Thermal leptogenesis in the type-I Dirac seesaw extension to the DFSZ axion model for dark matter

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A mi madre, por darme la vida y enseñarme a vivirla.

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Abstract

Thermal leptogenesis mechanism is proposed for a type-I Dirac seesaw extension to the DFSZ axion model, presenting a detailed construction of the model, with special emphasis in the Peccei-Quinn mechanism, 5-dimensional Weinberg effective operator construction, and neutrino mass generation. Dirac neutrino effective mass matrix computation is performed explicitly through the canonical seesaw mechanism for the (1+1)- and (3+3)-schemes. For the former, it is found that the light active neutrino mass is given by $m_\nu \approx v f_a/\Lambda_{UV}$, result which relates the three energy scales involved in the model: the electroweak scale determined by the Higgs vacuum expectation value v, the Peccei-Quinn symmetry breaking scale determined by the axion decay constant f_a, and the energy scale for the heavy sterile fermion mass Λ_{UV}, which was introduced through the type-I Dirac seesaw extension. From this result, it was also found that: the two energy scales f_a and Λ_{UV} are related by $10^3\,\mathrm{GeV}\, f_a \sim \Lambda_{UV}$, and that the Yukawa coupling associated to the gauge-singlet scalar, which is responsible for the QCD axion, candidate to dark matter, is highly supressed in comparison with the Yukawa coupling associated to Higgs doublet, exactly by a factor which goes from 10^{-4} to 10^{-10}, because of the dependence with the axion decay constant f_a. From the canonical seesaw mechanism performed in the (3+3)-scheme it was found that the two Yukawa terms associated to the gauge-singlet scalar and the Higgs doublet are linked together in a single Dirac neutrino effective mass matrix, whose nine components are calculated explicitly, and they depend on active-sterile mixing parameters. In addition, it was found that if this Dirac neutrino effective mass matrix is hermitian, then the lepton mixing matrix U_{PMNS} is exactly unitary in this model. After this, the CP asymmetry factor is computed considering unflavoured thermal leptogenesis and that the decay of the lightest sterile fermion N_1 is the main contribution for the lepton asymmetry, finding a dependence on QCD axion decay constant f_a and active-sterile mixing parameters through the Dirac neutrino effective mass matrix, hence new sources of CP violation are found thanks to the new nine CP-violating phases associated to this mixing. Finally, it was found an expression for the baryon-antibaryon density $Y_{\Delta B}$ in terms of the CP asymmetry factor computed before. Due to these results, this model links neutrino physics, QCD axion, and cosmological parameters into a same physical framework, hence future measurements and results in experiments related to any of these areas could indirectly provide valuable information about each other.

Keywords: Thermal leptogenesis, Peccei-Quinn mechanism, QCD axion, DFSZ axion model, dark matter, neutrino mass generation, Dirac neutrino effective mass matrix, CP violation

Resumen

Se propone un mecanismo de leptogénesis térmica para una extensión del modelo de axiones DFSZ via el modelo seesaw tipo-I de Dirac, presentando una construcción detallada del modelo, con especial énfasis en el mecanismo de Peccei-Quinn, la construcción del operador 5-dimensional efectivo de Weinberg y la generación de masa de neutrinos. El cálculo de la matriz de masa efectiva para neutrinos de Dirac se realiza explícitamente a través del mecanismo seesaw canónico para los esquemas (1+1) y (3+3). Para el primero, se encuentra que la masa del neutrino activo ligero viene dada por $m_\nu \approx v f_a/\Lambda_{UV}$, resultado que relaciona las tres escalas de energía implicadas en el modelo: la escala electrodébil determinada por el valor esperado en el vacío del Higgs v, la escala de ruptura de la simetría Peccei-Quinn determinada por la constante de decaimiento del axión f_a, y la escala de energía para la masa del fermión estéril pesado Λ_{UV}, que se introdujo a través de la extensión via el modelo seesaw tipo-I de Dirac. A partir de este resultado, también se encontró que: las dos escalas de energía f_a y Λ_{UV} están relacionadas de la forma $10^3\,\mathrm{GeV}\, f_a \sim \Lambda_{UV}$, y que la constante de acoplamiento de Yukawa asociada al singlete gauge escalar, que es responsable del axión de la QCD, candidato a materia oscura, está muy suprimida en comparación con la constante de acoplamiento de Yukawa asociada al doblete de Higgs, exactamente por un factor comprendido entre 10^{-4} y 10^{-10}, debido a la dependencia con la constante de decaimiento del axion f_a. A partir del mecanismo seesaw canónico realizado en el esquema (3+3) se encontró que los dos términos de Yukawa asociados al singlete gauge escalar y al doblete de Higgs están unidos en una única matriz de masa efectiva para neutrinos de Dirac, cuyas nueve componentes se calculan explícitamente, y dependen de los parámetros de mezcla entre el sector activo y estéril. Adicionalmente, se encontró que si esta matriz de masa efectiva para neutrinos de Dirac es hermíticia, entonces la matriz de mezcla de leptones U_{PMNS} es exactamente unitaria en este modelo. Luego de esto, se calcula el factor de asimetría CP considerando leptogénesis térmica sin tener en cuenta el sabor de las partículas y que el decaimiento del fermión estéril más liviano N_1 es la contribución principal para la asimetría leptónica, encontrando una dependencia con la constante de decaimiento del axión de la QCD f_a y los parámetros de mezcla activo-estéril a través de la matriz de masa efectiva para neutrinos de Dirac, por lo que se encuentran nuevas fuentes de violación de CP gracias a las nuevas nueve fases de violación de CP asociadas a esta mezcla. Finalmente, se encontró una expresión para la densidad barión-antibari'on $Y_{\Delta B}$ en términos del factor de asimetría CP calculado anteriormente. Debido a estos resultados, este modelo une parámetros de la física de neutrinos, los axiones de la QCD, y la cosmología, en un mismo marco físico, por lo que futuras medidas y resultados en experimentos relacionados a cualquiera de estas áreas podría proveer de forma indirecta información valiosa sobre las otras. Debido a estos resultados, los parámetros de la física de los neutrinos pueden relacionarse directamente con los parámetros de los axiones de la QCD. Finalmente, se propone un mecanismo de leptogénesis térmica a

través de los decaimientos de los fermiones pesados estériles.

Palabras clave: Leptogénesis térmica, mecanismo de Peccei-Quinn, axiones de la QCD, el modelo de axiones DFSZ, materia oscura, generación de masa para los neutrinos, matriz de masa efectiva para neutrinos de Dirac, violación CP.

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1. Introduction

In 2002 it was discovered that neutrinos have mass, because of their oscillation among flavour states [1]. This is one of the biggest and still open problems in particle physics, since it cannot be explained within the Standard Model of particle physics. Even so, in 1962 Maki, Nakagawa, y Sakata [2], and later in 1969 Pontecorvo [3], had already proposed and described mathematically lepton flavour mixing, inspired by Cabbibo [4], and Kobayashi y Maskawa [5] works about flavour mixing in the quark sector. Although get a mathematical description of neutrino masses and oscillation was possible, the mechanism behind neutrino mass generation remains a mystery [6]. It is clear that it cannot be within the Standard Model, because it leads to the well-known *"hierarchy problem"* [7], where the Yukawa coupling associated to neutrinos would be extremely small compared with Yukawa couplings to charged leptons and quarks. Neutrino mass generation mechanisms beyond the Standard Model are well-justified thanks to the so-called Weinberg effective operator [8], which describes the Yukawa interactions that lead to the neutrino mass generation, in an effective approach. Thus, neutrino mass generation mechanisms arise from the UV-completion of this operator [9]. One of these is the type-I seesaw model, which is a Majorana high energy extension to the Standard Model via the introduction of new heavy sterile (gauge-singlet) fermions [10, 11, 12, 13]. After the spontaneous symmetry breaking of the Standard Model gauge symmetry $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ into the electromagnetism gauge symmetry $\mathrm{U}(1)_{EM}$, in a very similar way to the Higgs mechanism, through canonical seesaw mechanism neutrinos adquire mass inversely proportional to the mass of the heavy sterile fermions introduced in the model [11]. Mixing between light active neutrinos and heavy sterile fermions appeared, giving rise to new CP-violating phases [13].

Neutrino oscillations are well-described by the lepton flavour mixing matrix, the so-called PMNS (Pontecorvo-Maki-Nakagawa-Sakata) matrix [3], where CP violation is described and quantified through its components [13], specifically via complex phases. Those components are measured through neutrino oscillation experiments, especially in those which look for compairing a process and its CP-conjugated, in order to test the CP symmetry [14].

On the other hand, from observations made in the Cosmic Microwave Background (CMB) anisotropies by the Planck colaboration, there exists a baryon asymmetry in the observable universe [15], which cannot be explained within the Standard Model [16]. Furthermore, Sakaharov conditions demand for new sources of CP-violation [17], thus CP violation in the lepton sector is a good way to generate those new sources. In addition, because of the (B+L) anomaly and sphaleron processes that transform lepton number asymmetry into a baryon number asymmetry [18], the generation of the baryonic asymmetry of the universe via a lepton asymmetry [19] is one of the most attractive models on the market.

In 1977 R. D. Peccei and Helen R. Quinn proposed a new global U(1) broken symmetry in the Standard Model, through chiral transformations, in order to solve the strong CP problem [20]. This Peccei-Quinn invariance requires the introduction of a new Higgs doublet in the Standard Model [20]. Thus, it was thought that Peccei-Quinn and electroweak symmetry breaking took place at the same energy scale [21], generating mass for quarks and charged leptons in the Standard Model and, as a consequence of the Nambu-Goldstone theorem [22], a new pseudo-scalar particle [21], the QCD axion. However, later it was found that the Peccei-Quinn symmetry breaking must happen on a higher energy scale than the electroweak energy scale, leading to the birth of the so-called "*invisible axion models*" [23]. One of them was the DFSZ axion model [24, 25], which introduces a new gauge-singlet scalar field whose vacuum expectation value sets the Peccei-Quinn symmetry breaking energy scale above the electroweak energy scale. This model could explain dark matter existence as weakly interacting massive scalar particles [24, 25], providing thus a solution to other of the biggest and open problems in cosmology and physics, dark matter [26].

In this work it will be shown connections between neutrino masses, the baryonic asymmetry of the universe, and dark matter, some of the biggest current problems in cosmology and particle physics. Neutrino mass generation arises through a 5-dimensional Weinberg effective operator built in the type-I Dirac seesaw extension to the DFSZ axion model [27]. Couplings between QCD axions and neutrinos provide a new theoretical and experimental approach to impose and test constraints on the parameter space of neutrino masses and mixing, baryonic asymmetry in the universe, and QCD axion parameters.

The content of this document is organised as follows:

In chapter 2 a brief overview of the Standard Model of particle physics is presented, emphasising in the Yukawa sector: mass generation for fermions through the Higgs mechanism, flavour mixing in quark sector and the CKM matrix, CP violation in the Standard Model, and the Kobayashi-Maskawa mechanism.

In chapter 3 the basics of neutrino physics are presented, starting by neutrino mass generation, and its formal justification thanks to the 5-dimensional Weinberg effective operator. Then, neutrino flavour mixing in leptonic sector is reviewed, remarking the parallelism with quark sector. Then, a review of the canonical (type-I) seesaw mechanism for (1+1)- and (3+3)-schemes is presented. The chapter finishes with a brief summary of neutrino oscillations and CP-violating phases in the leptonic sector through the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix. The differences between Dirac and Majorana neutrinos and their associated physical consequences are stressed throughout the chapter.

In chapter 4 a brief introduction to foundations of cosmology is made, starting with the ΛCDM cosmological model and basics of thermodynamics in the early universe. Then, an overview of baryogenesis is presented where all the required concepts and tools for leptogenesis, such as Sakharov conditions and (B-L) conservation, are explained. Finally a toy model for thermal leptogenesis is presented. Boltzmann equations for lepton number density evolution in time are derivated, and then the standard estimation for the baryon asymmetry of the observable universe is performed.

The chapter 5 is dedicated exclusively to the Peccei-Quinn symmetry and its consequences, such as the need to introduce another Higgs doublet in order to preserve this symmetry, solving the strong CP problem, and the emergence of QCD axions. A particular emphasis in the DFSZ axion model is made, performing all the necessary mathematical aspects to build this model, since it will be the basis model for implementing thermal leptogenesis through a Dirac seesaw extension in the following chapter. Some important aspects related to gauge theories, spontaneous symmetry breaking, and Nambu-Goldstone theorem are mentioned.

The chapter 6 is the most important of all, as it takes the most important results of the previous chapters and applies them to the neutrino mass generation mechanism, as well as to the implementation of thermal leptogenesis, for an extension to the model that has been built in chapter 4: the DFSZ axion model. This is made through the construction of a 5-dimensional Weinberg effective operator in the Dirac extension to the DFSZ axion model. Then, the effective lagrangian is UV-complete through the type-I Dirac seesaw extension to the DFSZ axion model. The Dirac neutrino effective mass matrix computation is performed explicitly in the (1+1)- and (3+3)-schemes, where new CP-violating phases appears because of the active-sterile mixing terms in leptonic sector. Finally, thermal leptogenesis is implemented through the decays of the heavy sterile fermions introduced in the type-I Dirac seesaw extension to the model.

2. The Standard Model of particle physics

2.1. Higgs mechanism and fermion masses

To give mass to quarks and charged leptons, i.e., to fermions, as well as to vector electroweak bosons $W^{\pm}$ and Z, it is necessary to implement the so-called *Higgs mechanism* [28]. The argument of this mechanism relies on the implementation of the *spontaneous symmetry breaking* through the potential of a scalar doublet, the Higgs field [22, 28].
The Higgs field Φ is introduced to the Standard Model as a $SU(2)_L$ doublet of complex scalar fields, into a gauge symmetry $SU(2)_L \times U(1)_Y$. Furthermore, this Higgs doublet carries hypercharge $Y = \frac{1}{2}$ in order to preserve the electric charge conservation ($U(1)_{EM}$ gauge symmetry) after the spontaneous symmetry breaking, i.e., the generator Q defined as [29]:

$$Q \equiv T^3 + Y \tag{2-1}$$

Leaves **invariant** the vacuum expectation value of the Higgs field v. T^3 is the projection in the third (diagonal) component of weak isospin generator and the Q generator is called the electric charge. Explicitly, it reads as [29]:

$$\begin{aligned} Q &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{2-2}$$

In other words, this generator annihilates the vacuum:

$$\begin{aligned} Q \langle \Phi \rangle &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \\ &= 0 \end{aligned} \tag{2-3}$$

Note that it was necessary to choose a particular direction in the internal $SU(2)_L$ space for the vacuum expectation value of Φ (v) to implement the spontaneous symmetry breaking [29]:

$$\langle \Phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{2-4}$$

Now, let us use the Higgs field to add mass terms for fermions in the lagrangian. These terms must be gauge and Lorentz invariant, and also to link left- and right-handed components of fermions in a convenient way, actually in one way such that the chiral symmetry be preserved [29]. For this, it is useful to remember the invariants for a Dirac spinor. Let be a Dirac spinor written in terms of its Weyl components [30]:

$$\psi = \begin{pmatrix} \chi_L \\ \zeta_R \end{pmatrix} \tag{2-5}$$

Where two possible Lorentz invariants are [30]:

$$I_1 = (\chi_L)^\dagger \zeta_R \tag{2-6}$$

$$I_2 = (\zeta_R)^\dagger \chi_L \tag{2-7}$$

Then, it is possible to write these Lorentz invariants using Dirac spinors as follows [30]:

$$\begin{aligned} \overline{\psi}\psi &= \overline{\psi}_L \psi_R + \overline{\psi}_R \psi_L \\ &= \psi_L^\dagger \gamma^0 \psi_R + \psi_R^\dagger \gamma^0 \psi_L \\ &= \begin{pmatrix} \chi_L^\dagger & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \zeta_R \end{pmatrix} + \begin{pmatrix} 0 & \zeta_R^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \chi_L \\ 0 \end{pmatrix} \\ &= (\chi_L)^\dagger \zeta_R + (\zeta_R)^\dagger \chi_L \end{aligned} \tag{2-8}$$

This is known as a Dirac mass term. Explicitly it has the following form:

$$\mathcal{L}_D = \frac{1}{2} m \overline{\psi} \psi c + h.c. \tag{2-9}$$

However, this cannot be the mass term, because it always combines the right- and left-handed fields, and it is known that these components transform different under SU$(2)_L$ gauge symmetry, i.e., they belong to different representations of SU$(2)_L$ [31]. In fact, the left-handed fields transform as doublets of this group, while the right-handed ones transform as singlets.

In order to obtain a SU$(2)_L$, U$(1)_Y$, and Lorentz invariant term, it is coupled a spin 0 doublet (the Higgs field) to the spin $\frac{1}{2}$ fields [32]:

$$\overline{\Psi}_L \Phi \psi_R \tag{2-10}$$

This term is known as the **Yukawa coupling** and satisfies the gauge invariance requirements. Let us see:

Under a SU(2) transformation this term transforms as:

$$\begin{aligned} \overline{\Psi}'_L \Phi' \psi'_R &= \overline{\Psi}_L e^{-ia_i(x)T_i} e^{ia_i(x)T_i} \Phi \psi_R \\ &= \overline{\Psi}_L \Phi \psi_R \end{aligned} \tag{2-11}$$

And under a U(1) transformation:

$$\begin{aligned}\overline{\Psi}'_L \Phi' \psi'_R &= \overline{\Psi}_L e^{-ib(x)} e^{ib(x)} \Phi \psi_R \\ &= \overline{\Psi}_L \Phi \psi_R\end{aligned} \tag{2-12}$$

The Lorentz invariance is immediate because of scalar nature of the Higgs field and the left-right-handed combination of spinors [32].
Therefore, considering just the first generation of fermions, the mass terms for the electron-neutrino are added to the Lagrangian as Yukawa couplings, and have the following form:

$$\mathcal{L}_m^e = -\lambda_e \overline{E}_L \Phi e_R + h.c. \tag{2-13}$$

Where e_R is a SU(2)$_L$ singlet, while the doublets have the form:

$$E_L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \tag{2-14}$$

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \tag{2-15}$$

And λ_e is a dimensionless complex coupling constant.
Notice that the hypercharge of each Yukawa term is zero. In fact, the hypercharge of E_L is [33, 34]:

$$Y_{E_L} = -\frac{1}{2} = -Y_{\overline{E}_L} \tag{2-16}$$

The hypercharge of the Higgs field is [33, 34]:

$$Y_\Phi = \frac{1}{2} = -Y_{\Phi^\dagger} \tag{2-17}$$

And the singelt has an hypercharge [33, 34]:

$$Y_{e_R} = -1 = -Y_{\overline{e}_R} \tag{2-18}$$

So, the hypercharge of each Yukawa term sum zero:

$$\begin{aligned}\overline{E}_L \Phi e_R : Y_{\overline{E}_L} + Y_\Phi + Y_{e_R} &= \frac{1}{2} + \frac{1}{2} + (-1) \\ &= 0\end{aligned} \tag{2-19}$$

$$\begin{aligned}\overline{e}_R \Phi^\dagger E_L : Y_{\overline{e}_R} + Y_{\Phi^\dagger} + Y_{E_L} &= 1 + \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right) \\ &= 0\end{aligned} \tag{2-20}$$

On the other hand, recalling that the Higgs field can be written around its *vev* as [29]:

$$\Phi(x) = U^{-1}(x)\begin{pmatrix} 0 \\ \frac{v+H(x)}{\sqrt{2}} \end{pmatrix} \tag{2-21}$$

Where U is a $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ gauge transformation, i.e., $U(x) = \exp\left(-i\frac{a_i(x)T_i}{v}\right)\exp\left(-i\frac{b(x)}{v}\right)$, and $v = \frac{\mu}{\sqrt{\lambda}}$. In this way it is being implemented the spontaneous symmetry breaking. Now, in the unitary gauge it is possible to remove the factor $U^{-1}(x)$ by applying another gauge transformation [29], then:

$$\Phi(x) = \begin{pmatrix} 0 \\ \frac{v+H(x)}{\sqrt{2}} \end{pmatrix} \tag{2-22}$$

Therefore, the Yukawa terms in this unitary gauge read as [29]:

$$\begin{aligned}
\mathcal{L}_m^e &= -\lambda_e \overline{E}_L \Phi e_R - \lambda_e^* \overline{e}_R \Phi^\dagger E_L \\
&= -\frac{\lambda_e}{\sqrt{2}}\begin{pmatrix} \overline{\nu}_L & \overline{e}_L \end{pmatrix}\begin{pmatrix} 0 \\ v+H \end{pmatrix} e_R - \frac{\lambda_e^*}{\sqrt{2}}\overline{e}_R \begin{pmatrix} 0 & v+H \end{pmatrix}\begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \\
&= -\frac{\lambda_e}{\sqrt{2}}\overline{e}_L(v+H)e_R - \frac{\lambda_e^*}{\sqrt{2}}\overline{e}_R(v+H)e_L \\
&= -\frac{v+H}{\sqrt{2}}(\lambda_e \overline{e}_L e_R + \lambda_e^* \overline{e}_R e_L) \\
&= -\frac{\lambda_e(v+H)}{\sqrt{2}}(\overline{e}_L e_R + \overline{e}_R e_L) \\
&= -\frac{\lambda_e(v+H)}{\sqrt{2}}\overline{e}e
\end{aligned} \tag{2-23}$$

Notice that it has been taken λ to be real and positive, by making the redefinition of the complex constant λ_e in such way that $\lambda_e \to e^{-i\varphi}\lambda_e$ and $e_R \to e^{i\varphi}e_R$. Then, from the last result it is found an explicit mass term for the electron, due to [29]:

$$\mathcal{L}_m^e = -m_e \overline{e}e - \frac{\lambda_e}{\sqrt{2}}H\overline{e}e \tag{2-24}$$

Where the mass term correspond to:

$$m_e = \frac{\lambda_e v}{\sqrt{2}} \tag{2-25}$$

Thus, the coupling constant which determines the strenght of interaction between the Higgs field and the electron is given by:

$$\lambda_e = \frac{\sqrt{2}m_e}{v} \tag{2-26}$$

i.e., it is proportional to the mass of the fermion, just as the bosons case.
Similarly, it is also possible to generate mass terms for the quarks via the Yukawa coupling [28]. For example, for the down quark [29]:

$$\mathcal{L}_m^d = -\lambda_d \overline{Q}_L \Phi d_R + h.c. \tag{2-27}$$

which is $\mathrm{U}(1)_Y$ invariant since its hypercharge is null:

$$\begin{aligned} Y_{\overline{Q}_L} + Y_\Phi + Y_{d_R} &= -\frac{1}{6} + \frac{1}{2} - \frac{1}{3} \\ &= 0 \end{aligned} \tag{2-28}$$

Now, in the unitary gauge these Yukawa terms read as [29]:

$$\begin{aligned} \mathcal{L}_m^d &= -\lambda_d \overline{Q}_L \Phi d_R - \lambda_d^* \overline{d}_R \Phi^\dagger Q_L \\ &= -\frac{\lambda_d}{\sqrt{2}} \begin{pmatrix} \overline{u}_L & \overline{d}_L \end{pmatrix} \begin{pmatrix} 0 \\ v+H \end{pmatrix} d_R - \frac{\lambda_d^*}{\sqrt{2}} \overline{d}_R \begin{pmatrix} 0 & v+H \end{pmatrix} \begin{pmatrix} u_L \\ d_L \end{pmatrix} \\ &= -\frac{\lambda_d}{\sqrt{2}} \overline{d}_L (v+H) d_R - \frac{\lambda_d^*}{\sqrt{2}} \overline{d}_R (v+H) d_L \\ &= -\frac{v+H}{\sqrt{2}} (\lambda_d \overline{d}_L d_R + \lambda_d^* \overline{d}_R d_L) \\ &= -\frac{\lambda_d (v+H)}{\sqrt{2}} \overline{d} d \end{aligned} \tag{2-29}$$

And again, this last term can be written as:

$$\mathcal{L}_m^d = -m_d \overline{d} d - \frac{\lambda_d}{\sqrt{2}} H \overline{d} d \tag{2-30}$$

With the mass term equals to:

$$m_d = \frac{\lambda_d v}{\sqrt{2}} \tag{2-31}$$

Where the coupling constant will be proportional to the quark down mass as it was expected.
On the other hand, for the mass of the up quark it is used the following Yukawa terms [29]:

$$\mathcal{L}_m^u = -\lambda_u \overline{Q}_L \Phi^c u_R + h.c. \tag{2-32}$$

Where [31]:

$$\Phi^c = \epsilon \Phi^* = i\tau_2 \Phi^* \tag{2-33}$$

i.e., Φ^c is the charge-conjugated Higgs field. In the unitary gauge:

$$\begin{aligned} \Phi^c &= i\tau_2 \Phi^* \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v+H \end{pmatrix} \\ &= \begin{pmatrix} v+H \\ 0 \end{pmatrix} \end{aligned} \tag{2-34}$$

Then, the explicit Yukawa terms will be:

$$\begin{aligned} \mathcal{L}_m^u &= -\frac{\lambda_u}{\sqrt{2}}\overline{u}_L(v+H)u_R - \frac{\lambda_u^*}{\sqrt{2}}\overline{u}_R(v+H)u_L \\ &= -\frac{v+H}{\sqrt{2}}(\lambda_u\overline{u}_L u_R + \lambda_u^*\overline{u}_R u_L) \\ &= -\frac{\lambda_u(v+H)}{\sqrt{2}}\overline{u}u \end{aligned} \tag{2-35}$$

Therefore, the mass term for the quark up is:

$$m_u = \frac{\lambda_u v}{\sqrt{2}} \tag{2-36}$$

And again the coupling constant will be proportional to its mass.
Thus, the sum of all the Yukawa terms for the first generation fermions is given by [29]:

$$\begin{aligned} \mathcal{L}_m^f &= \mathcal{L}_m^e + \mathcal{L}_m^u + \mathcal{L}_m^d \\ &= -\lambda_e\overline{E}_L\Phi e_R - \lambda_u\overline{Q}_L\Phi^c u_R - \lambda_d\overline{Q}_L\Phi d_R + h.c. \end{aligned} \tag{2-37}$$

Which in the unitary gauge, and with the redefinition of the complex coupling constants, i.e., with λ_e,λ_u,$\lambda_d > 0$ read as [29]:

$$\mathcal{L}_m^f = -m_e\overline{e}e - m_u\overline{u}u - m_d\overline{d}d - \frac{\lambda_e}{\sqrt{2}}H\overline{e}e - \frac{\lambda_u}{\sqrt{2}}H\overline{u}u - \frac{\lambda_d}{\sqrt{2}}H\overline{d}d \tag{2-38}$$

2.2. C, P, and CP violation in the Standard Model

It is known that in QED and QCD the discrete symmetries: charge-conjugation (C), parity (P) and time reversal (T) are preserved [35, 36]. However, in the electroweak theory it does not occur, and this is due to the nature of the weak interaction which couples only to left-handed (left-chiral) particles. This implies the violation of parity in weak interactions which was experimentally confirmed by Madame Wu and collaborators in 1957 studying the radioactive decay ${}^{60}_{27}\mathrm{Co} \to {}^{60}_{28}\mathrm{Ni} + e^- + \overline{\nu_e} + 2\gamma$ [37]. Later, in the same year, Garwing, Lederman and Weinrich confirmed that charge-conjugation is also violated in weak interactions [38], studying the decay of muons to electrons and anti-muons to positrons and then analyzing the polarization of the electrons and positrons. It was found that muons decayed to left-handed electrons but not to right-handed electrons, and that anti-muons decayed to right-handed positrons but not to left-handed positrons.

The above facts implies that a left-handed fermion described by the Dirac spinor ψ_L, which couples to the gauge boson W, under parity transforms as [30]:

$$\begin{aligned} P^{-1}\psi_L(x)P &= \hat{P}\psi_L(x) \\ &= \gamma^0\frac{(1-\gamma_5)}{2}\psi(x) \\ &= \frac{(1+\gamma_5)}{2}\gamma^0\psi(x) \\ &= \frac{(1+\gamma_5)}{2}\hat{P}\psi(x) \\ &= \frac{(1+\gamma_5)}{2}\psi(x') \\ &= \psi_R(x') \end{aligned} \tag{2-39}$$

Where it was used the fact that the matrix γ_5 anticommutes with all the others gamma matrices, i.e., $\{\gamma^\mu, \gamma_5\} = 0$ for $\mu = 0, 1, 2, 3$, and:

$$x' \equiv x'_\mu = -g_{\mu\nu}x^\nu \tag{2-40}$$

However, the right-handed fermion ψ_R is not coupled to the gauge boson W, i.e., it is a singlet under $\mathrm{SU}(2)_L$, and this is the way how the Standard Model describes parity violation [33, 34].

Similarly, if now a charge-conjugation transformation is applied to a left-handed fermion [30]:

$$\begin{aligned} C\psi_L(x)C^{-1} = (\psi_L(x))^c &= i\gamma^2(\psi_L(x))^* \\ &= i\gamma^2\left[\frac{(1-\gamma_5)}{2}\psi(x)\right]^* \\ &= i\gamma^2\frac{(1-\gamma_5)^*}{2}\psi^*(x) \\ &= i\gamma^2\frac{(1-\gamma_5)}{2}\psi^*(x) \\ &= \frac{(1+\gamma_5)}{2}(i\gamma^2\psi^*(x)) \\ &= \frac{(1+\gamma_5)}{2}(\psi(x))^c \\ &= (\psi^c(x))_R \end{aligned} \tag{2-41}$$

Which is also a right-handed field, and therefore it is not coupled to the gauge boson W, hence charge-cojungation (C) violation is described by the Standard Model [33, 34].

Now, regarding to the CP symmetry, it is necessary recall that during that time when parity and charge-conjugation violation were confirmed, it was believed that CP must be a preserved discrete symmetry since otherwise time reversal would be also violated due to the CPT theorem. Thus, under CP a left-handed fermion will transform as [30]:

$$\begin{aligned}
(CP)^{-1}\psi_L(x)CP &= \hat{P}(\psi_L(x))^c \\
&= \hat{P}(\psi^c(x))_R \\
&= \gamma^0\frac{(1+\gamma_5)}{2}\psi^c(x) \\
&= \frac{(1-\gamma_5)}{2}\gamma^0\psi^c(x) \\
&= \frac{(1-\gamma_5)}{2}\hat{P}\psi^c(x) \\
&= \frac{(1-\gamma_5)}{2}\psi^c(x') \\
&= (\psi^c(x'))_L
\end{aligned} \tag{2-42}$$

Hence, under CP a left-handed fermions is transformed into a left-handed field again, which couples to the gauge boson W and therefore it interacts weakly according to the Standard Model [33, 34]. This implies that CP would be a preserved discrete symmetry in the model. However, in 1964, Cronin and Fitch discovered CP violation in experiments involving decays of the neutral kaons system [39].

2.2.1. Phenomenology of the neutral kaons system

Kaons are particular cases of mesons in the Standard Model, and it is known that a meson is a bound state of a valence quark and anti-quark [31]. In the case of neutral kaons, there are two types [31]:

$$K^0 = (d\overline{s}) \rightarrow S = 1 \tag{2-43}$$

$$\overline{K}^0 = (\overline{d}s) \rightarrow S = -1 \tag{2-44}$$

Where S refers to the **strangeness** which is preserved in the strong nuclear interaction but not in weak nuclear interaction. They were discovered in strong reactions, for example a typical one is:

$$\pi^- + p \rightarrow K^0 + \Lambda \tag{2-45}$$

And further:

$$K^0(d\overline{s}) + p \rightarrow n + K^+(u\overline{s}) \tag{2-46}$$

$$\overline{K}^0(\overline{d}s) + n \rightarrow p + K^-(\overline{u}s) \tag{2-47}$$

Preserving the strangeness in both reactions. This fact allows to classify the CP violation in neutral kaons system according to the change of strangeness, i.e., in purely electroweak processes.

From the study of some strong processes it was experimentally proved, and theoretically deduced, that K^0 and $\overline{K}^0$ are pseudoscalar mesons ($l = 0$), and therefore under parity they behave like [31]:

$$\begin{aligned} P(q\overline{q}) &= P(q)P(\overline{q}) \cdot (-1)^l \\ &= (+1)(-1)(-1)^0 \\ &= -1 \end{aligned} \tag{2-48}$$

Where P denotes the parity eigenvalue, and the term $(-1)^l$ is the symmetry of the orbital wavefunction. Hence, the lightest mesons (with $l = 0$) have odd intrinsic parities. Thus, for the neutral kaons in particular:

$$\hat{P}\left|K^0\right\rangle = -\left|K^0\right\rangle \tag{2-49}$$

$$\hat{P}\left|\overline{K}^0\right\rangle = -\left|\overline{K}^0\right\rangle \tag{2-50}$$

Since the neutral kaons are the lightest hadrons with non-zero strangeness ($m_{K^0,\overline{K}^0} \sim$ 494MeV) [31], they can only decay weakly to final states of either leptons ($e^\pm$ or $\mu^\pm$) or pions ($\pi^\pm$ and π^0) owing to the kinematics (energy-momentum conservation). Therefore, these decays will violate parity (P), charge-conjugation (C), and of course, strangeness. Furthermore, weak interactions make possible the $K^0 - \overline{K}^0$ mixing, or also called, the $K^0 - \overline{K}^0$ system [31], which is described by the box diagrams shown in the figure **2-1**.

Therefore, the physical states which are measurable in experiments are a mixed of K^0 and $\overline{K}^0$ kaons. In other words, the physical neutral kaon states are the eigenstates of the hamiltonian for the $K^0 - \overline{K}^0$ system. These physical states are known as K-short (K_S) and K-long (K_L) and would be linear combinations of the $|K^0\rangle$ and $\left|\overline{K}^0\right\rangle$ states [31]. They have similar masses ($\sim$ 498Mev $\sim m_{K^0,\overline{K}^0}$), and their names are suggested by their lifetimes [31]:

$$\tau(K_S) \approx 0{,}9 \times 10^{-10}\,\mathrm{s} \tag{2-51}$$

$$\tau(K_L) \approx 0{,}5 \times 10^{-7}\,\mathrm{s} \tag{2-52}$$

Under charge-conjugation, the neutral kaon states transform as [31]:

$$\hat{C}\left|K^0\right\rangle = \xi_k \left|\overline{K}^0\right\rangle \tag{2-53}$$

$$\hat{C}\left|\overline{K}^0\right\rangle = \xi_k^* \left|K^0\right\rangle \tag{2-54}$$

Because of:

$$(d\overline{s})\hat{C}(\overline{d}s) \tag{2-55}$$

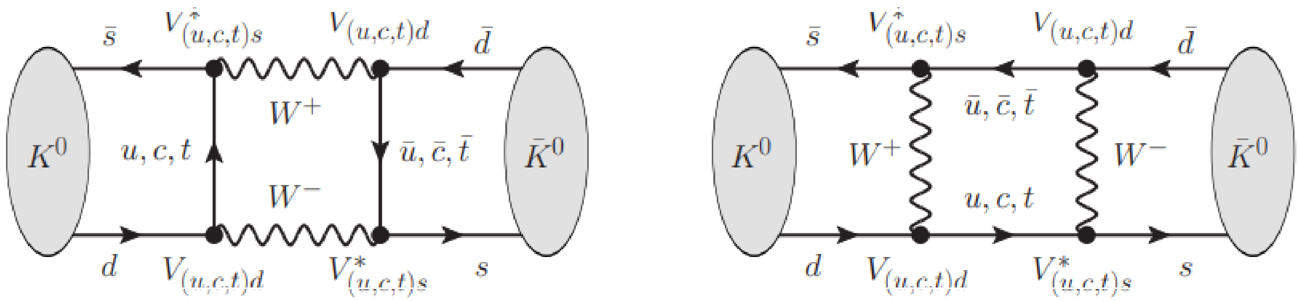

Figure 2-1.: Box (one-loop) diagrams for the $K^0 - \overline{K}^0$ system. Images taken from [40].

Where ξ_k is just a phase. By convention and convenience, it is chosen to be $\xi_k = -1$, and therefore:

$$\hat{C}\left|K^0\right\rangle = -\left|\overline{K}^0\right\rangle \tag{2-56}$$

$$\hat{C}\left|\overline{K}^0\right\rangle = -\left|K^0\right\rangle \tag{2-57}$$

Thus, under the combined discrete operation CP, the neutral kaons transform as [31]:

$$\begin{aligned}\hat{C}\hat{P}\left|K^0\right\rangle &= \hat{C}\left(-\left|K^0\right\rangle\right) = -\left(-\left|\overline{K}^0\right\rangle\right)\\ &= \left|\overline{K}^0\right\rangle\end{aligned} \tag{2-58}$$

$$\begin{aligned}\hat{C}\hat{P}\left|\overline{K}^0\right\rangle &= \hat{C}\left(-\left|\overline{K}^0\right\rangle\right) = -\left(-\left|K^0\right\rangle\right)\\ &= \left|K^0\right\rangle\end{aligned} \tag{2-59}$$

This result makes possible to define CP eigenstates, which of course will be linear combinations of the neutral kaons [31]:

$$\left|K_1\right\rangle = \frac{1}{\sqrt{2}}\left(\left|K^0\right\rangle + \left|\overline{K}^0\right\rangle\right) \tag{2-60}$$

And:

$$\left|K_2\right\rangle = \frac{1}{\sqrt{2}}\left(\left|K^0\right\rangle - \left|\overline{K}^0\right\rangle\right) \tag{2-61}$$

Where the factor $1/\sqrt{2}$ is for the normalization. In fact, $\left|K_1\right\rangle$ is an eigenstate of the CP operator with eigenvalue +1:

$$\begin{aligned}\hat{C}\hat{P}\left|K_1\right\rangle &= \frac{1}{\sqrt{2}}\hat{C}\hat{P}\left(\left|K^0\right\rangle + \left|\overline{K}^0\right\rangle\right) \\ &= \frac{1}{\sqrt{2}}\left(\hat{C}\hat{P}\left|K^0\right\rangle + \hat{C}\hat{P}\left|\overline{K}^0\right\rangle\right) \\ &= \frac{1}{\sqrt{2}}\left(\left|\overline{K}^0\right\rangle + \left|K^0\right\rangle\right) \\ &= \left|K_1\right\rangle\end{aligned} \tag{2-62}$$

And $|K_2\rangle$ is an eigenstate of CP with eigenvalue -1:

$$\begin{aligned}\hat{C}\hat{P}\left|K_2\right\rangle &= \frac{1}{\sqrt{2}}\hat{C}\hat{P}\left(\left|K^0\right\rangle - \left|\overline{K}^0\right\rangle\right) \\ &= \frac{1}{\sqrt{2}}\left(\hat{C}\hat{P}\left|K^0\right\rangle - \hat{C}\hat{P}\left|\overline{K}^0\right\rangle\right) \\ &= \frac{1}{\sqrt{2}}\left(\left|\overline{K}^0\right\rangle - \left|K^0\right\rangle\right) \\ &= -\left|K_2\right\rangle\end{aligned} \tag{2-63}$$

This suggests that if CP were a conserved symmetry, it has already been found their eigenstates and therefore the physical states [31].
Experimentally, the physical states K_L and K_S have well-defined masses and decay widths. In the hadronic decays, K_S mostly decays to $\pi\pi$ products, whereas K_L mostly decays to $\pi\pi\pi$, i.e.,

$$\frac{\Gamma(K_S \to \pi\pi\pi)}{\Gamma(K_S \to \pi\pi)} << 1 \tag{2-64}$$

And:

$$\frac{\Gamma(K_L \to \pi\pi)}{\Gamma(K_L \to \pi\pi\pi)} << 1 \tag{2-65}$$

These differences in the partial decay widths are due precisely to the CP violation in weak interactions.
Now, in order to link these experimental facts with the results above about the eigenstaes of CP, let us see the following:

1. Two-body decays in the rest frame.

   a) Consider the decay $K \to \pi^0\pi^0$:

   First, by analyzing the parity it was found that the lightest mesons (with $l = 0$), such as the K and π mesons, have intrinsic odd parity. Then, because of angular momentum conservation the relative angular momentum of the product $\pi^0\pi^0$ must also be zero [31], therefore the parity will be:

$$\begin{aligned} P(\pi^0\pi^0) &= P(\pi^0)P(\pi^0)(-1)^l \\ &= (-1)(-1)(-1)^0 \\ &= +1 \end{aligned} \tag{2-66}$$

Now, considering the charge-conjugation, it has been chosen by convention in equations (2-56) and (2-57) that $C(K) = -1$. On the other hand, the neutral pion has the following flavour wave function [31] :

$$\left|\pi^0\right\rangle = \frac{1}{\sqrt{2}}\left(|u\overline{u}\rangle - \left|d\overline{d}\right\rangle\right) \tag{2-67}$$

Then $C(\pi^0) = +1$, and consequently:

$$\begin{aligned} C(\pi^0\pi^0) &= C(\pi^0)C(\pi^0) \\ &= (+1)(+1) \\ &= +1 \end{aligned} \tag{2-68}$$

The above results lead us to conclude that under CP it is satisfied that:

$$\begin{aligned} CP(\pi^0\pi^0) &= C((+1)\pi^0\pi^0) \\ &= +1 \end{aligned} \tag{2-69}$$

$b)$ Now, consider the decay $K \to \pi^+\pi^-$:

In the same way as in the previous case, parity analysis leads to $P(\pi^+\pi^-) = +1$. Now, considering the flavour wave functions for the charged pions [31]:

$$\left|\pi^+\right\rangle = \left|u\overline{d}\right\rangle \tag{2-70}$$

$$\left|\pi^-\right\rangle = |d\overline{u}\rangle \tag{2-71}$$

Therefore, by applying the charge-conjugation operator to the product in the final state:

$$\begin{aligned} \hat{C}\left|\pi^+\pi^-\right\rangle &= \hat{C}\left|\pi^+\right\rangle \hat{C}\left|\pi^-\right\rangle \\ &= \left|\pi^-\right\rangle\left|\pi^+\right\rangle \end{aligned} \tag{2-72}$$

Thus, it has been just found that $C(\pi^+\pi^-) = +1$. Hence, from these results it is possible to conclude that under a CP transformation:

$$\begin{aligned} CP(\pi^+\pi^-) &= C((+1)\pi^+\pi^-) \\ &= +1 \end{aligned} \tag{2-73}$$

Therefore owing to (a) and (b) results, it is possible to conclude that if CP were conserved then the decay $K \to \pi\pi$ could only occur if K is an eigenstate of CP with eigenvalue +1, i.e., if $|K\rangle = |K_1\rangle$.

