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Sufficient conditions for Benford's law

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ABSTRACT

We present two sufficient conditions for an absolutely continuous random variable to obey Benford's law for the distribution of the first significant digit. These two sufficient conditions suggest that Benford's law will not often be observed in everyday sets of numerical data. On the other hand, we recall that there are two processes by way of which a random variable can come close to following Benford's law. The first of these is the multiplication of independent random variables and the second is the exponentiation of a random variable to a large power. Our working tool is the Poisson sum formula of Fourier analysis. Like the central limit theorem, Benford's law has an asymptotic nature.
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1. Introduction

Let $\beta \geq 2$ be an integer. Let ξ be a random variable taking values in $\mathbf{R}$ and such that $1 = \mathbf{P}\{0 < \xi < \infty\}$. Let $r = r_\xi \in \mathbf{R}$ and $k = k_\xi \in \mathbf{Z}$ be such that

$$\xi = r \times \beta^k \quad \text{and} \quad 0 < r < \beta. \tag{1}$$

We write $X = \log_\beta \xi$ if and only if $\beta^X = \xi$. Let d be an integer such that $1 \leq d < \beta$. Benford's law for the distribution of the first significant digit asserts that

$$\mathbf{P}\{d \leq r < d+1\} = \log_\beta \left\{ \frac{d+1}{d} \right\}.$$

See Hill (1995) for an introduction to Benford's law. The aim of this note is to discuss Benford's law by using the Poisson sum formula as a light shedding tool.

In Section 3 we show how to construct density functions of random variables satisfying Benford's law. The examples constructed from Theorems 2 and 3 satisfy some symmetry relations and thus these examples seem rather artificial. One might not expect that lists of numbers occurring in everyday experience come from random variables with such symmetry relations. More precisely, Theorems 2 and 3 give sufficient conditions for a random variable in order for it to follow Benford's law. If it were possible to prove that the density function of a random variable satisfying Benford's law necessarily satisfies some symmetry relations, then this would cast serious doubts on the universality of Benford's law. The hypothetical symmetry relations necessary for random variables to obey Benford's law exactly would make these random variables rather special. Thus, Theorems 2 and 3 suggest (in a weak fashion) the non-universality of Benford's law. This non-universality would account for the difficulty research has had in finding a good explanation for the law.

On the positive side we recall that there are two processes by way of which a random variable can come close to following Benford's law. The first of this is the multiplication of independent random variables and the second is the exponentiation of a random variable to a large power. These two processes are asymptotic in nature and are discussed in Section 4.

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In Section 5 we show how to use the central limit theorem in order to produce a counterexample to Benford's law. In Example 5 we present a second counterexample to Benford's law. As the statistician becomes aware of the distinct cases when the law does not occur, greater care will be exerted when deciding whether or not a deviation from Benford's law is due to artificial causes.

2. Application of the Poisson formula

Let us write

$$\mathrm{M}_g(s) = \int_0^\infty g(x)x^s \mathrm{d}x$$

for the Mellin transform at $s+1$ of a function $g(x)$. We also write $\mathrm{M}(g, s)$ instead of $\mathrm{M}_g(s)$. For the natural logarithm of a number x we write $\log x$ instead of $\log_e x$.

Let $g(x)$ be the density function of ξ and let $f_0(x)$ be the density function of $\log_\beta \xi$ so that

$$f_0(x) = \log(\beta)\beta^x g(\beta^x).$$

Given a Lebesgue measurable set $A \subset [0, 1]$, we want to compute $\mathbf{P}\left\{\log_\beta r \in A\right\}$. Note however that

$$\mathbf{P}\left\{\log_\beta r \in A\right\} = \sum_{j\in\mathbf{Z}} \mathbf{P}\left\{\log_\beta \xi \in j + A\right\} = \int_A f(x)\mathrm{d}x$$

where

$$f(x) = \sum_{j\in\mathbf{Z}} f_0(x+j).$$

Since f is a periodic function, $f(x) = f(x+1)$ for all $x \in \mathbf{R}$; then we can use the Poisson sum formula and write

$$f(x) = \sum_{j\in\mathbf{Z}} \mathrm{e}^{2\pi \mathrm{i}jx} \int_{-\infty}^{+\infty} f_0(x)\mathrm{e}^{-2\pi \mathrm{i}jx}\mathrm{d}x.$$

Now we can state the following result.

