that does not give such fine-detail results, but can tolerate more general flow patterns.

Exercise

Exercise 2.5 Suppose that customers at each queue in a stationary series of $M/M/1$ queues are served in the order of their arrival. Note that, from a sample path such as that illustrated in Figure 2.1, each arrival time can be matched to a departure time, corresponding to the same customer. Argue that the sojourn time of a customer in queue 1 is independent of departures from queue 1 prior to her departure. Deduce that in equilibrium the sojourn times of a customer at each of the J queues are independent.

2.3 Closed migration processes

In this section, we will analyze a generalization of the series of queues example. It is simplest to just give a Markovian description. The state space

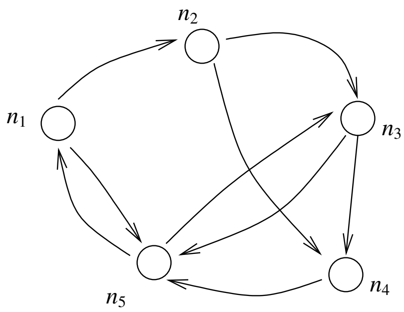

Figure 2.3 Closed migration process.

of the Markov process is $S = \{n \in \mathbb{Z}_+^J : \sum_{j=1}^J n_j = N\}$. Each state is written as $n = (n_1, \dots, n_J)$, where n_j is the number of *individuals* in *colony* j.

For two different colonies j and k, define the operator T_{jk} as

$$T_{jk}(n_1, \dots, n_J) = \begin{cases} (n_1, \dots, n_j - 1, \dots, n_k + 1, \dots, n_J), & j < k, \\ (n_1, \dots, n_k + 1, \dots, n_j - 1, \dots, n_J), & j > k. \end{cases}$$

That is, T_{jk} transfers one individual from colony j to colony k.

We now describe the rate at which transitions occur in our state space. We will only allow individuals to move one at a time, so transitions can only occur between a state n and $T_{jk}n$ for some j, k. We will assume that the transition rates have the form

$$q(n, T_{jk}(n)) = \lambda_{jk}\phi_j(n_j), \quad \phi_j(0) = 0.$$

That is, it is possible to factor the rate into a product of two functions: one depending only on the two colonies j and k, and another depending only on the number of individuals in the "source" colony j.

We will suppose that n is irreducible in S (in particular, that it is possible for individuals to get from any colony to any other colony, possibly in several steps). In this case, we call n a *closed migration process*.

We can model an s-server queue at colony j by taking $\phi_j(n) = \min(n, s)$. Each of the customers requires an exponential service time with parameter $\lambda_j = \sum_k \lambda_{jk}$; and once service is completed, the individual goes to colony k with probability λ_{jk}/λ_j.

Another important example is $\phi_j(n) = n$ for all j. These are the transition rates we get if individuals move independently of one another. This can be thought of as a network of *infinite-server* queues (corresponding to $s = \infty$ above), and is an example of a *linear migration process*; we study these further in Section 2.6. If $N = 1$, the single individual performs a random walk on the set of colonies, with equilibrium distribution (α_j), where the α_j satisfy

$$\alpha_j > 0, \qquad \sum_j \alpha_j = 1,$$

$$\alpha_j \sum_k \lambda_{jk} = \sum_k \alpha_k \lambda_{kj}, \qquad j = 1, 2, \ldots, J. \tag{2.1}$$

We refer to these equations as the *traffic equations*, and we use them to define the quantities (α_j) in terms of (λ_{jk}) for a general closed migration process.

Remark 2.3 Note that the quantities λ_{jk} and $\phi_j(\cdot)$ are only well defined up to a constant factor. In particular, the (α_j) are only well defined after we have picked the particular set of (λ_{jk}).

Theorem 2.4 *The equilibrium distribution for a closed migration process is*

$$\pi(n) = G_N^{-1} \prod_{j=1}^{J} \frac{\alpha_j^{n_j}}{\prod_{r=1}^{n_j} \phi_j(r)}, \qquad n \in S.$$

Here, G_N is a normalizing constant, chosen so the distribution sums to 1, and (α_j) are the solution to the traffic equations (2.1).

Remark 2.5 Although this expression looks somewhat complicated, its form is really quite simple: the joint distribution factors as a product over individual colonies.

Proof In order to check that this is the equilibrium distribution, it suffices to verify that for each n the full balance equations hold:

$$\pi(n)\sum_j\sum_k q(n, T_{jk}n) \stackrel{?}{=} \sum_j\sum_k \pi(T_{jk}n)q(T_{jk}n, n).$$

These will be satisfied provided the following set of *partial balance* equations hold:

$$\pi(n)\sum_k q(n, T_{jk}n) \stackrel{?}{=} \sum_k \pi(T_{jk}n)q(T_{jk}n, n), \qquad \forall j.$$

