Banded Matrices with Banded Inverses and $A = LPU$

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It is unusual for the inverse of a banded matrix to be banded. A block diagonal matrix F (or a product of $F's$) has this property, which allows fast multiplication by A and A^{-1}. We show that if $A_{ij} = 0$ and also $A_{ij}^{-1} = 0$ for $|i - j| > w$, then the matrix A can be factored into block diagonal $F_1 \dots F_N$ with $N < Cw^2$ (the main point is that N is independent of the matrix size n).

Examples include wavelet matrices (with $N = w$) and banded permutation matrices (with $N < 2w$). Those can be infinite! But factorization of other banded infinite matrices is still to be proved. We begin with the familiar $A = LU$, lower times upper triangular, including a permutation P to exchange rows. The question is whether P comes before L or after. Numerical analysts put P first, we follow algebraists for whom P is unique in $A = LPU$. With finite matrices, the four starting points in the corners give four inequivalent factorizations.