Damped Vibrating String Project
Numerical methods for Physics (Prof. Dr. Alain Dereux)

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Abstract

This project analyzes the damped oscillations of a vibrating string with fixed endpoints, using both analytical and numerical methods. By solving the damped wave equation, it is determined the string vibrational modes and explore how damping affects the string natural frequencies and time evolution. Employing Fourier series decomposition, the initial displacement and velocity distributions are expanded in terms of eigenfunctions, allowing for a precise approximation of the system's dynamics. Through MATLAB computations and visualizations, including a slow-motion simulation, it is illustrated the influence of damping on wave propagation along a one dimensional string. This study emphasizes the utility of numerical methods for modeling damped oscillatory systems and provides insights into the practical implications of damping in vibrational phenomena.

1 Introduction

The aim of this project is to show an analysis of a damped vibrating one-dimensional string fixed at its endpoints.

For this purpose, in section 2 is briefly presented the theoretical framework which modeled this problem, and specially where physical quantities to study are defined.

In section 3 is the most important and robust part of this documents, since there is showed results of the numerical approach and computations performed. Starting by defining the input parameters, and then computing all quantities necessary to perform a well detailed simulation of the damped vibrating string.

Finally, a short conclusions section is included in order to highlight the main results of this analysis.

2 Theoretical framework [1]

A damped vibrating string can be modeled by the following differential equation:

$$\frac{\partial^2\Psi(x,t)}{\partial x^2} - \Gamma\frac{\partial\Psi(x,t)}{\partial t} - \frac{1}{c^2}\frac{\partial^2\Psi(x,t)}{\partial t^2} = 0 \tag{1}$$

Where Γ is known as the damping factor.

For this analysis it will be considered the one-dimensional problem of a string fixed at its boundaries. Thus, after separation of variables, there is found the following two differential ordinary equations:

$$\frac{d^2\Phi(x)}{dx^2} + k^2\Phi(x) = 0 \tag{2}$$

$$\frac{d^2T(t)}{dt^2} + 2\gamma\frac{dT(t)}{dt} + \omega^2T(t) = 0 \tag{3}$$

Where the dispersion relation is given by:

$$\omega = kc \tag{4}$$

And the factor γ is proportional to the damping factor:

$$\gamma = \frac{\Gamma c^2}{2} \tag{5}$$

Thus, after considering the boundary conditions of fixed extreme points for the string, the solution for equation in space 2 is given by the following eigen modes:

$$\Phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \tag{6}$$

Which constitutes a Fourier basis for this problem, and satisfy the orthonormalisation relation given by:

$$\langle \Phi_n(x) | \Phi_m(x) \rangle = \delta_{n,m} \tag{7}$$

Thus, the general solution will be written as a linear combination of this eigen modes. Also, each of them has associated physical quantities which depends of n: angular wavenumber k_n, angular oscillation frequency ω, linear frequency ν_n, and period T_n, given by the following relations:

$$k_n = \frac{n\pi}{L} \tag{8}$$

$$\omega_n = \frac{n\pi c}{L} \tag{9}$$

$$\nu_n = \frac{\omega_n}{2\pi} \tag{10}$$

$$T_n = \nu_n^{-1} \tag{11}$$

On the other hand, the solution for time part is found to be a decaying oscillation:

$$T(t) = e^{-\gamma t}\left[\alpha_\omega e^{i\Omega t} + \beta_\omega e^{-i\Omega t}\right] \tag{12}$$

Where the frequency of oscillation Ω depend on ω by the following relation:

$$\Omega_n = \sqrt{\omega_n^2 - \gamma^2} \tag{13}$$

Thus, the solution to equation (1) can be written as:

$$\Psi(x,t) = \sum_{n=1}^{\infty} \Phi_n(x)\left[\langle \Phi_n(x) | f(x) \rangle \cos \omega_n t + \omega_n^{-1} \langle \Phi_n(x) | g(x) \rangle \sin \omega_n t\right] \tag{14}$$

Notice that the Fourier coefficients for this problem are given by:

$$a_n' = (\langle \Phi_n(x) | f(x) \rangle) \tag{15}$$

$$b_n' = \omega_n^{-1} \langle \Phi_n(x) | g(x) \rangle \tag{16}$$

3 Analysis

In the present, it is presented a physical and computacional analysis of a single damped string fixed at its extremes.

3.1 Initial input data

with the following physical characteristics:

- **Lenght**: L[m]=0.328.
- **Density**: ρ[kg/m]=0.660$\times 10^{-3}$.
- **Tension**: F[N]=55.0.