That is, it suffices to check that, from any state, the rate of individuals leaving a given colony j is the same as the rate of individuals arriving into it. We now recall

$$q(n, T_{jk}n) = \lambda_{jk}\phi_j(n_j), \qquad q(T_{jk}n, n) = \lambda_{kj}\phi_k(n_k + 1),$$

and from the claimed form for π we have that

$$\pi(T_{jk}n) = \pi(n)\frac{\phi_j(n_j)}{\alpha_j}\frac{\alpha_k}{\phi_k(n_k + 1)}.$$

($T_{jk}(n)$ has one more customer in colony k than n does, hence the appearance of $n_k + 1$ in the arguments.) After substituting and cancelling terms, we see that the partial balance equations are equivalent to

$$\sum_k \lambda_{jk} \stackrel{?}{=} \frac{1}{\alpha_j}\sum_k \alpha_k\lambda_{kj},$$

which is true by the definition of the α_j. □

Remark 2.6 The full balance equations state that the total probability flux into and out of any state is the same. The detailed balance equations state that the total probability flux between any pair of states is the same. Partial balance says that, for a fixed state, there is a subset of the states for which the total probability flux into and out of the subset is equal.

Example 2.7 A telephone banking facility has N incoming lines and a single (human) operator. Calls to the facility are initiated as a Poisson process of rate ν, but calls initiated when all N lines are in use are lost. A call finding a free line has to wait for the operator to answer. The operator deals with waiting calls one at a time, and takes an exponentially distributed length of time with parameter λ to check the caller's identity, after which the call is passed to an automated handling system for the caller to transact banking business, and the operator becomes free to deal with another caller. The automated handling system is able to serve up to N callers simultaneously, and the time it takes to serve a call is exponentially distributed with parameter μ. All these lengths of time are independent of each other and of the Poisson arrival process.

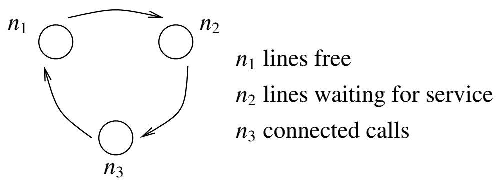

Figure 2.4 Closed migration process for the telephone banking facility.

We model this system as a closed migration process as in Figure 2.4. The transition rates correspond to

$$\begin{aligned} \lambda_{12} &= \nu, & \phi_1(n_1) &= I[n_1 > 0], \\ \lambda_{23} &= \lambda, & \phi_2(n_2) &= I[n_2 > 0], \\ \lambda_{31} &= \mu, & \phi_3(n_3) &= n_3. \end{aligned}$$

We can easily solve the traffic equations

$$\alpha_1 : \alpha_2 : \alpha_3 = \frac{1}{\nu} : \frac{1}{\lambda} : \frac{1}{\mu}$$

because we have a random walk on three vertices, and these are the average amounts of time it spends in each of the vertices. Therefore, by Theorem 2.4,

$$\pi(n_1, n_2, n_3) \propto \frac{1}{\nu^{n_1}} \frac{1}{\lambda^{n_2}} \frac{1}{\mu^{n_3}} \frac{1}{n_3!}.$$

For example, the proportion of incoming calls that are lost is, by the PASTA property,

$$\mathbb{P}(n_1 = 0) = \sum_{n_2+n_3=N} \pi(0, n_2, n_3).$$

Exercises

Exercise 2.6 For the telephone banking example, show that the proportion of calls lost has the form

$$\frac{H(N)}{\sum_{n=0}^{N} H(n)},$$

where

$$H(n) = \left(\frac{\nu}{\lambda}\right)^n \sum_{i=0}^{n} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!}.$$

Exercise 2.7 A restaurant has N tables, with a customer seated at each table. Two waiters are serving them. One of the waiters moves from table to table taking orders for food. The time that he spends taking orders at each table is exponentially distributed with parameter μ_1. He is followed by the wine waiter who spends an exponentially distributed time with parameter μ_2 taking orders at each table. Customers always order food first and then wine, and orders cannot be taken concurrently by both waiters from the same customer. All times taken to order are independent of each other. A customer, after having placed her two orders, completes her meal at rate ν, independently of the other customers. As soon as a customer finishes her meal, she departs, and a new customer takes her place and waits to order. Model this as a closed migration process. Show that the stationary probability that both waiters are busy can be written in the form

$$\frac{G(N-2)}{G(N)} \cdot \frac{\nu^2}{\mu_1 \mu_2}$$

for a function $G(\cdot)$, which may also depend on ν, μ_1, μ_2, to be determined.

2.4 Open migration processes

It is simple to modify the previous model so as to allow customers to enter and exit the system. Define the operators

$$T_{j\to} n = (n_1, \ldots, n_j - 1, \ldots, n_J), \qquad T_{\to k} n = (n_1, \ldots, n_k + 1, \ldots, n_J),$$

where $T_{j\to}$ corresponds to an individual from colony j departing the system; $T_{\to k}$ corresponds to an individual entering colony k from the outside world. We assume that the transition rates associated with these extra possibilities are

$$q(n, T_{jk} n) = \lambda_{jk} \phi_j(n_j); \quad q(n, T_{j\to} n) = \mu_j \phi_j(n_j); \quad q(n, T_{\to k} n) = \nu_k.$$