2. Three-body decays in the rest frame.

   a) For the decay $K \to \pi^0\pi^0\pi^0$:

   First, by angular momentum conservation it must be satisfied that $\vec{L}_i = \vec{L}_f = 0$, with $\vec{L}_f = \vec{L}_1 + \vec{L}_2$ [31], where $\vec{L}_1$ can be thought of as the relative angular momentum for two pions whereas $\vec{L}_2$ would be the angular momentum for the third one. Thus [31]:

$$\begin{aligned} P(\pi^0\pi^0\pi^0) &= P(\pi^0\pi^0)P(\pi^0)(-1)^{l_2} \\ &= P(\pi^0)P(\pi^0)P(\pi^0)(-1)^{l_1+l_2} \\ &= (-1)(-1)(-1)(-1)^0 \\ &= -1 \end{aligned} \tag{2-74}$$

   Because of $l_1 + l_2 = 0$, condition in order to conserve the angular momentum.

   Computation of the charge-conjugation is quite easy, because the previous results can be used, and moreover the pions are neutral, then:

$$\begin{aligned} C(\pi^0\pi^0\pi^0) &= C(\pi^0\pi^0)C(\pi^0) \\ &= (+1)(+1) \\ &= +1 \end{aligned} \tag{2-75}$$

   Therefore, under CP it must be satisfied that:

$$\begin{aligned} CP(\pi^0\pi^0\pi^0) &= C((-1)\pi^0\pi^0\pi^0) \\ &= -C(\pi^0\pi^0\pi^0) \\ &= (-1)(+1) \\ &= -1 \end{aligned} \tag{2-76}$$

   b) Now, for the decay $K \to \pi^+\pi^-\pi^0$:

   The parity analysis is exactly in the same way as in the last case, so $P(\pi^+\pi^-\pi^0) = -1$ [31].

   And for the charge-conjugation it is also possible to use previous results:

$$\begin{aligned} C(\pi^+\pi^-\pi^0) &= C(\pi^+\pi^-)C(\pi^0) \\ &= (+1)(+1) \\ &= +1 \end{aligned} \tag{2-77}$$

Thus, under a CP transformation:

$$\begin{aligned} CP(\pi^+\pi^-\pi^0) &= C((-1)\pi^+\pi^-\pi^0) \\ &= -C(\pi^+\pi^-\pi^0) \\ &= (-1)(+1) \\ &= -1 \end{aligned} \tag{2-78}$$

Therefore, it is concluded that if CP were conserved, then the decay $K \to \pi\pi\pi$ could occur only if K is an eigenstate of CP with eigenvalue (-1), i.e., if $|K\rangle = |K_2\rangle$.

Thus, the general conclusion for this analysis is: If CP were conserved, the decays $K_1 \to \pi\pi$ and $K_2 \to \pi\pi\pi$ would be allowed, but the decays $K_1 \to \pi\pi\pi$ and $K_2 \to \pi\pi$ would be totally forbidden.

Hence, the above results, and equations (2-64) and (2-65), allow to conclude that: If CP were a preserved discrete symmetry in the Standard Model, then it would be satisfied that:

$$|K_S\rangle = |K_1\rangle \tag{2-79}$$

And:

$$|K_L\rangle = |K_2\rangle \tag{2-80}$$

However, it is known that in weak interactions CP is violated, just a tiny amount, but it is. Thereby, the above equations are just approximations.

2.3. Flavour mixing, the CKM matrix and the Kobayashi-Maskawa mechanism

Now, let us generalize the Yukawa terms to the case of multiple fermion generations, specifically for three generations which is the case of the Standard Model [31]:

$$\begin{aligned} \mathcal{L}_m^f &= -[\Lambda_e]_{ml}\overline{e}_{R,m}\Phi^\dagger E_{L,l} - [\Lambda_d]_{ml}\overline{d}_{R,m}\Phi^\dagger Q_{L,l} - [\Lambda_u]_{ml}\overline{u}_{R,m}\Phi^{c\dagger} Q_{L,l} + h.c. \\ &= -\overline{e}_R\Lambda_e\Phi^\dagger E_L - \overline{d}_R\Lambda_d\Phi^\dagger Q_L - \overline{u}_R\Lambda_u\Phi^{c\dagger}Q_L + h.c. \end{aligned} \tag{2-81}$$

Where Λ_e, Λ_d and Λ_u are 3×3 complex matrices, and $l, m = 1, 2, 3$ are generation indices. The aim is writing the lagrangian in terms of the **mass eigenstates** so it is necessary to diagonalize these matrices. For that reason, it can be applied a change of basis from flavour states to mass states through unitary transformations [31]:

$$e_R \to U_1 e_R \tag{2-82}$$

$$u_R \to U_2 u_R \tag{2-83}$$

$$d_R \to U_3 d_R \tag{2-84}$$

$$E_L \to V_1 E_L \tag{2-85}$$

$$Q_L \to V_2 Q_L \tag{2-86}$$

Where U_1, U_2, U_3, V_1 and V_2 are 3×3 unitary matrices, which do not depend on the position. Then, performing this transformation for the Yukawa lagrangian in (2-81):

$$\begin{aligned}\mathcal{L}'^f_m &= -\overline{e}'_R\Lambda_e\Phi^\dagger E'_L - \overline{d}'_R\Lambda_d\Phi^\dagger Q'_L - \overline{u}'_R\Lambda_u\Phi^{c\dagger}Q'_L + h.c.\\ &= -(\overline{e}_R U_1^\dagger)\Lambda_e\Phi^\dagger(V_1E_L) - (\overline{d}_R U_3^\dagger)\Lambda_d\Phi^\dagger(V_2Q_L) - (\overline{u}_R U_2^\dagger)\Lambda_u\Phi^{c\dagger}(V_2Q_L) + h.c.\\ &= \overline{e}_R(U_1^\dagger\Lambda_e V_1)\Phi^\dagger E_L - \overline{d}_R(U_3^\dagger\Lambda_d V_2)\Phi^\dagger Q_L - \overline{u}_R(U_2^\dagger\Lambda_u V_2)\Phi^{c\dagger}Q_L + h.c.\end{aligned} \tag{2-87}$$

It is clear that with this change of basis the following rules of transformation for the coupling matrices are found [31]:

$$\Lambda_e \to U_1^\dagger\Lambda_e V_1 \tag{2-88}$$

$$\Lambda_u \to U_2^\dagger\Lambda_u V_2 \tag{2-89}$$

$$\Lambda_d \to U_3^\dagger\Lambda_d V_2 \tag{2-90}$$

Now, since Λ_e, Λ_d and Λ_u are complex matrices, then $\Lambda_e\Lambda_e^\dagger$, $\Lambda_u\Lambda_u^\dagger$ and $\Lambda_d\Lambda_d^\dagger$ will be positive-definite Hermitian matrices, so they can be diagonalized. Let uss see:

- For $\Lambda_e\Lambda_e^\dagger$ it is satisfied that under the change of basis:

$$\begin{aligned}(\Lambda_e\Lambda_e^\dagger)' &= \Lambda'_e\Lambda'^\dagger_e\\ &= (U_1^\dagger\Lambda_e V_1)(U_1^\dagger\Lambda_e V_1)^\dagger\\ &= U_1^\dagger\Lambda_e\Lambda_e^\dagger U_1\end{aligned} \tag{2-91}$$

  Where it has been applied the unitary nature of the V_1 matrix.

  Therefore, by a suitable choice of U_1, it is possible to diagonalize this term:

$$U_1^\dagger\Lambda_e\Lambda_e^\dagger U_1 = \begin{pmatrix}\lambda_e^2 & 0 & 0\\ 0 & \lambda_\mu^2 & 0\\ 0 & 0 & \lambda_\tau^2\end{pmatrix} \tag{2-92}$$

  With λ_e, λ_μ, $\lambda_\tau \geq 0$.

Then, $U_1^\dagger \Lambda_e$ can be written as:

$$U_1^\dagger \Lambda_e = \begin{pmatrix} \lambda_e & 0 & 0 \\ 0 & \lambda_\mu & 0 \\ 0 & 0 & \lambda_\tau \end{pmatrix} W_1 \tag{2-93}$$

With $W_1 \in U(3)$. In particular, the condition $W_1 = V_1^\dagger$ can be set, then:

$$\begin{aligned} U_1^\dagger \Lambda_e &= \begin{pmatrix} \lambda_e & 0 & 0 \\ 0 & \lambda_\mu & 0 \\ 0 & 0 & \lambda_\tau \end{pmatrix} V_1^\dagger \\ U_1^\dagger \Lambda_e V_1 &= \begin{pmatrix} \lambda_e & 0 & 0 \\ 0 & \lambda_\mu & 0 \\ 0 & 0 & \lambda_\tau \end{pmatrix} \end{aligned} \tag{2-94}$$

Thus, by suitable choices of these matrices $U_1, V_1 \in U(3)$, it is possible to "absorb" the complex phases of the matrix Λ_e and just to establish that [31]:

$$\Lambda_e = \begin{pmatrix} \lambda_e & 0 & 0 \\ 0 & \lambda_\mu & 0 \\ 0 & 0 & \lambda_\tau \end{pmatrix} \tag{2-95}$$

- For $\Lambda_u \Lambda_u^\dagger$ it can be made a totally analogous procedure as above, and therefore, by a convenient choice of the matrices $U_2, V_2 \in U(3)$, it is possible to diagonalize and absorbe the complex phases in the matrix Λ_u [31]:

$$\Lambda_u = \begin{pmatrix} \lambda_u & 0 & 0 \\ 0 & \lambda_c & 0 \\ 0 & 0 & \lambda_t \end{pmatrix} \tag{2-96}$$

With λ_u, λ_c, $\lambda_t \geq 0$.

- On the other hand, for the remaining matrix Λ_d it is not possible to perform again the procedure above [31]. Let us see how to transform the term $\Lambda_d \Lambda_d^\dagger$:

$$\begin{aligned} (\Lambda_d \Lambda_d^\dagger)' &= \Lambda_d' \Lambda_d'^\dagger \\ &= (U_3^\dagger \Lambda_d V_2)(U_3^\dagger \Lambda_d V_2)^\dagger \\ &= U_3^\dagger \Lambda_d \Lambda_d^\dagger U_3 \end{aligned} \tag{2-97}$$

Then, by a suitable choice of $U_3 \in U(3)$ it is found that:

$$U_3^\dagger \Lambda_d \Lambda_d^\dagger U_3 = \begin{pmatrix} \lambda_d^2 & 0 & 0 \\ 0 & \lambda_s^2 & 0 \\ 0 & 0 & \lambda_b^2 \end{pmatrix} \tag{2-98}$$

And consequently:

$$U_3^\dagger \Lambda_d = \begin{pmatrix} \lambda_d & 0 & 0 \\ 0 & \lambda_s & 0 \\ 0 & 0 & \lambda_b \end{pmatrix} W_3 \tag{2-99}$$

With λ_d, λ_s, $\lambda_b \geq 0$, and $W_3 \in U(3)$.

Nevertheless, $W_3 \neq V_2^\dagger$ because the condition $V_2 = W_2^\dagger$ had been already set. Then, this complex term (matrix) remains, and in consequence:

$$U_3^\dagger \Lambda_d V_2 = \begin{pmatrix} \lambda_d & 0 & 0 \\ 0 & \lambda_s & 0 \\ 0 & 0 & \lambda_b \end{pmatrix} W_3 V_2 \tag{2-100}$$

Therefore, the matrix Λ_d finally can be written as [31]:

$$\Lambda_d = V \begin{pmatrix} \lambda_d & 0 & 0 \\ 0 & \lambda_s & 0 \\ 0 & 0 & \lambda_b \end{pmatrix} V^\dagger \tag{2-101}$$

Where the matrix $V \in U(3)$ and is known as the **Cabibbo-Kobayashi-Maskawa (CKM) matrix** [4, 5].

Now, in particular notice that the diagonalized form of Λ_u and Λ_d remains unchanged under the change of basis only if the following is assumed:

$$\begin{aligned} V_2 &= U_2 = U_3 \\ &= U_\varphi = \begin{pmatrix} e^{i\varphi_1} & 0 & 0 \\ 0 & e^{i\varphi_2} & 0 \\ 0 & 0 & e^{i\varphi_3} \end{pmatrix} \in U(3) \end{aligned} \tag{2-102}$$

With φ_1, φ_2 and φ_3 arbitrary phases.
Let us see explicitly:

$$\Lambda_u = \begin{pmatrix} \lambda_u & 0 & 0 \\ 0 & \lambda_c & 0 \\ 0 & 0 & \lambda_t \end{pmatrix} \to U_\varphi^\dagger \lambda_u U_\varphi = \Lambda_u \tag{2-103}$$

$$\Lambda_d = V \begin{pmatrix} \lambda_d & 0 & 0 \\ 0 & \lambda_s & 0 \\ 0 & 0 & \lambda_b \end{pmatrix} V^\dagger \to U_\varphi^\dagger V \lambda_d V^\dagger U_\varphi = V' \Lambda_u V'^\dagger \tag{2-104}$$

Where $V' = U_\varphi^\dagger V$. Furthermore, the form of this matrix Λ_d alson remains unchanged if V multiplies from the right with another unitary diagonal matrix U_χ, i.e.:

$$V \to V U_\chi \tag{2-105}$$

With $U_\chi \in U(3)$, which depends on three arbitrary phases χ_1, χ_2 and χ_3:

$$U_\chi = \begin{pmatrix} e^{i\varphi_1} & 0 & 0 \\ 0 & e^{i\varphi_2} & 0 \\ 0 & 0 & e^{i\varphi_3} \end{pmatrix} \tag{2-106}$$

Thus [31]:

$$\Lambda_d = V \begin{pmatrix} \lambda_d & 0 & 0 \\ 0 & \lambda_s & 0 \\ 0 & 0 & \lambda_b \end{pmatrix} V^\dagger \to U_\varphi^\dagger V U_\chi \lambda_d U_\chi^\dagger V^\dagger U_\varphi = V'' \Lambda_u V''^\dagger \tag{2-107}$$

In other words, the CKM matrix can be rewritten as $V'' = V_{CKM} = U_\varphi^\dagger V U_\chi$.
Now, from the results in Appendix A, it was found that the mass terms, after implementing Higgs mechanism, for fermions can finally be written as [31]:

$$\mathcal{L}_m^f = -\left[\begin{pmatrix} \overline{e} & \overline{\mu} & \overline{\tau} \end{pmatrix} \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix} + \begin{pmatrix} \overline{u} & \overline{c} & \overline{t} \end{pmatrix} \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix} \begin{pmatrix} u \\ c \\ t \end{pmatrix} \right.$$
$$\left. + \begin{pmatrix} \overline{d}' & \overline{s}' & \overline{b}' \end{pmatrix} V \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} V^\dagger \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} \right] \left(1 + \frac{H}{v}\right)$$

Where d', s' and b' are flavour mixed states. The mass eigenstates for the down-type quarks are given through the CKM matrix Halzen:1984mc:

$$\begin{pmatrix} d \\ s \\ b \end{pmatrix} = V^\dagger \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} \tag{2-108}$$

Which components are commonly labeled as Halzen:1984mc:

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{2-109}$$

In order to specify each one of the possible mixings between up and down-type quarks from the same or different generations.

2.4. CKM matrix parameterizations

In the last section it was found that the CKM matrix V is a 3×3 unitary matrix which can be parameterized by just 4 free real parameters. In this section it is pretended to present the most common of the possible parameterizations which are of great relevance in the study of CP violation in the Standard Model.

2.4.1. Kobayashi-Maskawa parameterization

The first parameterization made for this quark mixing matrix was the presented by Kobayashi and Maskawa in their paper [5] in 1973, also known as the canonical one. Inspired by the previous work done by Cabibbo [4], they introduced three mixing angles defined into the first quadrant and a complex phase, writing the CKM matrix as [5]:

$$V = \begin{pmatrix} c_1 & s_1 c_3 & s_1 s_3 \\ -s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} & c_1 c_2 s_3 + s_2 c_3 e^{i\delta} \\ -s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 e^{i\delta} & c_1 s_2 s_3 - c_2 c_3 e^{i\delta} \end{pmatrix} \tag{2-110}$$

With:

$$c_i \equiv \cos\theta_i \tag{2-111}$$

$$s_i \equiv \sin\theta_i \tag{2-112}$$

For $i, j = 1, 2, 3$, and:

$$0 \leq \theta_i \leq \frac{\pi}{2} \tag{2-113}$$

$$0 \leq \delta \leq 2\pi \tag{2-114}$$

2.4.2. Standard Parameterization

If the CKM matrix V was purely real, i.e., if V∈O(3) was satisfied, it is well-known that it could be parameterized in terms of the three Euler angles. Nevertheless, V has complex components which have to be included too. Despite this, it is possible to make use of these Euler angles, in the following form: Let us write V as the product of three real matrices which correspond to rotations in a 3-dimensional space (flavour space), and therefore these matrices are written in terms of Euler angles. The complex part will be include through the multiplication of the above real part with pure complex matrices written in terms of the remaining parameter, i.e., the CKM matrix can be written as the product of successive rotations in the 3-dimensional real flavor space and rotations in a complex plane, thus [41]:

$$V = R_1 I_2^{-1} R_2 I_2 R_3 \tag{2-115}$$

Where R_1 will represent a rotation a certain angle θ_{23} with respect to the axis 1, and analogously with R_2 y R_3; I_2 would be a rotation by a phase $\delta_{13}/2$ in a complex plane formed by the axis 1 and 3.
Explicitly [41]:

$$
\begin{aligned}
V &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & C_{23} & S_{23} \\ 0 & -S_{23} & C_{23} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\delta_{13}}{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\frac{\delta_{13}}{2}} \end{pmatrix} \begin{pmatrix} C_{13} & 0 & S_{13} \\ 0 & 1 & 0 \\ -S_{13} & 0 & C_{13} \end{pmatrix} \begin{pmatrix} e^{i\frac{\delta_{13}}{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\frac{\delta_{13}}{2}} \end{pmatrix} \begin{pmatrix} C_{12} & S_{12} & 0 \\ -S_{12} & C_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & C_{23} & S_{23} \\ 0 & -S_{23} & C_{23} \end{pmatrix} \begin{pmatrix} C_{13} & 0 & S_{13}e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -S_{13}e^{i\delta_{13}} & 0 & C_{13} \end{pmatrix} \begin{pmatrix} C_{12} & S_{12} & 0 \\ -S_{12} & C_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} C_{12}C_{13} & S_{12}C_{13} & S_{13}e^{-i\delta_{13}} \\ -S_{12}C_{23} - C_{12}S_{23}S_{13}e^{i\delta_{13}} & C_{12}C_{23} - S_{12}S_{23}S_{13}e^{i\delta_{13}} & S_{23}C_{13} \\ S_{12}S_{23} - C_{12}C_{23}S_{13}e^{i\delta_{13}} & -C_{12}S_{23} - S_{12}C_{23}S_{13}e^{i\delta_{13}} & C_{23}C_{13} \end{pmatrix}
\end{aligned} \tag{2-116}
$$

Where $C_{ij} = \cos\theta_{ij}$ and $S_{ij} = \sin\theta_{ij}$, with θ_{12}, θ_{13} and θ_{23} the Euler angles, and δ_{13} a complex phase. These angles θ_{ij} are chosen to lie in the first quadrant, i.e.:

$$0 \leq \theta_{ij} \leq \frac{\pi}{2} \tag{2-117}$$

And therefore:

$$S_{ij}, C_{ij} \geq 0 \tag{2-118}$$

This parameterization is known as the *Standard Parameterization* and it is appropriate because of the indices i, j which refer to each on of the three generations. For example, notice that in particular, if $\theta_{23} = \theta_{13} = 0$, i.e., if there is not quark mixing between the first two generations and the third one, it is found that:

$$V_c = \begin{pmatrix} C_{12} & S_{12} & 0 \\ -S_{12} & C_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2-119}$$

Which is the Cabbibo mixing matrix for the first two generations [4], with $\theta_{12} = \theta_c$ the Cabbibo angle. This result was to be expected.

2.4.3. Wolfenstein parameterization and the unitary triangle

Another relevant parameterization is the so-called *Wolfenstein parameterization* [42], which is motivated by the experimental data. From the Particle Data Group, and neglecting the uncertainty, the absolute values of the CKM matrix are [43]:

$$\begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \approx \begin{pmatrix} 0{,}97401 & 0{,}22650 & 0{,}00361 \\ 0{,}22636 & 0{,}97320 & 0{,}04053 \\ 0{,}00854 & 0{,}03978 & 0{,}999172 \end{pmatrix} \tag{2-120}$$

It is clear that the diagonal elements are closer to 1 than the off-diagonal ones. This is common to said that they are "suppressed" with respect to the diagonal. Therefore, it is convenient to parameterize in such a way to highlight this pattern. Then, let us introduce a new set of parameters λ, A, ρ and η in the following way [42]:

$$\sin\theta_c = \sin\theta_{12} \equiv \lambda \tag{2-121}$$

$$\sin\theta_{23} \equiv A\lambda^2 \tag{2-122}$$

$$\sin\theta_{13}e^{i\delta_{13}} \equiv A\lambda^3(\rho + i\eta) = A\lambda^3\sqrt{\rho^2+\eta^2}e^{i\delta_{13}} \tag{2-123}$$

With the experimental known hierarchy:

$$\left|A\lambda^3(\rho+i\eta)\right| << \left|A\lambda^2\right| << |\lambda| < 1 \tag{2-124}$$

Experimentally, the Particle Data Group has reported the values for this parameter to be [43]:

$$\lambda \approx 0{,}225 \tag{2-125}$$

$$A \approx 0{,}81 \tag{2-126}$$

$$\rho \approx 0{,}14 \tag{2-127}$$

$$\eta \approx 0{,}35 \tag{2-128}$$

Therefore, with this new parameterization the higher order elements to λ^3 are neglected because they would not appear in the elements of the CKM matrix and would therefore be strongly suppressed.
Let us apply this Wolfenstein parameterization in the Standard CKM matrix term by term:

- V_{ud}:

$$\begin{aligned} C_{12}C_{13} &= \sqrt{1-S_{12}^2}\sqrt{1-S_{13}^2} \\ &= \sqrt{1-\lambda^2}\sqrt{1-A^2\lambda^6(\rho^2+\eta^2)} \\ &\approx \left(1-\frac{1}{2}\lambda^2+\cdots\right)\left(1-\frac{1}{2}A^2\lambda^6(\rho^2+\eta^2)+\cdots\right) \\ &\approx 1-\frac{\lambda^2}{2}+\mathcal{O}(\lambda^6) \end{aligned} \tag{2-129}$$

- V_{us}:

$$\begin{aligned} S_{12}C_{13} &= \lambda\sqrt{1-A^2\lambda^6(\rho^2+\eta^2)} \\ &\approx \lambda\left(1-\frac{1}{2}A^2\lambda^6(\rho^2+\eta^2)+\cdots\right) \\ &\approx \lambda+\mathcal{O}(\lambda^7) \end{aligned} \tag{2-130}$$

- V_{ub}:

$$S_{13}e^{-i\delta_{13}} = A\lambda^3(\rho - i\eta) \tag{2-131}$$

- V_{cd}:

$$\begin{aligned} -S_{12}C_{23} - C_{12}S_{23}S_{13}e^{i\delta_{13}} &= -\lambda\sqrt{1-A^2\lambda^4} - \sqrt{1-\lambda^2}A\lambda^2 A\lambda^3(\rho + i\eta) \\ &\approx -\lambda\left(1 - \frac{1}{2}A^2\lambda^4 + \cdots\right) + \mathcal{O}(\lambda^6) \\ &\approx -\lambda + \mathcal{O}(\lambda^5) \end{aligned} \tag{2-132}$$

- V_{cs}:

$$\begin{aligned} C_{12}C_{23} - S_{12}S_{23}S_{13}e^{i\delta_{13}} &= \sqrt{1-\lambda^2}\sqrt{1-A^2\lambda^4} - \lambda A\lambda^2 A\lambda^3(\rho + i\eta) \\ &\approx \left(1 - \frac{1}{2}\lambda^2 + \cdots\right)\left(1 - \frac{1}{2}A^2\lambda^4 + \cdots\right) + \mathcal{O}(\lambda^6) \\ &\approx 1 - \frac{\lambda^2}{2} + \mathcal{O}(\lambda^6) \end{aligned} \tag{2-133}$$

- V_{cb}:

$$\begin{aligned} S_{23}C_{13} &= A\lambda^2\sqrt{1 - A^2\lambda^6(\rho^2+\eta^2)} \\ &\approx A\lambda^2\left(1 - \frac{1}{2}A^2\lambda^6(\rho^2+\eta^2) + \cdots\right) \\ &\approx A\lambda^2 + \mathcal{O}(\lambda^8) \end{aligned} \tag{2-134}$$

- V_{td}:

$$\begin{aligned} S_{12}S_{23} - C_{12}C_{23}S_{13}e^{i\delta_{13}} &= \lambda A\lambda^2 - \sqrt{1-\lambda^2}\sqrt{1-A^2\lambda^4}A\lambda^3(\rho + i\eta) \\ &\approx A\lambda^3 - \left(1 - \frac{1}{2}\lambda^2 + \cdots\right)\left(1 - \frac{1}{2}A^2\lambda^4 + \cdots\right)A\lambda^3(\rho + i\eta) \\ &\approx A\lambda^3 - A\lambda^3(\rho + i\eta) + \mathcal{O}(\lambda^9) \\ &\approx A\lambda^3(1 - \rho - i\eta) \end{aligned} \tag{2-135}$$

- V_{ts}:

$$\begin{aligned} -C_{12}S_{23} - S_{12}C_{23}S_{13}e^{i\delta_{13}} &= -\sqrt{1-\lambda^2}A\lambda^2 - \lambda\sqrt{1-A^2\lambda^4}A\lambda^3(\rho + i\eta) \\ &\approx -\left(1 - \frac{1}{2}\lambda^2 + \cdots\right)A\lambda^2 + \mathcal{O}(\lambda^6) \\ &\approx -A\lambda^2 + \mathcal{O}(\lambda^4) \end{aligned} \tag{2-136}$$

- $V_{tb} \approx 1$.

Therefore, the CKM matriz can be written as [42]:

$$V = \begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4) \tag{2-137}$$

Which is the explicit form for the Wolfenstein parameterization. This is actually an approximation made as an expansion in λ, being this the most remarkable parameter.
On the other hand, recall that because of the unitary of the CKM matrix, there are imposed nine conditions over its components:

$$V^\dagger V = I \to V^*_{ik}V_{ij} = \delta_{jk} \tag{2-138}$$

$$VV^\dagger = I \to V_{ji}V^*_{ki} = \delta_{jk} \tag{2-139}$$

Then, for the off-diagonal elements ($j \neq k$), the above conditions give the following six equations:

$$V^*_{ik}V_{ij} = 0 \tag{2-140}$$

$$V_{ji}V^*_{ki} = 0 \tag{2-141}$$

Each of one of these six conditions are the sum of complex numbers vanishing, therefore it is possible to represent them as a unitary triangle in the complex plane [31].
Let us see it explicitly. In particular, one of the six possibilities is the most commonly used relation, which is the following:

$$V_{ud}V^*_{ub} + V_{cd}V^*_{cb} + V_{td}V^*_{tb} = 0 \tag{2-142}$$

And this is owing to the appereance of the element V_{cb} which is especially important in the study of B meson decays. Furthermore, this term is of order λ^3:

- $$\begin{aligned} V_{ud}V^*_{ub} &= \left(1-\frac{\lambda^2}{2}\right) A\lambda^3(\rho + i\eta) \\ &\approx A\lambda^3(\rho + i\eta) + \mathcal{O}(\lambda^4) \end{aligned} \tag{2-143}$$

- $$\begin{aligned} V_{cd}V^*_{cb} &= -\lambda A\lambda^2 \\ &= -A\lambda^3 \end{aligned} \tag{2-144}$$

- $$\begin{aligned} V_{td}V^*_{tb} &= A\lambda^3(1-\rho-i\eta)(1) \\ &= A\lambda^3(1-\rho-i\eta) \end{aligned} \tag{2-145}$$

Now, as a convention, the lenght of the sides of the unitary triangle in (2-142) are normalized to the middle term $V_{cd}V_{cb}^*$, thus [31]:

$$1 + \frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*} + \frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*} = 0 \tag{2-146}$$

The sketch of this expression found above is shown in the figure **2-2**.

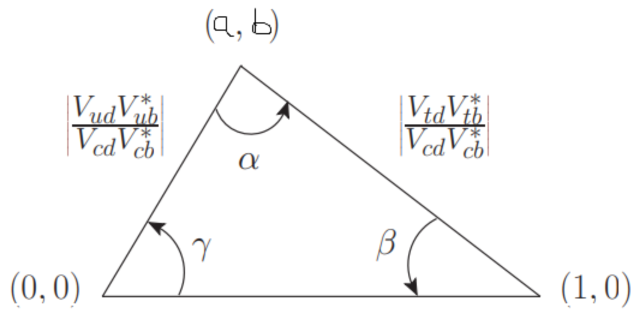

Figure 2-2.: Unitary triangle [31].

With [31]:

$$\begin{aligned} a + ib &= 1 + \frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*} \\ &= -\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*} \end{aligned} \tag{2-147}$$

Where it was used the equation (2-146).
Thus [31]:

$$\begin{aligned} a = \mathrm{Re}\left(-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right) &\approx \mathrm{Re}\left(\frac{-A\lambda^3(\rho + i\eta)}{-A\lambda^3}\right) \\ &= \mathrm{Re}(\rho + i\eta) \\ &= \rho \end{aligned} \tag{2-148}$$

And [31]:

$$\begin{aligned} b = \mathrm{Im}\left(-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right) &\approx \mathrm{Im}\left(\frac{-A\lambda^3(\rho + i\eta)}{-A\lambda^3}\right) \\ &= \mathrm{Im}(\rho + i\eta) \\ &= \eta \end{aligned} \tag{2-149}$$

The angles between sides of this unitary angle will be [31]:

- $$\gamma = \arg\left(\frac{-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}}{1}\right) = \arg\left(-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right) \tag{2-150}$$

- $$\beta = \arg\left(\frac{-1}{\frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*}}\right) = \arg\left(-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right) \tag{2-151}$$

- $$\alpha = \arg\left(\frac{-\frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*}}{\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}}\right) = \arg\left(-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}\right) \tag{2-152}$$

Now, another interesting quantity, which would be of relevance in the study of CP violation, is the area A of this unitary triangle. To compute it is simple:

$$\begin{aligned} A &= \frac{1}{2}(1)(b) \\ &= \frac{1}{2}\,\mathrm{Im}\left((1)\left(-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right)\right) \end{aligned} \tag{2-153}$$

And returning to the no-normalized expression:

$$\begin{aligned} A &= \frac{1}{2}\,\mathrm{Im}\left((1)(V_{cd}V_{cb}^*)\left(-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right)(V_{cd}V_{cb}^*)\right) \\ &= \frac{1}{2}\,\mathrm{Im}\left(-V_{ud}V_{ub}^*V_{cd}V_{cb}^*\right) \end{aligned} \tag{2-154}$$

This leads us to the definition of the **Jarlskog invariant** which is defined as [44]:

$$J = \mathrm{Im}(V_{ij}V_{kl}V_{il}^*V_{kj}^*); i \neq k, j \neq l \tag{2-155}$$

Or in another form [44]:

$$J \sum_{m,n=1}^{3} \epsilon_{ikm}\epsilon_{jln} = \mathrm{Im}(V_{ij}V_{kl}V_{il}^*V_{kj}^*) \tag{2-156}$$

Let us see why it is called invariant. Consider the particular case $i = j = 1$ and $k = l = 2$:

$$\begin{aligned} J &= \mathrm{Im}(V_{11}V_{22}V_{12}^*V_{21}^*) \\ &= \mathrm{Im}(V_{ud}V_{cs}V_{us}^*V_{cd}^*) \end{aligned} \tag{2-157}$$

Now, if a transformation of the following form is applied:

$$\begin{pmatrix} u \\ c \end{pmatrix} \to \begin{pmatrix} u' \\ c' \end{pmatrix} = e^{i\alpha_i}\begin{pmatrix} u \\ c \end{pmatrix} \tag{2-158}$$

$$\begin{pmatrix} d \\ s \end{pmatrix} \to \begin{pmatrix} d' \\ s' \end{pmatrix} = e^{-i\beta_i} \begin{pmatrix} d \\ s \end{pmatrix} \tag{2-159}$$

I.e., a general redefinition of the phases for the first two generations quarks is made. Then, the parameter J will be transformed as:

$$\begin{aligned} J' &= \mathrm{Im}(V'_{ud}V'_{cs}{V'_{us}}^{*}{V'_{cd}}^{*}) \\ &= \mathrm{Im}\left[(e^{i(\alpha_i-\beta_i)}V_{ud})(e^{i(\alpha_i-\beta_i)}V_{cs})(e^{-i(\alpha_i-\beta_i)}V^{*}_{us})(e^{-i(\alpha_i-\beta_i)}V^{*}_{cd})\right] \\ &= \mathrm{Im}(V_{ud}V_{cs}V^{*}_{us}V^{*}_{cd}) \\ &= J \end{aligned} \tag{2-160}$$

Then, J is an invariant under quark phases redefinitions.
Now, let us see the explicit form for this Jarlskog invariant defined in (2-157), hence writing the components using the standard parameterization for the CKM matrix (equation (2-116)), it is found that:

$$\begin{aligned} J &= \mathrm{Im}(V_{ud}V_{cs}V^{*}_{us}V^{*}_{cd}) \\ &= \mathrm{Im}\left[C_{12}C_{13}(C_{12}C_{23} - S_{12}S_{23}S_{13}e^{i\delta_{13}})(S_{12}S_{C_{13}})^{*}(-S_{12}C_{23} - C_{12}S_{23}S_{13}e^{i\delta_{13}})^{*}\right] \\ &= \mathrm{Im}\left(-C^2_{12}S^2_{12}C^2_{13}C^2_{23} - C^3_{12}S_{12}C^2_{13}S_{23}C_{23}S_{13}e^{-i\delta_{13}} + C_{12}S^3_{12}C^2_{13}S_{23}C_{23}S_{13}e^{i\delta_{13}} + C^2_{12}S^2_{12}C^2_{13}S^2_{23}S^2_{13}\right) \\ &= \mathrm{Im}\left(iC^3_{12}S_{12}C^2_{13}S_{23}C_{23}S_{13}\sin\delta_{13} + iC_{12}S^3_{12}C^2_{13}S_{23}C_{23}S_{13}\sin\delta_{13}\right) \\ &= C^3_{12}S_{12}C^2_{13}S_{23}C_{23}S_{13}\sin\delta_{13} + C_{12}S^3_{12}C^2_{13}S_{23}C_{23}S_{13}\sin\delta_{13} \\ &= C_{12}C^2_{13}C_{23}S_{12}S_{13}S_{23}\sin\delta_{13}(C^2_{12} + S^2_{12}) \\ &= C_{12}C^2_{13}C_{23}S_{12}S_{13}S_{23}\sin\delta_{13} \end{aligned} \tag{2-161}$$

On the other hand, to see the order of this Jarlskog invariant in the λ parameter it is necessary to use the Wolfenstein parameterization. For this purpose it is convenient to use another unitary triangle, in particular by making $i = 2, j = 1, k = 3$ and $l = 3$ in (2-156):

$$\begin{aligned} J\epsilon_{231}\epsilon_{132} &= \mathrm{Im}(V_{21}V_{33}V^{*}_{23}V^{*}_{31}) \\ J &= -\,\mathrm{Im}(V_{cd}V_{tb}V^{*}_{cb}V^{*}_{tu}) \\ &= -\,\mathrm{Im}\left[(-\lambda)(1)(A\lambda^2)(A\lambda^3(1-\rho+i\eta))\right] \\ &= A\lambda^6\eta \end{aligned} \tag{2-162}$$

And from the reported experimental values (2-125)-(2-128), the approximated experimental value for this J would be:

$$J \approx (0{,}81)(0{,}225)^6(0{,}35) = 3{,}6783 \times 10^{-5} \tag{2-163}$$

And in general, for the other cases $J \sim \mathcal{O}(10^{-5})$. This is one of the reasons why it is very difficult to make experiments to measure the CP violation.