Theorem 1. *Let $\eta > 0$ be a fixed real number. Let ξ be a random variable with values in $(0, \infty)$ and with $g(x)$ as its density function. Let $f(x)$ be the density function of $\eta \log_\beta \xi \pmod 1$. Then*

$$f(x) = \sum_{j\in\mathbf{Z}} \mathrm{e}^{2\pi \mathrm{i}jx} \mathrm{M}_g \left\{ -\frac{2\pi \mathrm{i}j\eta}{\log \beta} \right\}.$$

Proof. If we assume that $\eta = 1$ then it is enough to write $\log(\beta)\beta^x g(\beta^x)$ in place of $f_0(x)$ in the Poisson sum formula and then perform a change of variable. Theorem 1 is also true for $\eta \neq 1$ since

$$\int_0^\infty \frac{1}{\eta} x^{\frac{1}{\eta}-1} g(x^{\frac{1}{\eta}}) x^s \mathrm{d}x = \int_0^\infty g(x)x^{\eta s}\mathrm{d}x$$

and the left hand side is the Mellin transform of the density function of ξ^η. □

Corollary. *If $\sigma > 0$, and ξ has $g(x/\sigma)/\sigma$ as its density function, then $\eta \log_\beta \xi \pmod 1$ has*

$$f(x) = \sum_{j\in\mathbf{Z}} \exp\left\{ 2\pi \mathrm{i}j \left(x - \eta \frac{\log \sigma}{\log \beta} \right) \right\} \mathrm{M}_g \left\{ -\frac{2\pi \mathrm{i}j\eta}{\log \beta} \right\}$$

as its density function.

Before we draw consequences from Theorem 1, we recall one fact about the product of random variables. If for $\ell = 1, 2$ the random variable ξ_ℓ has $g_\ell(x)$ as its density function, then $\xi_1\xi_2$ has

$$g(x) = \int_0^\infty g_1\left(\frac{x}{y}\right) g_2(y) \frac{\mathrm{d}y}{y}$$

as its density function. This in turn implies that $\mathrm{M}(g, s) = \mathrm{M}(g_1, s)\mathrm{M}(g_2, s)$.

If the density function $g(x)$ is such that $0 = \mathrm{M}_g(-2\pi \mathrm{i}j/\log\beta)$ for all $j \in \mathbf{Z} \setminus \{0\}$, then $f(x) = 1$ for all $x \in [0, 1]$. Hence, writing $A = [\log_\beta d, \log_\beta d + 1)$,

$$\mathbf{P}\{d \le r < d+1\} = \mathbf{P}\left\{\log_\beta d \le \log_\beta r < \log_\beta d + 1\right\} = \int_A \mathrm{d}x = \log_\beta \left\{ \frac{d+1}{d} \right\}.$$

Thus $0 = \mathrm{M}_g(-2\pi \mathrm{i}j/\log\beta)$ is sufficient for the random variable ξ to obey Benford's law. Moreover, if ξ_1 obeys Benford's law exactly, then the product $\xi_1\xi_2$ also follows Benford's law exactly, regardless of the density function of ξ_2.

3. Exact Benford

Leemis et al. (2000) presented three examples of random variables satisfying Benford's law exactly. The examples considered in Leemis et al. (2000) are of the form 10^X, where X is a random variable taking values in $\mathbf{R}$. In this section, we deduce from Theorem 1 simple sufficient conditions for a random variable of the form β^X to satisfy Benford's law exactly. From Theorem 1 we know that in order for β^X to satisfy Benford's law exactly it is enough if

$$0 = \mathrm{M}_g \left\{ -\frac{2\pi \mathrm{i} j\eta}{\log \beta} \right\} = \int_{-\infty}^{+\infty} f_0(x) \mathrm{e}^{-2\pi \mathrm{i} j\eta x} \mathrm{d}x \tag{2}$$

for all $j \in \mathbf{Z} \setminus \{0\}$. Note that the right hand side does not depend on β.