With the above characteristics, it is possible to compute the wave speed of this string as:

$$\begin{aligned} c &= \sqrt{F/\rho} \\ &= 288 \text{ m/s} \end{aligned} \tag{17}$$

Now, to implement the computational analysis it was fixed the value for γ in 400 Hz. The reason for this choice will be fully explained in section 3.3 when comparing the values of ω_n and Ω_n displayed in table 1.

Then, in order to set a time interval of analysis, it was computed the time t_{max} in which the initial amplitude decay by a factor of 10^{-2}:

$$\begin{aligned} t_{max} &= \frac{\ln(100)}{\gamma} \\ &= 0.0115 \text{ s} \end{aligned} \tag{18}$$

Thus, it was defined an array of time, from 0 to $t_{max} = 0.0115$ s, in `nst=13` equally spaced points. On the other hand, for space coordinates it was defined an array from 0 to L=0.328 m with `npt=200` equally spaced points.

3.2 Initial conditions

One can consider an initial deformation of the string, in such a way that:

$$\Psi(x, t=0) = f(x) \tag{19}$$

In particular, if the string is pinched at a point $x = p$, with an amplitude z, this deformation will be given by the function:

$$f(x) = \begin{cases} \frac{zx}{p} & \text{if } 0 \leq x \leq p \\ \frac{z(x-L)}{p-L} & \text{if } p < x \leq L \end{cases} \tag{20}$$

In the figure 1 is plotted this one pinch initial deformation for $p = L/4 = 0.082$ m and amplitude $z = 0.15$ m.

On the other hand, if it is pinched in two points p_1 and p_2, with $p_2 > p_1$, and z_1 and z_2 being the respective amplitudes, the deformation can be modeled by:

$$f(x) = \begin{cases} \frac{z_1 x}{p_1} & \text{if } 0 \leq x \leq p_1 \\ \frac{(z_1 - z_2)}{p_1 - p_{p2}}(x - p_1) + z_1 & \text{if } p_1 < x \leq p_2 \\ \frac{z_2(x-L)}{p_2 - L} & \text{if } p_2 < x \leq L \end{cases} \tag{21}$$

In figure 2 is plotted an initial deformation consisting of two pinches: one at point $p_1 = 0.082$ m, and the other one at $p_2 = 0.246$ m, with amplitudes $z_1 = 0.150$ m and $z_2 = -0.150$ m respectively.

