From Quantized Chern Transport to Quantum Technologies: Nonlinear Topological Pumping

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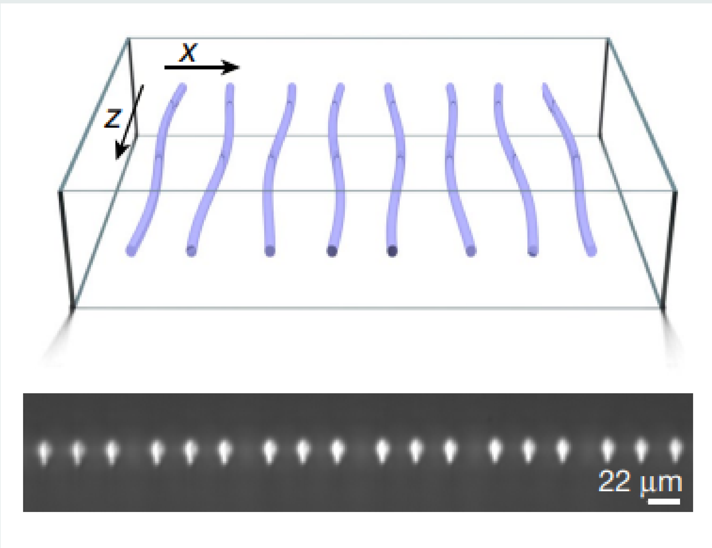
ABSTRACT

Simulations on *nonlinear bosonic* topological transport is presented in a fully quantum approach for a double periodic 1D lattice, computing *Chern number* (C) from Zak phases of time-evolved ground states over full pump cycles to predict quantized displacement per cycle. The approach is based on soliton solutions to the Hubbard model for the Thouless pump (Bohm et al. 2025). These results delineate transport/no-transport regimes versus pumping frequency and interaction strength, identifying adiabatic windows with integer pumping ($C\approx 1$), and degradation under stronger nonlinearity or nonadiabatic drive, in line with recent observations of nonlinear Thouless pumping of solitons (Jürgensen, M. et al. 2021 & 2023). It was found that the frequency of pumping must be smaller than the energy gap density in order to carry out non-trivial topological quantum transport, characterizing topological robust regimes with potential applications to quantum technologies (Zhang et al. 2025).

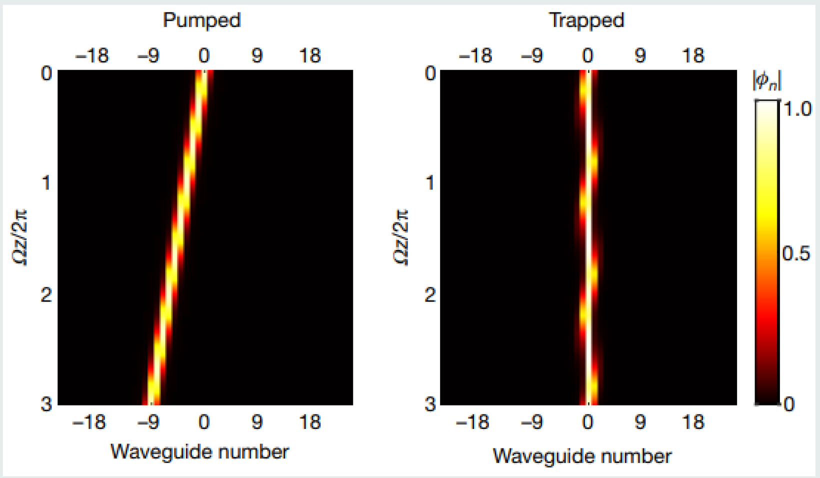
1. Introduction

(a) Nonlinear Topological pumping:

Quantized transport can be achieved in insulators, when the evolution of a full filled Bloch band is done in an adiabatic way (Thouless 1983). This can be achieved in a 1D lattice with two periodic parameters, where transport phenomena is fully governed by the topological protection of the Chern number (eq. (2)). An extension of the single particle transport to a many-body behaviour has been proposed with solitons (Lumer et al. 2013), many-particle bounded states that stick together due to a local attractive interaction, leading to a nonlinear model. Recent experiments (Jürgensen, M. et al. 2021 & 2023) has demonstrated integer and fractional quantized nonlinear transport, by modulating a 1D array of evanescently coupled waveguides (see Fig. 1) with a Kerr nonlinearity.



(a) Illustration of the 1D lattice implementation of pumping.



(b) Normalized soliton wavefunction in the case of topological pumping (Transport), and in the trivial topological case (Trapped).

Figure 1. Experimental realization of nonlinear topological pumping. Taken from (Jürgensen et al. 2021).