We now present our first example of a random variable satisfying Benford's law exactly. Example 1 is not new; we recall it here as a simple application of condition (2) and also because it will serve as a building block for Example 3.

Example 1. Let X have the uniform density function $f_0(x) = 1$ if $|x| \le 1/2$ and $f(x) = 0$ if $|x| > 1/2$. Then

$$\mathrm{M}_g \left\{ -\frac{2\pi \mathrm{i} j}{\log \beta} \right\} = \int_{-1/2}^{+1/2} \mathrm{e}^{-2\pi \mathrm{i} jx} \mathrm{d}x = 0$$

for all $j \in \mathbf{Z} \setminus \{0\}$. Hence β^X satisfies Benford's law exactly. More generally, if X is uniformly distributed in an interval of the form $[\mu, \mu + h]$, where $\mu \in \mathbf{R}$ and $h \in \mathbf{N}$, then β^X satisfies Benford's law exactly. Recall that if ξ_1 obeys Benford's law exactly, then the product $\xi_1\xi_2$ also follows Benford's law exactly. In particular, if X has $f_0(x) = 1 - |x|$, for $|x| \le 1$, as its density function, then β^X satisfies Benford's law exactly.

Theorem 2. *Let X be an absolutely continuous random variable with a density $f^0(x)$ satisfying the following conditions*

$$\begin{cases} f^0(x) = f^0(-x) & \text{for all } x, \\ f^0(x) = 0 & \text{if } |x| > 1, \\ f^0\left(\frac{1}{2} + x\right) = c - f^0\left(\frac{1}{2} - x\right) & \text{if } 0 \le x \le \frac{1}{2} \end{cases}$$

for a suitable constant c. Let $\beta \ge 2$ be an integer. Let $\xi = \beta^X$. Then ξ satisfies Benford's law, i.e.,

$$\mathbf{P}\{d \le r < d+1\} = \log_\beta \left\{ \frac{d+1}{d} \right\}$$

where r is as in (1) and $1 \le d < \beta$.

Proof. Let $j \in \mathbf{Z} \setminus \{0\}$. It is enough to note that

$$\int_{-\infty}^{+\infty} f^0(x) \mathrm{e}^{-2\pi \mathrm{i} jx} \mathrm{d}x = 2(-1)^j \int_0^{1/2} \left[f^0\left(\frac{1}{2} + x\right) + f^0\left(\frac{1}{2} - x\right) \right] \cos(2\pi jx) \mathrm{d}x = 0.$$

Hence $f(x) = 1$ is the density function of $\log_\beta r$ in the unit interval $[0, 1]$. □

Example 2. If $f^0(x)$ is as in Theorem 2, then it is completely determined once we know its values in the interval $[0, 1/2]$. For example, taking $f^0(x) = \left(1 - 4\left(2 - 2^{\frac{2}{3}}\right)x^3\right)^3$ for $x \in [0, 1/2]$, and taking $c = 2f^0(1/2)$, then we obtain a density function f^0 as shown in Fig. 1. If X has this f^0 as its density function, then β^X satisfies Benford's law exactly.

Theorem 3. *Let $\beta \ge 2$ be an integer. Let Y be a random variable taking values in $\mathbf{R}$ and $f^0(x)$ be its density function. Assume that β^Y satisfies Benford's law exactly. Let X have a density $f(x)$, with $f_0 = f^0 + f^1$, where f^1 satisfies the followingconditions*

$$\begin{cases} f^1(x) = -f^1(-x) & \text{for all } x, \\ f^1(x) = 0 & \text{if } |x| > 1, \\ f^1\left(\frac{1}{2} + x\right) = f^1\left(\frac{1}{2} - x\right) & \text{if } 0 \le x \le \frac{1}{2}. \end{cases}$$

Let $\xi = \beta^X$. Then ξ satisfies Benford's law.