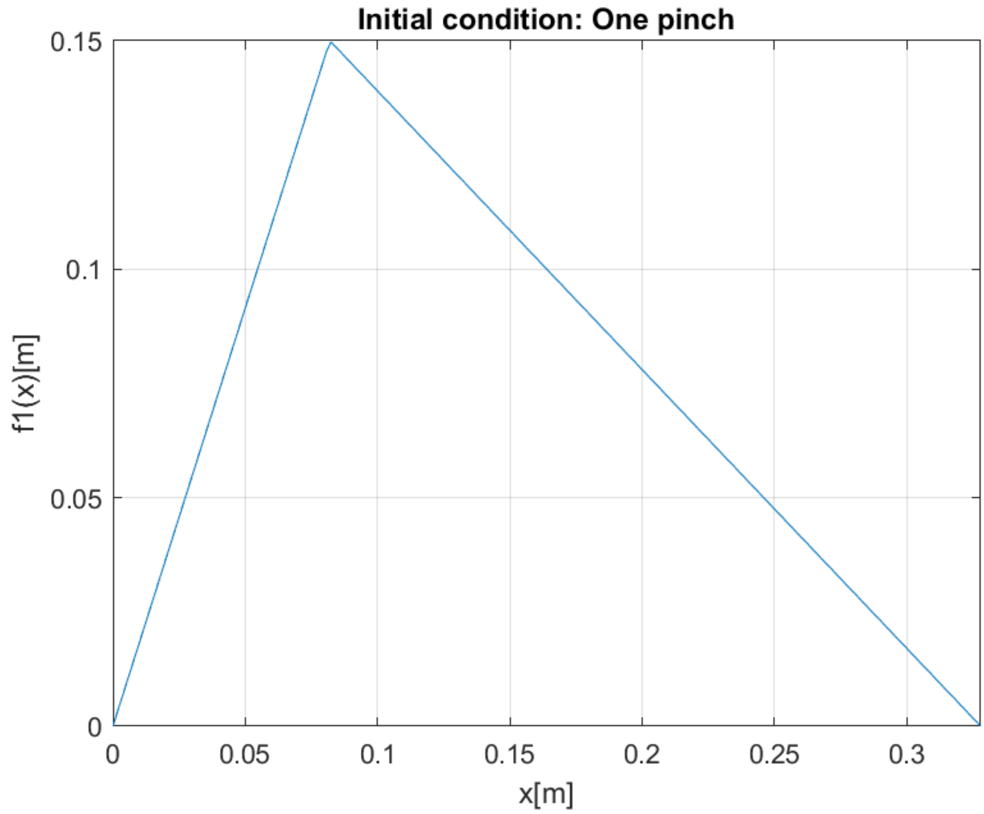

Figure 1: One pinch at $p = 0.082$ m as the initial deformation of the string.

3.3 Eigen modes and frequencies analysis

From the solution in the spatial part for the fixed string found in (6), there were computed the first `nmax`=30 eigen modes, which corresponds to normal modes of oscillation when there is no damping. These eigen modes are the Fourier basis of the damped string problem. In figure 3 the first seven eigen modes are plotted. Notice that they all are periodic modes such that $\Phi(x = 0) = \Phi(x = L) = 0$, as the fixed extremes boundary condition required. Also, notice that their amplitude is much larger than string length, since is given by $\sqrt{2/L} = 2.47$ m.

Also, for `nmax`=30 were computed the numerical values for the physical variables described in equations (8) - (11). This was made respecting the significant digit rule, i.e., three significant digits for this analysis. The numerical values are displayed in the table 1. Notice that when comparing the values for the frequencies ω_n and Ω_n, the only difference are found in the first two modes, since here only three significant digits are considered. This is why γ was set to be 400 Hz for this analysis, to be able to notice this slight difference.

A further analysis in the difference between the for the frequencies ω_n and Ω_n was performed, by setting $n = 1$, and varying the factor γ between 10 Hz and $\omega_1/2$:

$$\begin{aligned} \omega_1 &= \frac{\pi c}{L} \\ &= 2760\text{Hz} \end{aligned} \tag{22}$$

Thus, $\gamma \in [10, 1380]$ Hz.

In figure 4 it is shown the discrepancy of frequencies ω_n and Ω_n with respect to γ. As expected from eq. 13, for values of γ closer to 0 there is almost no discrepancy, but if γ increases the difference increases considerably. Notice that for $\gamma = 100$ there is no considerable difference, which justifies the choice of a greater value of γ to perform this analysis.

Then, using the tool `trapz()`, it was checked numerically the orthogonality relation for the eigen modes

n	$\mathbf{k}_n[m^{-1}]$	$\omega_n[Hz]$	$\Omega_n[Hz]$	$\nu_n[Hz]$	$\mathbf{T}_n[s]$
1	9.58	2.76E+03	2.74E+03	435.	0.00230
2	19.2	5.53E+03	5.52E+03	878.	0.00114
3	28.7	8.29E+03	8.29E+03	1.32E+03	0.000758
4	38.3	1.11E+04	1.11E+04	1.76E+03	0.000568
5	47.9	1.38E+04	1.38E+04	2.20E+03	0.000455
6	57.5	1.66E+04	1.66E+04	2.64E+03	0.000379
7	67.0	1.94E+04	1.94E+04	3.08E+03	0.000325
8	76.6	2.21E+04	2.21E+04	3.52E+03	0.000284
9	86.2	2.49E+04	2.49E+04	3.96E+03	0.000253
10	95.8	2.76E+04	2.76E+04	4.40E+03	0.000227
11	105.	3.04E+04	3.04E+04	4.84E+03	0.000207
12	115.	3.32E+04	3.32E+04	5.28E+03	0.000189
13	125.	3.59E+04	3.59E+04	5.72E+03	0.000175
14	134.	3.87E+04	3.87E+04	6.16E+03	0.000162
15	144.	4.15E+04	4.15E+04	6.60E+03	0.000152
16	153.	4.42E+04	4.42E+04	7.04E+03	0.000142
17	163.	4.70E+04	4.70E+04	7.48E+03	0.000134
18	172.	4.98E+04	4.98E+04	7.92E+03	0.000126
19	182.	5.25E+04	5.25E+04	8.36E+03	0.000120
20	192.	5.53E+04	5.53E+04	8.80E+03	0.000114
21	201.	5.81E+04	5.81E+04	9.24E+03	0.000108
22	211.	6.08E+04	6.08E+04	9.68E+03	0.000103
23	220.	6.36E+04	6.36E+04	1.01E+04	9.88E-05
24	230.	6.64E+04	6.64E+04	1.06E+04	9.47E-05
25	239.	6.91E+04	6.91E+04	1.10E+04	9.09E-05
26	249.	7.19E+04	7.19E+04	1.14E+04	8.74E-05
27	259.	7.47E+04	7.47E+04	1.19E+04	8.42E-05
28	268.	7.74E+04	7.74E+04	1.23E+04	8.12E-05
29	278.	8.02E+04	8.02E+04	1.28E+04	7.84E-05
30	287.	8.29E+04	8.29E+04	1.32E+04	7.57E-05