3. Neutrino mass generation and CP violation in the lepton sector

3.1. Neutrino mass terms

Based on the Higgs mechanism and Yukawa lagrangian, it is plausible to suggest a mass term for neutrinos in the following form [45]:

$$\begin{aligned}\mathcal{L}^{\nu}_{Yukawa} &= \lambda_\nu \bar{E}_L \Phi^c \nu_R \\ &= -\lambda_\nu \bar{E}_L \phi^c \nu_R - \lambda_\nu^\dagger \bar{\nu}_R \Phi^\dagger E_L\end{aligned} \tag{3-1}$$

Then, after the electroweak spontaneous symmetry breaking it will lead to [45]:

$$\begin{aligned}\mathcal{L}^{\nu}_{Yukawa} &= -\frac{\lambda_\nu}{\sqrt{2}}(\bar{\nu}_L, \bar{e}_L)\begin{pmatrix} v+H \\ 0\end{pmatrix}\nu_R - \frac{\lambda_\nu^\dagger}{\sqrt{2}}\bar{\nu}_R(v+H,0)\begin{pmatrix}\nu_L \\ e_L\end{pmatrix} \\ &= -\frac{\lambda_n u}{\sqrt{2}}\bar{\nu}_L(v+h)\nu_R - \frac{\lambda_n u^\dagger}{\sqrt{2}}\bar{\nu}_R(v+H)\nu_L \\ &= -\frac{\lambda_\nu}{\sqrt{2}}(v+H)(\bar{\nu}_L\nu_R + \bar{\nu}_R\nu_L) \text{ with } \lambda_\nu > 0 \text{ due to } \lambda_\nu \to e^{-i\phi}\lambda_n u; \quad \lambda_\nu \to e^{i\phi}_n u \\ &= -m_\nu \bar{\nu}\nu - \frac{\lambda_\nu}{\sqrt{2}}H\bar{\nu}\nu \quad ; m_\nu = \frac{\lambda_\nu}{\sqrt{2}}v\end{aligned} \tag{3-2}$$

Therefore, because of (2-9), this would be a Dirac mass term. However, notice that, in order to preserve the $U(1)_Y$ invariance, which is the symmetry involved here, the Yukawa term introduced in (3-1) must have null hypercharge [31]:

$$Y_{\bar{E}_L} + Y_{\Phi^c} + Y_{\nu_R} = 0 \tag{3-3}$$

$$\frac{1}{2} + \left(-\frac{1}{2}\right) + Y_{\nu_R} = 0 \Rightarrow Y_{\nu_R} = 0 \tag{3-4}$$

In consequence, the right-handed ν_R carries no Standard Model gauge charge, that is why it is often called to be "**sterile**". Despite this fact, these neutrinos would have mass and then only couple weakly to the Higgs. This scenario is known as a minimal Dirac extension to the Standard Model.

One important issue is: Mass hierarchy proble more severe. This is because $m_\nu \sim 10^{-6} m_e$ [6], then $\lambda_\nu \sim 10^{-6}\lambda_e$, and it is known that $\lambda_e \sim 10^{-4}\lambda_\tau$. Hence:

$$\lambda_\nu \sim 10^{-10}\lambda_\tau \tag{3-5}$$

Which is a huge difference in orders of magnitude.
Furthermore, according to interactions known up to now, there would be no way of detecting directly these sterile neutrinos [6].
Recall that a Dirac spinor can be written in terms of Weyl components as [30]:

$$\Psi_D = \begin{pmatrix} \chi_L \\ G_R \end{pmatrix} \tag{3-6}$$

With $\chi_L \in (1/2, 0)$ and $G_R \in (0, 1/2)$. Thus, $\Psi \in (1/2, 0) \bigoplus (0, 1/2)$.
In equation (2-8) it had been already shown that:

$$\bar{\Psi}_D \Psi_D = (\chi_L)^\dagger G_R + (G_R)^\dagger \chi_L \tag{3-7}$$

Therefore, a Dirac mass term can be written as:

$$\mathcal{L}^D_{mass} = -m\bar{\Psi}_D\Psi_D = -m\left((\chi_L)^\dagger G_R + (G_R)^\dagger \chi_L\right) \tag{3-8}$$

Thus, if the neutrino field is a spin$-\frac{1}{2}$ Dirac field, then a mass term may be generated in the Standard Model by a Yukawa term of the form [45]:

$$\mathcal{L}^{\nu_i}_{Yukawa} = -\lambda_{\nu_i}\bar{E}_{i,L}\Phi^c\nu_{i,R} + h.c. \tag{3-9}$$

Where $i = e, \mu, \tau$ is the flavour index.
Thus, after the spontaneous electroweak symmetry breaking a Dirac mass term is generated [45]:

$$\mathcal{L}^D_{mass} = -m_{\nu_i}(\bar{\nu}_{i,L}\nu_{i,R} + \bar{\nu}_{i,R}\nu_{i,L}) \tag{3-10}$$

With $m_{\nu_i} = \lambda_{\nu_i}\frac{v}{\sqrt{2}}$.

3.2. Majorana mass terms:

A majorana particle field (spin$-\frac{1}{2}$) is its own antiparticle, which means that [45]:

$$C\Psi_m u(x) = \Psi_\mu(x) \Leftrightarrow \Psi^c_\mu(x) = \Psi_\mu(x) \tag{3-11}$$

And can be written using a single Weyl bispinor [45]:

$$\Psi_\mu = \begin{pmatrix} \chi \\ i\sigma_2\chi^* \end{pmatrix} \tag{3-12}$$

Indeed, recalling that the charge-conjugation operation of spinors is [45]:

$$
\begin{aligned}
\Psi^c_\mu(x) &= i\gamma^2\Psi^*_\mu(x) \\
&= -i\gamma^2\begin{pmatrix}\chi \\ i\sigma_2\chi^*\end{pmatrix}^* \quad ; \quad \text{Applying the ansatz proposed in (3-12)} \\
&= -i\begin{pmatrix}0 & \sigma_2 \\ -\sigma_2 & 0\end{pmatrix}\begin{pmatrix}\chi^* \\ -i\sigma_2^*\chi\end{pmatrix} \\
&= \begin{pmatrix}0 & -i\sigma_2 \\ i\sigma_2 & 0\end{pmatrix}\begin{pmatrix}\chi^* \\ i\sigma_2\chi\end{pmatrix} \quad ; \quad \sigma_2^* = -\sigma_2 \\
&= \begin{pmatrix}-i(\sigma_2)^2\chi \\ i\sigma_2\chi^*\end{pmatrix} = \begin{pmatrix}-(-1)\chi \\ i\sigma_2\chi^*\end{pmatrix} \quad ; \quad (\sigma_2)^2 = 1_{2\times 2}
\end{aligned}
\tag{3-13}
$$

$$
\Rightarrow \Psi^c_\mu(x) = \begin{pmatrix}\chi \\ i\sigma_2\chi^*\end{pmatrix} = \Psi_\mu(x) \tag{3-14}
$$

Thus, a mass term for a Majorana fermion would be:

$$
\begin{aligned}
\mathcal{L}^\mu_{mass} &= -m\bar{\psi}_\mu\Psi_\mu \\
&= -m((\chi)^\dagger(i\sigma_2\chi^*) + (i\sigma_2\chi^*)^\dagger\chi) \quad ; \quad \text{Lorentz's invariant} \\
&= -m(\chi^\dagger i\sigma_2\chi^* - i\chi^T\sigma_2\chi) \quad ; \quad \sigma_2^\dagger = \sigma_2
\end{aligned}
\tag{3-15}
$$

Therefore, if the neutrino field is a spin$-\frac{1}{2}$ Majorana field, it is not necessary to introduce additional degrees of freedom like right-handed neutrinos.

Consequently, the neutrino field has the following form [45]:

$$
\nu_{\mu,i} = \begin{pmatrix}\nu_{i,L} \\ i\sigma_2\nu^*_{i,L}\end{pmatrix} \quad ; \quad \text{where } i = e, \mu, \tau \text{ the flavour index} \tag{3-16}
$$

With:

$$
\nu^c_{\mu,i} = .i\gamma^2\nu^*_{\mu,i} = \nu_{\mu,i} \tag{3-17}
$$

And then:

$$
\bar{\nu}_{\mu,i} = \nu^\dagger_{\mu,i}\gamma^0 = (\nu^\dagger_{i,L} - i\nu^\dagger_{i,L}\sigma_2)\gamma^0 \quad ; \quad (\nu^*_{i,L})^\dagger = \nu^T_{i,L} \tag{3-18}
$$

Thus, the Majorana mass term for the neutrinos can be written as [45]:

$$\begin{aligned}
\mathcal{L}^{\mu}_{mass} &= -\frac{m_{\nu_i}}{2}\bar{\nu}_{\mu,i}\nu_{\mu,i} \\
&= \frac{m_{\nu_i}}{2}\nu^{\dagger}_{\mu,i}\gamma^{0}\nu_{\mu,i} \\
&= -\frac{m_{\nu_i}}{2}(\nu^{\dagger}_{i,L} - i\nu^{T}_{i,L}\sigma_2)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\nu_{i,L}\\ i\sigma_2\nu^{*}_{i,L}\end{pmatrix} \\
&= -\frac{m_{\nu_i}}{2}(\nu^{\dagger}_{i,L} - i\nu^{T}_{i,L}\sigma_2)\begin{pmatrix}i\sigma_2\nu^{*}_{i,L}\\ \nu_{i,L}\end{pmatrix} \\
&= -\frac{m_{\nu_i}}{2}\left[(\nu^{\dagger}_{i,L}i\sigma_2\nu^{*}_{i,L}) - (i\nu^{T}_{i,L}\sigma_2\nu_{i,L})\right] \\
&= -\frac{m_{\nu_i}}{2}\left[(\nu^{\dagger}_{i,L})(i\sigma_2\nu^{*}_{i,L}) + (-\nu^{T}_{i,L}i\sigma_2)(\nu_{i,L})\right]
\end{aligned} \tag{3-19}$$

Now, it is desirable to redefine:

$$\nu_{i,L} \equiv \begin{pmatrix}\nu_{i,L}\\ 0\end{pmatrix} \quad , \quad \bar{\nu}_{i,L}(0, \nu^{\dagger}_{i,L}) \tag{3-20}$$

$$\nu^{c}_{i,L} \equiv \begin{pmatrix}0\\ i\sigma_2\nu^{*}_{i,L}\end{pmatrix} \quad , \quad \bar{\nu}^{c}_{i,L}(-\nu^{T}_{i,L}i\sigma_2, 0) \tag{3-21}$$

Where $\nu^{c}_{i,L}$ is a right-handed field.
Therefore, the involved terms can be rewritten as:

$$(\nu^{\dagger}_{i,L})(i\sigma_2\nu^{*}_{i,L}) = (0, \nu^{\dagger}_{i,L})\begin{pmatrix}0\\ i\sigma_2\nu^{*}_{i,L}\end{pmatrix} = \bar{\nu}_{i,L}\nu^{c}_{i,L} \tag{3-22}$$

$$(-\nu^{T}_{i,L}i\sigma_2)(\nu_{i,L}) = (-\nu^{T}_{i,L}i\sigma_2, 0)\begin{pmatrix}\nu_{i,L}\\ 0\end{pmatrix} = \bar{\nu}^{c}_{i,L}\nu_{i,L} \tag{3-23}$$

And the Majorana mass term will be [45]:

$$\begin{aligned}
\mathcal{L}^{M}_{mass} &= -\frac{m_{\nu_i}}{2}(\bar{\nu}_{i,L}\nu^{c}_{i,L} + \bar{\nu}^{c}_{i,L}\nu_{i,L}) \\
&= -\frac{m_{\nu_i}}{2}(\bar{\nu}_{i,L}\nu^{c}_{i,L} + h.c.)
\end{aligned} \tag{3-24}$$

On the other hand, a Majorana spinor can also be written with right-handed two-component Weyl bispinors [45]:

$$\nu_{\mu,i} = \begin{pmatrix}-i\sigma\nu^{*}_{i,R}\\ \nu_{i,R}\end{pmatrix} \tag{3-25}$$

From which it is possible to construct a Majorana mass term out of right-handed fields only [45]:

$$\mathcal{L}^{M}_{mass} = -\frac{m_{\nu_i}}{2}(\bar{\nu}_{i,R}\nu^{c}_{i,R} + \bar{\nu}^{c}_{i,R}\nu_{i,R}) \tag{3-26}$$

Where the spinors in the above expression should interpreted as:

$$\nu_{i,R} \equiv \begin{pmatrix} 0 \\ \nu_{i,R} \end{pmatrix} \quad , \quad \bar{\nu}_{i,R} = (\nu_{i,R}^{\dagger}, 0) \tag{3-27}$$

$$\nu_{i,R}^{c} \equiv \begin{pmatrix} -i\sigma_2 \nu_{i,R}^{*} \\ 0 \end{pmatrix} \quad , \quad \bar{\nu}_{i,R}^{c} = (0, i\nu_{i,R}^{T}\sigma_2) \tag{3-28}$$

In this case, $\nu_{i,R}^{c}$ is a left-handed field.
From the above results, it is concluded then:

- For Majorana fermions, it is only necessary to write either left-handed fields or right-handed ones, but not both, and thus obtain a Lorentz invariant mass term.
- On the other side, for Dirac fermions it is necessary to write both components, left handed and right-handed, and it is possible to obtain a mass term which preserves the Standard Model symmetries (Lorentz and SU$(2)_L\times$SU$(1)_Y$ gauge symmetries).

It is important to remark that, due to neutrino oscillations, the neutrino have (small) mass [1]. Then, flavour states are no the physical states, and this leads to neutrino flavour mixing [2, 3].

3.3. Neutrino flavour mixing

Since neutrinos have mass, the real physical states are those mass eigenstates and not the flavour eigenstates. Thus, the Yukawa Lagrangian for the lepton sector is not written in terms of mass eigenstates [45]:

$$\begin{aligned} \mathcal{L}_Y^{leptons} &= -[\Lambda_e]_{ml}\overline{e}_{m,R}\Phi^{\dagger}E_{l,L} - [\Lambda_\nu]_{ml}\overline{\nu}_{m,R}\Phi^{c\dagger}E_{l,L} + h.c. \\ &= -\overline{e}_R\Lambda_e\Phi^{\dagger}E_L - \overline{\nu}_R\Lambda_\nu\Phi^{c\dagger}E_L + h.c. \end{aligned} \tag{3-29}$$

Where Λ_e y Λ_ν are respectively the 3×3 complex Yukawa's matrices for charged leptons and neutrinos, E_L and Φ are the $SU(2)_L \times U(1)_Y$ doublets for leptons and Higgs field respectively and, e_R and ν_R are the $SU(2)_L \times U(1)_Y$ singlets for charged leptons and neutrinos, respectively again. The latin indices $m, l = 1, 2, 3$, are the generation indices.
In a similar way to the quark sector studied in the chapter 2, in order to write the Yukawa Lagrangian in terms of the mass eigenstates, it is necessary to diagonalize these matrices. For that purpose, it is possible to apply a change of basis between flavour states to mass states through unitary transformations:

$$e_R \rightarrow e'_R = U_1 e_R \tag{3-30}$$

$$\nu_R \rightarrow \nu'_R = U_2 \nu_R \tag{3-31}$$

$$E_L \to E'_L = V_1 E_L \tag{3-32}$$

Being U_1, U_2 and V_1 3×3 unitary matrices.
In a similar way to the quark sector, there will be a flavour mixing for the neutrinos, and this is due to the remaining complex phases which can not be totally vanished for field redefinitions. This can be seen as follows: appplying the change of basis to the Yukawa Lagrangian (3-29), it is found the following rules of transformation for the Λ's matrices:

$$\Lambda_e \to \Lambda'_e = U_1^\dagger \Lambda_e V_1 \tag{3-33}$$

$$\Lambda_\nu \to \Lambda'_\nu = U_2^\dagger \Lambda_\nu V_1 \tag{3-34}$$

Then, under a suitable choice of the matrices U_1 and V_1, it is possible to diagonalize the matrix Λ_e vanishing all complex phases. However, for the matrix Λ_ν this does not occur, i.e., it is not possible to choose a suitable matriz U_2 because V_1 has already set before. In consequence [45]:

$$\Lambda_\nu = U^\dagger \begin{pmatrix} \lambda_{\nu_e} & 0 & 0 \\ 0 & \lambda_{\nu_\mu} & 0 \\ 0 & 0 & \lambda_{\nu_\tau} \end{pmatrix} U \tag{3-35}$$

Where the 3×3 unitary matrix U is known as the **PMNS** (Pontecorvo-Maki-Nakagawa-Sakata) **matrix** [3, 45], and λ_{ν_l} the Yukawa coupling constants for each flavour $l = e, \mu, \tau$. Thus, the flavour eigenstates can be written in terms of the mass eigenstates, through this matrix, leading to the flavour mixing for the neutrinos, which will be relevant in the charged-current processes for weak interactions [45]:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \tag{3-36}$$

Or in another equivalent way:

$$\nu_\alpha = \sum_{i=1}^{3} U_{PMNS,\alpha i} \nu_i \tag{3-37}$$

With ν_i the mass eigenstates of which are expected to have defined masses m_i, and l the lepton flavour indice.
Notice that the assumption that neutrinos are Dirac particles allows to make a parallelism with the quark flavour mixing in the Standard Model. Hence, the PMNS matrix can be fully

parameterized, in the standard way, in terms of three mixing angles between generations (θ_{12}, θ_{23} and θ_{13}) and one CP-violating phase (δ_{CP}) [45]:

$$\begin{aligned} U_{PMNS} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & C_{23} & S_{23} \\ 0 & -S_{23} & C_{23} \end{pmatrix} \begin{pmatrix} C_{13} & 0 & \hat{S}_{13}^* \\ 0 & 1 & 0 \\ -\hat{S}_{13} & 0 & C_{13} \end{pmatrix} \begin{pmatrix} C_{12} & S_{12} & 0 \\ -S_{12} & C_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} D_M \\ &= \begin{pmatrix} C_{12}C_{13} & S_{12}C_{13} & \hat{S}_{13}^* \\ -S_{12}C_{23} - C_{12}S_{23}\hat{S}_{13} & C_{12}C_{23} - S_{12}S_{23}\hat{S}_{13} & C_{13}S_{23} \\ S_{12}S_{23} - C_{12}C_{23}\hat{S}_{13} & -C_{12}S_{23} - S_{12}C_{23}\hat{S}_{13} & C_{13}C_{23} \end{pmatrix} D_M \end{aligned} \tag{3-38}$$

Where the diagonal matrix is for the Majorana phases $D_M = \mathrm{diag}(1, e^{i\rho}, e^{i\sigma})$.
Where $C_{ij} = \cos\theta_{ij}$ and $S_{ij} = \sin\theta_{ij}$, with $i = 1, 2$ and $j = 2, 3$. These angles θ_{ij} are chosen to lie in the first quadrant, i.e.:

$$0 \leq \theta_{ij} \leq \frac{\pi}{2} \tag{3-39}$$

And therefore:

$$S_{ij}, C_{ij} \geq 0 \tag{3-40}$$

Thus, in a totally analogous way that it was made in the quark sector, five out of six possible complex phases have been removed [45].

3.4. Weinberg effective operator

3.4.1. On the Weinberg effective Lagrangian for the generation of neutrino masses.

The only possible effective Lagrangian which generates a neutrino mass term, assuming only left.handed neutrino fields $\nu'_{i,l}$ (components of the lepton doublets E_L), has the form [8]:

$$\begin{aligned} \mathcal{L}_M^{eff} &= -\frac{1}{\Lambda_{UV}} \sum_{i,m} \overline{(\tilde{\phi}^\dagger E_{L,i})} X'_{i,m} (\tilde{\phi}^\dagger E_{L,m})^c + h.c. \\ &= -\frac{1}{\Lambda_{UV}} \sum_{i,m} \overline{E}_{L,i} \tilde{\phi} X'_{i,m} \tilde{\phi}^T E_{L,m}^c + h.c. \end{aligned} \tag{3-41}$$

where $i, m = e, \mu, \tau$ the flavour indices, and $\tilde{\phi} = i\gamma^2\phi^*$ is the charge conjugated Higgs doublet. X' is a symmetrical non-diagonal matrix, and its Feynmann diagram is shown in the Figure **3-1**.
Notice that $[\mathcal{L}_M^{eff}] = M^5$, because of the extra Higgs doublet, therefore, in order to recover the usual dimensions, it must be satisfied that $[\Lambda_{UV}] = M \Rightarrow [\mathcal{L}_M^{eff}] = M^4$ [46].

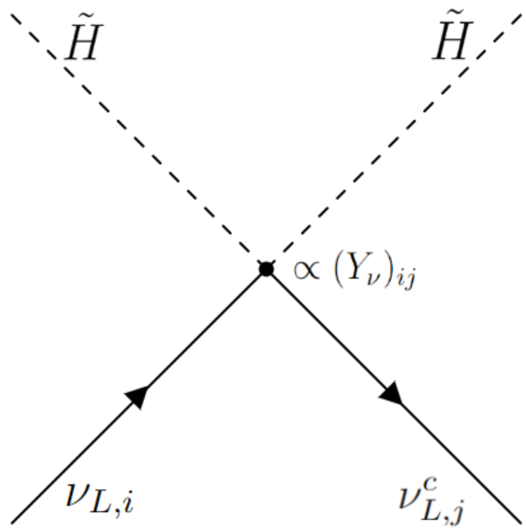

Figure 3-1.: Feynman diagram for the Weinberg effective operator in the Majorana extension.

After the electroweak symmetry breaking it is found that [46]:

$$(\tilde{\phi}^\dagger E_{L,i}) = \frac{v}{\sqrt{2}}\nu'_{L,i} \quad ; \quad \langle\phi\rangle_0 = v \tag{3-42}$$

Thus, the following mass term is obtained from the effective Lagrangian [46]:

$$\mathcal{L}^M = -\frac{v}{2\Lambda_{UV}}\sum_{i,m}\bar{\nu}'_{L,i}X'_{i,m}\nu'^c_{L,m} + h.c. \tag{3-43}$$

Where $\nu'_{L,i}$ are actually the mass eigenstates neutrinos, the flavour states of neutrinos will be given by [45]:

$$\nu_{L,i} = \sum_j U_{ij}\nu'_{L,j} \quad ; \quad j = 1,2,3 \quad i = e,\mu,\tau \tag{3-44}$$

$$\Rightarrow \bar{\nu}_{L,i} = \sum_j \nu'_{L,j}U^\dagger_{ji} \tag{3-45}$$

Thus, rewriting the mass term, as it was expected, a Majorana mass term arises [8, 46]:

$$\mathcal{L}^M = -\frac{1}{2}\sum_{i,m}\bar{\nu}_{L,i}M_{i,m}\nu^c_{L,m} + h.c. \tag{3-46}$$

In which the Majorana matrix M is given by [46]:

$$M = \frac{v^2}{\Lambda_{UV}}X \tag{3-47}$$

With $X = U^{\dagger}X'(U^{\dagger})^{T} = U^{\dagger}X'U^{*}$, which can be diagonalized. Thus, writting the Majorana mass term in its usual form:

$$\mathcal{L}^{M} = -\frac{1}{2}\sum_{i=1}^{3} m_i\bar{\nu}_i\nu_i + h.c. \tag{3-48}$$

It is found that:

$$m_i = \frac{u}{\Lambda_{UV}}(vx_i) \quad ; \quad x_i = X_{ii} \tag{3-49}$$

Therefore, it is possible to estimate the energy scale of Λ_{UV}, which characterizes the scale of physics beyond the Standard Model, i.e., new physics.
From experimental data is know that [6]:

$$m_3 \approx \sqrt{\Delta m_A^2} \approx 5\times 10^{-2} eV \tag{3-50}$$

And assuming $x_3 \lesssim 1$ ($\sim$ by analogy with Yukawa coupling of particles of the third family) [46], then:

$$\Lambda \approx \frac{v^2}{m_3}x_3 \Rightarrow \Lambda \lesssim \frac{(246)^2}{5\times 10^{-11}}\text{GeV} \tag{3-51}$$

$$\Rightarrow \Lambda \lesssim 1{,}210\times 10^{15}\text{GeV} \tag{3-52}$$

This imposes an energy scale for which is expected to show up new physics, and coincides with GUT energy scale [47]. Furthermore, this opens the door to study the different mechanisms to generate mass terms, e.g., the see-saw mechanisms (I, II and III) and a wide class of radiative models (higher dimensions) where the scale of energy is lower [6].

3.5. Type-I seesaw model

In the type-I seesaw model the Standard Model is extended by introducing (one to) three right-handed Majorana fermion fields $N_{R,i}$ (for $i = 1, 2, 3$), therefore these fermions satisfy (3-11) and leads to a lepton number violation, and with a sufficiently high mass scale [10, 11, 12, 13]. Sometimes these new heavy Majorana fermion fields are called heavy Majorana neutrino fields.
The $SU(2)_L\times U(1)_Y$ gauge invariant lagrangian for the type-I seesaw model is given by [10, 11, 12, 13]:

$$\mathcal{L}_{type-I} = -\overline{E}_{L,\alpha}y_{\alpha\beta}\tilde{H}N_{\beta,R} - \frac{1}{2}\overline{N}_{R,\alpha}M^{R}_{\alpha\beta}N^{c}_{\beta,R} + h.c. \tag{3-53}$$

Notice that the first term is a Yukawa interaction between the light active neutrino ν_L, through the leptonic doublet, and the heavy sterile Majorana fermion $N_{R,i}$. The Feynman

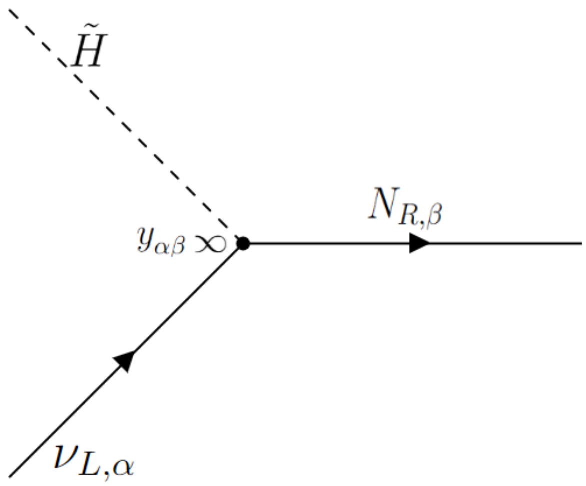

Figure 3-2.: Feynman diagram for the New Yukawa interaction given by the type-I seesaw model.

diagram at tree-level for this new Yukawa interaction has been drawn in Figure **3-2**. The second term corresponds to a Majorana mass term because it has the same form of (3-24).

Since this model is an UV-completion for the Weinberg effective operator, it is possible to show that if masses of the heavy sterile fermions M_i^R are much bigger than the Higgs vacuum expectation value v, i.e., $M_i^R >> v$, the Weinberg effective operator is obtained through the lagrangian (3-53) at second order of perturbation theory [46]. The Feynman diagram for this second order interaction is shown in Figure **3-3**.

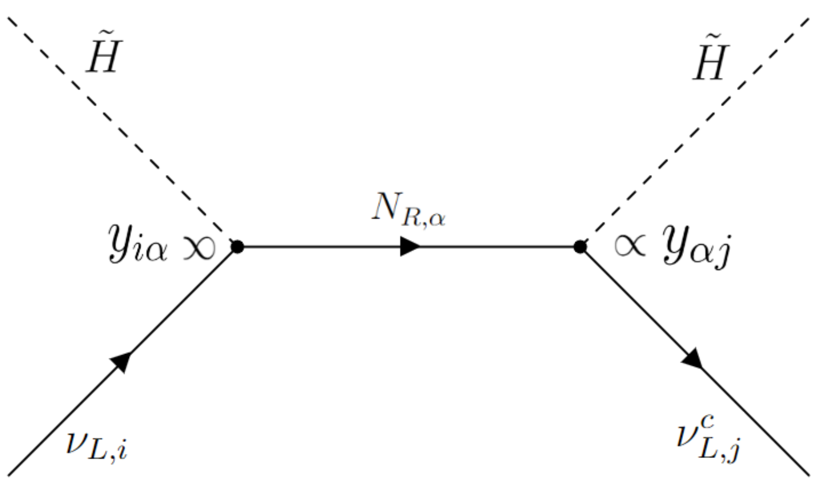

Figure 3-3.: Feynman diagram for the neutrino mass generation through heavy sterile fermion exchanges.

In the (1+1)-scheme it can be found that the light active neutrino mass is given by [11]:

$$m_\nu \propto \frac{v^2}{m^R} \tag{3-54}$$

Thus, taking the experimental value for neutrino mass about $\sim 10^{-1}$ eV [46], and recalling that $m_R \sim \Lambda_{UV}$, it is found that:

$$\Lambda_{UV} \lesssim 1 \times 10^{14}\,\text{GeV} \tag{3-55}$$

3.5.1. (3+3)-scheme

In the (3+3)-scheme the neutrino mass matrix can be approximated to [11]:

$$m_v \approx -M_D\,(M_R)^{-1}\,M_D^T \tag{3-56}$$

3.5.2. Reconstruction of the PMNS matrix

Let us write the unitary 6×6 matrix U as:

$$U = \begin{pmatrix} I & 0 \\ 0 & U'_{PMNS} \end{pmatrix} \begin{pmatrix} I & R \\ S & I \end{pmatrix} \begin{pmatrix} U_{PMNS} & 0 \\ 0 & I \end{pmatrix} \tag{3-57}$$

As U is a 6×6 unitary matrix it can be fully parameterized in terms of 15 mixing angles θ_{ij}, and 10 complex phases δ_{ij} which quantify the CP-violation [7]. However, for the sake of ease, there will be considered 15 complex phases as well, in order to apply the Euler-like parameterization.
From (3-57) after matrix multiplication it follows that:

$$U = \begin{pmatrix} U_{PMNS} & R \\ Q & U'_{PMNS} \end{pmatrix} \tag{3-58}$$

With Q and R active-sterile mixing matrices.
From the diagonalization relation:

$$M = UmU^\dagger \tag{3-59}$$

The explicit multiplication is made in Appendix C, obtaining:

$$M = \begin{pmatrix} U^\dagger_{PMNS} D_\nu U_{PMNS} + (U'_{PMNS} S U_{PMNS})^\dagger\, D_N\, (U'_{PMNS} S U_{PMNS}) & U^\dagger_{PMNS} D_\nu R + (U'_{PMNS} S U_{PMNS})^\dagger\, D_N U'_{PMNS} \\ R^\dagger D_\nu U_{PMNS} + U'^\dagger_{PMNS} D_N\, (U'_{PMNS} S U_{PMNS}) & R^\dagger D_\nu R + U'^\dagger_{PMNS} D_N U'_{PMNS} \end{pmatrix} \tag{3-60}$$

Furthermore, from experimental measures it is stablished the smallness of the nine active-sterile flavour mixing angles θ_{ij} (for $i = 1, 2, 3$ and $j = 4, 5, 6$). Therefore, it allows to make the following approximation [7]:

$$R \approx -S^\dagger = \begin{pmatrix} \hat{S}^*_{14} & \hat{S}^*_{15} & \hat{S}^*_{16} \\ \hat{S}^*_{24} & \hat{S}^*_{25} & \hat{S}^*_{26} \\ \hat{S}^*_{34} & \hat{S}^*_{35} & \hat{S}^*_{36} \end{pmatrix} \tag{3-61}$$

Where:

$$\hat{S}^*_{ij} = e^{i\delta_{ij}} \sin\theta_{ij} \tag{3-62}$$

And the terms of $\mathcal{O}(S^2_{ij})$ have been neglected.
Now, it is possible to take the basis where the heavy sterile flavour states of fermions correspond to the same as their mass states. Thus, the mass matrix M_R is diagonal, and in consequence:

$$U'_{PMNS} \approx I \tag{3-63}$$

Which means that there is no flavour mixing in the sterile sector.
Substituting these considerations into the calculation of $U^\dagger m U$ leads to a simplification of the matrix:

$$U^\dagger \begin{pmatrix} D_\nu & 0 \\ 0 & D_N \end{pmatrix} U = \begin{pmatrix} U^\dagger_{PMNS} D_\nu U_{PMNS} + \left(R^\dagger U_{PMNS}\right)^\dagger D_N \left(R^\dagger U_{PMNS}\right) & U^\dagger_{PMNS}\left(D_\nu R - R D_N\right) \\ \left(R^\dagger D_\nu - D_N R^\dagger\right) U_{PMNS} & R^\dagger D_\nu R + D_N \end{pmatrix} \tag{3-64}$$

3.6. Neutrino oscillations

3.6.1. Plane waves approach

Neutrinos are always produced by charged-current weak interactions with their associated charged lepton. Thus, by the reactions:

$$W^+ + l^- \to \nu_l \tag{3-65}$$

It is possible to create neutrinos source which starts off in a specific flavour eigenstate ν_l. Now, neutrino oscillations occurs because of their masses. Then, the subsequent time evolution of the produced neutrino simply involves time dependent phases for the neutrino mass eigenstates ν_i [45].
After that, neutrinos can be detected by charged-current mean interactions in the form:

$$\nu_i \to W^+ + l^- \tag{3-66}$$

Then the final flavour state will not be necessarily the same as the initial flavour. Therefore, the difference between flavour and mass basis allows the possibility of time dependent oscillation phenomena such as the $K^0 - \bar{K}^0$ system which was studied in section 2.2.1.
Thus, the $\nu_l \to \nu_{l'}$ flavour oscillation may take place after the ν_i source with an average energy $E \gg m_i$ travels a proper distance L in a vacuum. In general, the amplitude for these processes is [45]:

$$\langle \nu_{l'}(\vec{x}, t) | \nu_l(0,0) \rangle = \langle \nu_l | \exp\Big\{ (-i\hat{H}t + \hat{p}\cdot\vec{x}) \Big\} | \nu_{l'} \rangle \tag{3-67}$$

Where $\hat{H}$ and $\hat{p}$ are the time evolution and translation operators, $\vec{x}$ is the traveled distance and t is the time of propagation. Now, it is possible to insert a complete basis of mass eigenstates [45]:

$$\langle \nu_{l'}(\vec{x},t)|\nu_l(0,0)\rangle = \sum_{i,\sigma} \int d^3\vec{k} e^{-iE_i t+\vec{k}\cdot\vec{x}} \langle \nu_{l'}|\nu_i(\vec{k},\sigma)\rangle \langle \nu_i(\vec{k},\sigma)|\nu_l\rangle \tag{3-68}$$

If $|\vec{x}| = t$ and $E = |\vec{k}|$, which means neutrinos are massless, then the phase from the temporal and spatial propagation would cancel exactly. However, the phase arises because different kinds of neutrinos have slightly different dispersion relation:

$$E_i = \sqrt{|\vec{k}|^2 + m_i^2} \tag{3-69}$$

In typical applications the energy can not be perfectly measured in detectors, but for our purposes we can take E to be known. Furthermore, a very good approximation can b e used because neutrinos are ultrarelativistic particles [45]:

$$\begin{aligned}
E = \sqrt{|\vec{k}|^2 + m_i^2} &= \sqrt{|\vec{k}|^2 \left(1 + \frac{m_i^2}{2|\vec{k}|^2}\right)} \\
&= |\vec{k}|\sqrt{1 + \left(\frac{m_i^2}{2|\vec{k}|}\right)} \qquad ; \qquad m_i \ll |\vec{k}| \\
&\approx |\vec{k}|\left(1 + \frac{m_i^2}{2|\vec{k}|} + \cdots\right) \\
&= |\vec{k}| + \frac{m_i^2}{2E} + \cdots \qquad ; \qquad E = |\vec{k}|
\end{aligned} \tag{3-70}$$

$$\Rightarrow E - |\vec{k}| \approx \frac{m_i^2}{2E} \tag{3-71}$$

From the above result, and considering $t \sim |\vec{x}|$, it is possible to write the space-dependent part of the phases as [45]:

$$e^{-im_i^2|\vec{x}|/2E} \tag{3-72}$$

And in consequence, the amplitude becomes:

$$\langle \nu_{l'}(\vec{x},t)|\nu_l(0,0)\rangle = e^{i\phi} \sum_i e^{-im_i^2|\vec{x}|/2E} \langle \nu_{l'}|\nu_i\rangle \langle \nu_i|\nu_l\rangle \tag{3-73}$$

Where ϕ is an overall phase.
Therefore, each neutrino energy eigenstate picks up a relative phase $e^{-im_i^2|\vec{x}|/2E}$.

Now, the amplitude will depend explicitly on the PMNS matrix elements. Indeed, for the process $l \to \alpha$, $l' \to \beta$, the amplitude can be computed as [45]:

$$\begin{aligned}
\langle \nu_i | \nu_\alpha \rangle &= \langle \nu_i | \nu_\alpha(0,0) \rangle \\
&= \langle \nu_i | \left(\sum_j U^*_{\alpha j} | \nu_j \rangle \right) \\
&= \sum_j U^*_{\alpha j} \langle \nu_i | \nu_j \rangle \\
&= \sum_j U^*_{\alpha j} \delta_{ij} \\
&= U^*_{\alpha i}
\end{aligned} \tag{3-74}$$

Where $|\nu_\alpha\rangle = \sum_j U^*_{\alpha j} |\nu_j\rangle$; $\alpha = e, \mu, \tau$; $j = 1, 2, 3$ and $\hat{H}|\nu_j\rangle = E_j |\nu_j\rangle$.
Similarly [45]:

$$\begin{aligned}
\langle \nu_\beta | \nu_i \rangle &= \left(\sum_j \langle \nu_j | U_{\beta j} \right) | \nu_i \rangle \\
&= \sum_j U_{\beta j} \langle \nu_j | \nu_i \rangle \\
&= \sum_j U_{\beta j} \delta_{ij} \\
&= U_{\beta j}
\end{aligned} \tag{3-75}$$

Thus, the amplitude becomes in [45]:

$$\langle \nu_\beta(\vec{x}, t) | \nu_\beta(0,0) \rangle = e^{i\phi} \sum_i e^{-im_i^2 |\vec{x}|/2E} U_{\beta i} U^*_{\alpha_i} \tag{3-76}$$

Now, by making the energy E not to large and the oscillation length L very large, the probability of the neutrino being produced as the flavour ν_α and being detected in the flavour state ν_β is therefore [45]:

$$\begin{aligned}
P(E,L)_{\nu_\alpha \to \nu_\beta} &= \left| \langle \nu_\beta(L,t) | \nu_\alpha(0,0) \rangle \right|^2 \\
&= \langle \nu_\beta(L,t) | \nu_\alpha(0,0) \rangle \, \langle \nu_\beta(L,t) | \nu_\alpha(0,0) \rangle^* \\
&= \left(e^{i\phi} \sum_i e^{-im_i^2 L/2E} U_{\beta i} U^*_{\alpha i} \right) \left(e^{i\phi} \sum_j e^{-im_j^2 L/2E} U_{\beta j} U^*_{\alpha j} \right)^* \\
&= \left(e^{i\phi} \sum_i e^{-im_i^2 L/2E} U_{\beta i} U^*_{\alpha i} \right) \left(e^{-i\phi} \sum_j e^{im_j^2 L/2E} U^*_{\beta j} U_{\alpha j} \right) \\
P(E,L)_{\nu_\alpha \to \nu_\beta} &= \sum_{i,j} e^{-i(m_i^2 - m_j^2) L/2E} U_{\beta i} U^*_{\beta j} U_{\alpha j} U^*_{\alpha i}
\end{aligned} \tag{3-77}$$

And defining $\Delta m_{ij}^2 = m_i^2 - m_j^2$, it is found that:

$$P(E,L)_{\nu_\alpha \to \nu_\beta} = \sum_{i,j} e^{-i\Delta m_{ij}^2 L/2E} U_{\beta i} U_{\alpha i}^* U_{\beta j}^* U_{\alpha j} \tag{3-78}$$

Notice that the oscillation probability involves only the difference of squares of the neutrino masses and not the masses directly. And it depends directly on PMNS matrix components.