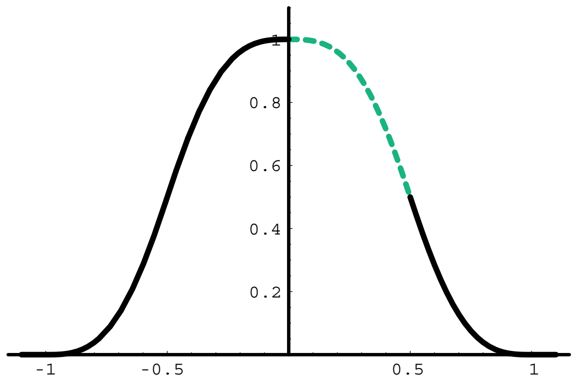

Fig. 1. The density function of Example 2, produced according to Theorem 2.

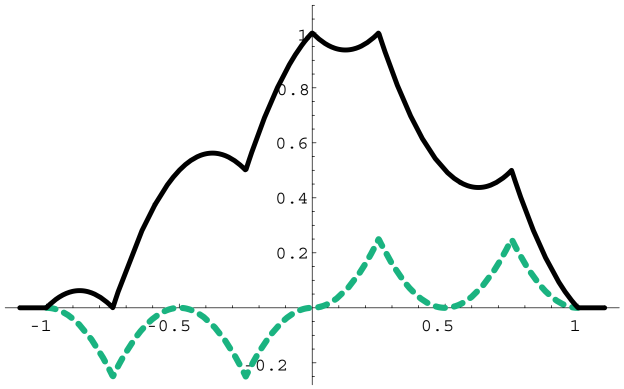

Fig. 2. The density function of Example 3, produced according to Theorem 3.

Proof. When computing the Fourier coefficients of $f_0 = f^0 + f^1$ one can disregard f^0 because we know that β^Y follows Benford's law. On the other hand, for all $j \in \mathbf{Z}$ we have

$$\int_{-\infty}^{+\infty} f^1(x) \mathrm{e}^{-2\pi \mathrm{i} jx} \mathrm{d}x = 2\mathrm{i}(-1)^j \int_0^{1/2} \left[f^1\left(\frac{1}{2} + x\right) - f^1\left(\frac{1}{2} - x\right) \right] \sin(2\pi jx) \mathrm{d}x = 0.$$

Hence $f(x) = 1$ is the density function of $\log_\beta r$ in the unit interval $[0, 1]$. $\square$

Example 3. In Fig. 2 we can see a density $f_0 = f^0 + f^1$, where $f^0(x) = 1 - |x|$ has a triangular shape and f^1 is plotted as the dashed line. As we remarked in Example 1, $f^0(x) = 1 - |x|$ is the density of a random variable Y such that β^Y does follow Benford's law. The function f^1 satisfies the three conditions of Theorem 3. If X has f_0 as its density function, then β^X satisfies Benford's law exactly.

It is clear that one can use Theorems 2 and 3 to produce an infinite number of random variables satisfying Benford's law. The examples so produced would look rather artificial because Theorems 2 and 3 set restrictive conditions on the density functions.

4. Approximate Benford

Assume that ξ does not follow Benford's law exactly. There are two processes by way of which ξ can come close to obeying Benford's law. The first such process is the multiplication of random variables. Indeed, let $\xi = \xi_1 \xi_2$ be the product of two random variables. For $\ell = 1, 2$, let g_ℓ be the density function of ξ_ℓ. Then

$$\left| \mathrm{M}\left\{ g_\ell, -\frac{2\pi \mathrm{i} j}{\log \beta} \right\} \right| < 1$$

for all $j \in \mathbf{Z} \setminus \{0\}$; see Lemma 3 in Feller (1971, page 500). Hence, if $j \in \mathbf{Z} \setminus \{0\}$, then the jth coefficient in the Fourier series in Theorem 1 for the density $f(x)$ of $\log_\beta \xi \pmod 1$ is strictly smaller than either those of $\log_\beta \xi_1 \pmod 1$ or $\log_\beta \xi_2 \pmod 1$. The smaller the Fourier coefficients of $f(x)$ the closer ξ will follow Benford's law. Actually, Benford's law holds when the jth Fourier coefficient of $f(x)$ equals zero for $j \in \mathbf{Z} \setminus \{0\}$.