Table 1: Numerical values for the main physical variables describing the damped vibrating string.

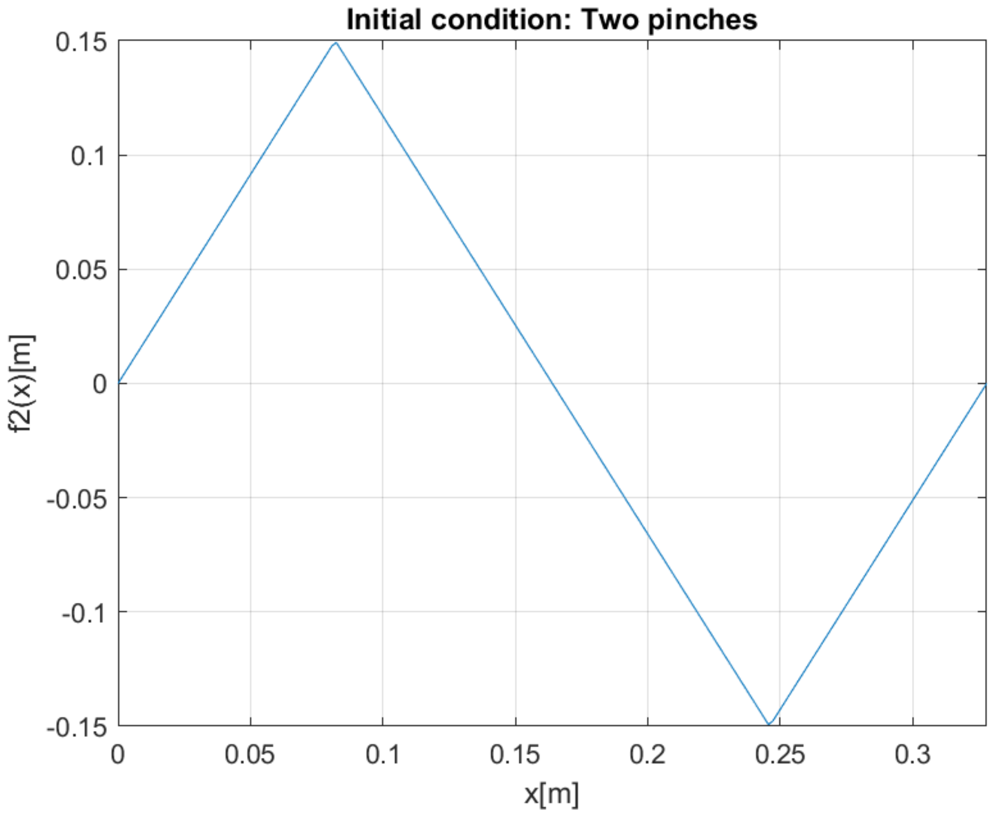

Figure 2: Initial deformation of two pinches at $p_1 = 0.082$ m and $p_2 = 0.246$ m.

given by eq. (7). The computed values are displayed in the table 2, when it is evident that orthonormalisation relation is satisfied by the `nmax`=30 eigen modes considered.

3.4 Fourier coefficients: Overlap computations

In order to solve numerically the damped vibrating string problem, it is necessary to compute the coefficients in equations (15) and (16). For this purpose, it is used again the tool `trapz()` which computes the overlap integral between the eigen modes Φ_n and the initial conditions $f(x)$, for the initial deformation of two pinches described showed in figure 2, and $g(x)$ for the initial distribution of velocities, which was considered to be 0. The numerical values of these overlaps are displayed in table 3.

