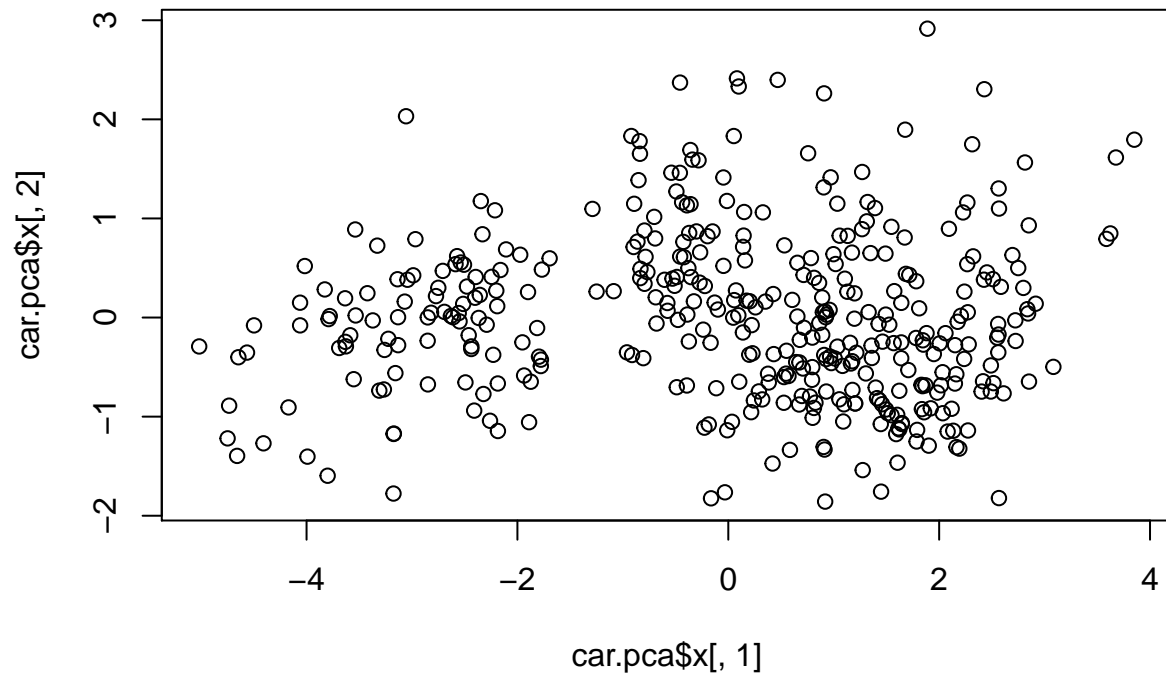
```
car.pca$rotation[,1]
```

```
## displacement  horsepower      weight acceleration      mpg
##   -0.4832332   -0.4844417   -0.4712207    0.3352350    0.4442640
```

```
car.pca$rotation[,2]
```

```
## displacement  horsepower      weight acceleration      mpg
##    0.1347900   -0.1242676    0.3263218    0.8761089   -0.3038692
```

```
plot(car.pca$x[,1],car.pca$x[,2])
```

2.5

```
library(boot)
```

```
##
## Attaching package: 'boot'
##
## The following object is masked from 'package:car':
##
##      logit
```

```
data = c(8,10,7,12,13,8,10,50)
mean_data = function(x,indices){
  return(mean(x[indices]))
}
boot.out = boot(data, mean_data,1000000)
boot.ci(boot.out)
```

```
## Warning in boot.ci(boot.out): bootstrap variances needed for studentized
## intervals
```

```
## BOOTSTRAP CONFIDENCE INTERVAL CALCULATIONS
## Based on 1000000 bootstrap replicates
##
## CALL :
## boot.ci(boot.out = boot.out)
##
## Intervals :
## Level      Normal          Basic
## 95%    ( 5.41, 24.09 )    ( 4.25, 20.88 )
##
```

```
## Level      Percentile      BCa
## 95%    ( 8.62, 25.25 )    ( 9.12, 30.50 )
## Calculations and Intervals on Original Scale
```