Example 4. Let $k \geq 2$ be an integer. Let $X_1, X_2, \ldots, X_k$ be independent random variables. Assume that each X_j has a uniform distribution in the interval $(0, 1)$. Let $\xi = X_1 \cdot X_2 \cdots X_k$. The density function of ξ is

$$g(x) = \frac{(-\log x)^{k-1}}{(k-1)!}$$

when $x \in (0, 1)$ and $g(x) = 0$ when $x \notin (0, 1)$; see Adhikari and Sarkar (1968). In this case, we have

$$\int_0^{\infty} g(x)x^s \mathrm{d}x = \frac{1}{(1+s)^k}.$$

Thus, for the density of $\log_\beta r$ we have

$$f(x) = \sum_{j\in\mathbf{Z}} \mathrm{e}^{2\pi \mathrm{i} jx} \left(\frac{\log \beta}{\log \beta - 2\pi \mathrm{i} j} \right)^k.$$

If $\beta = 10$, then we have

$$\left| \frac{\log 10}{\log 10 \pm 2\pi \mathrm{i} j} \right| \leq \frac{2}{5j} \quad \text{for } j \in \mathbf{N}.$$

Therefore

$$|f(x) - 1| \leq \frac{18}{5} \left(\frac{2}{5} \right)^k \quad \text{for all } x \in (0, 1).$$

Let $A = [\log_\beta d, \log_\beta d + 1)$. Then we have

$$\mathbf{P}\{d \leq r < d+1\} = \int_A f(x)\mathrm{d}x = \log_\beta \left\{ \frac{d+1}{d} \right\} + O\left(\frac{2}{5} \right)^k.$$

In Section 5 we expand our discussion on the topic of the product of random variables in the context of Benford's law.

The other process by means of which a random variable ξ can approximately follow Benford's law is exponentiation. Indeed, if X has $f_0(x)$ as its density function, then $\xi = \beta^X$ has

$$g(y) = \frac{1}{y \log \beta} f_0(\log_\beta y)$$

as its density function. Let $\eta > 0$. The jth Fourier coefficient of the density $f(x)$ for $\log_\beta \xi^\eta \pmod 1$ is

$$\mathrm{M}_g \left\{ -\frac{2\pi \mathrm{i} j\eta}{\log \beta} \right\} = \int_{-\infty}^{+\infty} f_0(x) \mathrm{e}^{-2\pi \mathrm{i} j\eta x} \mathrm{d}x.$$

From the Riemann–Lebesgue lemma we know that

$$\lim_{\eta \to \infty} \mathrm{M}_g \left\{ -\frac{2\pi \mathrm{i} j\eta}{\log \beta} \right\} = 0$$

for all $j \in \mathbf{Z} \setminus \{0\}$. Therefore ξ^η will follow approximately Benford's law whenever η is large.

The corollary to Theorem 1 is interesting because it reveals an invariance property in regard to the parameter σ. This invariance property is relevant whenever the random variable ξ does not follow Benford's law exactly. Let us write the density function $f(x)$ of the corollary to Theorem 1 as

$$f(x, \sigma, \beta) = \sum_{j\in\mathbf{Z}} \exp\left\{ 2\pi \mathrm{i} j \left(x - \eta \frac{\log \sigma}{\log \beta} \right) \right\} \mathrm{M}_g \left\{ -\frac{2\pi \mathrm{i} j\eta}{\log \beta} \right\}.$$

Let $\delta \in \mathbf{Z}$. Then we have $f(x, \sigma, \beta) = f(x, \sigma\beta^\delta, \beta)$ for all $x \in [0, 1]$. This invariance property shows that it is not justified to expect that a random variable ξ with density function $g(x/\sigma)/\sigma$ will satisfy Benford's law accurately, as σ increases in magnitude. This is opposite to what one might expect on heuristic grounds.