4. CP violation and Thermal Leptogenesis

4.1. Geodesic equation

It is well-kown that a free-point particle with mass $m \geq 0$ moves freely in a gravitational field following the geodesic equation [48]:

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0 \tag{4-1}$$

4.1.1. Continuity equation and the content of the universe

For a FLRW-type universe, in the perfect fluid approach, the continuity equation takes the form [49]:

$$\frac{\partial\rho}{\partial t} + 3H(t)(\rho + P) = 0 \tag{4-2}$$

Which is equivalent to the first law of thermodynamics for a closed expanding universe:

$$\begin{aligned}
\frac{\partial\rho}{\partial t} + 3\frac{\dot{a}(t)}{a(t)}(\rho + P) &= 0 \\
a^3(t)\frac{\partial\rho}{\partial t} + 3a^2(t)\dot{a}(t)(\rho + P) &= 0 \\
a^3(t)\frac{\partial\rho}{\partial t} + 3a^2(t)\dot{a}(t)\rho &= -3a^2(t)\dot{a}(t)P \\
\frac{d}{dt}\left(a^3(t)\rho\right) &= -P\frac{d}{dt}\left(a^3(t)\right)
\end{aligned} \tag{4-3}$$

And since the volume of the universe is directly proportional to $V \propto a^3(t)$, and the total energy will be proportional to the energy density times the universe volume $E \propto \rho a^3(t)$, the above equation implies that:

$$\begin{aligned}
\frac{dE}{dt} &= -P\frac{dV}{dt} \\
dE &= -PdV
\end{aligned} \tag{4-4}$$

Therefore, from the first law of thermodynamics, the above result implies that the entropy density of the universe should be constant:

$$ds = 0 \tag{4-5}$$

Now, to solve the continuity equation (4-2) it is necessary to recall the perfect fluid assumption, then state equations in which the energy density and the pressure are proportional are assumed [49]:

$$P = \omega\rho \tag{4-6}$$

Thus, substituting this purpose in continuity equation:

$$\begin{aligned} d\left(a^3(t)\rho\right) &= -\omega\rho d\left(a^3(t)\right) \\ \rho d(a^3) + a^3 d\rho &= -\omega\rho d\left(a^3\right) \\ a^3 d\rho &= -(\omega+1)\rho d\left(a^3\right) \\ \frac{d\rho}{\rho} &= -(\omega+1)\frac{d\left(a^3\right)}{a^3} \end{aligned} \tag{4-7}$$

Whose solution is given by [49]:

$$\begin{aligned} &\ln\rho = -(\omega+1)\ln\left(a^3\right) + c \\ &\rho(t) = c' a^{-3(\omega+1)}(t) \end{aligned} \tag{4-8}$$

Thus, the evolution in time for the energy density ρ depends on the evolution in time for the scale factor $a(t)$, which can be found through the Friedmann equations, and on the value for the parameter ω [49]. There is three possibilities for the state equation, which correspond to a universe full of radiation, or matter, or vacuum energy. In Table **4-1** it is summarized those possibilities.

Content	ω	State equation	Solution
Radiation	$\omega = 1/3$	$P = 1/3\rho$	$\rho(t) \propto a^{-4}(t)$
Matter	$\omega = 0$	$P = 0$	$\rho(t) \propto a^{-3}(t)$
Vacuum energy	$\omega = -1$	$P = -\rho$	$\rho(t) \propto cte$

Table 4-1.: Energy density solution depending on the content of the universe [49].

4.1.2. Thermodynamics in the early universe

The early universe ($T \geq 10^{12}$ GeV) can be considered as a system in a state of local equilibrium. This does not imply a sort of "infinite" thermal bath, rather it means that matter has maximal possible entropy, and entropy is well-defined no matter how far away the system is from the equilibrium [49]. In the early universe particles scattering each other abruptly, hence the entropy should reach its maximal value before the size of the universe changes significantly:

$$t_c < t_H \tag{4-9}$$

Where t_c is the so-called collision time, and t_H is the Hubble time. Now, since the amplitude for a scattering process is $\Gamma \sim t_c^{-1}$, and the Hubble time is defined as $t_H = H(t)^{-1}$, the above relation implies that:

$$\Gamma > H(t) \tag{4-10}$$

Roughly the collision time is given by [49]:

$$t_c \approx \frac{1}{\sigma n_A v} \tag{4-11}$$

Where σ is the cross-section of the interaction, n_A is the number density of the particles "A" involved, and v their relative velocity.
Now, it is expected that in the early universe particles are either fermions or bosons, or both. Therefore, their number density will be given by the Fermi Dirac or Bose-Einstein distribution [50]:

$$n_A = \frac{g}{e^{(E-\mu)/k_B T} \pm 1} = gf(p) \tag{4-12}$$

With g the number of internal degrees of freedom, i.e., the number of the total quantum states a particle may have within a cell in the momentum space of volume $\hbar^3$, and p is the momentum of particles.
Then, from statistical mechanics it is well known that [50]:

$$n_A = g \int f(p)dp \tag{4-13}$$

$$\rho_A = g \int \epsilon f(p)dp \tag{4-14}$$

$$\begin{aligned} P_A &= g \int \langle v_r p_r \rangle f(p)dp \\ &= \frac{g}{3} \int vpf(p)dp \\ &= \frac{g}{3} \int \frac{p^2}{\epsilon} f(p)dp \end{aligned} \tag{4-15}$$

Notice that for the computation of pressure P_A have been used an isotropy consideration and the relativistic relation $v = p/\epsilon$.

Now, due to the first law of thermodynamics:

$$\begin{aligned} TdS - PdV &= dE \\ Td(sV) - PdV &= d(\rho V) \\ sTdV + TVds - PdV &= \rho dV + d\rho V \\ sTdV &= PdV + \rho dV + d\rho V \\ sTdV &= (P+\rho)dV + Vd\rho \end{aligned} \tag{4-16}$$

Where it has been replaced tha fact that $dS = 0$. Now, considering a local equilibrium such that $d\rho \approx 0$, it is found that:

$$\begin{aligned} sTdV &= (P+\rho)dV \\ sT &= (P+\rho) \\ s &= \frac{P+\rho}{T} \end{aligned} \tag{4-17}$$

Now, for relativistic particles, i.e., considering the radiation-dominated epoch of the universe where baryogenesis is expected to take place, and performing the approximations $T >> m, \mu$, from statistical mechanics computations one gets [50]:

$$n_A \propto g T_A^3 \tag{4-18}$$

$$\rho_A \propto g T_A^4 \tag{4-19}$$

$$P_A = \frac{1}{3}\rho_A \tag{4-20}$$

Thus, substituting the above relations in the expression found for the entropy density in (4-17):

$$\begin{aligned} s &\propto \frac{\rho_A + \rho}{T_A} \\ s &\propto g\frac{T_A^4}{T_A} \\ s &\propto g T_A^3 \end{aligned} \tag{4-21}$$

And without loss of generality, in the radiation-dominated epoch there were several number of species i, therefore:

$$s = \sum_i g_i T_i^3 \tag{4-22}$$

And those species might have had different temperatures to those of the radiation thermal bath $T_{i\gamma} = T$. Then:

$$\begin{aligned} s &= \sum_i g_i \left(\frac{T_i}{T_\gamma}\right)^3 T_\gamma^3 \\ s &= g_{*s} T_\gamma^3 \end{aligned} \tag{4-23}$$

Where it has been defined the following quantity which accounts for the effective number of degrees of freedom contributing to the entropy density:

$$g_{*s} = \sum_i g_i \left(\frac{T_i}{T_\gamma}\right)^3 \tag{4-24}$$

In this scenario, it is also expected that the Standard Model symmetry is valid, and furthermore, the cross-sections for the electroweak and strong interactions have a similar energy dependence [49]. Thus, one may roughly estimate an only cross-section which will depend on an effective interaction coupling constant α:

$$\begin{aligned}\sigma &\approx \mathcal{O}(1)\frac{\alpha^2}{p^2} \\ &\propto \frac{\alpha^2}{T^2}\end{aligned} \tag{4-25}$$

Where the assumption $1/p \approx 1/E \approx 1/T$ have been applied since it is the radiation-dominated epoch, then content is relativistic particles.
Thus, substituting these results in (4-11), the collision time can be estimated as:

$$\begin{aligned}t_c &\approx \frac{1}{(\alpha^2/T^2)\,T^3(1)} \\ &\approx \frac{1}{\alpha^2 T}\end{aligned} \tag{4-26}$$

On the other hand, the Hubble time in the radiation-dominated epoch can be estimated as [49]:

$$\begin{aligned}t_H &= \frac{1}{H(t)} \\ &\approx \frac{1}{\sqrt{\rho}} \\ &\approx \frac{1}{\sqrt{T^4}} \\ &\approx \frac{1}{T^2}\end{aligned} \tag{4-27}$$

Therefore, condition (4-9) implies that:

$$\begin{aligned}\frac{1}{\alpha^2 T} &< \frac{1}{T^2} \\ T &< \alpha^2\end{aligned} \tag{4-28}$$

And since 10^{15} GeV $\leq \alpha^2 \leq 10^{17}$ GeV [49], the above result is in concordance with the baryogenesis epoch [51, 52]:

$$10^{12}\,\text{GeV} \leq T \leq 10^{15}\,\text{GeV} \tag{4-29}$$

4.2. Baryogenesis: A brief overview

From observations made in the Cosmic Microwave Background (CMB) anisotropies by the Planck colaboration, there exists a baryon asymmetry in the observable universe [15]:

$$\eta \equiv \left.\frac{n_B - n_{\overline{B}}}{n_\gamma}\right|_{T_0} = (6{,}21 \pm 0{,}16) \times 10^{-10} \tag{4-30}$$

Where η is defined as the baryon asymmetry quantity, n_B, $n_{\overline{B}}$ and n_γ are the number density of baryons, antibaryons, and photons respectively, and T_0 denotes that the computation is performed today, where the universe is in thermal equilibrium.
Other way to express the baryon asymmetry, and it is in terms of the entropy density s [52]:

$$Y_{\Delta B} \equiv \left.\frac{n_B - n_{\overline{B}}}{s}\right|_{T_0} = (8{,}75 \pm 0{,}23) \times 10^{-11} \tag{4-31}$$

$Y_{\Delta B}$ is called the baryon-antibaryon density.
There is also other way to present the baryon asymmetry and it is in terms of the baryonic function of the critical energy density [49]:

$$\Omega_B \equiv \frac{\rho_B}{\rho_c} \tag{4-32}$$

With the critical energy density given by:

$$\rho_c \equiv \frac{3H^2(t)}{8\pi G} \tag{4-33}$$

Thus, substituting this in the expression for Ω_B one gets:

$$\Omega_B = \frac{8\pi G}{3H^2(t)}\rho_B \tag{4-34}$$

And from CMB measurements it has been found that [52]:

$$\eta = 2{,}74 \times 10^{-8}\Omega_B h^2 \tag{4-35}$$

With h the so-called Present Hubble parameter:

$$h \equiv \frac{H_0}{100\,\mathrm{kms}^{-1}\,\mathrm{Mpc}^{-1}} = (0{,}701 \pm 0{,}013) \tag{4-36}$$

From CMB measurements there exists the following constraint [15]:

$$0{,}02149 \leq \Omega_B h^2 \leq 0{,}02397 \tag{4-37}$$

All this data is a proof of the baryon asymmetry of the observable universe. Therefore, there must be a physical model which explains this. A general approach will be presented in the following section, where general conditions must be satisfied in any model which pretends to explain the baryon asymmetry of the observable universe.

4.2.1. Sakharov conditions

To be physically and mathematically possible a dynamical origin of an acceptable baryon-antibaryon asymmetry there are three requirements known as the *Sakharov conditions* [17]:

1. **Baryon number violation**: This condition is required to be possible evolution from a initial state with no baryon asymmetry:

$$Y_{\Delta_B}\Big|_{t=0} = 0 \tag{4-38}$$

   To a state with baryon asymmetry in the observable universe today:

$$Y_{\Delta_B}\Big|_{\text{today}} \neq 0 \tag{4-39}$$

$$\Delta B = \Delta L = \pm 3 \tag{4-40}$$

   Therefore, only $(B - L)$ is exactly conserved.

   Despite this fact, this is not enough to generate the observed maximal baryon asymmetry of the universe [52]. Hence, new physics beyond the Standard Model is needed.

2. **C and CP violation**:

   Since a baryon is converted into its antiparticle under the charge-conjugation C, hence to generate a net baryon asymmetry C violation is required.

   Furthermore, the baryon number operator:

$$\hat{B} = \int d^3x \sum_{\text{quarks}} \left[\psi_i^\dagger(t,\vec{x})\psi_i(t,\vec{x})\right] \tag{4-41}$$

   Is odd under the charge-conjugation (C) transformation, is even under parity (P) and time-reversal (T) transformations [7]. Therefore, it is odd under CP and CPT transformations.

   Fortunately, as it was shown explicitly in section 2.2, weak interactions in the Standard Model violate C maximally, and they also violate CP symmetry via the Kobayashi-Maskawa mechanism.

   Nevertheless, it is impossible to generate the amount of baryon asymmetry (4-31) within the Standard Model [7, 52]. Therefore, new sources of CP violation are needed. Later it will be shown that in leptogenesis those new sources will be new CP-violating phases in the leptonic sector.

3. **Departure from thermal equilibrium**:

   In the primordial plasma, if the whole universe stayed in thermal equilibrium and was described by a density operator [52]:

$$\hat{\rho} = \exp\Big\{(-\hat{H}/T)\Big\} \tag{4-42}$$

   With $\hat{H}$ the Hamiltonian operator, then:

$$\langle \hat{B} \rangle_T = \mathrm{Tr}[\hat{B} \exp\Big\{(-\hat{H}/T)\Big\}] \tag{4-43}$$

   Now, applying the CPT operators and demanding the CPT invariance of the Hamiltonian operator it follows that [52]:

$$\begin{aligned} \langle \hat{B} \rangle_T &= \mathrm{Tr}[(CPT)\hat{B}(CPT)^{-1}(CPT) \exp\Big\{(-\hat{H}/T)\Big\}(CPT)^{-1}] \\ &= \mathrm{Tr}[(-\hat{B}) \exp\Big\{(-\hat{H}/T)\Big\}] \\ &= -\langle \hat{B} \rangle_T \end{aligned} \tag{4-44}$$

   Therefore, in thermal equilibrium the following condition is satisfied [52]:

$$\langle \hat{B} \rangle_T = 0 \tag{4-45}$$

   This implies that if there exists any net baryon number excess and the universe is in thermal equilibrium, it would disappear.

Within the Standard Model the departure from thermal equilibrium takes place in the electroweak phase transition, which is consequence of the interactions of particles with the bubble wall as it sweeps through the primordial plasma [53]. However, experimental lower bound on the Higgs boson mass implies that this transition is not at first order at all, as required for succesful baryogenesis [52]. This is why a new and different kind of departure from thermal equilibrium is required, and in consequence, physics beyond the standard model is needed. There are a bunch of models in the literature which pretends to explain baryogenesis. Among them the most popular are: GUT baryogenesis [54, 55], Electroweak baryogenesis [56, 57], the Affleck-Dine mechanism [58, 59], and leptogenesis [60]. In the present work leptogenesis will be developed. For this purpose, it is necessary to recall sphaleron processes [18]:

$$\begin{aligned} \Delta B = \Delta L &= \frac{Ng^2}{32\pi^2} \int d^4x W^i_{\mu\nu} \tilde{W}^{i,\mu\nu} \\ &= N \Delta N_{cs} \end{aligned} \tag{4-46}$$

Which transform a lepton number asymmetry into a baryon number asymmetry, being N the number of families involved, and ΔN_{cs} an integer number, the Chern-Simons coefficient.

4.3. Thermal leptogenesis toy model

Leptogenesis outline is mainly based on the following ideas [60, 52]:

- New heavy sterile particles are introduced via the seesaw models.
- The new sources of CP violation are originated from the new Yukawa couplings associated to seesaw models, and hence through CP-violating phases in the leptonic sector.
- To depart from thermal equilibrium, the rate for Yukawa interactions must be lower than expansion universe rate: $\Gamma < H(t)$.
- CP-violating decays of heavy sterile particles take place, gemerating lepton number asymmetry.
- Standard Model sphaleron processes still play a crucial role in converting partially the lepton number asymmetry into a baryon asymmetry.

The present thesis focuses on type-I seesaw model, hence the new heavy sterile particles introduced correspond to the heavy singlet-gauge fermions N_i.
Notice that type-I seesaw model satisfies the Sakharov conditions [60, 52], since:

- The lagrangian in equation (3-53) explicitly violates the lepton number symmetry.
- New sources of CP-violation are introduced through neutrino Yukawa interactions.
- The new heavy gauge-singlet fermions N_i decay out of thermal equilibrium.

4.3.1. CP asymmetry factor

Now, without loss of generality it is always possible to choose a basis where the mass matrix of the heavy sterile fermions is diagonal [52]:

$$M^R = \text{diag}(M_1, M_2, M_3) \tag{4-47}$$

Therefore, the flavour states are the same of the mass states:

$$N_\alpha = N_i \tag{4-48}$$

And let us assume that the masses of the heavy sterile fermions are ordered hierarchically:

$$M_1 << M_2 << M_3 \tag{4-49}$$

Since the masses M_i are expected to be far above the electroweak symmetry breaking scale, the lepton-number violation decays of N_i into a pair of lepton doublet E_L and Higgs doublet

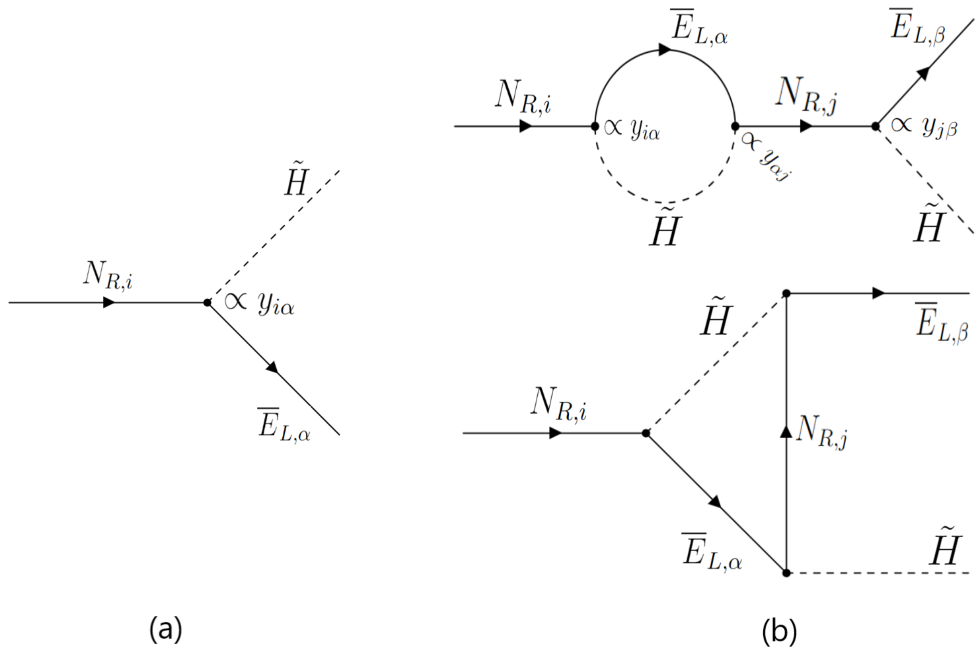

Figure 4-1.: Feynman diagrams at tree-level and one-loop level for N_i decays through Yukawa couplings with the $\mathrm{SU}(2)_L$ leptonic and Higgs doublets.

H can take place via the Yukawa interactions [60, 52]. For the sake of simplicity, let us consider just the decays at first and second order, i.e., at tree-level and one-loop level. The Feynman diagrams corresponding these decays are shown in Figure **4-1**, where in (a) is shown the tree-level feynmann diagram, and in (b) the two one-loop diagrams are shown. Thus, the amplitude associated to the decay at tree-level will be computed as [61]:

$$\Gamma_{i\alpha} = \frac{1}{32\pi}|y_{i\alpha}|^2 M_i^R \tag{4-50}$$

Being M_i^R the mass of the heavy sterile fermion $N_{R,i}$ decaying.
And the amplitude for the one-loop processes is proportional to:

$$\Gamma_{i\alpha} \propto |y_{i\alpha} y_{\alpha j} y_{j\beta}|^2 M_i^R \tag{4-51}$$

Thus, the total decay width for N_i can be computed as:

$$\Gamma_{N_i} = \sum_{\alpha}\left[\Gamma(N_{R,i} \to \overline{E}_{L,\alpha} + \tilde{H}) + \Gamma(N_{R,i}^c \to \overline{E}_{L,\beta}^c + \tilde{H}^c)\right] \tag{4-52}$$

Now, the CP-violation parameter is defined as the normalized interference for the decays through the $\mathrm{SU}(2)_L$ leptonic and Higgs doublets, described by the Yukawa interaction, and their CP-conjugated processes [52]:

$$\epsilon_{i\alpha} = \frac{\Gamma(N_{R,i} \to \overline{E}_{L,\alpha} + \tilde{H}) - \Gamma(N_{R,i}^c \to \overline{E}_{L,\alpha}^c + \tilde{H}^c)}{\Gamma_{N_i}} \tag{4-53}$$

Now, based on the Feynman diagram for the Weinberg effective operator in Figure **3-1**, it is possible to define an effective Yukawa matrix for neutrino mass interaction, in the following form:

$$(Y_\nu)_{ij} = y_{i\alpha} y^*_{\alpha j} \tag{4-54}$$

With this, the CP-violation parameter can be computed as [7, 61]:

$$\epsilon_{i\alpha} = \frac{1}{8\pi (Y_\nu^\dagger Y_\nu)_{ii}} \sum_{i \neq j} \left\{ \mathrm{Im}\left[(Y_\nu^*)_{\alpha i} (Y_\nu)_{\alpha j} (Y_\nu^\dagger Y_\nu)_{ij} \right] f(x_{ji}) + \mathrm{Im}\left[(Y_\nu^*)_{\alpha i} (Y_\nu)_{\alpha j} (Y_\nu^\dagger Y_\nu)^*_{ij} \right] g(x_{ji}) \right\} \tag{4-55}$$

Where $x_{ji} = M_j^N / M_i^N$ and, $f(x)$ and $g(x)$ are the so-called "one-loop functions", which are given by [61]:

$$f(x) = x \left[1 + \frac{1}{(1 - x^2)} + (1 + x^2) \ln \left(1 + \frac{1}{x^2} \right) \right] \tag{4-56}$$

$$g(x) = \frac{1}{1 - x^2} \tag{4-57}$$

Therefore, the nine different CP-violating asymmetry parameters $\epsilon_{i\alpha}$ depends on the nine CP-violating phase differences $(\delta_{i4} - \delta_{i5})$, $(\delta_{i4} - \delta_{i6})$, and $(\delta_{i5} - \delta_{i6})$, for $i = 1, 2, 3$, among which six of them are independent because of the result found in section 3.5.1.
It is possible to simplify the CP-violating mechanism considering that the lepton asymmetry is produced in a single flavour α, which is known as unflavoured leptogenesis, since the lepton asymmetry took place before the baryogenesis, and therefore the temperature of the universe was about $T \geq 10^{12}$ GeV [7, 61]. Thus, the CP-asymmetry factor is reduced to:

$$\begin{aligned} \epsilon_i &= \sum_\alpha \epsilon_{i\alpha} \\ &= \frac{1}{8\pi (Y_\nu^\dagger Y_\nu)_{ii}} \sum_{j \neq i} \mathrm{Im}\left[(Y_\nu^\dagger Y_\nu)^2_{ij} \right] f(x_{ji}) \end{aligned} \tag{4-58}$$

Furthermore, because of the mass hierarchy asumption for $M_{R,i}$, the lepton-number-violating decays of N_1 may be fast enough to wash out the lepton-antilepton number asymmetry generated by decays of N_2 and N_3 [7], thus the majority of the baryon asymmetry will be produced in the out-of-equilibrium decay of the lightest sterile fermion N_1 [61]. Therefore, only one CP asymmetry factor will be relevant to thermal leptogenesis [7]:

$$\begin{aligned} \epsilon_1 &= \frac{\sum_\alpha \left[\Gamma(N_{R,1} \to \overline{E}_{L,\alpha} + \tilde{H}) - \Gamma(N^c_{R,1} \to \overline{E}^c_{L,\alpha} + \tilde{H}^c) \right]}{\Gamma_{N_1}} \\ &= \frac{1}{8\pi (Y_\nu^\dagger Y_\nu)_{11}} \sum_{j \neq 1} \mathrm{Im}\left[(Y_\nu^\dagger Y_\nu)^2_{1j} \right] f(x_{j1}) \\ &= \frac{1}{8\pi (Y_\nu^\dagger Y_\nu)_{11}} \left\{ \mathrm{Im}\left[(Y_\nu^\dagger Y_\nu)^2_{12} \right] f(x_{j2}) + \mathrm{Im}\left[(Y_\nu^\dagger Y_\nu)^2_{13} \right] f(x_{31}) \right\} \end{aligned} \tag{4-59}$$

The lepton asymmetry dynamically generated in an expanding universe can be written in terms of the partial decay widths for the heavy sterile fermion N_1 [16]:

$$\Gamma(N_{R,1} \to \overline{E}_{L,\alpha} + \tilde{H}) = \frac{1}{2}(1+\epsilon_1)\Gamma_{N_i} \tag{4-60}$$

$$\Gamma(N^c_{R,1} \to \overline{E}^c_{L,\alpha} + \tilde{H}^c) = \frac{1}{2}(1-\epsilon_1)\Gamma_{N_i} \tag{4-61}$$

And let us define: n_{N_1} as the number density of N_1, n_l the number density of leptons, and $n_{\bar{l}}$ the number density of anti-leptons. These number densities evolve out-of-equilibrium due to the conditions assumed above. The evolution in time of these density distributions is given by the Boltzmann equations, and to apply them it is necessary to assume first [62, 63]:

- Between scatterings the particles move freely in the gravitational field of an expanding universe.
- Interactions are described by quantum field theory at T=0.

In the following section it will be reviewed the necessary tools for this computation.

4.3.2. Boltzmann equations

To start with, let us rewritte the geodesic equation (4-1) in terms of the 4-momentum [63]:

$$\begin{aligned} \frac{d}{d\tau}\left(\frac{dx^\alpha}{d\tau}\right) + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} &= 0 \\ \frac{dp^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma}p^\beta p^\gamma &= 0 \end{aligned} \tag{4-62}$$

On the other hand, the Liouville operator is defined as [63]:

$$\hat{L} = p^\alpha \frac{\partial}{\partial x^\alpha} - \Gamma^\alpha_{\beta\gamma}p^\beta p^\gamma \frac{\partial}{\partial p^\alpha} \tag{4-63}$$

Thus, the geodesic equation can be written as [63]:

$$\frac{dp^\alpha}{d\tau} - \hat{L}(p^\alpha) = 0 \tag{4-64}$$

Now, considering a non-interacting gas of particles of type "ψ" which obeys this geodesic equation, i.e., it evolves freely in thermal equilibrium, then the phase space density has to be constant [63]:

$$\frac{df_\psi(x,p)}{d\tau} = 0 \tag{4-65}$$

Or in terms of the Liouville-operator restricted to the mass-shell of these particles ($p^2 = m^2$), hence the temporal equation of motion is a null geodesic and only spatial components are relevant [63]:

$$\hat{L}_m = p^\alpha \frac{\partial}{\partial x^\alpha} - \Gamma^i_{\beta\gamma} p^\beta p^\gamma \frac{\partial}{\partial p^i} \tag{4-66}$$

Therefore, equation (4-65) can be written as:

$$\hat{L}_m \left(f_\psi(x,p) \right) = 0 \tag{4-67}$$

In the FLRW cosmology model, presented in section 4.1, the phase space density $f_\psi(x,p)$ can only be a function of time t and momentum $|\vec{p}_\psi|$, and the above equation is reduced to [63]:

$$\hat{L}_m \left(f_\psi(x,p) \right) = E_\psi \frac{\partial f_\psi}{\partial t} - H(t)|\vec{p}_\psi|^2 \frac{\partial f_\psi}{\partial E_\psi} = 0 \tag{4-68}$$

Where $E_\psi = \sqrt{|\vec{p}_\psi|^2 + m_\psi^2}$ and $H(t)$ is the Hubble parameter.
However, at the departure from equilibrium ($t_c >> t_H$) the interactions can be included as a collision term $C[f_\psi]$ leading to the Boltzmann equation [63]:

$$\begin{aligned} \hat{L}_m \left(f_\psi(x,p) \right) &= C[f_\psi] \\ E_\psi \frac{\partial f_\psi}{\partial t} - H(t)|\vec{p}_\psi|^2 \frac{\partial f_\psi}{\partial E_\psi} &= C[f_\psi] \end{aligned} \tag{4-69}$$

And recalling that the number density n_ψ is given in terms of the phase space density by [50]:

$$n_\psi = g_\psi \int \frac{d^3 p_\psi}{(2\pi)^3 (2E_\psi)} f_\psi(x,p) \tag{4-70}$$

From the Boltzmann equation (4-69) it follows that [63]:

$$\frac{dn_\psi}{dt} + 3H(t) n_\psi = \frac{g_\psi}{(2\pi)^3} \int \frac{d^3 p_\psi}{E_\psi} C[f_\psi] \tag{4-71}$$

Therefore, in order to solve the above equation it is necessary to estimate the collision term first. This term counts the number of collision a particle ψ undergoes in a time and volume element [63]. Thus, let us suppose the process:

$$\psi + a + b + \cdots \to i + j + \cdots$$

And define the width decay as [63]:

$$\Gamma(\psi + a + b + \cdots \to i + j + \cdots) \equiv -\frac{g_\psi}{(2\pi)^3} \int \frac{d^3 p_\psi}{E_\psi} C[f_\psi]$$

$$= \int d\tilde{p}_\psi d\tilde{p}_a \cdots d\tilde{p}_i \cdots (2\pi)^4 \delta^4(p_\psi + p_a + p_b + \cdots - p_i - p_j - \cdots) |\mathcal{M}(\psi + a + b + \cdots \to i + j + \cdots)|^2 f_\psi f_a f_b (1 \pm f_i)(1 \pm f_j) \tag{4-72}$$

Where $+$ stands for bosons and $-$ for fermions.
Now, it is possible to make an approximation considering a dilute gas such that the number of particles per unit volume is very small [64]:

$$n_\psi V_\psi << 1 \tag{4-73}$$

Thus, the factors corresponding to fermionic or bosonic nature (1_i) can be neglected, and only decays of the form:

$$\psi \to i + j + \cdots$$

And two particles scattering:

$$\psi + a \to i + j + \cdots$$

And their corresponding back reactions, are considered to take place [64].
Thus, the Boltzmann equation for the number density n_ψ reads as follows:

$$\begin{aligned} \frac{dn_\psi}{dt} + 3H(t) n_\psi = &- \sum_{i,j} [\Gamma(\psi \to i + j + \cdots) - \Gamma(i + j + \cdots \to \psi)] \\ &- \sum_{a,i,j} [\Gamma(\psi + a \to i + j + \cdots) - \Gamma(i + j + \cdots \to \psi + a)] \end{aligned} \tag{4-74}$$

If elastic scatterings are assumed to occur at a higher rate than inelastic scatterings, then it is possible to assume kinetick equilibrium [64], i.e.:

$$f_\psi(E_\psi, T) = \frac{n_\psi}{n_\psi^{eq.}} e^{-E_\psi/T} \tag{4-75}$$

And in consequence, the Boltzmann equation takes the following form [64]:

$$\begin{aligned} \frac{dn_\psi}{dt} + 3H(t) n_\psi = &- \sum_{i,j} \left[\frac{n_\psi}{n_\psi^{eq.}} \Gamma^{eq.}(\psi \to i + j + \cdots) - \frac{n_i n_j}{n_i^{eq.} n_j^{eq.}} \Gamma^{eq.}(i + j + \cdots \to \psi) \right] \\ &- \sum_{a,i,j} \left[\frac{n_\psi n_a}{n_\psi^{eq.} n_a^{eq.}} \Gamma^{eq.}(\psi + a \to i + j + \cdots) - \frac{n_i n_j}{n_i^{eq.} n_j^{eq.}} \Gamma(i + j + \cdots \to \psi + a) \right] \end{aligned} \tag{4-76}$$

Where $\Gamma^{eq.}$ denotes that collisions take place in kinetic equilibrium.
Notice that kinetic equilibrium is not thermal equilibrium, it is still out-of-equilibrium dynamics, since when applying the Liouville operator (4-66) on the density function $f_\psi(x, p)$, given by (4-75), the result is different from 0.

Now, it is convenient to introduce the particle density Y_ψ as:

$$Y_\psi \equiv \frac{n_\psi}{s} \tag{4-77}$$

With s the entropy density, hence this quantity is not affected by the expansion of the universe because of the result in (4-5) ($ds = 0$). Thus, the particle density Y_ψ can be understood as the number of particles in a co-moving volume element. With this definition, the Boltzmann equation can be rewritten as [64]:

$$\begin{aligned} s\frac{dY_\psi}{dt} + 3sH(t)Y_\psi = &-\sum_{i,j}\left[\frac{n_\psi}{n_\psi^{eq.}}\Gamma^{eq.}(\psi \to i+j+\cdots) - \frac{n_i n_j}{n_i^{eq.}n_j^{eq.}}\Gamma^{eq.}(i+j+\cdots \to \psi)\right] \\ &-\sum_{a,i,j}\left[\frac{n_\psi n_a}{n_\psi^{eq.}n_a^{eq.}}\Gamma^{eq.}(\psi + a \to i+j+\cdots) - \frac{n_i n_j}{n_i^{eq.}n_j^{eq.}}\Gamma(i+j+\cdots \to \psi + a)\right] \end{aligned} \tag{4-78}$$

4.3.3. Lepton number asymmetry

Based on the results from the last section, the Boltzmann equations for the number densities of heavy neutrinos (n_{N_1}), leptons (n_L), and antileptons ($n_{\overline{L}}$) are given respectively by [64]:

$$\begin{aligned} \frac{dn_{N_1}}{dt} + 3H(t)n_{N_1} = \sum_\alpha \Big[&-\Gamma(N_{R,1} \to \overline{E}_{L,\alpha} + \tilde{H}) + \Gamma(\overline{E}_{L,\alpha} + \tilde{H} \to (N_{R,1}) \\ &-\Gamma(N_{R,1} \to \overline{E}^c_{L,\alpha} + \tilde{H}^c) + \Gamma(\overline{E}^c_{L,\alpha} + \tilde{H}^c \to N_{R,1})\Big] \end{aligned} \tag{4-79}$$

$$\begin{aligned} \frac{dn_L}{dt} + 3H(t)n_L = \sum_\alpha \Big[&\Gamma(N_{R,1} \to E_{L,\alpha} + H) - \Gamma(E_{L,\alpha} + H \to N_{R,1}) \\ &+\Gamma(\overline{E}_{L,\alpha} + \tilde{H} \to E_{L,\alpha} + H) - \Gamma(E_{L,\alpha} + H \to \overline{E}_{L,\alpha} + \tilde{H})\Big] \end{aligned} \tag{4-80}$$

$$\begin{aligned} \frac{dn_{\overline{L}}}{dt} + 3H(t)n_{\overline{L}} = \sum_\alpha \Big[&\Gamma(N_{R,1} \to \overline{E}_{L,\alpha} + \tilde{H}) - \Gamma(\overline{E}_{L,\alpha} + \tilde{H} \to N_{R,1}) \\ &-\Gamma(E_{L,\alpha} + H \to \overline{E}_{L,\alpha} + \tilde{H}) + \Gamma(\overline{E}_{L,\alpha} + \tilde{H} \to E_{L,\alpha} + H)\Big] \end{aligned} \tag{4-81}$$

And taking the difference between Boltzmann equations for n_L (4-80) and $n_{\overline{L}}$ (4-81), and dividing by the entropy density s, it follows that:

$$\begin{aligned} \frac{dY_{\Delta L}}{dt} + 3H(t)Y_{\Delta L} &= \frac{\Gamma_{N_1}\epsilon_1}{s} \\ &= \frac{\Gamma_{N_1}}{s}\frac{1}{8\pi(Y_\nu^\dagger Y_\nu)_{11}}\left\{ \mathrm{Im}\left[(Y_\nu^\dagger Y_\nu)^2_{12}\right] f(x_{j2}) + \mathrm{Im}\left[(Y_\nu^\dagger Y_\nu)^2_{13}\right] f(x_{31})\right\} \end{aligned} \tag{4-82}$$

The above equation cannot be solved analytically at all, but under some considerations, such as kinetic equilibrium in a quasistatic evolution, its solution has been estimated to [19, 65]:

$$\begin{aligned} Y_{\Delta L} &\approx k\frac{\epsilon_1}{g_{*s}} \\ &\approx \frac{k}{g_{*s}}\frac{1}{8\pi(Y_\nu^\dagger Y_\nu)_{11}}\left\{ \text{Im}\left[(Y_\nu^\dagger Y_\nu)^2_{12}\right] f(x_{j2}) + \text{Im}\left[(Y_\nu^\dagger Y_\nu)^2_{13}\right] f(x_{31})\right\} \end{aligned} \tag{4-83}$$

Where g_{*s} is given in terms of the quotient between fermion temperatures and thermal bath (radiation) temperature (recall equation (4-24)), and the factor k is an efficiency factor which stands for the effect of washout processes. This factor must be determined and fixed according to the model, and for thermal leptogenesis via type-I seesaw model is expected to be about $10^{-3} \leq k \leq 1$ [66]. Also, in the type-I seesaw model is taken the estimation $g_{*s} \approx 3{,}9$ [52].
Then, sphalerons convert effectively the lepton-number asymmetry into a baryon-number asymmetry because of the result (4-46) [19], and through the thermal leptogenesis it is found that [63, 67]:

$$Y_{\Delta B} = -CY_{\Delta L} \tag{4-84}$$

With the coefficient C given by [63, 67]:

$$C = \frac{8N_f + 4N_H}{22N_f + 13N_H} \tag{4-85}$$

Being N_f the number of fermion families, and N_H the number of Higgs doublets considered in the model. For this case $N_f = 3$, and $N_H = 1$, therefore $C = 28/79 \approx 0{,}354$. Thus, substituting the result for the lepton asymmetry found in (4-83) into $Y_{\Delta B}$ is found that:

$$Y_{\Delta B} \approx (-0{,}0908)k\epsilon_1 \tag{4-86}$$

Therefore, the above expression is a direct way to compute the baryon asymmetry of the universe. However, since the CP-asymmetry parameter ϵ_1 depends on active-sterile mixing parameters which have not been measured properly, this estimation cannot be done accurately. Future experiments in neutrino oscillations and CP violation will help us to compute the above result and get a result to contrast with the observed in (4-31).