(b) Theoretical model:

A fully quantum description to construct the Chern number for the many-particle wavefunction of a soliton is possible (Bohm et al. 2025). This can be done by a 1D bosonic Hubbard model extension: A The non-linear off-diagonal Aubry - André - Harper (AAH) model in a bipartite unit cell:

$$\hat{H}(t)=\sum_{j=1}^{L} J_A(t)\hat{b}^\dagger_{B,j}\hat{b}_{A,j}+J_B(t)\hat{b}^\dagger_{A,j+1}\hat{b}_{B,j}+\text{h.c.}+\Delta(t)\left(\hat{b}^\dagger_{A,j}\hat{b}_{A,j}-\hat{b}^\dagger_{B,j}\hat{b}_{Bj}\right)+\frac{U_0}{2}\hat{n}_{A,B,j}(\hat{n}_{A,B,j}-1) \quad (1)$$

Where $J_A(t)$ and $J_B(t)$ are the time-dependent hopping parameters, $\Delta(t)$ is the time-dependent on-site potential, and U_0 is the strength of the non-linear interaction.
Quantum transport is fully characterized by the Chern-number (Thouless 1983), which in a periodic lattice model can be computed in terms of the Zak phase (φ_n^{ZAK}), and therefore in terms of the Bloch bands:

$$\Delta X=\frac{1}{2\pi}\oint d\vec{\lambda}\frac{\partial\varphi_n^{ZAK}}{d\vec{\lambda}}=\frac{1}{2\pi}\int_{BZ}dk\int_0^T dt\frac{\partial}{\partial t}\langle u_n(k,t)|\partial_k u_n(k,t)\rangle \quad (2)$$

2. Methodology

- **Two-sites Soliton Ansatz:** When considering strong interacting regime $U_0 \geq J_A, J_B$ and many-body excitations, a two-sites soliton ansatz can be assummed:

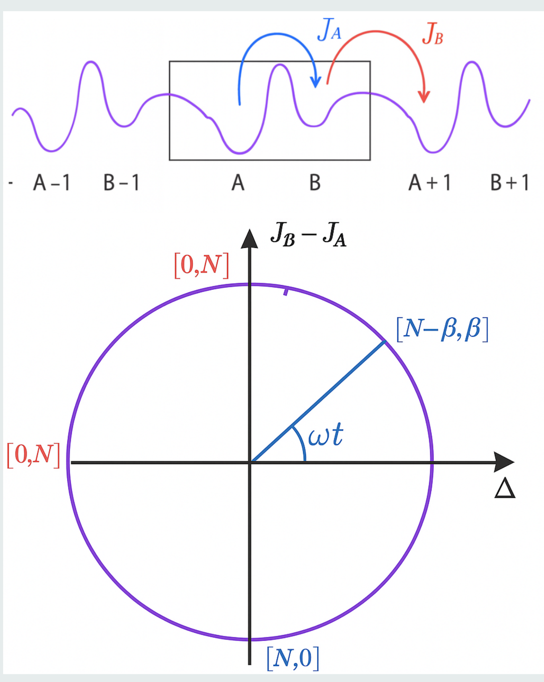
$$|\Phi_N(k)\rangle=\sum_{\alpha=1}^{2}\sum_{j=1}^{l}(e^{ik}\hat{T})^{j-1}\sum_{\beta=0}^{N-1}C_{\alpha\beta}|N-\beta,\beta\rangle_\alpha \quad (3)$$

With N the number of bosons in the lattice, and $2\cdot l$ its lenght.

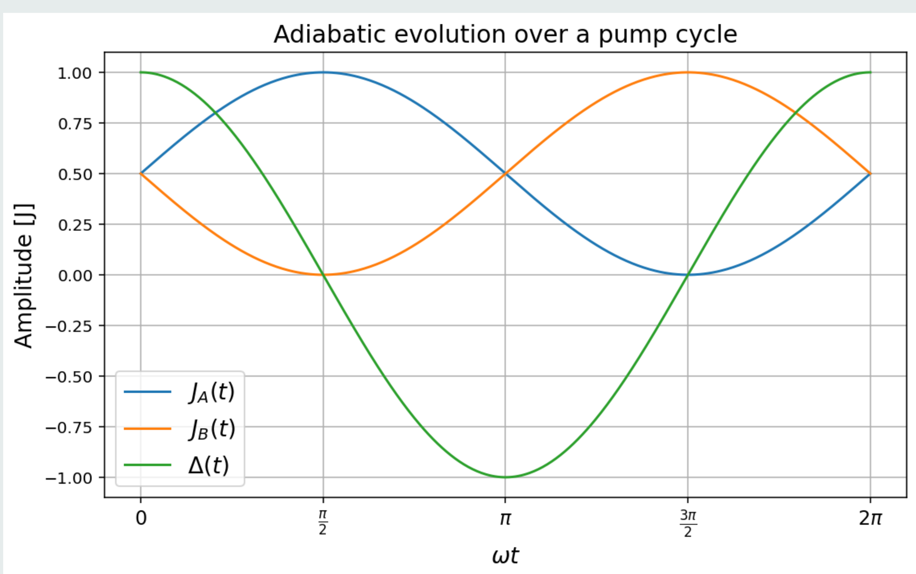
- **Adiabatical Pumping:** In order to ensure the non-trivial topological transport, an adiabatical pumping is needed, which can be parameterized by exploiting the periodicity of the parameters:

$$J_A(t)=\frac{1}{2}(1+\sin\omega t),\qquad J_B(t)=\frac{1}{2}(1-\sin\omega t),\qquad \Delta(t)=\cos\omega t \quad (4)$$

Where ω is the pump frequency which carries out the quantum transport.



(a) Parameter space winding.



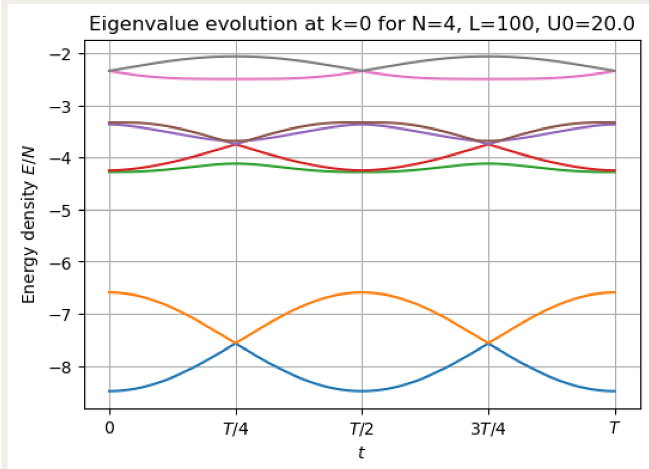
(b) Evolution of the parameters in adiabatical pumping.

Figure 2. Parameter space evolution.

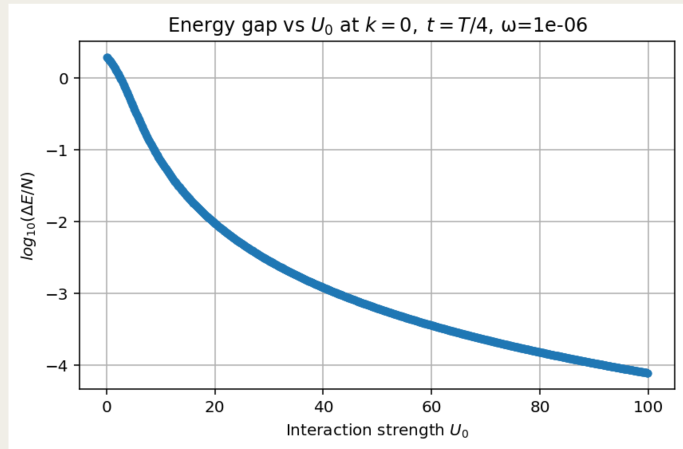
3. Results

Time evolution and topological protection:

The AHH model in (1) was solved numerically by time-evolve the lattice soliton ansatz in (3) with $N=4$. In Fig. 3a is plotted the energy density evolution of the time-evolved states, where the ground state in blue represents the case where all bosons are seated on the lattice-site A. An apparent degeneracy occurs, but it is ruled out by Fig. 3b where a **non-zero** energy gap ΔE is spotted, proving that the quantum transport is topologically protected over the pumped cycle.



(a) Energy density (in units of [J]) evolution over a pumped cycle.



(b) Energy gap density ($\Delta E/N$[J]) dependence.

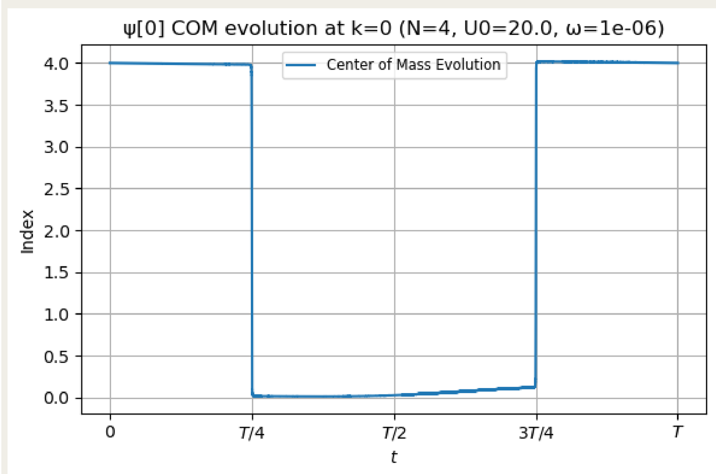
Figure 3. Energy gap analysis and topological protection.