From the numerical values displayed in table 3, it is possible to establish what are the eigen modes that best resembling the initial conditions, in particular, the initial deformation $f(x)$, since they are directly proportional to Fourier coefficients given by equations (15) and (16). Notice that for n **even** the overlap with $f(x)$ is considerably greater than for those with n odd, specifically, by 10 to 16 orders of magnitude. Thus, the most relevant eigen modes to resembling the initial conditions are those corresponding to an **even** n. Also, notice that the most relevant eigen mode, which corresponds to the greatest absolute numerical value of the overlap integral, corresponds to the value of $n = 2$. This is due to the fact that the shape of initial deformation showed in 2 is quite similar to the eigen mode $\Phi_2(x)$ since it has a peak and a valley equally spaced from the center of the length and with the same amplitude.

Then, it was computed the approximation of the initial conditions described by functions $f(x)$ and $g(x)$ by truncating the Fourier series, taking the first n_{max}= 30 terms. In the graph showed in figure 5 it is plotted the original two pinches initial deformation $f(x)$ and its Fourier series approximation, as well as the initial distribution of the velocity field $g(x)$ set to be 0. Original functions are depicted as blue solid lines, while their Fourier series approximations as red dotted lines. It is evident graphically that truncating the series at n_{max}= 30 resembles perfectly the initial conditions.

In order to check if this truncation is satisfactory enough, it is performed the goodness of fit $f(x)$ and

n	$\langle \phi_n, \phi_n \rangle$
1	1.00
2	1.00
3	1.00
4	1.00
5	1.00
6	1.00
7	1.00
8	1.00
9	1.00
10	1.00
11	1.00
12	1.00
13	1.00
14	1.00
15	1.00
16	1.00
17	1.00
18	1.00
19	1.00
20	1.00
21	1.00
22	1.00
23	1.00
24	1.00
25	1.00
26	1.00
27	1.00
28	1.00
29	1.00
30	1.00

Table 2: Normalization Values for ϕ_n

n	$\langle \phi_n, f \rangle$	$\langle \phi_n, g \rangle$
1	-7.95E-18	0.00
2	0.0492	0.00
3	1.01E-17	0.00
4	-4.84E-08	0.00
5	4.11E-18	0.00
6	-0.00547	0.00
7	-2.78E-18	0.00
8	9.69E-08	0.00
9	-2.22E-18	0.00
10	0.00197	0.00
11	3.25E-18	0.00
12	-1.46E-07	0.00
13	8.35E-18	0.00
14	-0.00100	0.00
15	-3.69E-18	0.00
16	1.94E-07	0.00
17	3.84E-18	0.00
18	0.000607	0.00
19	3.10E-18	0.00
20	-2.43E-07	0.00
21	1.11E-17	0.00
22	-0.000406	0.00
23	-4.40E-19	0.00
24	2.93E-07	0.00
25	1.48E-18	0.00
26	0.000291	0.00
27	-8.08E-18	0.00
28	-3.43E-07	0.00
29	-1.69E-17	0.00
30	-0.000218	0.00

Table 3: Overlap values with the initial conditions: initial deformation $f(x)$ and initial velocity distribution $g(x)$