5. Peccei-Quinn symmetry and QCD axions

5.1. Strong CP problem and Peccei-Quinn symmetry

The Standard Model preserves CP symmetry despite of explicit parity (P) and change-conjugated (C) violation. This is because of gauge couplings to chiral currents in the electroweak sector, CP violation is performed by Kobayashi Mashawa mechanism. The above means that QCD (at least to perturbative level) is also CP-invariant. Now, on the other hand, in non Abelian gauge theories there exist topological solutions to the field equations, labeled by θ, which has the form [68]:

$$S = \int d^4x(\mathcal{L}(\psi, B_\mu, t) + i\theta\dot{\psi}) \tag{5-1}$$

Where $i\theta\dot{\psi}$ is a new phase.
This implies a more complicated vacuum state, with infinty of possible states, labeled by the parameter θ [68].
Thus, the vacuum-to-vacuum transition amplitude is as follows [69]:

$$\langle 0|0\rangle = \sum_{a=0}^{\infty} \int (d\beta_\mu)_a \int d\phi \exp\{[\mathcal{L}(\psi, B_\mu, t)]\} e^{iq\theta} \tag{5-2}$$

Where the integration is over all the configurations. This surface term satisfy the bondary condition [69, 70]:

$$\frac{g^2}{32\pi^2} \int d^4x F^a_{\mu\nu} F^{\nu a\mu\nu} = q \propto \Delta N_{cs} \tag{5-3}$$

This solution can be gotten from an extra term on the lagrangian [69, 70]:

$$\mathcal{L}_{eff} = \mathcal{L} + i\theta \frac{g^2}{32\pi^2} \int d^4x F^a_{\mu\nu} F^{\nu a\mu\nu} \tag{5-4}$$

Now, given the fact that CP violation has never been observed on any strong interactions, it is natural to ask if such a CP-violating term, which its origin is topological, should in exist in the Standard Model. This is a big puzzle in theoretical physics known as the strong CP problem.

Explicitly, the QCD lagrangian is given by [31]:

$$\mathcal{L}_{QCD} = \sum_i (i\bar{q}_i\gamma^\mu D_\mu q_i + i\bar{q}'_i\gamma^\mu D_\mu q'_i) - \sum_{i,j}(\bar{q}_{i,L}M^u_{i,j}q_{j,R} + \bar{q}'_{i,L}M^d_{i,j}q'_{j,R}) + \left(-\frac{1}{4} + \theta\frac{g^2}{32\pi^2}\right) G^a_{\mu\nu}G^{a,\mu\nu} + h.c. \tag{5-5}$$

Where the topological θ-vacuum is irrelevant to the classical equations of motion and perturbative extensions of QCD. However it may produce non-perturbative effects at high energies, the well known instanton and sphaleron processes [69, 70].
Nevertheless, θ should be null because of the no evidence of CP violation in strong interactions. Furthermore, the θ-term produces an electric dipole moment for the neutron of the order [71]:

$$d_n \approx \frac{em_q}{m_n^2}\theta \approx 10^{-16}\theta e.m \tag{5-6}$$

And from experimental constraints, this implies that $\theta \leq 10^{-10}$ [72].
This issue goes worse if the effect of chiral transformations on the θ-vacuum it is considered. Diagonalizing the quarks mass matrices in $\mathcal{L}_{QCD}$ [31]:

$$\mathcal{L}_{q-mass} = \overline{\begin{pmatrix} u & c & t \end{pmatrix}}_L D_u \begin{pmatrix} u \\ c \\ t \end{pmatrix}_R + \overline{\begin{pmatrix} d & s & b \end{pmatrix}}_L D_d \begin{pmatrix} d \\ s \\ b \end{pmatrix}_R + h.c. \tag{5-7}$$

Where:

$$D_u = \mathrm{diag}(m_u, m_c, m_t)$$
$$D_d = \mathrm{diag}(m_d, m_s, m_b)$$

Recall that CP violation has been shifted from Yukawa-interaction term to charged current-interactions.
Now, performing the chiral transformations for the fields:

$$q_i \to \exp\{(i\alpha_i\gamma_s)\}q_i \tag{5-8}$$

For $i = u, c, t, d, s, b$, and by using Taylor expansion:

$$\exp\{(i\alpha_i\gamma_s)\} = \sum_{n=0}^{\infty}\frac{(i\alpha_i\gamma_s)^n}{n!} \tag{5-9}$$

$$= \sum_{n=0}^{\infty}\frac{(i\alpha_i\gamma_s)^{2n}}{2n!} + \sum_{n=1}^{\infty}\frac{(i\alpha_i\gamma_s)^{2n-1}}{(2n-1)!} \tag{5-10}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(\alpha_i)^{2n}\mathbb{I}^n}{2n!} + i\gamma_s\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(\alpha_i)^{2n-1}\mathbb{I}^{n-1}}{(2n-1)!} \tag{5-11}$$

$$= \cos\alpha_i + i\gamma_s\sin\alpha_i \tag{5-12}$$

Thus, performing these chiral transformations on quark fields:

$$q_{i,L} \rightarrow \exp\{(i\alpha_i\gamma_s\}q_{i,L} \tag{5-13}$$
$$= \cos\alpha_i q_{i,L} + i\gamma_s \sin\alpha_i q_{i,L} \tag{5-14}$$
$$= \cos\alpha_i q_{i,L} + i\sin\alpha_i \frac{\gamma_s - 1}{2} q_i \tag{5-15}$$
$$= \cos\alpha_i q_{i,L} - i\sin\alpha_i \frac{1-\gamma_s}{2} q_i \tag{5-16}$$
$$= e^{i\alpha_i} q_{i,L} \tag{5-17}$$

And hence $q_{i,L} \rightarrow e^{i\alpha_i} q_{i,L}$.

$$q_{i,R} \rightarrow \exp\{(i\alpha_i\gamma_s\}q_{i,R} \tag{5-18}$$
$$= \cos\alpha_i q_{i,R} + i\gamma_s \sin\alpha_i q_{i,R} \tag{5-19}$$
$$= \cos\alpha_i q_{i,R} + i\sin\alpha_i \frac{\gamma_s - 1}{2} q_i \tag{5-20}$$
$$= \cos\alpha_i q_{i,R} - i\sin\alpha_i \frac{1-\gamma_s}{2} q_i \tag{5-21}$$
$$= e^{i\alpha_i} q_{i,R} \tag{5-22}$$

And therefore $q_{i,R} \rightarrow e^{i\alpha_i} q_{i,R}$.
Now, applying this chiral transformation in the mass terms:

$$\mathcal{L}_{q-mass} = \overline{\begin{pmatrix} u & c & t \end{pmatrix}}_L e^{i\alpha_i} D_u e^{i\alpha_i} \begin{pmatrix} u \\ c \\ t \end{pmatrix}_R + \overline{\begin{pmatrix} d & s & b \end{pmatrix}}_L e^{i\alpha_i} D_d e^{i\alpha_i} \begin{pmatrix} d \\ s \\ b \end{pmatrix}_R + h.c. \tag{5-23}$$

$$= \overline{\begin{pmatrix} u & c & t \end{pmatrix}}_L D'_u \begin{pmatrix} u \\ c \\ t \end{pmatrix}_R + \overline{\begin{pmatrix} d & s & b \end{pmatrix}}_L D'_d \begin{pmatrix} d \\ s \\ b \end{pmatrix}_R + h.c. \tag{5-24}$$

With $D'_u = e^{2i\alpha_i} D_u$. $D'_d = e^{2i\alpha_i} D_d$ and $D'^{\dagger}_u = e^{2i\alpha_i} D^{\dagger}_u$, $D'^{\dagger}_d = e^{2i\alpha_i} D^{\dagger}_d$, which is equivalent to:

$$m_i \rightarrow e^{2i\alpha_i} m_i \tag{5-25}$$

This means that:

$$arg(detD_u) + arg(detD_d) \rightarrow arg(detD_d) + 2\sum \alpha_i \tag{5-26}$$
$$arg(detM_u) + arg(detM_d) \rightarrow arg(detM_d) + 2\sum \alpha_i \tag{5-27}$$

Quark masses explicitly breaks the chiral symetry. Thus recalling the well known Adler-Bell-Jackiew chiral canocaly realtion [73]:

$$\partial_\mu(\bar{q}_i\gamma^\mu\gamma_s q_i) = 2im_i\bar{q}_i\gamma_s q_i + \frac{g_s^2}{16\pi} G^a_{\mu,\nu}\tilde{G}^{a,\mu\nu} \tag{5-28}$$

From which follows that [20]:

$$\theta \to \theta - 2\sum_i \alpha_i \tag{5-29}$$

Therefore, for the above results, the quantity defined as :

$$\bar{\theta} \equiv \theta + arg(detM_u) + arg(det(M_d) \tag{5-30}$$

Must be covariant under chiral transformations of the quark fields [20].
Moreover, notice that $\bar{\theta}$ came from strong CP violation (θ) and weak CP violation (through phases on M_d).
Thus the QCD lagrangian can be written in the quark mass basis as [31]:

$$\mathcal{L}'_{QCD} = \sum_{i=1}^{6}(\bar{q}_i - m_i)q_i - \frac{1}{4}G\mu\nu^a G^{a,\mu\nu} + \theta\frac{\alpha_s}{8\pi}G^a_{\mu,\nu}\tilde{G}^{a,\mu\nu} \tag{5-31}$$

Where $\alpha_s \equiv \frac{g_s^2}{4\pi}$ is the analogy to fine structure constant of QED. In principle, a comparison between the strengths of weak and strong CP violation in the Standard Model should make sense. However, on practice it is very difficult to choose a proper measure of either of them, because energy scales and flavor parameters. Even so, it can be found at least the dependence of the CP-violation for each case [7]:

$$\begin{aligned} Q_{weak} \sim & \frac{1}{\Lambda^6_{EW}}(m_u - m_c)(m_C - m_t)(mt - m_u)(m_d - m_s)(m_s - m_b)(m_b - m_d)\mathcal{J}_q \\ & \sim 10^{-13} \end{aligned} \tag{5-32}$$

$$Q_{strong} \sim \frac{1}{\Lambda^6_{QCD}} m_u m_c m_t m_d m_s m_b \sin\bar{\theta} \sim 10^4 \sin\bar{\theta} < 10^{-16} \tag{5-33}$$

In which $\Lambda_{EW} \sim 10^2 GeV$, $\Lambda_{QCD} \sim 0{,}2GeV$, and the Jarlskog invariant $\mathcal{J}_q \cong 3{,}2 \times 10^{-5}$.
Thus, it has been shown one posible solution to the strong CP problem. A chiral symmetry drives $\theta \to 0$, which introduces a new global, spontaneously broken symmetry,the Peccei-Quinn symmetry [20].
The lagrangian $\mathcal{L}_{QCD}$ must posses a chiral $U(1)$ invariance, such that changes in θ are equivalent to changes in the definitions of the fields divided in the lagrangian leading to a $\theta = 0$ theory, and this has no strong P and CP violations.
Let us consider a toy model of the strong interactions in which there is only a single fermion flauvour which acquires its mass from a Yukawa coupling to a color-singlet complex scalar [74]:

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu} + \bar{\Psi}D\Psi + \bar{\Psi}(G\phi P_R + \dagger G \dagger \phi P_L)\Psi + |\partial_\mu \phi|^2 - V(\phi) \tag{5-34}$$

With the scalar potential is $V(\phi) = \mu^2|\psi|^2 + h|\phi|^4$, $\mu^2 > 0$, and the chiral projectors are $P_L = \frac{1-\gamma_S}{2}$, $P_R = \frac{1+\gamma_S}{2}$.
The scalar vacuum expectation value will be [74]:

$$\langle\phi\rangle_0 = \lambda e^{i\beta} \sim \frac{1}{Z_\theta}\frac{\partial Z_\theta}{\partial\rho}|_{J=J^*=0} \tag{5-35}$$

Where λ and β are real constants to be determined. Considering perturbations around this minimum:

$$\begin{aligned}\phi &= \langle\phi\rangle_0 + \delta\phi \\ &= e^{i\beta}(\lambda + \rho + i\sigma)\end{aligned}$$

And imposing that the minimum remains same, then [20, 74]:

$$0 = \langle\rho\rangle = \int d\rho \int d\sigma\rho[A_0 + \sum_n F_n \cos n\alpha] \tag{5-36}$$

$$0 = \langle\sigma\rangle = \int d\rho \int d\sigma\sigma^2[\sum_n F_n \sin n\alpha] \tag{5-37}$$

Where [20, 74]:

$$\begin{aligned}F_n &= Re(A_n|G|^n(\lambda + \rho + i\sigma)) \\ \sigma G_n &= Im(A_n|G|^n(\lambda + \rho + i\sigma))\end{aligned}$$

A_n are polynomials of the form [20, 74]:

$$A_n(\phi\phi^*) = \sum_m \prod_{i=1}^{m} \left[\int dx_i \int dy_i \phi(x_i)\phi^*(y_i)\right] C : m^n(x_i, y_i) \tag{5-38}$$

And α has been expresed as [20, 74]:

$$\alpha = \arg G e^{c(\alpha+\beta)} \tag{5-39}$$

Thus, a solution of the equation (5-37) is arrived from $\alpha = 0, \pi$, which also can be a solution of equation 5-36 by setting A_0 [20, 74].

5.2. Peccei-Quinn mechanism and QCD axion models

Let us consider an extension to the Standard Model with two Higgs doublets H_u and H_d in such a way that H_u couples to up-type quarks and H_d to down-type quarks.

Considering just the quark sector, the Higgs-particle-fermion interaction lagrangian is of the form [21, 31]:

$$\begin{aligned}\mathcal{L}_{\text{int}} &= \mathcal{L}_Y + \mathcal{L}_{H_u,H_d} \\ &= \left(-y_u \bar{Q}_L H_u u_R - y_d \bar{Q}_L H_d^c d_R + h.c.\right) + \mathcal{L}_{H_u,H_d}\end{aligned} \tag{5-40}$$

Where:

$$\mathcal{L}_{H_u,H_d} = (D^\mu H_u)(D_\mu H_u)^\dagger + (D^\mu H_d)(D_\mu H_d)^\dagger - V(H_u, H_d) \tag{5-41}$$

And:

$$\begin{aligned}V(H_u, H_d) &= -\rho_u^2 H_u H_u^\dagger + \lambda_u \left(H_u H_u^\dagger\right)^2 - \rho_d^2 H_d H_d^\dagger + \lambda_d \left(H_d H_d^\dagger\right)^2 \\ &+ \mathcal{O}(H_u H_d + H_u H_d^\dagger + h.c. + \text{ other powers})\end{aligned} \tag{5-42}$$

Notice that the extra Higgs doublet allows to ensure the Peccei-Quinn symmetry, which is built thanks to the chiral symmetry in $\mathcal{L}_{int}$:

- $$\begin{aligned}Q_{L,i} \to Q'_{L,i} &= e^{i\alpha_i\gamma_5} Q_{L,i} \\ &= e^{-i\alpha_i} Q_{L,i}\end{aligned} \qquad \Leftrightarrow \qquad \overline{Q_{L,i}} \to \overline{Q'_{L,i}} = e^{i\alpha_i}\overline{Q_{L,i}}$$

- $$\begin{aligned}u_{R,i} \to u'_{R,i} &= e^{i\alpha_i\gamma_5} u_{R,i} \\ &= e^{i\alpha_i} u_{R,i}\end{aligned} \qquad \Leftrightarrow \qquad \overline{u_{R,i}} \to \overline{u'_{R,i}} = e^{-i\alpha_i}\overline{u_{R,i}}$$

- $$H_u \to H'_u = e^{-2i\alpha_i} H_u \qquad \Leftrightarrow \qquad H_u^\dagger \to H_u^{\dagger\prime} = e^{2i\alpha_i} H_u^\dagger$$

- $$\begin{aligned}d_{R,i} \to d'_{R,i} &= e^{i\beta_i\gamma_5} d_{R,i} \\ &= e^{i\beta_i} d_{R,i}\end{aligned} \qquad \Leftrightarrow \qquad \overline{d_{R,i}} \to \overline{d'_{R,i}} = e^{-i\beta_i}\overline{d_{R,i}}$$

-

$$\begin{aligned} H_d \to H'_d &= e^{i(\alpha_i+\beta_i)} H_d \\ &\equiv e^{i\delta_i} H_d \end{aligned} \qquad \Leftrightarrow \qquad H^c_d \to H'^c_d = e^{-i\delta_i} H^c_d$$

Notice that these rules enforce to make:

$$\mathcal{O}\left(H_u H_d + H_u H_d^\dagger + h.c. + \text{other powers}\right) = 0 \tag{5-43}$$

AS there will always remain a phase of the form:

$$a\alpha_i + b\beta_i \qquad \text{with } a, b \in \mathbb{N} \tag{5-44}$$

And since α_i and β_i are independent phases, their linear combination will never be zero. This is imposed in order to preserve the Peccei-Quinn symetry in the whole lagrangian. However, notice that terms of the form:

$$H_u H_u^\dagger h_d H_d^\dagger + \text{ higher powers} \tag{5-45}$$

Would be allowed in the potential:

$$\Rightarrow \text{Independent phases} = H_u \text{ and } H_d\ U(1)_{PQ} \text{ charge differ} \tag{5-46}$$

Is this a necessary condition? From [75]:

> “Although $V(H_u, H_d)$, even corrected by all radiactive correctors due to the local interactions in the lagrangian, is insensitive to the relative phase of H_u and H_d, this degeneracy is lifted when instantons effects are taken into account (fig. **5-1**).
>
> Thus, the relative phase og H_u and H_d is therefore dynamically determined by minimizing the Higgs potential $V(H_u, V_d)$”.

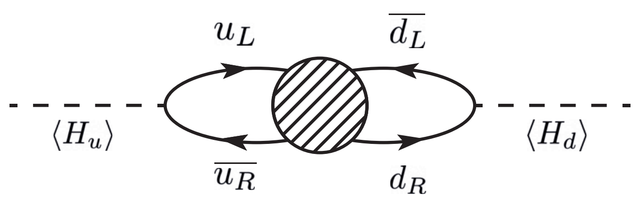

Figure 5-1.: Instanton interaction generating the axion mass.

Now, considering the Higgs potential to be:

$$V(H_u, H_d) = -\rho_u^2 H_u H_u^\dagger + \lambda_u \left(H_u H_u^\dagger\right)^2 - \rho_d^2 H_d H_d^\dagger + \lambda_d \left(H_d H_d^\dagger\right)^2 \tag{5-47}$$

$$= V(H_u) + V(H_d) \to \text{Two independent Higgs potentials} \tag{5-48}$$

Since it can be divided into the sum of two independent potentials (complex factors), its minimum will be achieved where both potentials reach their minimum:

$$V_{\min}(H_u, H_d) = V_{\min}(H_u) + V_{\min}(H_d) \tag{5-49}$$

And assuming:

$$H_u = \begin{pmatrix} h_u^0 \\ h_u^\dagger \end{pmatrix} \qquad H_d = \begin{pmatrix} h_d^0 \\ h_d^\dagger \end{pmatrix} \tag{5-50}$$

After minimization of the Higgs potential it is found that:

- $$H_{u,\min} = \langle H_u \rangle = \begin{pmatrix} \langle h_u^0 \rangle \\ \langle h_u^\dagger \rangle \end{pmatrix} \tag{5-51}$$
  $$= \begin{pmatrix} \sqrt{\frac{\rho_u^2}{2\lambda}} e^{i\alpha_u} \\ 0 \end{pmatrix} \tag{5-52}$$
  $$\Rightarrow \langle H_u \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} v_u \\ 0 \end{pmatrix} e^{i\alpha_u} \qquad ; \qquad v_u = \sqrt{\frac{\rho_u^2}{\lambda_u}} \tag{5-53}$$
  $$\equiv \langle h_u^0 \rangle \tag{5-54}$$

- $$H_{d,\min} = \langle H_d \rangle = \begin{pmatrix} \langle h_d^0 \rangle \\ \langle h_d^\dagger \rangle \end{pmatrix} \tag{5-55}$$
  $$= \begin{pmatrix} \sqrt{\frac{\rho_d^2}{2\lambda}} e^{i\alpha_d} \\ 0 \end{pmatrix} \tag{5-56}$$
  $$\Rightarrow \langle H_d \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} v_d \\ 0 \end{pmatrix} e^{i\alpha_d} \qquad ; \qquad v_d = \sqrt{\frac{\rho_d^2}{\lambda_d}} \tag{5-57}$$
  $$\equiv \langle h_d^0 \rangle \tag{5-58}$$

Thus, quarks acquire masses through Yukawa couplings after spontaneous symmetry breaking [21]:

$$m_u \sim \frac{|y_u| v_u}{\sqrt{2}} \tag{5-59}$$

$$m_d \sim \frac{|y_d| v_d}{\sqrt{2}} \tag{5-60}$$

Notice that the two separete phase rotations associated to the $U(1)_{PQ}$ symetry α_1 and $\delta_i = \alpha_i + \beta_i$, give two Nambo-Goldstone bosons [22], since the vacuum states $\langle h_u^0 \rangle$ and $\langle h_d^0 \rangle$ are not invariant under them.
Furthermore, those two independent phase rotations are closely related to α_u and α_d. Therefore:

$$H_u = \begin{pmatrix} \langle h_u^o \rangle + h_u \\ 0 \end{pmatrix}$$

$\Rightarrow$ The fields can be written around their minimum (vacuum) state

$$H_d = \begin{pmatrix} \langle h_d^o \rangle + h_d \\ 0 \end{pmatrix}$$

In such a way that $\langle h_u \rangle = 0 = \langle h_d \rangle$, with h_u and h_d real scalar fields.
Now, since $\langle h_u^0 \rangle$ and $\langle h_d^0 \rangle$ are independent states, then H_u and H_d form a linearly independent basis for the scalar sector in the SM.
Therefore, the Standard Model Higgs field can be written in terms of H_u and H_d [76]. Let us proceed as follows:
Performing a change of basis $(H_u, H_d) \to (H, H')$ with [76]:

$$\begin{aligned} H &\equiv \frac{\langle H_u \rangle H_u + \langle H_d \rangle H_d}{\sqrt{\langle H_u \rangle^2 + \langle H_d \rangle^2}} \\ H' &=\equiv \frac{\langle H_d \rangle H_u + \langle H_u \rangle H_d}{\sqrt{\langle H_u \rangle^2 + \langle H_d \rangle^2}} \end{aligned} \tag{5-61}$$

Thus, explicitly [76]:

- $$\begin{aligned} H &= \frac{\langle h_u^0 \rangle H_u + \langle h_d^0 \rangle H_d}{\sqrt{\langle h_u^0 \rangle^2 + \langle h_d^0 \rangle^2}} \\ &= \frac{v_u H_u e^{i\alpha_u} + v_d H_d e^{i\alpha_d}}{\sqrt{v_u^2 e^{i2\alpha_u} + v_d^2 e^{i2\alpha_d}}} \end{aligned} \tag{5-62}$$

  And in the vacuum state:

  $$\begin{aligned} \langle H \rangle &= \frac{v_u \langle H_u \rangle e^{i\alpha_u} + v_d \langle H_d \rangle e^{i\alpha_d}}{\sqrt{v_u^2 e^{i2\alpha_u} + v_d^2 e^{i2\alpha_d}}} \\ &= \frac{v_u \left(\frac{1}{\sqrt{2}} v_u e^{i\alpha_u} \right) e^{i\alpha_u} + v_d \left(\frac{1}{\sqrt{2}} v_d e^{i\alpha_d} \right) e^{i\alpha_d}}{\sqrt{v_u^2 e^{i2\alpha_u} + v_d^2 e^{i2\alpha_d}}} \\ &= \frac{1}{\sqrt{2}} \frac{v_u^2 e^{2i\alpha_u} + v_d^2 e^{2i\alpha_d}}{\sqrt{v_u^2 e^{i2\alpha_u} + v_d^2 e^{i2\alpha_d}}} \\ &= \frac{1}{\sqrt{2}} \sqrt{v_u^2 e^{2i\alpha_u} + v_d^2 e^{2i\alpha_d}} \end{aligned} \tag{5-63}$$

Now, in order to recover the Standard Model Higgs field it is necessary to choose a particular direction for each vacuum $\langle h_u^0 \rangle$ and $\langle h_d^0 \rangle$, i.e., make $\alpha_u = 0 = \alpha_d$ (for convenience).

Thus:

$$\begin{aligned}\langle H \rangle &= \frac{1}{\sqrt{2}}\sqrt{v_u^2 + v_d^2} \\ &= \frac{1}{\sqrt{2}} v\end{aligned} \tag{5-64}$$

- H' is defined as [76]:

$$\begin{aligned}H' &= \frac{\langle h_d^0 \rangle H_u - \langle h_u^0 \rangle H_d}{\sqrt{\langle h_u^0 \rangle^2 + \langle h_d^0 \rangle^2}} \\ &= \frac{v_d H_u - v_u H_d}{\sqrt{v_u^2 + v_d^2}}\end{aligned} \tag{5-65}$$

And in the vacuum state:

$$\begin{aligned}\langle H' \rangle &= \frac{v_d \langle H_u \rangle - v_u \langle H_d \rangle}{\sqrt{v_u^2 + v_d^2}} \\ &= \frac{1}{\sqrt{2}} \frac{v_d v_u - v_u v_d}{\sqrt{v_u^2 + v_d^2}}\end{aligned} \tag{5-66}$$

$$\Rightarrow \langle H' \rangle = 0 \tag{5-67}$$

Thus, H would correspond to the usual Standard Model Higgs doublet, and H' would be a massless doublet scalar which remains after spontaneous electroweak symmetry breaking. Let us define [76]:

$$\tan\beta = \frac{\langle H_u \rangle}{\langle H_d \rangle} = \frac{v_u}{v_d} \tag{5-68}$$

Hence the fields H and H' can be rewritten as [76]::

- $$H = \frac{v_u}{\sqrt{v_u^2}} H_u + \frac{v_d}{\sqrt{v_u^2}} H_d \tag{5-69}$$

$$H = \sin\beta H_u + \cos\beta H_d \tag{5-70}$$

- $$H' = \frac{v_d}{\sqrt{v_u^2}} H_u - \frac{v_u}{\sqrt{v_u^2}} H_d \tag{5-71}$$

$$H' = \cos\beta H_u - \sin\beta H_d \tag{5-72}$$

Now, it is straightforward to rewritte the whole lagrangian $\mathcal{L}_{int}$ in eq. (5-40) in terms of the scalar doublets H and H'. For that purpose, let us find the inverse change of basis from the above relations (5-70) and (5-72):

$$H_u = \sin\beta H + \cos\beta H' \tag{5-73}$$

$$H_d = \cos\beta H - \sin\beta H' \tag{5-74}$$

By substituting the results (5-73) and (5-74) in equation (5-40) it is found that:

$$\mathcal{L}_{int} = -y_u \bar{Q}_L (\sin\beta H + \cos\beta H') u_R - y_d \bar{Q}_L (\cos\beta H - \sin\beta H') d_R + h.c. + \mathcal{L}_{H,H'} \tag{5-75}$$

Now, to implement the Higgs mechanism, first recall that [76]::

$$H = \langle H \rangle + \delta H = \frac{1}{\sqrt{2}} \begin{pmatrix} v + h(x) \\ 0 \end{pmatrix} \tag{5-76}$$

$$H' = \langle H' \rangle + \delta H = \frac{1}{\sqrt{2}} \begin{pmatrix} h'(x) \\ 0 \end{pmatrix} \tag{5-77}$$

With $h(x)$ and $h'(x)$ real scalar fields.
Therefore, replacing this in the Yukawa interactions, it follows that:

$$\begin{aligned}
\mathcal{L}_y = & - y_u (\bar{u}_L \bar{d}_L) \left(\frac{\sin\beta}{\sqrt{2}} \begin{pmatrix} v + h(x) \\ 0 \end{pmatrix} + \frac{\cos\beta}{\sqrt{2}} \begin{pmatrix} h'(x) \\ 0 \end{pmatrix} \right) u_R \\
& - y_d \bar{Q}_L \left(\frac{\cos\beta}{\sqrt{2}} \begin{pmatrix} v + h(x) \\ 0 \end{pmatrix} - \frac{\sin\beta}{\sqrt{2}} \begin{pmatrix} h'(x) \\ 0 \end{pmatrix} \right) d_R + h.c. \\
= & \underbrace{-y_u \frac{\sin\beta}{\sqrt{2}} v \bar{u}_L u_R - y_d \frac{\cos\beta}{\sqrt{2}} v \bar{d}_L d_R}_{\text{mass-terms}} \underbrace{-y_u \frac{\sin\beta}{\sqrt{2}} \bar{u}_L h u_R - y_d \frac{\cos\beta}{\sqrt{2}} \bar{d}_L h d_R}_{\text{Higgs coupling}} \\
& \underbrace{-y_u \frac{\cos\beta}{\sqrt{2}} \bar{u}_L h' u_L + y_d \frac{\sin\beta}{\sqrt{2}} \bar{d}_L h' d_L}_{\text{New couplings to a scalar field } h'} + h.c.
\end{aligned} \tag{5-78}$$

Now, let us look the scalar potential:

$$\begin{aligned}
V(H_u, H_d) = & - \rho_u^2 (\sin\beta H + \cos\beta H')(\sin\beta H + \cos\beta H')^\dagger \\
& - \rho_d^2 (\cos\beta H - \sin\beta H')(\cos\beta H - \sin\beta H')^\dagger + \mathcal{O}(v^4)
\end{aligned} \tag{5-79}$$

And searching for mass terms, i.e., after Higgs mechanism was implemented:

$$
\begin{aligned}
=& -\rho_u^2\left(\frac{\sin\beta}{\sqrt{2}}\begin{pmatrix}h\\0\end{pmatrix}+\frac{\cos\beta}{\sqrt{2}}\begin{pmatrix}h'\\0\end{pmatrix}\right)\left(\frac{\sin\beta}{\sqrt{2}}(h\quad 0)+\frac{\cos\beta}{\sqrt{2}}(h'\quad 0)\right)\\
&-\rho_d^2\left(\frac{\cos\beta}{\sqrt{2}}\begin{pmatrix}h\\0\end{pmatrix}-\frac{\sin\beta}{\sqrt{2}}\begin{pmatrix}h'\\0\end{pmatrix}\right)\left(\frac{\cos\beta}{\sqrt{2}}(h\quad 0)-\frac{\sin\beta}{\sqrt{2}}(h'\quad 0)\right)\\
=& -\frac{\rho_u^2\sin^2\beta}{2}h^2-\frac{\rho_u^2\cos^2\beta}{2}h'^2-\frac{\rho_d^2\cos^2\beta}{2}h^2-\frac{\rho_d^2\sin^2\beta}{2}h'^2\\
&+\frac{2\rho_u^2\sin\beta\cos\beta}{2}hh'+\frac{2\rho_d^2\sin\beta\cos\beta}{2}hh'\\
=& -\frac{1}{2}\left(\rho_u^2\sin^2\beta+\rho_d^2\cos^2\beta\right)h^2-\frac{1}{2}\left(\rho_u^2\cos^2\beta+\rho_d^2\sin^2\beta\right)h'^2\\
&+\left(\rho_u^2+\rho_d^2\right)\sin\beta\cos\beta hh'
\end{aligned}
\tag{5-80}
$$

Thus, the two real scalar fields h and h' acquire their masses in terms of the energy parameters (ρ_u, ρ_d, λ_u, $\lambda_d \leftrightarrow v_u$, v_d) of the theory.
h is expected to be the Standard Model Higgs boson, since $\langle h\rangle = v \neq 0$. On the other hand, h' would be a new scalar field of the theory, with $\langle h'\rangle = 0$, and with a mass of the order of h.
Wilczek called to this extra scalar real field h' the "axion", since it clears the strong CP through the Peccei-Quinn mechanism [75], which is shown in Figure **5-2**.

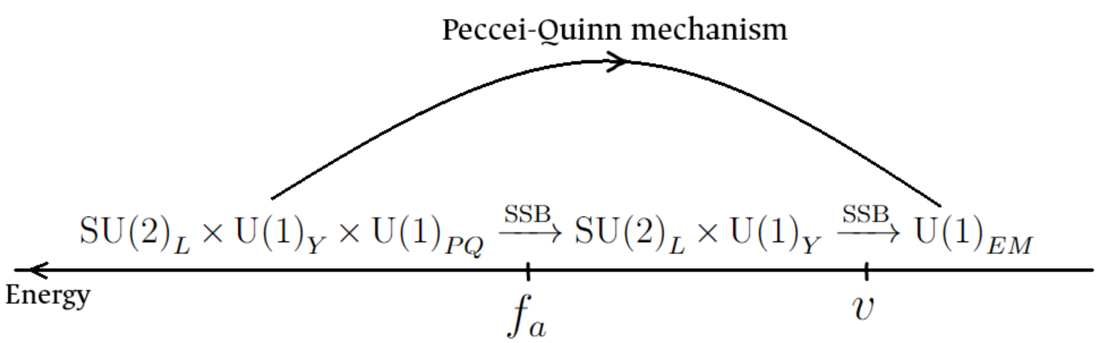

Figure 5-2.: Peccei-Quinn mechanism illustration.

From QCD axion coupling to charged-currents (through $D^\mu H_u$ and $D^\mu H_d$), one can show that the axion decay constant [77]:

$$
f_a = \frac{v}{6}\sin(2\beta) \propto \mathcal{O}(v) \tag{5-81}
$$

In consequence, the action coupling to Standard Model fields is not sufficiently suppressed, which is a problem because of the experimental constraints.

Therefore, the Weinberg-Wilczek model [75, 21] has been ruled out since experiments suggests [77]:

$$f_a \gg v \tag{5-82}$$

Thus, axion's interactions are then parametrically suppressed as:

$$\frac{1}{f_a} \ll \frac{1}{v} \tag{5-83}$$

This led to the so-called "*Invisible Axion Models*" [77], in which the Peccei-Quinn symmetry breaking decoupled from the eletroweak energy scale via the introduction of a new Standard Model singlet scalar field $a(x)$, which acquires a vacuum expectation value:

$$\langle a \rangle = v_a \sim f_a \gg v \tag{5-84}$$

Invisible Axion Models can be divided in two large classes [77]:

- Dine-Fischler-Srednicmi-Zhitnitsku (DSFZ) type:

  Anomaly is carried by Standard Model quarks (as in the Weinberg-Wilczek model), and a new scalar field $a(x)$ is introduced such as that $\langle a \rangle = v_a \sim f_a \gg v$.

- Kim-Shifman-Vainshtein-Zanharov (KSVZ) type:

  Anomaly is carried by new coloured fermions Q, such that $\langle Q \rangle = f_a \gg v \Rightarrow$ Heavier quarks.