Example 5. Let us consider the random variable $|\xi|$, where ξ is normally distributed with mean 0 and variance σ^2. From the corollary to Theorem 1 we know that $\log_\beta |\xi| \pmod 1$ has

$$f(x, \sigma, \beta) = \frac{1}{\sqrt{\pi}} \sum_{j\in\mathbf{Z}} \exp\left\{ 2\pi \mathrm{i} j \left(x - \frac{\log \sigma}{\log \beta} - \frac{\log 2}{2 \log \beta} \right) \right\} \Gamma\left(\frac{1}{2} - \frac{\pi \mathrm{i} j}{\log \beta} \right)$$

as its density function. Let $\delta \in \mathbf{Z}$. Note that $f(x, \sigma, \beta) = f(x, \sigma\beta^\delta, \beta)$ for all $x \in [0, 1]$. If we assume that $\beta = 10$ then we can see from Table 1 how far can $|\xi|$ be from obeying Benford's law.

Table 1
If ξ is normally distributed as in Example 5, then $|\xi|$ does not follow Benford's law.

d	1	2	3	4	5	6	7	8	9
Ben	0.301	0.176	0.124	0.096	0.079	0.066	0.057	0.051	0.045
Nor	0.385	0.173	0.085	0.065	0.061	0.059	0.057	0.055	0.053

Table 2
The random variable $\xi = 10^X$, with ψ_9 as the density of X, does not follow Benford's law.

d	1	2	3	4	5	6	7	8	9
Ben	0.301	0.176	0.124	0.096	0.079	0.066	0.057	0.051	0.045
ψ_9	0.496	0.003	0.000	0.003	0.019	0.058	0.109	0.148	0.159
Obs	0.493	0.002	0.000	0.002	0.018	0.059	0.109	0.155	0.158

Indeed, Table 1 records in the third row the values of $\mathbf{P}\{d \le r < d+1\}$ when $|\xi| = r \times \beta^k$ and ξ has a normal distribution with mean 0 and variance $\sigma_0^2 = (0.1328)^2$. The discrepancy reported in Table 1 between $\mathbf{P}\{d \le r < d+1\}$ and Benford will be observed whenever ξ has a variance σ^2 such that $\sigma = \sigma_0 10^\delta$ with $\delta \in \mathbf{Z}$.

The corollary to Theorem 1 also suggests that invariance of the density $f(x)$ under transformations in σ occurs only when $0 = \mathrm{M}_g(-2\pi \mathrm{i}j/\log\beta)$ for all $j \in \mathbf{Z}\setminus\{0\}$. Thus, independence of $f(x)$ from the parameter σ seems to imply that the random variable ξ does follow Benford's law exactly.

5. A central limit theorem

In this section we use the central limit theorem in order to construct a counterexample to Benford's law. With this aim, let us consider the case of a random variable β^X, where $X = (Y_1+\cdots+Y_n)/\sigma\sqrt{n}$ is the sum of n independent and identically distributed random variables Y_ℓ and $\sigma \in (0,\infty)$. The Fourier coefficients of the density function $f(x)$ of $X \pmod 1$, as given in the Poisson sum formula, are

$$\left\{\widehat{f_Y}\left(\frac{j}{\sigma\sqrt{n}}\right)\right\}^n \quad \text{where } \widehat{f_Y}(\xi) = \int_{-\infty}^{+\infty} f_Y(x)\mathrm{e}^{-2\pi\mathrm{i}\xi x}\mathrm{d}x$$

where $f_Y(x)$ is the density function of Y_ℓ. Let us assume that $\widehat{f_Y}(j) \ne 0$ for all $j \in \mathbf{N}$. Let us assume also that

$$0 = \int_{-\infty}^{+\infty} x f_Y(x)\mathrm{d}x, \qquad 1 = \int_{-\infty}^{+\infty} x^2 f_Y(x)\mathrm{d}x \quad \text{and} \quad \int_{-\infty}^{+\infty} |x|^3 f_Y(x)\mathrm{d}x < \infty.$$