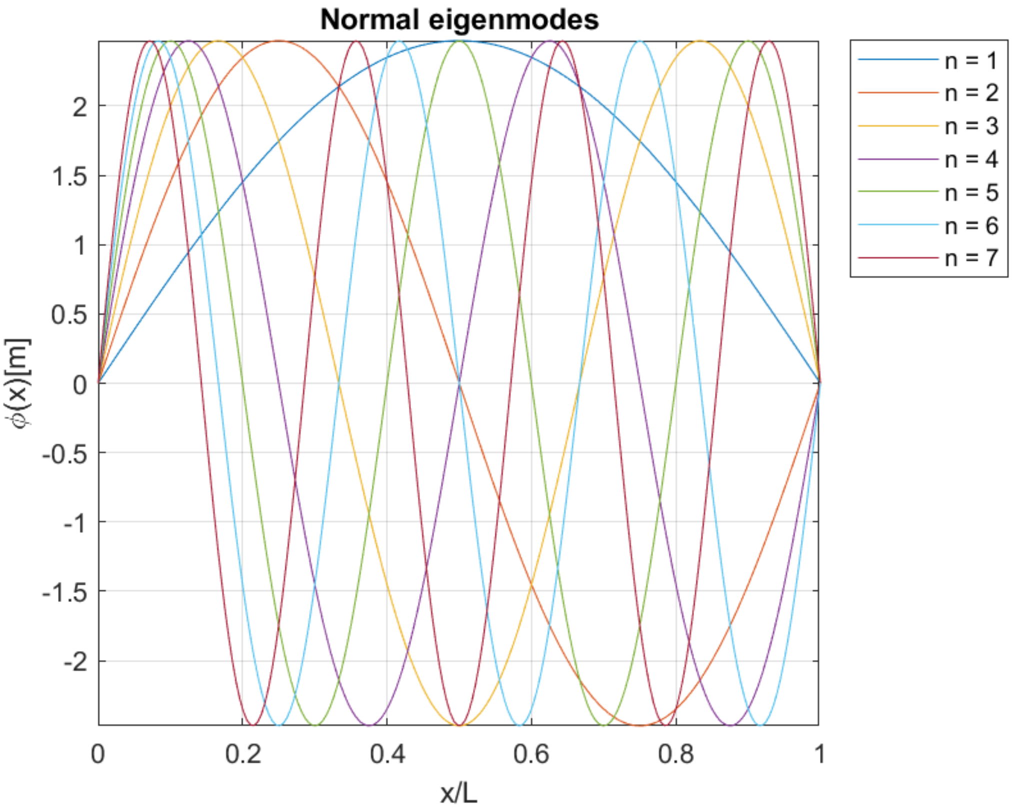

Figure 3: First seven eigen modes of the string fixed at its extremes.

$g(x)$ using the function `gof.m`. The found values are displayed in table 4, where is found a R^2 value of 1.00 in both cases, which corresponds to an approximately perfect fitness in the precision number of digits given by this function.

Metric	Goodness of Fit for $f(x)$	Goodness of Fit for $g(x)$
MEAN	-9.8554e-18	0.00
TSS	1.4925	0.00
SSE	5.8441e-05	0.00
SSR	1.4924	0.00
SZ	1.3461e-17	0.00
NSSR	1.0000	1.00
NSZ	9.0194e-18	0.00
MSE	2.9220e-07	0.00
NRMSE	0.0063	0.00
NMSE	3.9157e-05	0.00
Rsquare	1.0000	1.0000
FOM	1.00	1.00

Table 4: Goodness of fit metrics for $f(x)$ and $g(x)$ with their Fourier series expansions.

3.5 Time evolution: simulation

Then there were taken `nst`=13 values of time variable t, sampling the range $[0, t_{max} = 0.0115\text{s}]$, and for each one of them it was computed the associated wave function $\Psi(x,t)$, which are depicted in the figure 6. In this graph it is shown how the amplitude of damped vibrating string transversal displacement decays due to damping. Notice that there is a scale of color, from blue to red, which allows to identify easier the `nst`=13