5.3. The DFSZ axion model

This axion model includes the three generations of fermions [24, 25]:

$$Q_L(3,2,+1/6,0) = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad E_L(1,2,-1/2,0) = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \tag{5-85}$$

$$u_R(1,1,+2/3,1), \quad d_R(1,1,-1/3,1), \quad e_R(1,1,-1,1) \tag{5-86}$$

Two Higgs doublets [24, 25]:

$$H_u(1,2,-1/2,-1) = \begin{pmatrix} \phi_1^0 \\ \phi_1^- \end{pmatrix} \tag{5-87}$$

$$H_d(1,2,+1/2,+1) = \begin{pmatrix} \phi_2^0 \\ \phi_2^- \end{pmatrix} \tag{5-88}$$

And a Standard Model gauge-singlet complex scalar field [24, 25]:

$$\chi(1,1,0,+1) = \mathrm{Re}(\chi) + i\mathrm{Im}(\chi) \tag{5-89}$$

Which extends the Weiberg-Wilczek model allowing to decouple the Peccei-Quinn symmetry breaking energy scale from the electroweak energy scale.
Now, the kinetic terms of the above fermions and scalars can be written as [24, 25]:

$$\begin{aligned}\mathcal{L}_K =& i\bar{Q}_L\gamma^\mu D^Q_\mu Q_L + i\bar{u}_R\gamma^\mu D^u_\mu u_R + i\bar{d}_R\gamma^\mu D^d_\mu d_R + i\bar{E}_L\gamma^\mu D^E_\mu E_L \\ &+ i\bar{e}_R\gamma^\mu D^e_\mu e_R + (D^E_\mu \Phi_1)(D^{E\mu}\Phi_1)^\dagger + (D^E_\mu \Phi_2)(D^{E\mu}\Phi_2)^\dagger + (\partial_\mu \chi)(\partial^\mu \chi)^\dagger\end{aligned} \tag{5-90}$$

With the covariant derivates:

$$D^Q_\mu = \partial_\mu - ig_s\frac{\lambda^a}{2}G^a_\mu - ig\frac{\sigma^a}{2}W^a_\mu - i\frac{1}{6}g'B_\mu \tag{5-91}$$

$$D^u_\mu = \partial_\mu - ig_s\frac{\lambda^a}{2}G^a_\mu - i\frac{2}{3}g'B_\mu \tag{5-92}$$

$$D^d_\mu = \partial_\mu - ig_s\frac{\lambda^a}{2}G^a_\mu - i\frac{1}{3}g'B_\mu \tag{5-93}$$

$$D^E_\mu = \partial_\mu - ig\frac{\sigma^a}{2}W^a_\mu - i\frac{1}{2}g'B_\mu \tag{5-94}$$

$$D^e_\mu = \partial_\mu + ig'B_\mu \tag{5-95}$$

On the other hand, the Yukawa interactions will be [24, 25]:

$$\mathcal{L}_Y = -y_u\bar{Q}_L H_u u_R - y_d\bar{Q}_L\Phi^c_2 d_R - y_e\bar{E}_L H^c_d e_R + h.c. \tag{5-96}$$

Notice that the kinetic terms $\mathcal{L}_k$ and the Yukawa couplings $\mathcal{L}_Y$ preserve the $U(1)_{PQ}$ symmetry since the $U(1)_{PQ}$ charge for each term sums zero.
Now, the renormalizable scalar potential is written in such a way to preserve the $U(1)_{PQ}$ symmetry [24, 25]:

$$\begin{aligned}V(H_u,H_d,\chi) =& \tilde{V}_{module}(|H_u|,|H_d|,|H_uH_d|,\chi) + \lambda_8(\chi^2 H^\dagger_d H_u + h.c.) \\ =& \mu_1^2 H_u^\dagger H_u + \mu_2^2 H_d^\dagger H_d + \mu_3^2\chi^\dagger\chi + \lambda_1(H_u^\dagger H_u)^2 + \lambda_2(H_d^\dagger H_d)^2 + \lambda_3(\chi^\dagger\chi)^2 \\ &+ \lambda_4 H_u^\dagger H_u H_d^\dagger H_d + \lambda_5 H_u^\dagger H_d H_d^\dagger H_u + \lambda_6 H_u^\dagger H_u \chi^\dagger\chi \\ &+ \lambda_7 H_d^\dagger H_d\chi^\dagger\chi + \lambda_8(\chi^2 H_d^\dagger H_u + H_u^\dagger H_d \chi^2)\end{aligned} \tag{5-97}$$

And this scalar potential ensures that all three scalar fields pick up a vacuum expectation value:

$$\langle H_u\rangle = \frac{v_1}{\sqrt{2}} = \frac{v_u}{\sqrt{2}} \tag{5-98}$$

$$\langle H_d\rangle = \frac{v_3}{\sqrt{2}} = \frac{v_d}{\sqrt{2}} \tag{5-99}$$

$$\langle \chi\rangle = \frac{v_\chi}{\sqrt{2}} \tag{5-100}$$

Thus [24, 25, 77]:

$$H_u \supset \frac{v_u}{\sqrt{2}} e^{i\frac{a_u}{v_u}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5-101}$$

$$H_d \supset \frac{v_d}{\sqrt{2}} e^{i\frac{a_d}{v_d}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5-102}$$

$$\chi \supset \frac{v_\chi}{\sqrt{2}} e^{i\frac{a_\chi}{v_\chi}} \tag{5-103}$$

With a_u, a_d and a_χ real scalar fields.
Now, it is expected to obtain the QCD axion field $a(x)$ in terms of $a_{u,d,\chi}$. For this purpose let us write down the Peccei-Quinn current [77]:

$$J_\mu^{PQ} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \delta\psi \tag{5-104}$$

With [77]:

$$\psi = \text{Fermion, vector, scalar fields} \tag{5-105}$$

$$\delta\psi = (U(1)_{PQ}\text{-charge})\psi \tag{5-106}$$

$$\Rightarrow J_\mu^{PQ} = -x_\chi \chi^\dagger i \overleftrightarrow{\partial_\mu} \chi - x_u \Phi_1^\dagger i \overleftrightarrow{\partial_\mu} \Phi_1 - x_d \Phi_2^\dagger i \overleftrightarrow{\partial_\mu} \Phi_2 + \text{Fermion contribution} \tag{5-107}$$

Then, the axion field $a(x)$ will be a linear combination of $a_{u,d,\chi}$. Therefore [77]:

$$J_\mu^{PQ}\big|_{a(x)} = \sum_{i=\chi,u,d} x_i v_i \partial_\mu a_i \tag{5-108}$$

Thus, the axion field is defined as [77]:

$$a(x) = \frac{1}{f_a} \sum_i x_i v_i a_i \tag{5-109}$$

With $v_a = \sum_i x_i^2 v_i^2$, so that:

$$J_\mu^{PQ}\big|_a = v_a \partial_\mu a \tag{5-110}$$

By convention the charge x_i are set to be [77]:

$$x_\chi = 1 \tag{5-111}$$

$$x_u = 2\cos^2\beta \tag{5-112}$$

$$x_d = 2\sin^2\beta \tag{5-113}$$

With:

$$\tan\beta = \frac{v_u}{v_d} \Rightarrow \cos\beta \frac{v_d}{\sqrt{v_u^2 + v_d^2}}, \quad \sin\beta \frac{v_u}{\sqrt{v_u^2 + v_d^2}} \tag{5-114}$$

Therefore, the following energy scale relation is found [77]:

$$\begin{aligned}
f_a^2 &= x_\chi^2 v_\chi^2 + x_u^2 v_u^2 + x_d^2 v_d^2 \\
&= v_\chi^2 + 4(\cos^4 \beta v_u^2 + \sin^4 \beta v_d^2) \\
&= v_\chi^2 + 4\left(\frac{v_d^4}{(v_u^2+v_d^2)^2} v_u^2 + \frac{v_u^4}{(v_u^2+v_d^2)^2} v_d^2\right) \\
&= v_\chi^2 + \frac{4}{v^4} v_u^2 v_d^2 (v_d^2 + v_u^2) \\
&= v_\chi^2 + 4v^2 \frac{v_u^2 v_d^2}{v^4} \\
&= v_\chi^2 + 4v^2 \frac{v_u^2}{v^2}\frac{v_d^2}{v^2} \\
&= v_\chi^2 + 4v^2 \sin^2 \beta \cos^2 \beta \\
&= v_\chi^2 + v^2 (2 \sin^2 \beta \cos^2 \beta)^2 \\
&= v_x^2 + \sin^2 (2\beta) v^2
\end{aligned} \tag{5-115}$$

6. Thermal leptogenesis in the DFSZ axion model

The model departs from the canonical choice for the axion quantum numbers, which is known as "*the standard DFSZ axion model*" [77, 78]. The field content in this model is: two Higgs doublets H_u and H_d with Peccei-Quinn charge 2; a gauge singlet scalar field χ with Peccei-Quinn charge 4, and the rest of particles are the fermions from the Standard Model. In the Table **6-1** are shown all the fields involved in the model as well as their gauge and Peccei-Quinn charges.

	Q_i	u_i^c	d_i^c	$E_{L,l}$	e_l^c	H_u	H_d	χ
$\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$	(2,1/6)	(1,-2/3)	(1,1/3)	(2,-1/2)	(1,1)	(2,-1/2)	(2,1/2)	(0,0)
$\mathrm{U}(1)_{PQ}$	1	1	1	1	1	2	2	4

Table 6-1.: Field content of the standard DFSZ axion model.

Now, in order to implement the thermal leptogenesis mechanism it is necessary first to generate the sources of CP violation, and this is consequence of the flavour mixing in the leptonic sector, which is described by the PMNS matrix, and the CP-violating decays emerging from this model. Therefore, a neutrino mass generation mechanism is required as a first step towards this purpose.

6.1. Neutrino mass generation in the DFSZ axion model

6.1.1. Weinberg effective operator in the DFSZ axion model

In the chapter 2 it was shown that in order to generate neutrino masses is necessary to build a 5-dimension Weinberg effective operator, which must preserve the $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ gauge symmetry. In that case, there was just only Higgs doublet since it was the Majorana extension to the Standard Model, and the form for this operator was found to be as in equation (3-41), where each coupling to each one of the two leptonic doublets involved was made in such a way to preserve the gauge symmetry. Now, in this context where there are two Higgs doublets one may expect that there are more than one way to build the Weinberg effective operator. In fact, there are exactly three forms for this effective operator which preserve the gauge symmetry [27], and they can be labeled by the type of Higgs doublet involved:

1. The **up-up** type Weinberg effective operator:

$$\mathcal{L}_W^{up-up} \sim \frac{1}{\Lambda} E_L \tilde{H}_u E_L \tilde{H}_u \tag{6-1}$$

Notice that this effective operator is quite similar to the Weinberg effective operator in the Majorana extension to the Standard Model in (3-41), but with the up-type Higgs doublet, which differs from the Standard Model Higgs doublet by a phase.

Now, besides the gauge symmetry these operators must preserve the $\mathrm{U}(1)_{PQ}$ symmetry, in order to provide a solution to the strong CP problem. Let us see if the up-up type effective operator satisfies this condition:

$$\begin{aligned} \left(\mathcal{L}_W^{up-up}\right)_{PQ} &= \left(\frac{1}{\Lambda} E_L \tilde{H}_u E_L \tilde{H}_u\right)_{PQ} \\ &= 2\left((E_L)_{PQ} + (\tilde{H}_u)_{PQ}\right) \\ &= 2\left((1) + (2)(-1)\right) \\ &= -2 \end{aligned} \tag{6-2}$$

Therefore, this up-up type effective operator does not preserve the $\mathrm{U}(1)_{PQ}$ symmetry since its associated charge is not zero, hence it cannot be included in the model.

2. The **up-down** type Weinberg effective operator:

$$\mathcal{L}_W^{up-down} \sim \frac{1}{\Lambda} E_L \tilde{H}_u E_L H_d \tag{6-3}$$

It is straightforward to see that this effective operator is gauge invariant since it involves the contraction between two leptonic and Higgs doublets, as in the previous case and in the Majorana extension to the Standard Model. Now, let us see its associated Peccei-Quinn charge:

$$\begin{aligned} \left(\mathcal{L}_W^{up-down}\right)_{PQ} &= \left(\frac{1}{\Lambda} E_L \tilde{H}_u E_L H_d\right)_{PQ} \\ &= 2(E_L)_{PQ} + (\tilde{H}_u)_{PQ} + (H_d)_{PQ} \\ &= 2(1) + (2)(-1) + (2) \\ &= 2 \end{aligned} \tag{6-4}$$

Hence, this up-down type effective operator does not preserve the $\mathrm{U}(1)_{PQ}$ symmetry and cannot be included in the model.

3. The **down-down** type Weinberg effective operator:

$$\mathcal{L}_W^{down-down} \sim \frac{1}{\Lambda} E_L H_d E_L H_d \tag{6-5}$$

As well as in the two above cases, this effective operator is gauge invariant since it is written as the contraction between leptonic and Higgs doublets. Its associated Peccei-Quinn charge is:

$$\begin{aligned}\left(\mathcal{L}_W^{down-down}\right)_{PQ} &= \left(\frac{1}{\Lambda} E_L H_d E_L H_d\right)_{PQ} \\ &= 2\left((E_L)_{PQ} + (H_d)_{PQ}\right) \\ &= 2(1+2) \\ &= 6\end{aligned} \tag{6-6}$$

Therefore, this down-down type effective operator is not invariant under the $\mathrm{U}(1)_{PQ}$ symmetry either.

Notice that although all of the three possible Weinberg effective operators are gauge invariant, none of them is $\mathrm{U}(1)_{PQ}$ invariant. One may build a 5-dimension effective operator which is $\mathrm{U}(1)_{PQ}$ invariant, for example:

$$\mathcal{L}_W^{PQ} \sim \frac{1}{\Lambda} \overline{(E_L \tilde{H}_u)} E_L H_d^\dagger \tag{6-7}$$

Which has Peccei-Quinn charge zero:

$$\begin{aligned}\left(\mathcal{L}_W^{PQ}\right)_{PQ} &= \left(\frac{1}{\Lambda} \overline{(E_L \tilde{H}_u)} E_L H_d^\dagger\right)_{PQ} \\ &= ((-1)(1+(-2)) + 1 + (-2)) \\ &= ((-1)(-1) - 1) \\ &= 0\end{aligned} \tag{6-8}$$

However, the term $E_L H_d^\dagger$ is not gauge invariant since the $\mathrm{SU}(2)_L$ gauge transformation associated to the leptonic doublet is:

$$E_L \to E_L' = e^{i a_i(x) T_i} E_L \tag{6-9}$$

And the associated to the Higgs doublet would be:

$$H_d \to H_d = e^{-i a_i(x) T_i} H_d \tag{6-10}$$

Then, the transformation associated to the adjoint will be exactly as the one associated to the leptonic doublet, hence the $\mathrm{SU}(2)_L$ phases will not cancel each other. Instead, they will be added.

Therefore, with only the two leptonic and Higgs doublets of the model there is no way to build a 5-dimension Weinberg effective operator gauge and $U(1)_{PQ}$ invariant at the same time [27]. But, recall that the model contents a gauge singlet scalar field χ whose Peccei-Quinn charge is 4, hence one might think at first sight that there is some way to include this gauge singlet scalar in such a way that the total Peccei-Quinn charge vanishes. However, considering all the possible gauge invariant contractions of the scalars and their associated Peccei-Quinn charges:

$$(\chi^n)_{PQ} = 4n$$

$$((H_u H_d)^n)_{PQ} = 4n$$

$$\left((H_u^\dagger H_u)^n\right)_{PQ} = 0$$

$$\left((H_d^\dagger H_d)^n\right)_{PQ} = 0$$

And recalling that the Peccei-Quinn charges associated for the only three possible gauge invariant 5-dimensional Weinberg effective operators (6-1), (6-3), and (6-5), are -2, 2, and 6 respectively, which are not multiples of 4, it is straightforward to conclude that there is no leptonic-scalar combination which can lead to the gauge and $U(1)_{PQ}$ invariance [27].
For the above reasons, it is necessary to add new degrees of freedom to the theory, i.e., to add new fields, because there is no way to build a Weinberg effective operator to provide mass to neutrinos. Those new fields should be added in such a way that the model is physically consistent. Therefore, the most natural way to extent the model including new degrees of freedom is with right-handed light neutrinos $\nu_{R,i}$, hence the mass terms for neutrinos will be built with the left and right handed parts, i.e., they will be Dirac-like mass terms. This is in agreement with the consistence of the model, since in the chapter 2 it was shown that if it is possible to build a 5-dimension Weinberg effective operator in the model, then lepton-number violation arises and neutrinos are Majorana particles, but since no such operator appears in this model, there is no lepton-number violation and the only option for neutrinos is to be Dirac particles [27]. We will call this extension of the model as the "*the Dirac extension to the DFSZ axion model*".
Although the Dirac extension to the DFSZ axion model is naturally viable and physically consistence, an explicit Dirac-like mass term is forbidden. Therefore, let us build the following 5-dimensional Weinberg effective operator [27, 78]:

$$\mathcal{L}_W^D = \frac{y^\nu_{\alpha\beta}}{\Lambda_{UV}} \overline{E}_{L,\alpha} H_u \nu_{R,\beta} \chi + h.c. \tag{6-11}$$

With the greek indices running over flavour as usual: $\alpha, \beta = e, \mu, \tau$.
$y^\nu_{\alpha\beta}$ is an effective Yukawa matrix for coupling between the neutrinos and the scalars of the model.

First, using the relations found in the Appendix B for scalar and dirac fields units in (B-3) and (B-4), respectively, let us check if it is a 5-dimensional operator:

$$\begin{aligned} \left[\mathcal{L}_W^D\right] &= \left[\overline{E}_{L,\alpha} H_u \nu_{R,\beta} \chi\right] \\ &= \left[\overline{E}_{L,\alpha}\right] \left[H_u\right] \left[\nu_{R,\beta}\right] \left[\chi\right] \\ &= \left[M^{3/2}\right] \left[M\right] \left[M^{3/2}\right] \left[M\right] \\ &= \left[M^5\right] \end{aligned} \tag{6-12}$$

Indeed it is. Hence the factor Λ_{UV} has units of mass as expected, since it indicates the scale energy above which the effective point-like interaction is in fact mediated by other particles and described by a higher energy model.
The Feynman diagram for this effective interaction which provides mass to neutrinos light active neutrinos is shown in the Figure **6-1**.

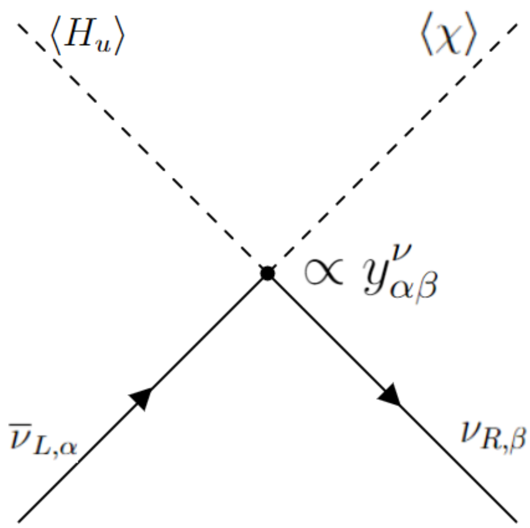

Figure 6-1.: Feynman diagram for the Weinberg effective operator in the Dirac extension.

Now, let us check the Peccei-Quinn charge for this 5-dimensional Weinberg effective operator in the Dirac extension:

$$\begin{aligned} \left(\mathcal{L}_W^D\right)_{PQ} &= \left(\frac{y^{\nu}_{\alpha\beta}}{\Lambda_{UV}} \overline{E}_{L,\alpha} H_u \nu_{R,\beta} \chi\right)_{PQ} \\ &= \left(\overline{E}_{L,\alpha}\right)_{PQ} + \left(H_u\right)_{PQ} + \left(\nu_{R,\beta}\right)_{PQ} + \left(\chi\right)_{PQ} \\ &= (-1) + (2) + \left(\nu_{R,\beta}\right)_{PQ} + (4) \\ &= 5 + \left(\nu_{R,\beta}\right)_{PQ} \end{aligned} \tag{6-13}$$

Therefore, in order to preserve the U(1)$_{PQ}$ symmetry, let us stablish that the heavy sterile fermions $\nu_{R,\alpha}$ must have Peccei-Quinn charge -5.

Now, the interest is on the UV-completion of this operator, which can be achieved through the type-I Dirac seesaw model [27, 78]. Just as in the Majorana extension to the Standard Model shown in the chapter 2 with the type-I seesaw model, this UV-completion states that the model at higher energies contents gauge singlet heavy fermions, except that this time they will be sterile Dirac fermions rather than sterile Majorana fermions. The reason for choose those sterile fermions to be Dirac rather than Majorana will be clear later.

Before state the lagrangian for the type-I Dirac seesaw model extension to the DFSZ axion model, let us see the asymptotical states for the interaction and their Feynman diagrams at tree-level, in order to analyze their gauge and $\mathrm{U}(1)_{PQ}$ invariance:

- **Initial state**:

  The term in the lagrangian for this interaction has the following Yukawa form:

$$\mathcal{L}^{h}_{Type-I} = y^{h}_{\alpha\beta}\overline{E}_{L,\alpha}H_u N_{R,\beta} + h.c. \tag{6-14}$$

  Which is clearly $\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y$ gauge invariant since it has been proved that the leptonic-scalar doublet contraction ($\overline{E}_{L,\alpha}H_u$) is gauge invariant, and the $N_{R,\beta}$ are heavy sterile Dirac fermions, and whose Feynman diagram at tree-level is shown in Figure **6-2**. The Yukawa matrix associated to the coupling between the light active neutrinos, the Higgs doublet and the heavy sterile Dirac fermions has been called $y^{h}_{\alpha\beta}$, where the h states for *Higgs* since it is the Yukawa coupling with one of the Higgs doublets and therefore it belongs to the electroweak sector. This special notation will be very useful later, because there will be three diferent Yukawa couplings in the model associated to neutrinos, and it is very important to distinguish between them.

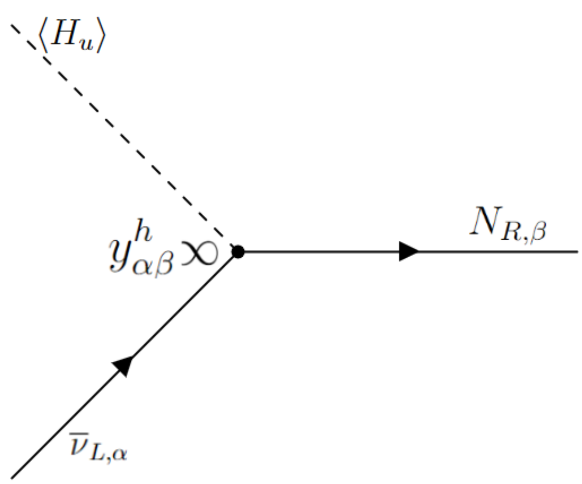

Figure 6-2.: Feynman diagram for the neutrino Yukawa interaction with the Higgs.

Now, in order to preserve the $U(1)_{PQ}$ symmetry in the model let us impose that the associated charge for this term is zero:

$$\begin{aligned} \left(\mathcal{L}^{h}_{Type-I}\right)_{PQ} &= 0 \\ \left(y^{h}_{\alpha\beta}\overline{E}_{L,\alpha}H_u N_{R,\beta}\right)_{PQ} &= 0 \\ \left(\overline{E}_{L,\alpha}\right)_{PQ} + (H_u)_{PQ} + (N_{R,\beta})_{PQ} &= 0 \\ (-1) + (2) + (N_{R,\beta})_{PQ} &= 0 \\ (N_{R,\beta})_{PQ} &= -1 \end{aligned} \tag{6-15}$$

Thus, it has been found that to preserve the $U(1)_{PQ}$ symmetry in the model, the associated charge to the heavy sterile Dirac fermions must be -1. Now, notice that this result was found for the right-handed part of the sterile Dirac fermion, however one may assume that it does not change for the left-handed part since the model is being built in such a way that all fermions included are Dirac fermions, specially the light active neutrinos and the heavy sterile gauge singlet fermions. To check this, let us look the final state.

- **Final state**:

At first sight, let us consider that the interaction term in the lagrangian, as in the initial state, will have the following Yukawa form:

$$\mathcal{L}^{\chi}_{Type-I} = y^{\chi}_{\alpha\beta}N_{R,\alpha}\chi\nu_{R,\beta} + h.c. \tag{6-16}$$

This term is $SU(2)_L \times U(1)_Y$ gauge invariant. Let us check its associated Peccei-Quinn charge:

$$\begin{aligned} \left(\mathcal{L}^{\chi}_{Type-I}\right)_{PQ} &= \left(y^{\chi}_{\alpha\beta}N_{R,\alpha}\chi\nu_{R,\beta}\right)_{PQ} \\ &= (N_{R,\alpha})_{PQ} + (\chi)_{PQ} + (\nu_{R,\beta})_{PQ} \\ &= (-1) + (4) + (-5) \\ &= -2 \end{aligned} \tag{6-17}$$

Hence, this term would not be Peccei-Quinn invariant.

Now, recall that for consistence the Peccei-Quinn charge associated to the left-handed part of a fermion must be the same for its right-handed part, then:

$$\begin{aligned} \left(\overline{N}_{L,\alpha}\right)_{PQ} &= (-1)\,(N_{R,\alpha})_{PQ} \\ &= 1 \end{aligned} \tag{6-18}$$

Hence, considering the interaction term in the following form:

$$\mathcal{L}^{\chi}_{Type-I} = y^{\chi}_{\alpha\beta}\overline{N}_{L,\alpha}\chi\nu_{R,\beta} + h.c. \tag{6-19}$$

Whose Feynman diagram at tree-level is shown in Figure **6-3**, and which is still trivially $\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y$ gauge invariant, it has been found the $\mathrm{U}(1)_{PQ}$ symmetry interaction to complete the model:

$$\begin{aligned}\left(\mathcal{L}^{\chi}_{Type-I}\right)_{PQ} &= \left(y^{\chi}_{\alpha\beta}\overline{N}_{L,\alpha}\chi\nu_{R,\beta}\right)_{PQ}\\ &= \left(\overline{N}_{L,\alpha}\right)_{PQ} + (\chi)_{PQ} + (\nu_{R,\beta})_{PQ}\\ &= (1) + (4) + (-5)\\ &= 0\end{aligned} \tag{6-20}$$

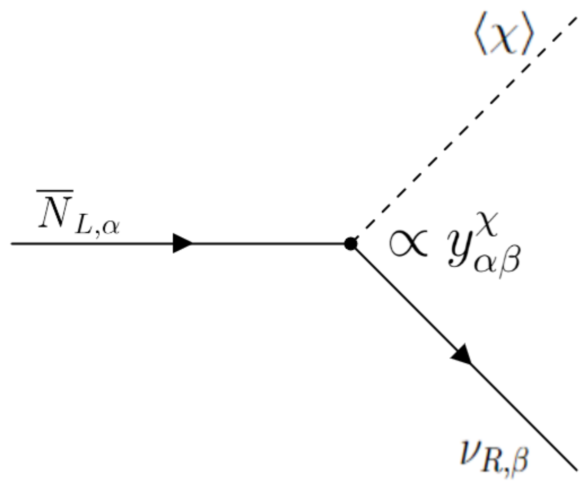

Figure 6-3.: Feynman diagram for the neutrino Yukawa interaction with the gauge singlet scalar.

Therefore, the Feynman diagram at tree-level for the UV-completion of the Weinberg effective operator in (6-11) through the type-I Dirac seesaw model looks like in Figure **6-4**. It is clear that the heavy sterile fermions N_α are Dirac particles, and this is why there is a mass propagator.

The above results allow to write the Yukawa lagrangian for the type-I Dirac seesaw model extension to the DFSZ axion model in the following form:

$$-\mathcal{L}_Y = y^u_{ij}\overline{Q}_iH_uu_j + y^d_{ij}\overline{Q}_iH_dd_j + y^l_{ij}\overline{E}_{L,i}H_de_j + h.c. + \mathcal{L}_{Type-I} \tag{6-21}$$

Where the lagrangian for the type-I Dirac seesaw model is given by:

$$-\mathcal{L}_{Type-I} = y^h_{\alpha\beta}\overline{E}_{L,\alpha}H_uN_{R,\beta} + y^{\chi}_{\alpha\beta}\overline{N}_{L,\alpha}\chi\nu_{R,\beta} + \frac{1}{2}\overline{N}_{L,\alpha}M^N_{\alpha\beta}N_{R,\beta} + h.c. \tag{6-22}$$

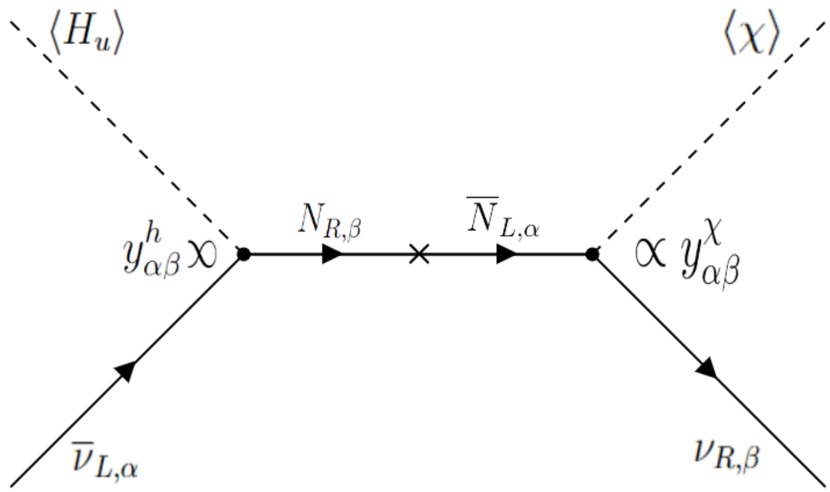

Figure 6-4.: Feynman diagram for the type-I Dirac seesaw extension to the DFSZ axion model.

Notice that it was included a Dirac-like mass term for the heavy sterile fermions N_α because of the results obtained before. Furthermore, notice that a Majorana mass term would explicitly break the $\mathrm{U}(1)_{PQ}$ symmetry. Recall the form for the Majorana mass term given by (**??**), then:

$$\begin{aligned} \left(\overline{N}_{R,\alpha} M^R_{\alpha\beta} N^c_{R,\beta}\right)_{PQ} &= \left(\overline{N}_{R,\alpha}\right)_{PQ} + \left(N^c_{R,\beta}\right)_{PQ} \\ &= (1) + (-1)(-1) \\ &= 2 \end{aligned} \tag{6-23}$$

6.1.2. Type-I Dirac seesaw mechanism in the DFSZ axion model

Once the UV-completed lagrangian for the model is known, let us proceed with the mass generation for the particle content of the model. Recall that in chapter 4 it was found the equations (5-73) and (5-74), which describe the relation between the Higgs doublets H_u and H_d with H and H', hence the two Higgs doublets model can be reduced to the Standard Model form again [76], with H being the standard Higgs doublet. Therefore, after spontaneous symmetry breaking, the first four terms in the Yukawa lagrangian (6-21) leads to the mass generation for quarks and charged leptons through the well-known Higgs mechanism, in such a way that the mass matrix for the latter is diagonal. Now, it is more relevant to study the extra Yukawa terms coming from the type-I Dirac seesaw model extension to the DFSZ axion model. In chapter 2 it was shown that after spontaneous symmetry breaking neutrinos also acquires mass through a dynamic mechanism totally analogous to the Higgs mechanism, and it was called the type-I Dirac seesaw mechanism. Let us apply explicitly this mechanism for this case:

- Scalar-leptonic doublet term:

$$\begin{aligned} y^h_{\alpha\beta}\overline{E}_{L,\alpha}H_u N_{R,\beta} &\longrightarrow \begin{pmatrix}\overline{\nu}_{L,\alpha} & \overline{e}_{L,\alpha}\end{pmatrix} y^h_{\alpha\beta}\begin{pmatrix}\frac{v_u+h_u}{\sqrt{2}}\\ 0\end{pmatrix} N_{R,\beta} \\ &= \frac{v_u}{\sqrt{2}}\overline{\nu}_{L,\alpha}y^h_{\alpha\beta}N_{R,\beta} + \frac{1}{\sqrt{2}}h_u\overline{\nu}_{L,\alpha}y^h_{\alpha\beta}N_{R,\beta} \\ &= \frac{\sin\beta}{\sqrt{2}}v\overline{\nu}_{L,\alpha}y^h_{\alpha\beta}N_{R,j\beta} + \frac{1}{\sqrt{2}}h_u\overline{\nu}_{L,\alpha}y^h_{\alpha\beta}N_{R,\beta}\end{aligned} \tag{6-24}$$

Where it is clear that the first term is a mass term while the second term describes the interaction between the Higgs boson h_u with the light active neutrino and the heavy sterile fermion.

- Scalar-leptonic singlet term:

$$\begin{aligned} y^\chi_{\alpha\beta}\overline{N}_{L,\alpha}\chi\nu_{R,\beta} &\longrightarrow \overline{N}_{L,\alpha}y^\chi_{\alpha\beta}\left(\frac{v_\chi+h_\chi}{\sqrt{2}}\right)\nu_{R,\beta} \\ &= \frac{v_\chi}{\sqrt{2}}\overline{N}_{L,\alpha}y^\chi_{\alpha\beta}\nu_{R,\beta} + \frac{1}{\sqrt{2}}h_\chi\overline{N}_{L,\alpha}y^\chi_{\alpha\beta}\nu_{R,\beta} \\ &= \sqrt{\frac{f_a^2-\sin^2(2\beta)v^2}{2}}\overline{N}_{L,\alpha}y^\chi_{\alpha\beta}\nu_{R,\beta} + \frac{1}{\sqrt{2}}h_\chi\overline{N}_{L,\alpha}y^\chi_{\alpha\beta}\nu_{R,\beta}\end{aligned} \tag{6-25}$$

Here it is also clear that the first term corresponds to a mass term, and the second term describes the interaction between the real scalar field h_χ with the light right-handed neutrino and the heavy sterile fermion.

Based on these two results above, it is convenient to define the following **Dirac mass matrices**:

$$M^{D1}_{\alpha\beta} = \sqrt{2}\sin\beta v y^h_{\alpha\beta} \tag{6-26}$$

$$M^{D2}_{\alpha\beta} = \sqrt{2\left(f_a^2-\sin^2(2\beta)v^2\right)}y^\chi_{\alpha\beta} \tag{6-27}$$

Thus, the terms left after implement the type-I Dirac seesaw mechanism correspond to the lagrangian for the neutrino mass generation:

$$\mathcal{L}^\nu_m = -\frac{\sin\beta}{\sqrt{2}}v\overline{\nu}_{L,\alpha}y^h_{\alpha\beta}N_{R,\beta} - \sqrt{\frac{f_a^2-\sin^2(2\beta)v^2}{2}}\overline{N}_{L,\alpha}y^\chi_{\alpha\beta}\nu_{R,\beta} - \frac{1}{2}\overline{N}_{L,\alpha}M^N_{\alpha\beta}N_{R,\beta} + h.c. \tag{6-28}$$

Which can be rewritten substituting the definitions for the Dirac matrices in (6-26) and (6-27):

$$\mathcal{L}^\nu_m = -\frac{1}{2}\overline{\nu}_{L,\alpha}M^{D1}_{\alpha\beta}N_{R,\beta} - \frac{1}{2}\overline{N}_{L,\alpha}M^{D2}_{\alpha\beta}\nu_{R,\beta} - \frac{1}{2}\overline{N}_{L,\alpha}M^N_{\alpha\beta}N_{R,\beta} + h.c. \tag{6-29}$$

These terms are clearly Dirac-like mass terms, but they mix the light neutrinos with heavy neutrinos through the Dirac mass matrices M^{D1} and M^{D2}.
Similarly to the canonical seesaw mechanism in the type-I seesaw model shown in the chapter 2, it is convenient to rewrite the above lagrangian in its matrix form:

$$\mathcal{L}_m^\nu = -\frac{1}{2}\overline{\nu}_L M^{D1} N_R - \frac{1}{2}\overline{N}_L M^{D2}\nu_R - \frac{1}{2}\overline{N}_L M^N N_R + h.c. \tag{6-30}$$

Where it has been defined the following 3-dimensional vectors:

$$\nu_L \equiv \left(\nu_{L,\alpha}\right)_{3\times 1} \tag{6-31}$$

$$\nu_R \equiv \left(\nu_{R,\alpha}\right)_{3\times 1} \tag{6-32}$$

$$N_L \equiv \left(N_{L,\alpha}\right)_{3\times 1} \tag{6-33}$$

$$N_R \equiv \left(N_{R,\alpha}\right)_{3\times 1} \tag{6-34}$$

Where the greek indice runs over the three flavours $\alpha = e, \mu, \tau$.
Now, let us rewritte the neutrino mass lagrangian $\mathcal{L}_m^\nu$ in a more simplified and compact matrix way:

$$\begin{aligned}\mathcal{L}_m^\nu &= -\frac{1}{2}\overline{\nu}_L M^{D1} N_R - \frac{1}{2}\overline{N}_L M^{D2}\nu_R - \frac{1}{2}\overline{N}_L M^N N_R + h.c. \\ &= -\frac{1}{2}\overline{\nu}_L M^{D1} N_R - \frac{1}{2}\overline{N}_L\left(M^{D2}\nu_R + M^N N_R\right) + h.c. \\ &= -\frac{1}{2}\left[\overline{\nu}_L M^{D1} N_R + \overline{N}_L\left(M^{D2}\nu_R + M^N N_R\right)\right] + h.c.\end{aligned} \tag{6-35}$$

The above expression can be written as the contraction of a row-vector and a column-vector:

$$\mathcal{L}_m^\nu = -\frac{1}{2}\begin{pmatrix}\overline{\nu}_L & \overline{N}_L\end{pmatrix}\begin{pmatrix} M^{D1} N_R \\ M^{D2}\nu_R + M^N N_R\end{pmatrix} + h.c. \tag{6-36}$$

And in the column-vector in turn can be written as the multiplication of a matriz by a more simplified column-vector. Thus, the most simplified matrix form for the neutrino mass lagrangian is found to be

$$\mathcal{L}_m^\nu = -\frac{1}{2}\begin{pmatrix}\overline{\nu}_L & \overline{N}_L\end{pmatrix}\begin{pmatrix} 0 & M^{D1} \\ M^{D2} & M^N\end{pmatrix}\begin{pmatrix}\nu_R \\ N_R\end{pmatrix} + h.c. \tag{6-37}$$

Where the matrices M^{D1}, M^{D2} and M^N are 3×3 complex non-diagonal matrices since they are Dirac mass matrices.
It is still possible to rewritte the matrix equation (6-37) in a more simplified way:

$$\mathcal{L}_m^\nu = -\frac{1}{2}\overline{n}_L M n_R + h.c. \tag{6-38}$$

With the following definitions:

$$n_L \equiv \begin{pmatrix} \nu_{L,\alpha} \\ N_{L,\alpha} \end{pmatrix}_{6\times1} \tag{6-39}$$

$$n_R \equiv \begin{pmatrix} \nu_{R,\alpha} \\ N_{R,\alpha} \end{pmatrix}_{6\times1} \tag{6-40}$$

$$M \equiv \begin{pmatrix} 0 & M^{D1} \\ M^{D2} & M^N \end{pmatrix}_{6\times6} \tag{6-41}$$

Thus, notice that the lagrangian for neutrinos mass written in the matrix form in (6-38) is a purely Dirac mass term.

Now, to find the physical states (mass states) for light neutrinos, and incidentally those for heavy fermions, it is necessary to separate the system in such a way that there is not mixing between them. This can be achieved in a directly analogous way to the method made in chapter for the canonical seesaw mechanism, i.e., through the diagonalization of the matrix M.

As it was seen in the canonical seesaw mechanism in the chapter 2, it is very useful for algebraic and physical purposes to study first the (1+1)-scheme, which is much easier to diagonalize.

6.1.3. (1+1)-scheme in the type-I Dirac seesaw extension to the DFSZ axion model

Considering the simplest case where there exists just one family (flavour), the lagrangian for the light active neutrino ν and the heavy sterile fermion N masses is:

$$\mathcal{L}_m^\nu = -\frac{1}{2} m^{D1} \overline{N}_R \nu_L - \frac{1}{2} m^{D2} \overline{\nu}_R N_L - \frac{1}{2} m^N \overline{N}_L N_R + h.c. \tag{6-42}$$

Which can be written in matrix form as:

$$\mathcal{L}_m^\nu = -\frac{1}{2} \begin{pmatrix} \overline{\nu}_R & \overline{N}_R \end{pmatrix}_{1\times2} \begin{pmatrix} 0 & m^{D1} \\ m^{D2} & m^N \end{pmatrix}_{2\times2} \begin{pmatrix} \nu_L \\ N_L \end{pmatrix}_{2\times1} + h.c. \tag{6-43}$$

Physical (mass) states will be found by diagonalizing the system. Thus, let us proceed first finding the eigenvalues $m_\pm$ for the 2×2 mass matrix:

$$\det \begin{pmatrix} -m_\pm & m^{D1} \\ m^{D2} & m^N - m_\pm \end{pmatrix} = 0 \tag{6-44}$$

And computing:

$$\begin{aligned} -m_\pm(m^N - m_\pm) - m^{D1} m^{D2} &= 0 \\ m_\pm^2 - m^N m_\pm - m^{D1} m^{D2} &= 0 \end{aligned} \tag{6-45}$$

This a second-order algebraic equation in $m_{\pm}$, therefore its solution is given by the quadratic formula:

$$m_{\pm} = \frac{m^N \pm \sqrt{(m^N)^2 + 4m^{D1}m^{D2}}}{2} \tag{6-46}$$

Now, taking into account that the heavy sterile fermion mass is much bigger than the Dirac masses $m^N >> m^{D1}$, m^{D2}, then it is possible to expand the square root as:

$$\begin{aligned} m_{\pm} &= \frac{m^N}{2}\left(1 \pm \sqrt{1 + \frac{4m^{D1}m^{D2}}{(m^N)^2}}\right) \\ m_{\pm} &\approx \frac{m^N}{2}\left[1 \pm \left(1 + \frac{2m^{D1}m^{D2}}{(m^N)^2}\right)\right] \end{aligned} \tag{6-47}$$

Thus, the two solutions will be:

- m_+:

$$\begin{aligned} m_+ &= \frac{m^N}{2}\left[1 + \left(1 + \frac{2m^{D1}m^{D2}}{(m^N)^2}\right)\right] \\ m_+ &= m^N + \frac{m^{D1}m^{D2}}{m^N} \end{aligned} \tag{6-48}$$

  And since $m^N >> m^{D1}$, m^{D2}, then:

$$m_+ \approx m^N \tag{6-49}$$

  Therefore, one may stablish that the physical state n_+ with mass m_+ will correspond approximately to the heavy sterile fermion:

$$n_+ \approx N \tag{6-50}$$

- m_-:

$$\begin{aligned} m_- &= \frac{m^N}{2}\left[1 - \left(1 + \frac{2m^{D1}m^{D2}}{(m^N)^2}\right)\right] \\ m_- &= -\frac{m^{D1}m^{D2}}{m^N} \end{aligned} \tag{6-51}$$

  And since $m^N >> m^{D1}$, m^{D2}, then:

$$m_- << 1 \tag{6-52}$$

  Therefore, the physical state n_- has a really tiny mass m_-, hence one may assign this to the light active neutrino:

$$n_- \approx \nu \tag{6-53}$$

Recalling the definition for the Dirac mass matrices (6-26) and (6-27), and writing them for the just one family case, the Yukawa matrices become into scalars. In consequence, the light neutrino mass m_- would be:

$$\begin{aligned} m_- \approx m_\nu &= -\frac{m^{D1}m^{D2}}{m^N} \\ &= -\frac{1}{m^N}\left(\sqrt{2}\sin\beta v y^h\right)\left(\sqrt{2\left(f_a^2-\sin^2(2\beta)v^2\right)}y^\chi\right) \\ &\propto \frac{\sin\beta v\sqrt{f_a^2-\sin^2(2\beta)v^2}}{m^N} \end{aligned} \tag{6-54}$$

And since $f_a >> v$ [77], then $f_a^2 >> \sin^2(2\beta)v^2$, and hence the light active neutrino mass will be proportional to:

$$m_\nu \propto \frac{vf_a}{m^N} \tag{6-55}$$

Notice that this result is quite similar to the one found in the (1+1)-scheme in the canonical seesaw mechanism in equation (3-54), except for the axion decay constant f_a which connects neutrino masses with QCD axions physics.