Recall that for all $x \in \mathbf{R}$ we have $\mathrm{e}^{\mathrm{i}x} = 1 + \mathrm{i}x - x^2/2 + O(|x|^3)$. Thus

$$\widehat{f_Y}(\xi) = \int_{-\infty}^{+\infty} f_Y(x)\left\{1 - 2\pi\mathrm{i}\xi x - 2(\pi\xi x)^2 + O(|\xi x|^3)\right\}\mathrm{d}x = 1 - 2(\pi\xi)^2 + O(|\xi|^3).$$

If $|x| \le 1/2$ then $\log(1+x) = x + O(x^2)$. Hence $\log\widehat{f_Y}(\xi) = -2(\pi\xi)^2 + O(|\xi|^3)$ as $\xi \to 0$. Thus

$$\lim_{n\to\infty}\left\{\widehat{f_Y}\left(\frac{j}{\sigma\sqrt{n}}\right)\right\}^n = \lim_{n\to\infty}\exp\left\{-2\left(\frac{\pi j}{\sigma}\right)^2 + O\left(\frac{|j|^3}{\sqrt{n}}\right)\right\} = \exp\left\{-2\left(\frac{\pi j}{\sigma}\right)^2\right\}.$$

We recognize these Fourier coefficients. Indeed, if we take $f_0(x) = \exp\{-x^2/2\}/\sqrt{2\pi}$, then $f(x)$, which is the periodic version of $f_0(x)$, has $\exp\left\{-2(\pi j)^2\right\}$ as its Fourier coefficients.

For $\sigma > 0$, let

$$\psi_\sigma(x) = \frac{\sigma}{\sqrt{2\pi}}\sum_{j\in\mathbf{Z}}\exp\left\{-\frac{\sigma^2}{2}(x+j)^2\right\}.$$

Let $\xi = \beta^X$. Then the above calculations show that $\log_\beta \xi \pmod 1$ has ψ_σ as its limit distribution.

In Fig. 3 we can see the graph of ψ_σ when $\sigma = 4$, $\sigma = 6$ and $\sigma = 9$. We see that ψ_σ deviates considerably from the uniform distribution when σ is large.

Let us consider, for example, the random variable $\xi = 10^X$, where

$$X = \frac{Y_1+\cdots+Y_{10}}{9\sqrt{10}}$$

is the sum of ten random variables. Here each Y_ℓ is uniformly distributed in $[-1/2, 1/2]$. In the third row of Table 2 we see the values of $\int_A \psi_9(x)\mathrm{d}x$, where $A = [\log_{10} d, \log_{10} d + 1)$ and $1 \le d < 10$. In the fourth row of Table 2 we can see the observed frequency of the occurrence of each digit d as the first significant figure of ξ when ξ was observed a total number of 1000 times. The observed data is close to the predicted values in the third row, but deviates considerably from Benford's law.

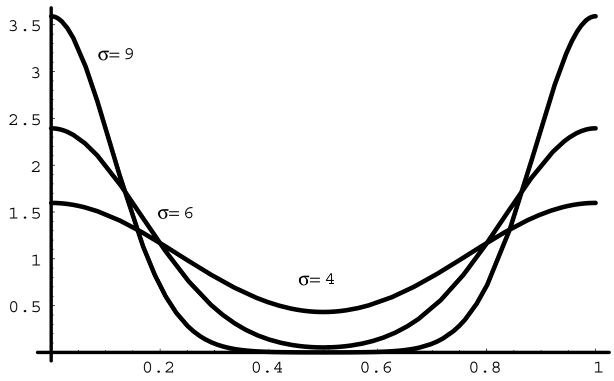

Fig. 3. If σ is large, then ψ_σ deviates from the uniform distribution.

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