The soliton ansatz simplifies the complexity dimension of the Hilbert space to a linear $2N$ problem. This is an advantage when time-evolving the system and computing numerically the associated Chern number by eq. (2).

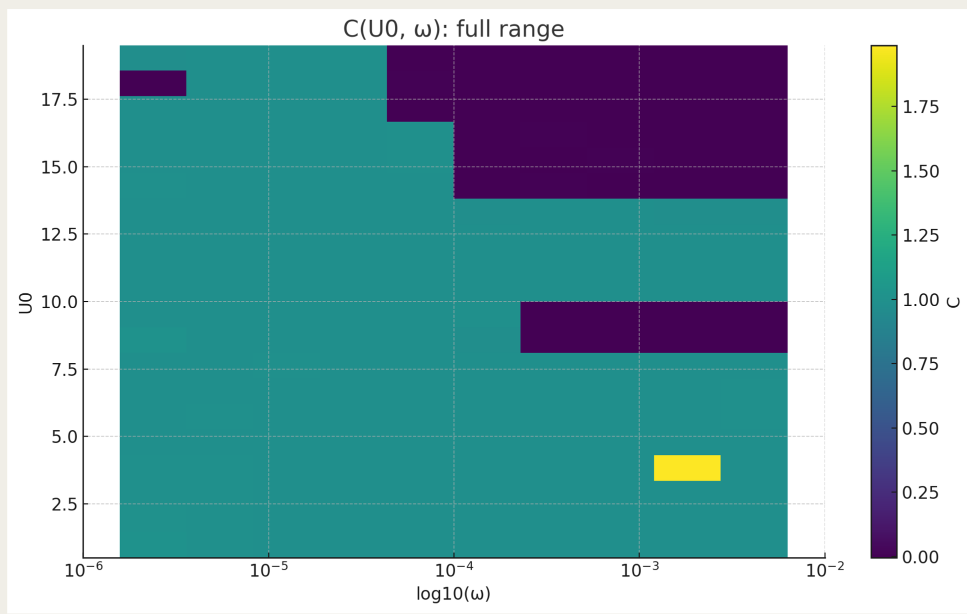
Quantum Transport condition:

In Fig. 4a is plotted the evolution of the Center-Of-Mass (COM) for the ground state. The index 4 means all four bosons are sitting in the lattice-site A, whereas the index 0 means all bosons are sitting in the lattice A+1, leading thus to a quantum transport for half of the pumped cycle period. The colormap in Fig. 4b shows a fully explored transport/no-transport analysis, by computing the Chern number (and therefore the COM evolution) for different values of the interaction strength (in units of [J]) and the pumping frequency (in units of [J/ℏ]), where the following **condition for non-trivial topological quantum transport** was found:

$$\omega << \frac{\Delta E}{N} \quad (5)$$



(a) Center-Of-Mass evolution over a pumped cycle.



(b) Color map for the Chern number.

Figure 4. Quantum Chern Transport analysis and characterization.

4. Outcomes and perspectives

- A robust theoretical and computational framework is presented, with a fully quantum and satisfactory computation of the Chern number for a many-body soliton wavefunction.
- A map of the quantum transport/no-transport regimes is explored, finding successful quantized transport when the pumping frequency is much smaller that the gap energy density.
- Nonlinear soliton pumping can enable quantized, disorder-resistant transport of localized excitations for reliable quantum information routing (Jürgensen, M. et al. 2023).
- Topological-protected state transfer in NV-center ensembles can support stable microwave-to-optical photon conversion for hybrid quantum networks (Zhang et al. 2025).

References

- Bohm, J., H. Gerlitz, C. Jörg, and M. Fleischhauer (2025). "Quantum theory of fractional topological pumping of lattice solitons". In: *arXiv preprint arXiv:2506.00090.*
- Zhang, T.-T., J.-L. Wu, Y. Wang, S.-L. Su, and C. Shan (2025). "Topologically protected chain-wise microwave-to-optical photon conversion interfaced by nitrogen-vacancy center ensembles". In: *Optics Express* 33.21, pp. 43865–43881.
- Jürgensen, M., S. Mukherjee, and M. C. Rechtsman (2021). "Quantized nonlinear Thouless pumping". In: *Nature* 596.7870, pp. 63–67.
- Lumer, Y., Y. Plotnik, M. C. Rechtsman, and M. Segev (Dec. 2013). "Self-Localized States in Photonic Topological Insulators". In: *Phys. Rev. Lett.* 111 (24), p. 243905.
- Thouless, D. (1983). "Quantization of particle transport". In: *Physical Review B* 27.10, p. 6083.