This result has remarkable physical consequences since one may estimate relations between the three energy scales involved, which are related to the three spontaneous symmetry breaking shown in Figure **5-2**. Considering that both Yukawa couplings are such that they do not deepen the hierarchy problem:

$$y^h, y^\chi \sim \mathcal{O}(1) \tag{6-56}$$

One may stablish, at first sight, that:

$$m_\nu \approx \frac{vf_a}{\Lambda_{UV}} \tag{6-57}$$

Where it has been replaced the heavy sterile fermion mass m^N for the scale of energy of the model Λ_{UV} since, as it was shown in chapter 2 for type-I seesaw model, the mass of the new particle introduced to the model is proportional to the scale of the energy in which the model is UV-complete.

Now, from current experimental data about neutrino physics, it is well known that there exists an upper limit for neutrino mass $m_\nu <$ 0,120 eV [6, 43]. Let us take the following estimation for the neutrino mass just for make a heuristic computation for the 6energy scales:

$$m_\nu \sim 1\times 10^{-1}\,\text{eV} \tag{6-58}$$

In consequence, from (6-57) it follows that:

$$\frac{vf_a}{\Lambda_{UV}} \sim 1\times 10^{-1}\,\text{eV} \tag{6-59}$$

And since tha value for the Higgs vacuum expectation value v is about 246 GeV [43], one may substitute it for $v \sim 1 \times 10^2$ GeV and find the following relation between the $\mathrm{U}(1)_{PQ}$ symmetry and type-I Dirac seesaw energy scales:

$$10^3\,\mathrm{GeV}\, f_a \sim \Lambda_{UV} \tag{6-60}$$

Or equivalently:

$$f_a < \Lambda_{UV} \tag{6-61}$$

This is an outstanding result since it imposes a strong condition in the energy scales, with the physical implication that the spontaneous Peccei-Quinn symmetry breaking occurs at lower energies than the mass of the heavy sterile particles N_i from the type-I Dirac seesaw model. From a cosmological point of view, and from the thermal leptogenesis toy model seen in chapter 3, it means that the decays of N_i take place before the spontaneous $\mathrm{U}(1)_{PQ}$ symmetry breaking. Now, due to the results from the capter 4, no pseudo-Goldstone boson appears, therefore the axion $a(x)$ is not in the particle spectrum yet. Hence, the baryon asymmetry of the universe is generated before the axion $a(x)$ existed, and if the axion is in fact the dark matter, before the dark matter existed and adquired mass.

Thus, the above results and the diagram pictured in Figure **5-2** tell us that a cosmological chronological order would be: the universe cools down enough until it reaches the energy Λ_{UV}, going from the type-I Dirac seesaw model described in (6-21) to an effective interaction described by (6-11) [46, 27], meanwhile the heavy sterile fermions N_i decay to lighter particles, and through CP-violation processes, generating the baryon asymmetry of the universe [61], and associated there must be a cosmological phase transition as it moved from a larger (unknown) symmetry of the type-I Dirac seesaw model to a smaller symmetry (DFSZ axion model); then, the universe continues cooling down and when it decreases by approximately 10^3 GeV it reaches the scale energy of the axion coupling constant f_a, where the spontaneous $\mathrm{U}(1)_{PQ}$ symmetry breaking occurs, hence a real scalar field (the axion $a(x)$) candidate to dark matter appears in the spectrum and adquires mass in this process [77], and a cosmological phase transition takes place again since the symmetry has been reduced from the DFSZ axion model to the Higgs doublet model, i.e., going back to the Standard Model gauge symmetry; finally, the universe cools down so much that it reaches the Higgs vacuum expectation value v, in principle far below f_a, giving rise to the Higgs mechanism [28] where all particles (except the axion as mentioned above) adquire mass, including light active neutrinos because of the seesaw mechanism, and going through the electroweak cosmological phase transition [51].

Furthermore, from the relation found above in (6-60), and from the fact that $10^6\,\mathrm{GeV} \leq f_a \leq 10^{12}\,\mathrm{GeV}$ [77], it follows that:

$$10^9\,\mathrm{GeV} \leq \Lambda_{UV} \leq 10^{15}\,\mathrm{GeV} \tag{6-62}$$

This is a remarkable result since it imposes a strong constraint in the energy range for the energy scale of the type-I Dirac seesaw model Λ_{UV}, and furthermore, succesful leptogenesis is guaranteed in this model [61]. Notice that the upper limit differs from the one found in (3-55) in chapter 2 by 10 GeV, and further, a lower limit was found. This lower limit is in agreement with the fact that the typer-I Dirac seesaw model energy scale must be far away the electroweak scale.
Let us prove now that the proposed relations in equations (6-50) and (6-53) are in fact valid. For this purpose it is necessary to find the physical states in the (1+1)-scheme, and this is made by solving the following eigenvalue equation:

$$\begin{pmatrix} 0 & m^{D1} \\ m^{D2} & m^N \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = m_k \begin{pmatrix} a \\ b \end{pmatrix} \tag{6-63}$$

- For $k=1$, $m_k = m_1 = -\frac{m^{D1}m^{D2}}{m^N}$, and in consequence:

$$b = -\frac{m^{D1}m^{D2}}{m^N} a \tag{6-64}$$

  Hence, the eigenvector will be:

$$\begin{aligned} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} a \\ -\frac{m^{D2}}{m^N} a \end{pmatrix} \\ &= a \begin{pmatrix} 1 \\ -\frac{m^{D2}}{m^N} \end{pmatrix} \end{aligned} \tag{6-65}$$

  And since a is any scalar, the eigenvector associated to the eigenvalue for the mass $m_1 = -\frac{m^{D1}m^{D2}}{m^N}$ is:

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{m^{D2}}{m^N} \end{pmatrix} \tag{6-66}$$

- For k=2, $m_k = m_2 = m^N$, from (6-63) follows that:

$$a = \frac{m^{D1}}{m^N} b \tag{6-67}$$

  Therefore:

$$\begin{aligned} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} \frac{m^{D1}}{m^N} b \\ b \end{pmatrix} \\ &= b \begin{pmatrix} \frac{m^{D1}}{m^N} \\ 1 \end{pmatrix} \end{aligned} \tag{6-68}$$

Hence the associated eigenvector to the mass eigenvalue $m_2 = m^N$:

$$\begin{pmatrix} \frac{m^{D1}}{m^N} \\ 1 \end{pmatrix} \tag{6-69}$$

Now, recalling the diagonalization relation for the 2×2 mass matrix M in equation (6-43):

$$M = SmS^{-1} \tag{6-70}$$

With m the diagonal matriz with the mass eigenvalues:

$$\begin{aligned} m &= \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{m^{D1}m^{D2}}{m^N} & 0 \\ 0 & m^N \end{pmatrix} \end{aligned} \tag{6-71}$$

And S the matriz whose columns are made up from the eigenvectors:

$$S = \begin{pmatrix} 1 & \frac{m^{D1}}{m^N} \\ -\frac{m^{D2}}{m^N} & 1 \end{pmatrix} \tag{6-72}$$

Then, after some approximations the inverse matrix will be (see Appendix D):

$$S^{-1} = \begin{pmatrix} 1 - \frac{m^{D1}}{m^N} & 1 - \frac{m^{D1}}{m^{D2}} \\ \frac{m^{D2}}{m^N} & 1 \end{pmatrix} \tag{6-73}$$

Thus, substituting these result in the mass lagrangian (6-43):

$$\begin{aligned} \mathcal{L}_m^\nu &= -\frac{1}{2} \begin{pmatrix} \overline{\nu}_R & \overline{N}_R \end{pmatrix} M \begin{pmatrix} \nu_L \\ N_L \end{pmatrix} + h.c. \\ &= -\frac{1}{2} \begin{pmatrix} \overline{\nu}_R & \overline{N}_R \end{pmatrix} \left(SmS^{-1}\right) \begin{pmatrix} \nu_L \\ N_L \end{pmatrix} + h.c. \\ &= -\frac{1}{2} \left(\begin{pmatrix} \overline{\nu}_R & \overline{N}_R \end{pmatrix} S \right) m \left(S^{-1} \begin{pmatrix} \nu_L \\ N_L \end{pmatrix} \right) + h.c. \end{aligned} \tag{6-74}$$

Where the physical states are found to be:

$$\begin{aligned} \begin{pmatrix} n_{1,L} \\ n_{2,L} \end{pmatrix} &= S^{-1} \begin{pmatrix} \nu_L \\ N_L \end{pmatrix} \\ &= \begin{pmatrix} 1 - \frac{m^{D1}}{m^N} & 1 - \frac{m^{D1}}{m^{D2}} \\ \frac{m^{D2}}{m^N} & 1 \end{pmatrix} \begin{pmatrix} \nu_L \\ N_L \end{pmatrix} \end{aligned} \tag{6-75}$$

I.e., the physical states are a linear combination from the flavour states:

$$n_{1,L} = \left(1 - \frac{m^{D1}}{m^N}\right)\nu_L + \left(1 - \frac{m^{D1}}{m^{D2}}\right)N_L \tag{6-76}$$

$$n_{2,L} = \frac{m^{D2}}{m^N}\nu_L + N_L \tag{6-77}$$

Now, since the heavy sterile mass is expected to be much larger than the Dirac masses $m^N >> m^{D1}, m^{D2}$, and these Dirac masses are of the same order of magnitude $m^{D1} \sim m^{D2}$, then the sterile component for $n_{1,L}$ is highly supressed by the coeficient, as well as the active component for $n_{2,L}$, therefore the above results can be approximated to:

$$n_{1,L} \approx \nu_L \tag{6-78}$$

$$n_{2,L} \approx N_L \tag{6-79}$$

Notice that this result is consistent with that proposed in equations (6-50) and (6-53).
Thus, in this (1+1)-scheme the flavour states are approximately the same physical states, with a weak active-sterile mixing, which correspond to a light active neutrino of tiny mass and to a heavy sterile fermion of large mass. The active neutrino interacts with other particles through charged currents, hence it will be possible to detect it in experiments, while the sterile femion does not participate in any gauge charged process and its detection would be indirectly.

6.1.4. (3+3)-scheme in the type-I Dirac seesaw extension to the DFSZ axion model

Let us consider the general Yukawa lagrangian obtained in (6-38) for the type-I Dirac seesaw extension to the DFSZ axion model in the most general case, i.e., for three families, which means three flavours for the light active neutrinos ν_α and three flavours for the heavy sterile fermions N_α. Therefore, this system corresponds to the (3+3)-scheme and it is necessary to diagonalize the 6×6 matrix M found in (6-41).
The three 3×3 Dirac matrices M^{D1}, M^{D2}, and M^N, are complex and must be positive definited since their eigenvalues (masses) are real and positive. Then, the matrix M will be complex and positive definited as well. Therefore, this matrix can be diagonalized through an unitary 6×6 matrix U in the following form:

$$M = U^\dagger m U \tag{6-80}$$

Where m a diagonal matrix with the masses eigenvalues $m = \mathrm{diag}(m_1, m_2, m_3, m_4, m_5, m_6)$.
By convenience let us writte it as:

$$m_{ij} = m_i \delta_{ij} \tag{6-81}$$

Where the indices i and j runs over the six physical states: $i, j = 1, 2, 3, 4, 5, 6$, and with $m_i \geq 0$ for every i.
Substituting (6-80) and (6-81) into the mass lagrangian for the (3+3)-scheme in its matrix form (6-38):

$$\begin{aligned}
\mathcal{L}_m^\nu &= -\frac{1}{2}\overline{n}_L M n_R + h.c. \\
&= -\frac{1}{2}\overline{n}_L \left(U^\dagger m U\right) n_R + h.c. \\
&= -\frac{1}{2}\overline{n}_{L,\alpha} U^\dagger_{\alpha i} m_{ij} U_{j\beta} n_{R,\beta} + h.c. \\
&= -\frac{1}{2}\left(n^\dagger_{L,\alpha}\gamma^0\right) U^\dagger_{\alpha i}\left(m_i \delta_{ij}\right) U_{j\beta} n_{R,\beta} + h.c. \\
&= -\frac{1}{2} m_i \left(n^\dagger_{L,\alpha} U^\dagger_{\alpha i}\right)\gamma^0 \delta_{ij} U_{j\beta} n_{R,\beta} + h.c. \\
&= -\frac{1}{2} m_i \left(U_{\alpha i} n_{L,\alpha}\right)^\dagger \gamma^0 \delta_{ij} U_{j\beta} n_{R,\beta} + h.c.
\end{aligned} \tag{6-82}$$

Notice that the matrix γ^0 can be moved throughout the expression since it has Dirac indices rather than flavour indices, or in other words, it multiplies each flavour component then it is seen as it was a scalar factor. Now, let us define the Dirac neutrinos ν_i^D as:

$$\nu_i^D = \nu_{L,i}^D + \nu_{R,i}^D \tag{6-83}$$

Where each chirality component is given by:

$$\nu_{L,i}^D \equiv U_{i\alpha} n_{L,\alpha} \tag{6-84}$$

$$\nu_{R,i}^D \equiv U_{i\alpha} n_{R,\alpha} \tag{6-85}$$

Where the latin indice runs over the six physical states $i = 1, 2, 3, 4, 5, 6$, and the greek indice runs over the three flavour states for the light active neutrino and the three flavour states for the heavy sterile neutrinos, i.e., $\alpha = e, \mu, \tau, e', \mu', \tau'$.
Substituting this definition into the mass lagrangian one gets:

$$\begin{aligned}
\mathcal{L}_m^\nu &= -\frac{1}{2} m_i \left(\nu_{L,i}^D\right)^\dagger \gamma^0 \delta_{ij} \nu_{R,j}^D + h.c. \\
&= -\frac{1}{2} m_i \overline{\nu}_{L,i}^D \delta_{ij} \nu_{R,j}^D + h.c. \\
&= -\frac{1}{2} m_i \overline{\nu}_{L,i}^D \nu_{R,i}^D + h.c.
\end{aligned} \tag{6-86}$$

Which is a purely Dirac mass term. Therefore, the system has been diagonalized, and it has been found that the 6 mass eigenstates, labeled by the latin indice i, are Dirac particles.

6.1.5. Effective neutrino mass matrix

As well as it was made in the (1+1)-scheme, to find neutrino masses in the (3+3)-scheme it is necessary to diagonalize the system, finding the eigenvalues for masses and their corresponding eigenvectors to get the physical states. However, the diagonalization of a 6×6 complex matrix can be unnecessarily tedious. Fortunately, there is a more ingenious way to proceed, and it is from the Dirac extension to the Weinberg effective lagrangian in (6-11). In the Figure **6-1** it is shown the Feynman diagram which corresponds to this effective interaction where the light active neutrino involved ν_i adquires mass. Thus, based on this effective interaction, the effective neutrino mass matrix will be:

$$\begin{aligned}
(m_\nu)_{\alpha\beta} &\approx y^\nu_{\alpha\beta}\frac{\langle H_u\rangle\langle\chi\rangle}{\Lambda_{UV}} \\
&= y^h_{\alpha\gamma}y^\chi_{\gamma\beta}\frac{(\sin\beta v)\left(\sqrt{f_a^2-\sin^2(2\beta)v^2}\right)}{\Lambda_{UV}} \\
&= \frac{1}{2}\frac{M^{D1}_{\alpha\gamma}M^{D2}_{\gamma\beta}}{\Lambda_{UV}}
\end{aligned} \tag{6-87}$$

Now, from the results found in (6-51) and (6-55) for the (1+1)-scheme, and from the fact that $|M^N| \sim \Lambda_{UV}$, it is straightforward to conclude that:

$$m_\nu \approx M^{D1}\left(M^N\right)^{-1}M^{D2} \tag{6-88}$$

Notice that this result is very similar to the obtained in the canonical seesaw mechanism in (3-56), but with two Dirac mass matrices rather than just one.
This result corresponds to the effective neutrino mass matrix in the flavour basis, and it is believed that these flavour states are approximately the same as the mass states (remember the results in (6-76)-(6-79)). Later in section 6.2.1 the Dirac neutrino effective masses will be computed.

6.2. Reconstruction of the PMNS matrix in the DFSZ axion model

Recalling the expressions proposed in (6-84) and (6-85), let us substitute them in (6-83):

$$\begin{aligned}
\nu_i^D &= U_{i\alpha}n_{L,\alpha} + U_{i\alpha}n_{R,\alpha} \\
&= U_{i\alpha}\left(n_{L,\alpha}+n_{R,\alpha}\right) \\
&= U_{i\alpha}n_\alpha
\end{aligned} \tag{6-89}$$

Where n_α involves both chiralities of the light active neutrinos and heavy sterile fermions:

$$
\begin{aligned}
n &= \begin{pmatrix} \nu_{L,l} \\ N_{L,l'} \end{pmatrix}_{6\times 1} + \begin{pmatrix} \nu_{R,l} \\ N_{R,l'} \end{pmatrix}_{6\times 1} \\
&= \begin{pmatrix} \nu_{L,l} + \nu_{R,l} \\ N_{L,l'} + N_{R,l'} \end{pmatrix} \\
&= \begin{pmatrix} \nu_l \\ N_{l'} \end{pmatrix}
\end{aligned}
\tag{6-90}
$$

Notice that since the greek indice α was redefined to run over $\alpha = e, \mu, \tau, e', \mu', \tau'$, it was necessary to introduce the indices l and l' which run over flavour indices in the active and sterile sector respectively, i.e., $l = e, \mu, \tau$, and $l' = e', \mu', \tau'$.
Now, consider that there are three light active physical states, and three heavy sterile physical states, i.e.:

$$
\nu^D = \begin{pmatrix} \nu_{i'} \\ N_{i'} \end{pmatrix}_{6\times 1} \tag{6-91}
$$

With the latin indice running over $i' = 1, 2, 3$.
Thus, substituting the two above results, and the parameterization of the unitary 6×6 matrix U proposed in (3-58), one gets:

$$
\begin{pmatrix} \nu_{i'} \\ N_{i'} \end{pmatrix} = \begin{pmatrix} U_{PMNS} & R \\ Q & U'_{PMNS} \end{pmatrix} \begin{pmatrix} \nu_l \\ N_{l'} \end{pmatrix} \tag{6-92}
$$

Which explicitly can be written as:

$$
\nu_{i'} = (U_{PMNS})_{i'l}\, \nu_l + R_{i'l'} N_{l'} \tag{6-93}
$$

$$
N_{i'} = Q_{i'l} \nu_l + (U'_{PMNS})_{i'l'}\, N_{l'} \tag{6-94}
$$

This result is exactly similar to the found in chapter 2 for the canonical seesaw mechanism but in that case physical states are Majorana particles rather than Dirac particles. It is important to remark again that if the matrix U_{PMNS} is exactly unitary, then the matrix R must be null and in consequence there would not be active-sterile mixing.
Now, let us make an analogy between the (1+1)- and (3+3)-schemes again. Notice that equations (6-70) and (6-80) state the same physics, find the mass eigenvalues, and the only difference are the degrees of freedom, or in other words, the number of families considered. Thus, one may establish the following proposal:

$$
U \sim S^{-1} \tag{6-95}
$$

It is necessary to remark that the above relation should not be taken as an exact result, since the analogy between the scalars components of S^{-1} and the matrix components of U

is not really direct. However, it is a good point to start to reconstruct the unitary matrix U in the (3+3)-scheme. Thus, some corrections will be added later in section 6.2.1, where an exact reconstruction of the mass matrix M will be done.
Now, the results found in (6-73) implies that:

$$U \sim \begin{pmatrix} I - M^{D1}\left(M^{N}\right)^{-1} & I - M^{D1}\left(M^{D2}\right)^{-1} \\ M^{D2}\left(M^{N}\right)^{-1} & I \end{pmatrix} \tag{6-96}$$

Therefore, from the above proposal and from the parameterization of U in (3-58), it follows that:

$$U_{PMNS} \sim I - M^{D1}\left(M^{N}\right)^{-1} \tag{6-97}$$

Which means that neutrino mixing depends only on the electroweak scale parameters, since $M^{D1}_{\alpha\beta} \propto v y^{h}_{\alpha\beta}$, and apparently there is no dependence on QCD axions parameters which are "carried" by the Dirac matrix M^{D2}.
On the other hand, one also gets that:

$$R \sim I - M^{D1}\left(M^{D2}\right)^{-1} \tag{6-98}$$

And recalling (3-61), it means that the mixing between light active neutrinos and heavy sterile fermions depends on the electroweak scale parameters as well as on the Peccei-Quinn symmetry scale parameters. Thus, QCD axions parameters will appear through this active-sterile mixing. Furthermore, notice that the matrix R is null if and only if:

$$M^{D1} = M^{D2} \tag{6-99}$$

Which is a relation very unlikely to be true, since for definition both Dirac matrices differ because they belong to different energy scales and symmetries. This equality could be possible only if a condition is forced on the Yukawa matrices, however this imposition would lead to a hierarchy problem in the Yukawa coupling values because of the great energy scales difference. In fact, in the following section it will be shown explicitly that the above equation cannot be true. Therefore, with the proposal made in (6-96), and within this Dirac seesaw extension to the DFSZ axion model, the neutrino mixing matrix U_{PMNS} cannot be exactly unitary.
Now, since the Peccei-Quinn symmetry energy scale is far above the electroweak scale ($v << f_a$):

$$|M^{D1}| << |M^{D2}| \tag{6-100}$$

Therefore:

$$|M^{D1}\left(M^{D2}\right)^{-1}| << 1 \tag{6-101}$$

And in consequence:

$$|R| < 1 \tag{6-102}$$

Hence, the heavy sterile component of the light neutrinos in (6-93) is highly supressed, and the relevant component will be associated to the light active neutrinos.
On the other hand, it also follows that:

$$Q \sim M^{D2} \left(M^{N}\right)^{-1} \tag{6-103}$$

And due to the result found in (6-61), the energy scale of the heavy sterile fermion masses is greater than the Peccei-Quinn symmetry energy scale, therefore:

$$|M^{D2}| < |M^{N}| \tag{6-104}$$

And in consequence:

$$|Q| < 1 \tag{6-105}$$

Thus, the light active component of the heavy fermions in (6-94) is highly supressed, hence the relevant component will be just the heavy sterile fermions. Furthermore, in this estimation it has been found that:

$$U'_{PMNS} \sim I \tag{6-106}$$

This suggests that there would not be fermion mixing in the sterile sector. Therefore, the fermion flavour states will be the same as the fermion mass states, and the matrix M^N will be diagonal.

6.2.1. Dirac neutrino effective masses

The neutrino mixing matrix U_{PMNS} has been already reconstructed in the previous section. As it was seen in chapter 2, its components are determined by neutrino oscillation experiments. Therefore, its components are input parameters in the model. Based on this, it is important to compute the Dirac neutrino effective masses given by the equation (6-80), where light active neutrino masses $m_{1,2,3}$ and heavy sterile fermion masses $m_{4,5,6}$ are input parameters as well. Explicitly:

$$\begin{pmatrix} 0 & M^{D1} \\ M^{D2} & M^{N} \end{pmatrix} = U^{\dagger} \begin{pmatrix} D_{\nu} & 0 \\ 0 & D_{N} \end{pmatrix} U \tag{6-107}$$

With $D_\nu = \mathrm{diag}(m_1, m_2, m_3)$ and $D_\nu = \mathrm{diag}(m_4, m_5, m_6)$, where $m_i \geq 0$ for every $i = 1, 2, 3, 4, 5, 6$. Hence, the matrix D_ν and D_N are real and positive definited.
Now, considering that there is no flavour mixing between the heavy sterile fermions and that the active-sterile flavour mixing angles δ_{ij} (for $i = 1, 2, 3$ and $j = 4, 5, 6$) are very small, it is

possible to compute the right-handed side of the equation exactly as in chapter 2. Therefore, using the result found in (3-64) for this case, and compairing each component, one finds that:

$$0 = U^\dagger_{PMNS} D_\nu U_{PMNS} + \left(R^\dagger U_{PMNS}\right)^\dagger D_N \left(R^\dagger U_{PMNS}\right) \tag{6-108}$$

$$M^{D1} = U^\dagger_{PMNS} \left(D_\nu R - R D_N\right) \tag{6-109}$$

$$M^{D2} = \left(R^\dagger D_\nu - D_N R^\dagger\right) U_{PMNS} \tag{6-110}$$

$$M^{N} = R^\dagger D_\nu R + D_N \tag{6-111}$$

Notice that from (6-109) and (6-110) follows that:

$$M^{D2} = \left(M^{D1}\right)^\dagger \tag{6-112}$$

Indeed,

$$\begin{aligned}\left(M^{D1}\right)^\dagger &= \left(U^\dagger_{PMNS} \left(D_\nu R - R D_N\right)\right)^\dagger \\ &= \left(D_\nu R - R D_N\right)^\dagger U_{PMNS} \\ &= \left(R^\dagger D_\nu^\dagger - D_N^\dagger R^\dagger\right) U_{PMNS} \\ &= \left(R^\dagger D_\nu - D_N R^\dagger\right) U_{PMNS} \\ &= M^{D2}\end{aligned} \tag{6-113}$$

This is a remarkable result since it means that, in fact, there is just one Dirac neutrino effective mass matrix. This has physical sense as the effective neutrino mass term in the interaction described by the Weinberg effective lagrangian (6-11) is just one, no matter if it is mediated by two different Yukawa couplings in a UV-complete model.
Substituting expressions (6-26) and (6-27), let us delve into this important result:

$$\begin{aligned}M^{D2} &= \left(M^{D1}\right)^\dagger \\ M^{D2}_{\alpha\beta} &= \left(M^{D1}_{\beta\alpha}\right)^* \\ \sqrt{f_a^2 - \sin^2(2\beta) v^2}\, y^\chi_{\alpha\beta} &= \sin\beta v (y^h_{\beta\alpha})^* \\ y^\chi_{\alpha\beta} &= \frac{\sin\beta v}{\sqrt{f_a^2 - \sin^2(2\beta) v^2}} (y^h_{\beta\alpha})^*\end{aligned} \tag{6-114}$$

And since $f_a >> v$, the above relation implies that:

$$|y^\chi_{\alpha\beta}| \approx \frac{v}{f_a} |y^h_{\alpha\beta}| \tag{6-115}$$

This means that there exists a scale relation between the Yukawa couplings, and in consequence couplings to the gauge singlet scalar χ are highly supressed compared to the Higgs doublet H_u couplings:

$$|y^{\chi}_{\alpha\beta}| << |y^{h}_{\alpha\beta}| \tag{6-116}$$

Explicitly, taking $v \approx 246\,\mathrm{GeV} \sim 10^2\,\mathrm{GeV}$ [43], and $10^6\,\mathrm{GeV} \leq f_a \leq 10^{12}\,\mathrm{GeV}$ [77], the Yukawa couplings are scaled as follows:

$$|y^{\chi}_{\alpha\beta}| \approx (10^{-4} - 10^{-10})|y^{h}_{\alpha\beta}| \tag{6-117}$$

Notice that the result (6-112) makes impossible for the matrix R to be null, unless the Dirac mass matrix M^{D1} is hermitian. However, there is no enough information in the model to conclude that M^{D1} is hermitian. The hermitian matrix is the complete mass matrix, i.e., M. And this can be understood as follows: the whole system is the (3+3)-scheme, containing three light active neutrinos and three heavy sterile fermions which must have real positive mass, therefore the mass matrix for the whole system is M, whose eigenvalues must be real and positive, and they are found by its diagonalization through the expression (6-80), expression that shows the hermiticity of the mass matrix M:

$$\begin{aligned} M^{\dagger} &= (U^{\dagger} m U)^{\dagger} \\ &= U^{\dagger} m^{\dagger} (U^{\dagger})^{\dagger} \\ &= U^{\dagger} m U \\ &= M \end{aligned} \tag{6-118}$$

But the above fact does not necessarily imply that the Dirac neutrino mass matrices M^{D1} and M^{D2} are hermitian. Therefore, the nullness of the matrix R cannot yet be ensured in the model, and in consequence, the unitary nature of the neutrino flavour mixing matrix U_{PMNS} remains an open question in the model as well. Once again, it is concluded that only the ongoing and future neutrino experiments will provide the answer to this question [6, 7]. Notice also that from result (6-111), the Dirac mass matrix for the heavy sterile fermions N_i is hermitian:

$$\begin{aligned} (M^{N})^{\dagger} &= \left(R^{\dagger} D_{\nu} R + D_N\right)^{\dagger} \\ &= \left(R^{\dagger} D_{\nu} R\right)^{\dagger} + (D_N)^{\dagger} \\ &= R^{\dagger} D_{\nu}^{\dagger} (R^{\dagger})^{\dagger} + D_N^{\dagger} \\ &= R^{\dagger} D_{\nu} R + D_N \\ &= M^{N} \end{aligned} \tag{6-119}$$

This result is not impressive since it was choosen the basis where the heavy sterile fermions do not mix between flavours, and therefore, the flavour states correspond approximately to the mass states.

Thus, based on the above results, it is convenient to define:

$$M^D \equiv M^{D1} \tag{6-120}$$

Therefore:

$$M^{D2} = (M^D)^\dagger \tag{6-121}$$

Thus, the 6×6 hermitian mass matrix M can be rewritten as:

$$M = \begin{pmatrix} 0 & M^D \\ (M^D)^\dagger & M^N \end{pmatrix} \tag{6-122}$$

Where its hermicity is shown more clearly.
Let us now compute explicitly the Dirac neutrino effective masses. From equation (6-109), after make matrix multiplications (see Appendix E), one gets the nine components of the Dirac neutrino effective mass matrix:

$$\begin{aligned}(M^D)_{ee} = (m_1 - M_1)C_{12}C_{13}\hat{S}^*_{14} - (m_2 - M_1)(S_{12}C_{23} + C_{12}S_{23}\hat{S}^*_{13})\hat{S}^*_{24} \\ + (m_3 - M_1)(S_{12}S_{23} - C_{12}C_{23}\hat{S}^*_{13})\hat{S}^*_{34}\end{aligned} \tag{6-123}$$

$$\begin{aligned}(M^D)_{e\mu} = (m_1 - M_2)C_{12}C_{13}\hat{S}^*_{15} - (m_2 - M_2)(S_{12}C_{23} + C_{12}S_{23}\hat{S}^*_{13})\hat{S}^*_{25} \\ + (m_3 - M_2)(S_{12}S_{23} - C_{12}C_{23}\hat{S}^*_{13})\hat{S}^*_{35}\end{aligned} \tag{6-124}$$

$$\begin{aligned}(M^D)_{e\tau} = (m_1 - M_3)C_{12}C_{13}\hat{S}^*_{16} - (m_2 - M_3)(S_{12}C_{23} + C_{12}S_{23}\hat{S}^*_{13})\hat{S}^*_{26} \\ + (m_3 - M_3)(S_{12}S_{23} - C_{12}C_{23}\hat{S}^*_{13})\hat{S}^*_{36}\end{aligned} \tag{6-125}$$

$$\begin{aligned}(M^D)_{\mu e} = (m_1 - M_1)S_{12}C_{13}\hat{S}^*_{14} + (m_2 - M_1)(C_{12}C_{23} - S_{12}S_{23}\hat{S}^*_{13})\hat{S}^*_{24} \\ - (m_3 - M_1)(C_{12}S_{23} + S_{12}C_{23}\hat{S}^*_{13})\hat{S}^*_{34}\end{aligned} \tag{6-126}$$

$$\begin{aligned}(M^D)_{\mu\mu} = (m_1 - M_2)S_{12}C_{13}\hat{S}^*_{15} + (m_2 - M_2)(C_{12}C_{23} - S_{12}S_{23}\hat{S}^*_{13})\hat{S}^*_{25} \\ - (m_3 - M_2)(C_{12}S_{23} + S_{12}C_{23}\hat{S}^*_{13})\hat{S}^*_{35}\end{aligned} \tag{6-127}$$

$$\begin{aligned}(M^D)_{\mu\tau} = (m_1 - M_3)S_{12}C_{13}\hat{S}^*_{16} + (m_2 - M_3)(C_{12}C_{23} - S_{12}S_{23}\hat{S}^*_{13})\hat{S}^*_{26} \\ - (m_3 - M_3)(C_{12}S_{23} + S_{12}C_{23}\hat{S}^*_{13})\hat{S}^*_{36}\end{aligned} \tag{6-128}$$

$$(M^D)_{\tau e} = (m_1 - M_1)\hat{S}_{13}\hat{S}^*_{14} + (m_2 - M_1)C_{13}S_{23}\hat{S}^*_{24} - (m_3 - M_1)C_{13}C_{23}\hat{S}^*_{34} \tag{6-129}$$

$$(M^D)_{\tau\mu} = (m_1 - M_2)\hat{S}_{13}\hat{S}^*_{15} + (m_2 - M_2)C_{13}S_{23}\hat{S}^*_{25} - (m_3 - M_2)C_{13}C_{23}\hat{S}^*_{35} \tag{6-130}$$

$$(M^D)_{\tau\tau} = (m_1 - M_3)\hat{S}_{13}\hat{S}^*_{16} + (m_2 - M_3)C_{13}S_{23}\hat{S}^*_{26} - (m_3 - M_3)C_{13}C_{23}\hat{S}^*_{36} \tag{6-131}$$

6.3. CP-violation and thermal leptogenesis in the type-I Dirac seesaw extension to the DFSZ axion model

In the last section it was found that the masses for the heavy sterile fermions N_i are far above the electroweak and Peccei-Quinn symmetry energy scale:

$$|M^N| \sim \Lambda_{UV} > f_a >> v \tag{6-132}$$

Thus, the heavy sterile fermions N_i can decay into lighter particles, and in particular, to those particles which they interact with through the Yukawa couplings. Therefore, the lepton-number-violating decays of N_i can take place via the Yukawa interactions described through the lagrangian for the type-I Dirac seesaw extension to the DFSZ axion model in (6-22). Recall that this lagrangian can be written as the following sum:

$$-\mathcal{L}_{Type-I} = \mathcal{L}^h_{Type-I} + \mathcal{L}^\chi_{Type-I} + \frac{1}{2}\overline{N}_{L,\alpha} M^N_{\alpha\beta} N_{R,\beta} + h.c. \tag{6-133}$$

Now, let us implement the thermal leptogenesis mechanism in the same way it was made in chapter 4. This will be very useful since the results obtained in chapter 4 will be applied to this model, and computations will be much easier. Thus, consider the decays of N_i at tree-level and second-order through the Yukawa coupling with the Higgs doublet H_u described by (6-14). The amplitude associated to the decay at tree-level, whose Feynman diagram is shown in Figure **6-2**, can be computed as [61]:

$$\begin{aligned}\Gamma_{\alpha\beta} &= \frac{1}{32\pi}|y^h_{\alpha\beta}|^2 M^N_i \\ &\propto \frac{1}{2\sin^2\beta v^2}\left[(M^{D1})^\dagger M^{D1}\right]_{\alpha\beta} M^N_i \\ &\propto \frac{1}{2\sin^2\beta v^2}\left[(M^{D})^\dagger M^{D}\right]_{\alpha\beta} M^N_i\end{aligned} \tag{6-134}$$

Where M^N_i is the mass of the heavy sterile fermion N_i that decays.
The decays of N_i described by (6-14) at second-order take place through two possible one-loop processes. The corresponding Feynman diagrams are shown in Figure **6-5**. The (a) process corresponds to a one-loop triangle decay, while the (b) process is a purely one-loop decay. Both diagrams have the same contribution to the amplitude:

$$\begin{aligned}\Gamma_{\alpha\beta} &\propto |y^h_{\alpha\gamma} y^h_{\gamma\delta} y^h_{\delta\beta}|^2 M^N_i \\ &\propto \frac{1}{8\sin^6\beta v^6}|M^{D1}_{\alpha\gamma} M^{D1}_{\gamma\delta} M^{D1}_{\delta\beta}|^2 M^N_i \\ &\propto \frac{1}{8\sin^6\beta v^6}|M^{D}_{\alpha\gamma} M^{D}_{\gamma\delta} M^{D}_{\delta\beta}|^2 M^N_i\end{aligned} \tag{6-135}$$

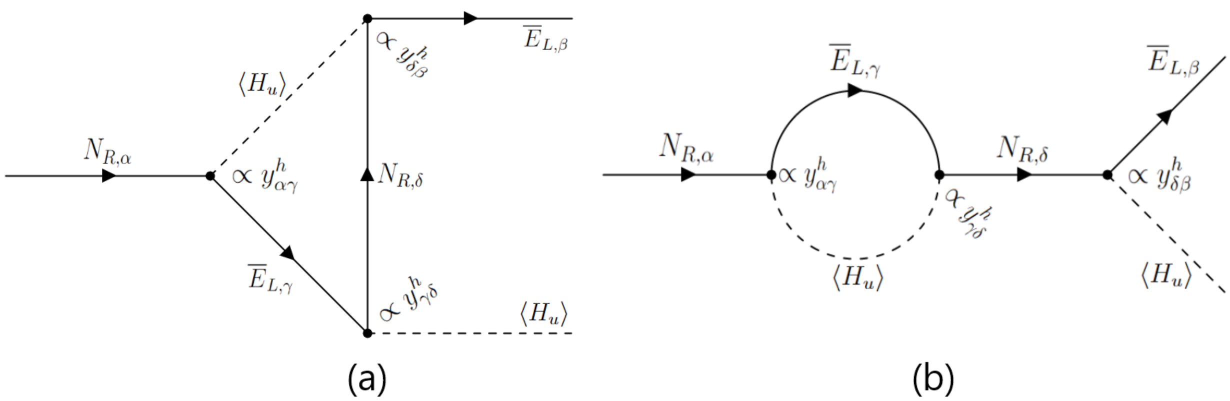

Figure 6-5.: Feynman diagrams for the decay of N_i through the Yukawa coupling with the $\mathrm{SU}(2)_L$ leptonic and Higgs doublets.