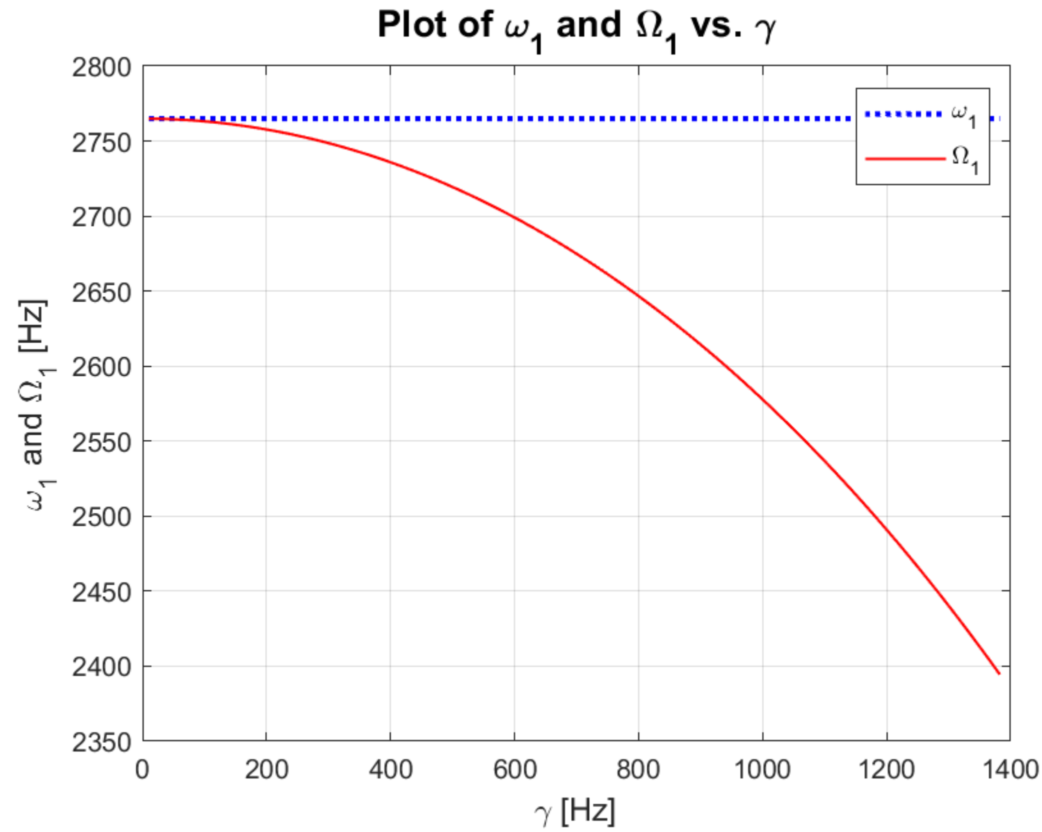

Figure 4: Graph of ω_1 and Ω_1 as a function of γ.

values considered for time.

Finally, in order to improve the visualization of how amplitude of the oscillation decays due to damping, it was made a movie which lasts for t_{max}=0.0115 s, with 283 frames in order to visualize it in slow motion and see clearly the damped vibrating string. This movie can be watched in the following link:

https://univbourgogne-my.sharepoint.com/:v:/g/personal/brayan-elian_castiblanco-ortigoza_
etu_u-bourgogne_fr/ER9JI23lXBhLtV40ASHF1GQBhLoSYFdjmEptrEwIpaZWOw?nav=eyJyZWZlcnJhbEluZm8iOnsic
e=8A1lIG

And also in the GitHub repository for this project:
https://github.com/bcastiblancoo/Damped-Vibrating-String-PROJECT

4 Conclusions

This project presented a detailed analysis of damped oscillations in a vibrating string, implementing both analytical and numerical techniques to model the effects of energy dissipation on wave dynamics. Fourier series decomposition enabled an accurate reconstruction of the initial deformation given by two symmetric pinches and null velocity distributions, by taking the first n_{max}=30 terms, with validation from goodness-of-fit metrics, finding $R^2 = 1$, thus showing high fidelity in the Fourier expansion.

It was found that, for values of γ less than 200 Hz, there is no such a discrepancy between the eigen modes oscillation frequencies and those of the damped decaying part, but if this factor increases, i.e., if damping increases, there is a significant difference and damping effects are more relevant.

Also, due to the symmetric two pinches initial deformation, and with a $\gamma = 400$ Hz, it was found that the eigen mode corresponding to $n = 2$ has the greatest weight in the series expansion. Furthermore, even eigen modes completely resembles the initial deformation since their contribution is 10 to 16 orders of magnitude greater.