Now, consider the decays of N_i at tree-level and second-order through the Yukawa coupling with the gauge-singlet scalar χ described by (6-19). The amplitude associated to the decay at tree-level, whose Feynman diagram is shown in Figure **6-3**, can be computed as [61]:

$$\begin{aligned} \Gamma_{\alpha\beta} &= \frac{1}{32\pi}|y^{\chi}_{\alpha\beta}|^2 M^N_i \\ &\propto \frac{1}{2\left(f_a^2-\sin^2(2\beta)v^2\right)}\left[(M^{D2})^{\dagger}M^{D2}\right]_{\alpha\beta} M^N_i \\ &\propto \frac{1}{2\left(f_a^2-\sin^2(2\beta)v^2\right)}\left[M^{D}(M^{D})^{\dagger}\right]_{\alpha\beta} M^N_i \end{aligned} \tag{6-136}$$

The decay processes at second order are shown in Figure **6-6**, and they correspond again to one-loop and one-loop triangle Feynman diagrams. Both diagrams have the same contribution to the amplitude:

$$\begin{aligned} \Gamma_{\alpha\beta} &\propto |y^{\chi}_{\alpha\gamma}y^{\chi}_{\gamma\delta}y^{\chi}_{\delta\beta}|^2 M^N_i \\ &\propto \frac{1}{8\left(f_a^2-\sin^2(2\beta)v^2\right)^3}|M^{D2}_{\alpha\gamma}M^{D2}_{\gamma\delta}M^{D2}_{\delta\beta}|^2 M^N_i \\ &\propto \frac{1}{8\left(f_a^2-\sin^2(2\beta)v^2\right)^3}|(M^D)^{\dagger}_{\alpha\gamma}(M^D)^{\dagger}_{\gamma\delta}(M^D)^{\dagger}_{\delta\beta}|^2 M^N_i \end{aligned} \tag{6-137}$$

Therefore, the vector-like heavy gauge-singlet fermions $N_i = N_{L,i} + N_{R,i}$ can have the two-body decays at tree-level and second-order shown before. Thus, the total decay width for N_i can be computed as:

$$\begin{aligned} \Gamma_{N_i} = \sum_{\alpha} \big[&\Gamma(N_{R,i}\to \overline{E}_{L,\alpha} + \langle H_u\rangle) + \Gamma(N_{L,i}\to \nu_{R,\alpha} + \langle \chi\rangle) \\ &+\Gamma(N^c_{R,i}\to \overline{E}^c_{L,\beta} + \langle H^c_u\rangle) + \Gamma(N^c_{L,i}\to \nu^c_{R,\alpha} + \langle \chi^c\rangle)\big] \end{aligned} \tag{6-138}$$

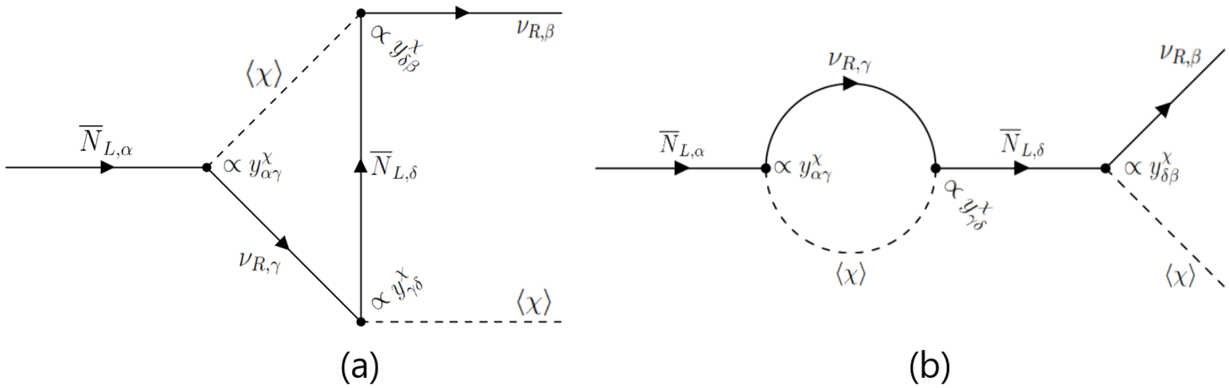

Figure 6-6.: Feynman diagrams for the decay of N_i through the Yukawa coupling with the gauge-singlet scalar χ and neutrino ν_R.

Where the CP-conjugated processes must be included in the computation, and correspond to the second line terms of the above equation. Furthermore, notice that the basis where the flavour states of the heavy sterile fermions is equal to their mass states has been chosen, hence the latin indice runs over $i = 1, 2, 3$.

Now, the interference between the tree-level and second order contributions result in a CP-violating asymmetry. However, remember that in leptogenesis the baryon asymmetry is generated through a lepton asymmetry, and it is thanks to the $\mathrm{SU}(2)_L$ sphaleron processes [61, 52]. Therefore, the right-handed neutrino asymmetry, computed through the interference between the processes whose amplitude is given by (6-136) and (6-137) with their CP-conjugated processes, will not affect the baryon asymmetry.

For the above reasons, CP-violation parameter will be defined just as the normalized interference for the decays through the $\mathrm{SU}(2)_L$ doublets, described by the Yukawa interaction (6-14), and their CP-conjugated processes:

$$\epsilon_{i\alpha} = \frac{\Gamma(N_{R,i} \to \overline{E}_{L,\alpha} + \langle H_u \rangle) - \Gamma(N^c_{R,i} \to \overline{E}^c_{L,\alpha} + \langle H^c_u \rangle)}{\Gamma_{N_i}} \tag{6-139}$$

Now, to use the result found in the chapter 3 for the CP-violation parameter (4-55) let us consider the effective Yukawa coupling matrix:

$$\begin{aligned}
(Y_\nu)_{\alpha i} &= y^\nu_{\alpha i} \\
&= y^h_{\alpha i} y^\chi_{\alpha i} \\
&= \left(\frac{\sqrt{2}}{2}\frac{1}{\sin\beta v} M^{D1}_{\alpha\beta}\right)\left(\frac{\sqrt{2}}{2}\frac{1}{\sqrt{f_a^2 - \sin^2(2\beta) v^2}} M^{D2}_{\beta i}\right) \\
&= \frac{1}{2}\frac{M^{D1}_{\alpha\beta} M^{D2}_{\beta i}}{\sin\beta v\sqrt{f_a^2 - \sin^2(2\beta)v^2}} \\
&\approx \frac{1}{2\sin\beta v f_a} M^{D1}_{\alpha\beta} M^{D2}_{\beta i} \\
&\approx \frac{1}{2\sin\beta v f_a} M^{D}_{\alpha\beta} (M^D)^\dagger_{\beta i}
\end{aligned} \tag{6-140}$$

Thus, substituting this Yukawa coupling matrix given by the type-I Dirac seesaw extension to the DFSZ axion model in (4-55), one gets the CP-violation parameter for this model:

$$\begin{aligned}
\epsilon_{i\alpha} &= \frac{(2\sin\beta v f_a)^2}{8\pi\left[(M^D(M^D)^\dagger)^\dagger(M^D(M^D)^\dagger)\right]_{ii}}\frac{1}{(2\sin\beta v f_a)^4}\sum_{i\neq j}\Big\{\mathrm{Im}\left[((M^D(M^D)^\dagger)^*)_{\alpha i}(M^D(M^D)^\dagger)_{\alpha j}((M^D(M^D)^\dagger)^\dagger M^D(M^D)^\dagger)_{ij}\right] f(x_{ji}) \\
&\qquad + \mathrm{Im}\left[((M^D(M^D)^\dagger)^*)_{\alpha i}(M^D(M^D)^\dagger)_{\alpha j}((M^D(M^D)^\dagger)^\dagger M^D(M^D)^\dagger)^*_{ij}\right] g(x_{ji})\Big\} \\
&= \frac{1}{32\pi\sin^2\beta v^2 f_a^2\left[M^D(M^D)^\dagger M^D(M^D)^\dagger\right]_{ii}}\sum_{i\neq j}\Big\{\mathrm{Im}\left[((M^D(M^D)^\dagger)^*)_{\alpha i}(M^D(M^D)^\dagger)_{\alpha j}(M^D(M^D)^\dagger M^D(M^D)^\dagger)_{ij}\right] f(x_{ji}) \\
&\qquad + \mathrm{Im}\left[((M^D(M^D)^\dagger)^*)_{\alpha i}(M^D(M^D)^\dagger)_{\alpha j}(M^D(M^D)^\dagger M^D(M^D)^\dagger)^*_{ij}\right] g(x_{ji})\Big\}
\end{aligned} \tag{6-141}$$

Whit $x_{ji} = M^N_j/M^N_i$, and $f(x)$ and $g(x)$ are the so-called "one-loop functions".
Now, the above result can be simplified considering the CP-violation mechanism in the unflavoured leptogenesis model. In this unflavoured model the lepton asymmetry is produced in a single flavour α, and this approximation is valid for $T \geq 10^{12}$ GeV. Since the heavy sterile fermion masses M^N_i are of the order of Λ_{UV}, and because of (6-62) the energy scale Λ_{UV} can be greater than 10^{12} GeV, hence this approximation is valid for this model. Thus, substituting the effective Yukawa coupling matrix found in (6-140) into equation (4-58):

$$\begin{aligned}
\epsilon_i &= \frac{(2\sin\beta v f_a)^2}{8\pi((M^D(M^D)^\dagger)^\dagger M^D(M^D)^\dagger)_{ii}}\frac{1}{(2\sin\beta v f_a)^4}\sum_{j\neq i}\mathrm{Im}\left[((M^D(M^D)^\dagger)^\dagger M^D(M^D)^\dagger)^2_{ij}\right] f(x_{ji}) \\
&= \frac{1}{32\pi\sin^2\beta v^2 f_a^2(M^D(M^D)^\dagger M^D(M^D)^\dagger)_{ii}}\sum_{j\neq i}\mathrm{Im}\left[(M^D(M^D)^\dagger M^D(M^D)^\dagger)^2_{ij}\right] f(x_{ji})
\end{aligned} \tag{6-142}$$

In fact, it is has been shown that if 10^8 GeV $\leq M_1 \leq 10^{12}$ GeV, CP violation necessary for successful leptogenesis can be provided entirely by the CP phases in the lepton mixing matrix U_{PMNS} [61]. Therefore, because of $\Lambda_{UV} \sim M_1$, and from the relation found in (6-62), succesful leptogenesis will be guaranteed in this model.
Moreover, because of the mass hierarchy asumption for $M_{R,i}$, the lepton-number-violating decays of N_1 may be fast enough to wash out the lepton-antilepton number asymmetry

generated by decays of N_2 and N_3 [7], therefore let us consider just one CP-violating factor, which is given by the result found in (4-59), and replacing the effective Yukawa matrix (6-140):

$$\begin{aligned}\epsilon_1 &= \frac{1}{8\pi(Y_\nu^\dagger Y_\nu)_{11}} \sum_{j\neq 1}^{3} \mathrm{Im}\left[(Y_\nu^\dagger Y_\nu)^2_{1j}\right] f(x_{j1}) \\ &= \frac{1}{32\pi \sin^2\beta v^2 f_a^2 (M^D(M^D)^\dagger M^D(M^D)^\dagger)_{11}} \sum_{j\neq 1}^{3} \mathrm{Im}\left[(M^D(M^D)^\dagger M^D(M^D)^\dagger)^2_{1j}\right] f(x_{j1})\end{aligned} \tag{6-143}$$

Now, to compute the lepton asymmetry, the associated Boltzmann equations for the number density of the heavy sterile fermions n_{N_1}, the leptons n_L, and the antileptons $n_{\overline{L}}$ will be exactly the same as those of the section 4.3.3, therefore the Boltzmann equation for the lepton-antilepton density $Y_{\Delta B}$ will have the same form, but this time with the CP-violating parameter found in this type-I Dirac seesaw extension to the DFSZ axion model in (6-143):

$$\begin{aligned}\frac{dY_{\Delta L}}{dt} + 3H(t)Y_{\Delta L} &= \frac{\Gamma_{N_1}\epsilon_1}{s} \\ &= \frac{\Gamma_{N_1}}{s}\frac{1}{32\pi \sin^2\beta v^2 f_a^2 (M^D(M^D)^\dagger M^D(M^D)^\dagger)_{11}} \sum_{j\neq 1}^{3} \mathrm{Im}\left[(M^D(M^D)^\dagger M^D(M^D)^\dagger)^2_{1j}\right] f(x_j\end{aligned} \tag{6-144}$$

In consequence, the solution can be estimated to [52, 19, 65]:

$$\begin{aligned}Y_{\Delta L} &\approx (0{,}26)\cdot k\epsilon_1 \\ &\approx (0{,}26)\cdot \frac{k}{32\pi \sin^2\beta v^2 f_a^2 (M^D(M^D)^\dagger M^D(M^D)^\dagger)_{11}} \sum_{j\neq 1}^{3} \mathrm{Im}\left[(M^D(M^D)^\dagger M^D(M^D)^\dagger)^2_{1j}\right] f(x_{j1})\end{aligned} \tag{6-145}$$

Now, since the model is a type-I Dirac extension, the efficiency factor k is expected to be about $10^{-3} \leq k \leq 1$ [66].
Then, this lepton asymmetry is converted into a baryon asymmetry through sphaleron processes [19], and from the thermal leptogenesis results (4-84) and (4-85) found in chapter 4, and since the DFSZ axion model has two Higgs doublets, it follows that [63, 67]:

$$\begin{aligned}C &= \frac{8N_f + 4N_H}{22N_f + 13N_H} \\ &= \frac{8(3)+4(2)}{22(3)+13(2)} \\ &= \frac{8}{23} \\ &\approx 0{,}347\end{aligned} \tag{6-146}$$

And in consequence:

$$\begin{aligned}
Y_{\Delta B} &= -CY_{\Delta L} \\
&\approx -C\frac{k}{g_{*s}}\epsilon_1 \\
&\approx -(0{,}0904)k\epsilon_1 \\
&\approx \frac{-(0{,}0904)k}{32\pi\sin^2\beta v^2 f_a^2 (M^D(M^D)^\dagger M^D(M^D)^\dagger)_{11}} \sum_{j\neq 1}^{3} \mathrm{Im}\left[(M^D(M^D)^\dagger M^D(M^D)^\dagger)^2_{1j}\right] f(x_{j1})
\end{aligned} \tag{6-147}$$

Therefore, it has been found an expression to compute directly the baryon asymmetry of the universe in the type-I Dirac seesaw extension to the DFSZ axion model. Nevertheless, notice that an estimation cannot be done accurately since the CP-asymmetry parameter ϵ_1 depends on active-sterile mixing parameters, besides QCD axion decay constant f_a, which have not been measured properly yet. Hence, it is necessary to wait for neutrino and QCD axion experiments to make a relevant calculation and be able to compare with the observed in (4-31). However, this is a remarkable result since it relates the three important quantities v, f_a, and $Y_{\Delta B}$, besides the Dirac neutrino mass matrix M^D, which belong to the three big problems in cosmology and particle physics covered along this thesis: neutrino mass, baryon asymmetry of the universe, and dark matter.

7. Conclusions

It was seen that the Peccei-Quinn invariance imposition in the UV-completion of the DFSZ axion model leads to Dirac neutrinos, which allows to make a type-I Dirac seesaw extension, finding that the light active neutrino mass is given by $m_\nu \approx v f_a/\Lambda_{UV}$, hence neutrino mass connects the three energy scales involved in the model: the Higgs vacuum expectation value v, the Peccei-Quinn symmetry breaking scale f_a, and the energy scale for the heavy sterile fermion mass Λ_{UV} introduced in the type-I Dirac seesaw extension. As a consequence of this, the following scale relation between the Peccei-Quinn symmetry energy scale f_a and the UV-completion scale energy Λ_{UV} was found: $10^3\,\mathrm{GeV}\, f_a \sim \Lambda_{UV}$. This fact led to find boundaries for the scale energy Λ_{UV} because of the experimental boundaries on f_a [77]: $10^9\,\mathrm{GeV} \leq \Lambda_{UV} \leq 10^{15}\,\mathrm{GeV}$, therefore succesful leptogenesis in this model is guaranteed because of Λ_{UV} energy scale spectrum [61]. This energy scale relation also implies that the Yukawa coupling associated to the gauge-singlet scalar, is exactly supressed by a factor which goes from 10^{-4} to 10^{-10}, in comparison with the Yukawa coupling associated to Higgs doublet, because of the dependence with the axion decay constant f_a.

It was implemented the canonical seesaw mechanism in the type-I Dirac seesaw extension to the DFSZ axion model, in the (1+1)- and (3+3)-schemes. For the latter it was found that the two Yukawa terms associated to the gauge-singlet scalar and the Higgs doublet are linked together in a single Dirac neutrino effective mass matrix. Their nine components were calculated explicitly, finding a dependence on active-sterile mixing parameters. Furthermore, it was found that if the Dirac neutrino effective mass matrix is hermitian, then the lepton mixing matrix U_{PMNS} is exactly unitary in this model. However, there is no way to know from the model if this Dirac neutrino effective mass matrix is hermitian or not, just future neutrino experiments have the answer, hence this model still leaves the questions about the unitary nature of the PMNS matrix open.

The CP asymmetry factor is computed in the simplest case for thermal leptogenesis: unflavoured and considering that the decay of the lightest sterile fermion N_1 is the principal contribution for the lepton asymmetry [7]. It was found that the CP asymmetry factor depends directly on the Dirac neutrino effective mass matrix, and therefore on active-sterile mixing parameters, and also on the QCD axion decay constant f_a. Thus, new sources of CP violation were found because of the new nine CP-violating phases associated to this mixing.

Finally, the baryon-antibaryon density $Y_{\Delta B}$ was computed, finding an expression in terms of the CP asymmetry factor computed previously. However, there was not possible to perform an estimation since this quantity depends on parameters which are still unknown, such as the active-sterile mixing angles or CP-violating phases, or they are not exactly determined, as the case of the axion decay constant f_a.

Based on the results found here, it is concluded that this model links in a same framework neutrino physics, QCD axion, and cosmological aspects, connecting three of the biggest problems in theoretical particle physics and cosmology: neutrino masses, the baryon asymmetry of the universe, and dark matter. Therefore, future measurements and results in experiments related to any of these areas could indirectly provide valuable information about each other.

A. CKM matrix characterisation

Let's see it explicitly: we start with the CKM matrix written as a general matrix $V \in U(3)$:

$$V = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} \tag{A-1}$$

Therefore, each component $V_{ij} \in \mathbb{C}$, so:

$$V_{ij} = a_i + ib_j \tag{A-2}$$

For some $i, j = 1, 2, 3$ and $a_i, b_j \in \mathbb{R}$. Then, V has $2 \times 9 = 18$ free real parameters to determine.
From the fact that V is unitary, i.e., $VV^\dagger = V^\dagger V = I$, we get 9 independent equations between the components:

$$|V_{11}|^2 + |V_{12}|^2 + |V_{13}|^2 = 1 \tag{A-3}$$

$$|V_{21}|^2 + |V_{22}|^2 + |V_{23}|^2 = 1 \tag{A-4}$$

$$|V_{31}|^2 + |V_{32}|^2 + |V_{33}|^2 = 1 \tag{A-5}$$

$$V_{ii}V_{ji}^* + V_{ij}V_{jj}^* + V_{ik}V_{jk}^* = 0 \tag{A-6}$$

where the last equation is actually 6 equations. Then, we have reduced the real parameters to determine to be just 9.
On the other hand, appplying the transformation with the phases:

$$\begin{aligned} V'' = U_\varphi^\dagger V U_\chi &= \begin{pmatrix} e^{-i(\varphi_1-\chi_1)}V_{11} & e^{-i(\varphi_1-\chi_2)}V_{12} & e^{-i(\varphi_1-\chi_3)}V_{13} \\ e^{-i(\varphi_2-\chi_1)}V_{21} & e^{-i(\varphi_2-\chi_2)}V_{22} & e^{-i(\varphi_2-\chi_3)}V_{23} \\ e^{-i(\varphi_3-\chi_1)}V_{31} & e^{-i(\varphi_3-\chi_2)}V_{32} & e^{-i(\varphi_3-\chi_3)}V_{33} \end{pmatrix} \\ &= \begin{pmatrix} e^{-i\phi_{11}}V_{11} & e^{-i\phi_{12}}V_{12} & e^{-i\phi_{13}}V_{13} \\ e^{-i\phi_{21}}V_{21} & e^{-i\phi_{22}}V_{22} & e^{-i\phi_{23}}V_{23} \\ e^{-i\phi_{31}}V_{31} & e^{-i\phi_{32}}V_{32} & e^{-i\phi_{33}}V_{33} \end{pmatrix} \end{aligned} \tag{A-7}$$

Where now we have that the components $V_{ij} \in \mathbb{R}$ and the complex part is on the phases $e^{-i\phi_{ij}}$, with $\phi_{ij} = \varphi_i - \chi_j$.

Then, we are going to invoke again the unitary of this matrix, but this time applied to the determinant, i.e.,

$$VV^\dagger = V^\dagger V = 1 \to (\det V)^2 = 1 \to \det V = \pm 1 \tag{A-8}$$

Therefore,

$$\begin{vmatrix} e^{-i\phi_{11}}V_{11} & e^{-i\phi_{12}}V_{12} & e^{-i\phi_{13}}V_{13} \\ e^{-i\phi_{21}}V_{21} & e^{-i\phi_{22}}V_{22} & e^{-i\phi_{23}}V_{23} \\ e^{-i\phi_{31}}V_{31} & e^{-i\phi_{32}}V_{32} & e^{-i\phi_{33}}V_{33} \end{vmatrix} = \pm 1 \tag{A-9}$$

$$e^{-i\phi_{11}}V_{11}\left(e^{-i(\phi_{22}+\phi_{33})}V_{22}V_{33} - e^{-i(\phi_{32}+\phi_{23})}V_{32}V_{23}\right) - e^{-i\phi_{12}}V_{12}\left(e^{-i(\phi_{21}+\phi_{33})}V_{21}V_{33} - e^{-i(\phi_{23}+\phi_{31})}V_{23}V_{31}\right)$$
$$+e^{-i\phi_{13}}V_{13}\left(e^{-i(\phi_{21}+\phi_{32})}V_{21}V_{32} - e^{-i(\phi_{22}+\phi_{31})}V_{22}V_{31}\right) = \pm 1$$

Then, it is possible to take as common factor the complex phases for one of the two terms between the brackets:

$$\begin{aligned} &e^{-i(\phi_{11}+\phi_{32}+\phi_{23})}\left(V_{11}V_{22}V_{33}e^{-i(\phi_{22}+\phi_{33}-\phi_{32}-\phi_{23})} - V_{11}V_{32}V_{23}\right) \\ &+e^{-i(\phi_{12}+\phi_{23}+\phi_{31})}\left(V_{12}V_{23}V_{31} - e^{-i(\phi_{21}+\phi_{33}-\phi_{23}-\phi_{31})}V_{12}V_{21}V_{33}\right) \\ &+e^{-i(\phi_{13}+\phi_{21}+\phi_{32})}\left(V_{13}V_{32}V_{21} - e^{-i(\phi_{22}+\phi_{31}-\phi_{21}-\phi_{32})}V_{13}V_{31}V_{22}\right) = \pm 1 \end{aligned}$$

Now, it is convenient to define:

$$\phi_{11} + \phi_{32} + \phi_{23} = \alpha \tag{A-10}$$

$$\phi_{12} + \phi_{23} + \phi_{31} = \beta \tag{A-11}$$

$$\phi_{13} + \phi_{21} + \phi_{32} = \gamma \tag{A-12}$$

In order to factorize a complex phase again:

$$\begin{aligned} &e^{-i\alpha}\left[\left(V_{11}V_{22}V_{33}e^{-i(\phi_{22}+\phi_{33}-\phi_{32}-\phi_{23})} - V_{11}V_{32}V_{23}\right)\right. \\ &+e^{-i(\beta-\alpha)}\left(V_{12}V_{23}V_{31} - e^{-i(\phi_{21}+\phi_{33}-\phi_{23}-\phi_{31})}V_{12}V_{21}V_{33}\right) \\ &\left.+e^{-i(\gamma-\alpha)}\left(V_{13}V_{32}V_{21} - e^{-i(\phi_{22}+\phi_{31}-\phi_{21}-\phi_{32})}V_{13}V_{31}V_{22}\right)\right] = \pm 1 \end{aligned}$$

Finally, to fulfill the above equation is necessary to make the imaginary part equals to zero, i.e., to make the 5 complex phases involved zero:

$$\phi_{22} + \phi_{33} - \phi_{32} - \phi_{23} = 0 \tag{A-13}$$

$$\beta - \alpha = 0 \tag{A-14}$$

$$\phi_{21} + \phi_{33} - \phi_{23} - \phi_{31} = 0 \tag{A-15}$$

$$\gamma - \alpha = 0 \tag{A-16}$$

$$\phi_{22} + \phi_{31} - \phi_{21} - \phi_{32} = 0 \tag{A-17}$$

In this way we have been able to absorb 5 complex phases. Therefore, at the end, due to the 9 equations (A-3)-(A-6) and the last 5 equations (A-13)-(A-17), it has been possible to reduce the initial 18 free real parameters to just 18-9-5=4. These are three angles and a remaining complex phase which will be the cause of CP violation in the SM.

B. Appendix: Lagrangian and field units

It is well-known that if one takes the natural units ($c = \hbar = 1$). velocities are adimensional, then space and time have the same dimensions; from the Plank hypothesis, the energy has units of $[t^{-1}]$; and from the energy relativistic relation for a particle of mass m at rest, energy and mas has the same units, let us denote it by $[M]$. Now, the action S is defined as the integral over time of the lagrangian L, or equivalently, the integral over space and time of the lagrangian density $\mathcal{L}$:

$$S = \int L dt = \int \mathcal{L} dx^4 \tag{B-1}$$

And this action must be dimensionless, which means that:

$$\begin{aligned}
[S] &= [1] \\
[\mathcal{L} dx^4] &= [1] \\
[\mathcal{L}] &= [dx^4]^{-1} \\
[\mathcal{L}] &= [t^4]^{-1} \\
[\mathcal{L}] &= [M^{-4}]^{-1} \\
[\mathcal{L}] &= [M^4]
\end{aligned} \tag{B-2}$$

Therefore, lagrangian densities have dimension $[M^4]$ in natural units. Now, taking into account this result is easy to find the dimension for the different fields involved in a lagrangian density. Thus, from the Klein-Gordon lagrangian density one finds the units for a scalar field ϕ:

$$[\phi] = [M] \tag{B-3}$$

From the Dirac lagrangian density one finds the units for a Dirac field ψ:

$$[\psi] = [M^{3/2}] \tag{B-4}$$

And from the Maxwell and Proca lagrangian density one finds the units for a vector field $[A^\mu]$:

$$[A^\mu] = [M] \tag{B-5}$$

For ease and without loss of generality, the Lagrangian density will be referred to throughout this document simply as the Lagrangian.

C. Appendix: Effective mass matrix M computation

Given the following parameterization for the unitary matrix U:

$$U = \begin{pmatrix} U_{PMNS} & R \\ U'_{PMNS} S U_{PMNS} & U'_{PMNS} \end{pmatrix} \tag{C-1}$$

It is possible to compute the components of the mass matrix M given the diagonalization relation:

$$M = U^{\dagger} m U \tag{C-2}$$

With:

$$m = \begin{pmatrix} D_{\nu} & 0 \\ 0 & D_N \end{pmatrix} \tag{C-3}$$

And, $D_{\nu} = \mathrm{diag}(m_1, m_2, m_3)$ and $D_N = \mathrm{diag}(m_4, m_5, m_6)$.
Thus, let us compute the adjoint of the unitary matriz U:

$$U^{\dagger} = \begin{pmatrix} U^{\dagger}_{PMNS} & (U'_{PMNS} S U_{PMNS})^{\dagger} \\ R^{\dagger} & {U'_{PMNS}}^{\dagger} \end{pmatrix} \tag{C-4}$$

Then, substituting this results in the diagonalization relation:

$$\begin{aligned}
M &= \begin{pmatrix} U^{\dagger}_{PMNS} & (U'_{PMNS} S U_{PMNS})^{\dagger} \\ R^{\dagger} & {U'_{PMNS}}^{\dagger} \end{pmatrix} \begin{pmatrix} D_{\nu} & 0 \\ 0 & D_N \end{pmatrix} \begin{pmatrix} U_{PMNS} & R \\ U'_{PMNS} S U_{PMNS} & U'_{PMNS} \end{pmatrix} \\
&= \begin{pmatrix} U^{\dagger}_{PMNS} & (U'_{PMNS} S U_{PMNS})^{\dagger} \\ R^{\dagger} & {U'_{PMNS}}^{\dagger} \end{pmatrix} \begin{pmatrix} D_{\nu} U_{PMNS} & D_{\nu} R \\ D_N (U'_{PMNS} S U_{PMNS}) & D_N U'_{PMNS} \end{pmatrix} \\
&= \begin{pmatrix} U^{\dagger}_{PMNS} D_{\nu} U_{PMNS} + (U'_{PMNS} S U_{PMNS})^{\dagger} D_N \left(U'_{PMNS} S U_{PMNS}\right) & U^{\dagger}_{PMNS} D_{\nu} R + (U'_{PMNS} S U_{PMNS})^{\dagger} D_N U'_{PMNS} \\ R^{\dagger} D_{\nu} U_{PMNS} + U'^{\dagger}_{PMNS} D_N \left(U'_{PMNS} S U_{PMNS}\right) & R^{\dagger} D_{\nu} R + U'^{\dagger}_{PMNS} D_N U'_{PMNS} \end{pmatrix}
\end{aligned} \tag{C-5}$$

D. Appendix: Computation of the inverse matrix of S in the (1+1)-scheme

Given the matrix:

$$S = \begin{pmatrix} 1 & \dfrac{m^{D1}}{m^N} \\ -\dfrac{m^{D2}}{m^N} & 1 \end{pmatrix} \tag{D-1}$$

Its inverse matrix can be found by:

$$\left(\begin{array}{cc|cc} 1 & \dfrac{m^{D1}}{m^N} & 1 & 0 \\ \dfrac{m^{D2}}{m^N} & 1 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 1 & \dfrac{m^{D1}}{m^N} & 1 & 0 \\ 0 & 1+\dfrac{m^{D1}m^{D2}}{(m^N)^2} & \dfrac{m^{D2}}{m^N} & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 1 & \dfrac{m^{D1}}{m^{D2}} & 1 & 0 \\ 0 & \dfrac{(m^N)^2+m^{D1}m^{D2}}{(m^N)^2}1 & \dfrac{m^{D2}}{m^N} & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|cc} 1 & \dfrac{m^{D1}}{m^{D2}} & 1 & 0 \\ 0 & 1 & \left(\dfrac{(m^N)^2}{(m^N)^2+m^{D1}m^{D2}}\right)\dfrac{m^{D2}}{m^N} & \dfrac{(m^N)^2}{(m^N)^2+m^{D1}m^{D2}} \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|cc} 1 & 0 & \dfrac{(m^N)^2+m^{D1}m^{D2}-m^Nm^{D1}}{(m^N)^2+m^{D1}m^{D2}} & \dfrac{\left((m^N)^2+m^{D1}m^{D2}\right)m^{D2}-m^{D1}(m^N)^2}{\left((m^N)^2+m^{D1}m^{D2}\right)m^{D2}} \\ 0 & 1 & \dfrac{m^Nm^{D2}}{(m^N)^2+m^{D1}m^{D2}} & \dfrac{(m^N)^2}{(m^N)^2+m^{D1}m^{D2}} \end{array}\right) \tag{D-2}$$

Therefore

$$S^{-1} = \begin{pmatrix} \dfrac{(m^N)^2+m^{D1}m^{D2}-m^Nm^{D1}}{(m^N)^2+m^{D1}m^{D2}} & \dfrac{\left((m^N)^2+m^{D1}m^{D2}\right)m^{D2}-m^{D1}(m^N)^2}{\left((m^N)^2+m^{D1}m^{D2}\right)m^{D2}} \\ \dfrac{m^Nm^{D2}}{(m^N)^2+m^{D1}m^{D2}} & \dfrac{(m^N)^2}{(m^N)^2+m^{D1}m^{D2}} \end{pmatrix} \tag{D-3}$$

And considering that:

$$\begin{aligned}(m^N)^2 + m^{D1}m^{D2} - m^N m^{D1} &\approx (m^N)^2 - m^N m^{D1} \\ (m^N)^2 m^{D2} + m^{D1}(m^{D2})^2 - (m^N)^2 m^{D1} &\approx (m^N)^2 \left(m^{D2} - m^{D1}\right)\end{aligned} \tag{D-4}$$

Thus:

$$\begin{aligned} S^{-1} &= \frac{1}{(m^M)^2 + m^{D1}m^{D2}} \begin{pmatrix} (m^N)^2 - m^N m^{D1} & \dfrac{(m^N)^2\left(m^{D2} - m^{D1}\right)}{m^{D2}} \\ \\ m^N m^{D2} & (m^N)^2 \end{pmatrix} \\ \\ &\approx \frac{1}{(m^N)^2} \begin{pmatrix} (m^N)^2 - m^N m^{D1} & \dfrac{(m^N)^2\left(m^{D2} - m^{D1}\right)}{m^{D2}} \\ \\ m^N m^{D2} & (m^N)^2 \end{pmatrix} \\ \\ &= \begin{pmatrix} 1 - \dfrac{m^{D1}}{m^N} & \dfrac{m^{D2} - m^{D1}}{m^{D2}} \\ \\ \dfrac{m^{D2}}{m^N} & 1 \end{pmatrix} \\ \\ &= \begin{pmatrix} 1 - \dfrac{m^{D1}}{m^N} & 1 - \dfrac{m^{D1}}{m^{D2}} \\ \\ \dfrac{m^{D2}}{m^N} & 1 \end{pmatrix} \end{aligned} \tag{D-5}$$

E. Appendix: Dirac neutrino effective masses computation

In (6-109) it was found that the Dirac mass matrix $M^{D1} = (M^{D2})^{\dagger} = M^{D}$ can be computed as:

$$M^{D1} = U^{\dagger}_{PMNS} \left(D_{\nu} R - R D_{N} \right) \tag{E-1}$$

Let us compute the term in brackets:

$$D_{\nu} R - R D_{N} = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \begin{pmatrix} \hat{S}^{*}_{14} & \hat{S}^{*}_{15} & \hat{S}^{*}_{16} \\ \hat{S}^{*}_{24} & \hat{S}^{*}_{25} & \hat{S}^{*}_{26} \\ \hat{S}^{*}_{34} & \hat{S}^{*}_{35} & \hat{S}^{*}_{36} \end{pmatrix} - \begin{pmatrix} \hat{S}^{*}_{14} & \hat{S}^{*}_{15} & \hat{S}^{*}_{16} \\ \hat{S}^{*}_{24} & \hat{S}^{*}_{25} & \hat{S}^{*}_{26} \\ \hat{S}^{*}_{34} & \hat{S}^{*}_{35} & \hat{S}^{*}_{36} \end{pmatrix} \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{pmatrix}$$

$$\begin{aligned} &= \begin{pmatrix} m_1\hat{S}^{*}_{14} & m_1\hat{S}^{*}_{15} & m_1\hat{S}^{*}_{16} \\ m_2\hat{S}^{*}_{24} & m_2\hat{S}^{*}_{25} & m_2\hat{S}^{*}_{26} \\ m_3\hat{S}^{*}_{34} & m_3\hat{S}^{*}_{35} & m_3\hat{S}^{*}_{36} \end{pmatrix} - \begin{pmatrix} M_1\hat{S}^{*}_{14} & M_2\hat{S}^{*}_{15} & M_3\hat{S}^{*}_{16} \\ M_1\hat{S}^{*}_{24} & M_2\hat{S}^{*}_{25} & M_3\hat{S}^{*}_{26} \\ M_1\hat{S}^{*}_{34} & M_2\hat{S}^{*}_{35} & M_3\hat{S}^{*}_{36} \end{pmatrix} \\ &= \begin{pmatrix} (m_1 - M_1)\hat{S}^{*}_{14} & (m_1 - M_2)\hat{S}^{*}_{15} & (m_1 - M_3)\hat{S}^{*}_{16} \\ (m_2 - M_1)\hat{S}^{*}_{24} & (m_2 - M_2)\hat{S}^{*}_{25} & (m_2 - M_3)\hat{S}^{*}_{26} \\ (m_3 - M_1)\hat{S}^{*}_{34} & (m_3 - M_2)\hat{S}^{*}_{35} & (m_3 - M_3)\hat{S}^{*}_{36} \end{pmatrix} \end{aligned} \tag{E-2}$$

Now, from (3-38) let us compute $U^{\dagger}_{PMNS}$:

$$U^{\dagger}_{PMNS} = \begin{pmatrix} C_{12}C_{13} & -S_{12}C_{23} - C_{12}S_{23}\hat{S}^{*}_{13} & S_{12}S_{23} - C_{12}C_{23}\hat{S}^{*}_{13} \\ S_{12}C_{13} & C_{12}C_{23} - S_{12}S_{23}\hat{S}^{*}_{13} & -C_{12}S_{23} - S_{12}C_{23}\hat{S}^{*}_{13} \\ \hat{S}_{13} & C_{13}S_{23} & C_{13}C_{23} \end{pmatrix} \tag{E-3}$$

Thus, substituting the above results:

$$M^{D} = U^{\dagger}_{PMNS} \begin{pmatrix} (m_1 - M_1)\hat{S}^{*}_{14} & (m_1 - M_2)\hat{S}^{*}_{15} & (m_1 - M_3)\hat{S}^{*}_{16} \\ (m_2 - M_1)\hat{S}^{*}_{24} & (m_2 - M_2)\hat{S}^{*}_{25} & (m_2 - M_3)\hat{S}^{*}_{26} \\ (m_3 - M_1)\hat{S}^{*}_{34} & (m_3 - M_2)\hat{S}^{*}_{35} & (m_3 - M_3)\hat{S}^{*}_{36} \end{pmatrix} \tag{E-4}$$

$$\begin{pmatrix} M^D_{ee} & M^D_{e\mu} & M^D_{e\tau} \\ M^D_{\mu e} & M^D_{\mu\mu} & M^D_{\mu\tau} \\ M^D_{\tau e} & M^D_{\tau\mu} & M^D_{\tau\tau} \end{pmatrix} = \begin{pmatrix} C_{12}C_{13} & -S_{12}C_{23} - C_{12}S_{23}\hat{S}^*_{13} & S_{12}S_{23} - C_{12}C_{23}\hat{S}^*_{13} \\ S_{12}C_{13} & C_{12}C_{23} - S_{12}S_{23}\hat{S}^*_{13} & -C_{12}S_{23} - S_{12}C_{23}\hat{S}^*_{13} \\ \hat{S}_{13} & C_{13}S_{23} & C_{13}C_{23} \end{pmatrix} \begin{pmatrix} (m_1 - M_1)\hat{S}^*_{14} & (m_1 - M_2)\hat{S}^*_{15} & (m_1 - M_3)\hat{S}^*_{16} \\ (m_2 - M_1)\hat{S}^*_{24} & (m_2 - M_2)\hat{S}^*_{25} & (m_2 - M_3)\hat{S}^*_{26} \\ (m_3 - M_1)\hat{S}^*_{34} & (m_3 - M_2)\hat{S}^*_{35} & (m_3 - M_3)\hat{S}^*_{36} \end{pmatrix} \tag{E-5}$$

Each component is computed explicitly in section 6.2.1 and corresponds to an effective Dirac neutrino mass.

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