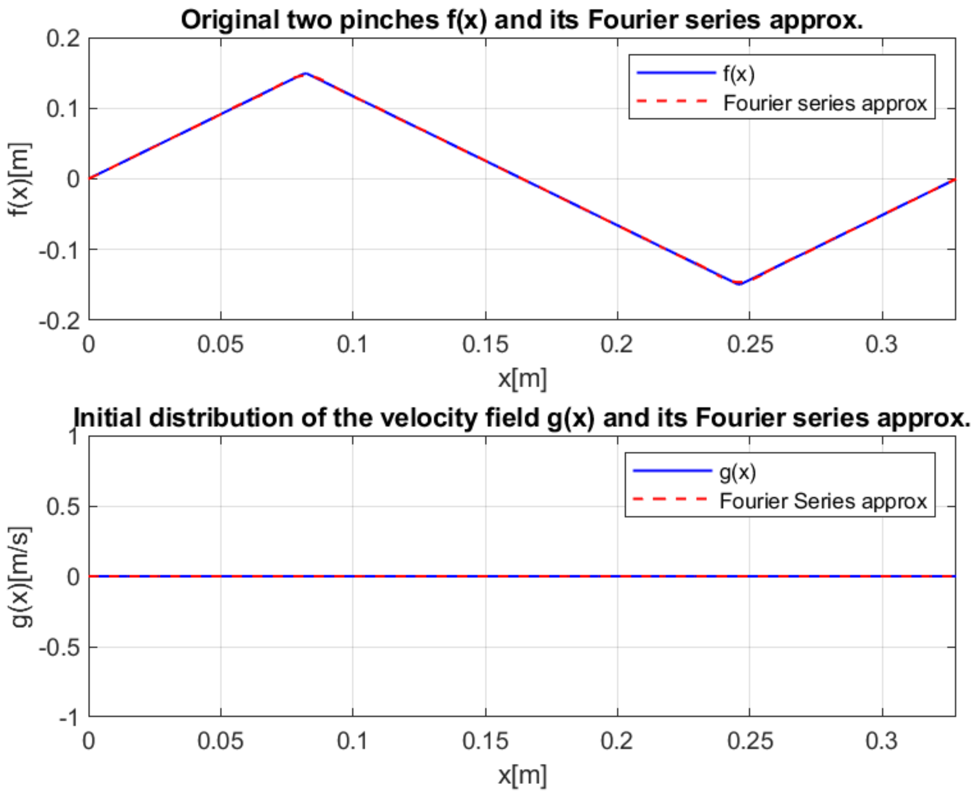

Figure 5: Initial conditions and their Fourier series approximations.

The results revealed that damping disproportionately affected lower frequency modes, leading to faster amplitude decay and slight frequency shifts in the vibrational spectrum. Visualization techniques, including the creation of a slow-motion video, demonstrated the temporal evolution and the significant role of damping in the string's dynamics. This study underscored the utility of computational simulations in analyzing complex wave behaviors and established a robust framework for investigating similar damped systems across fields where dissipative effects are important.

References

[1] *"Selected Chapters of Numerical Methods*, by Alain Dereux, Université de Bourgogne - Faculté des Sciences & Techniques (2024), Chaps. 1 and 2.

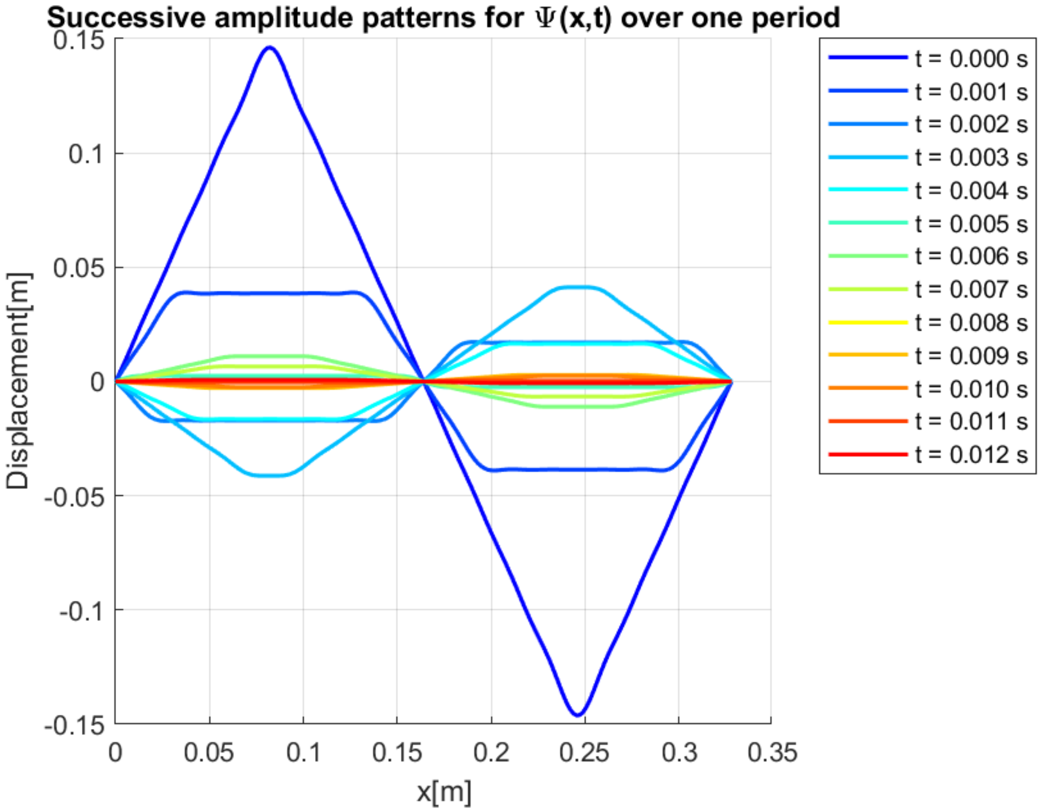

Figure 6: Graph for `nst=` 13 successive amplitude patterns for the damped vibrating string with damping factor proportional to γ=400 Hz.