Math 4610 Fundamentals of Computational Mathematics - Topic 12.

Most mathematical problems require the use of approximations as a part of the solution process. For example, the solution of the simple ordinary differential equation

$$\frac{dy}{dt} = -2\ y$$

with initial condition, $y(0) = 1$, is the function

$$y(t) = e^{-2t}$$

You can check this on your own. This is a single term that seems nice and tidy for predicting value of the solution to the differential equation. However, due to the fact that the number $e = 2.71828...$ is an irractional number. For certain values like $t = 0$ the exponential can be evaluated exactly. However, for just about any other number all we can do is approximate the value based on our mathematica knowledge.
From any standard second semester engineering calculus course, a series representation of the exponential function is given by

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Of course this is an infinite series and the best we can do is sum a finite number of terms.
Mathematically, we know that the series converges rapidly to a value. It may take a few terms, but a truncated series

$$e^x \approx \sum_{k=0}^{N} \frac{x^k}{k!}$$

for some, $N > 0$, can be used to approximate the exponential function. In this example, there are a couple of problems that still need to be addressed. First, if we truncate the infinite sum to a finite value, $N > 0$, how good is the approximation. The truncation of the series can be analyzed mathematically using Taylor series.
The second problem involves errors in number representation. That is, due to the finite number of digits available on a computer, we will run into problems for either very small values or very large values input to the exponential function.
The first problme is due to truncation error which is an artifact of approximating the mathematical model. Truncation error in any given problem needs to be analyzed mathematically. The second problem is really beyond our control since it is due to the particular computer and operating system we are using. We will treat the problem of truncation error in this topic and save the problem of round off error and machine precision for another topic in the near future.

A Brief Review of Taylor Series

In just about any calculus sequence, the topic of Taylor series is discussed. A definition for this concept is the following.

Definition 1 *Suppose that the function, f, is a function with derivatives of all orders at a point, a in the domain of f. Then the Taylor series of f about the point a is given by*

$$f(x) \sim f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}}{k!}\ (x-a)^k$$

where $f^{(k)}(a)$ is the k^{th} derivative of f at a and $k! = k(k-1)(k-2)\ldots(2)(1)$ with the assumption that $0! = 1$.

Students should be able to produce the Taylor series of simple functions like the trigonometric functions or examples like $f(x) = ln(1 + x)$. If you are a little foggy on the details, there are examples all over the internet or you can refer to any book the presents topics in engineering calculus.
It is also important to know how to apply the Taylor series with remainder. The definition we need is the following.

Definition 2 *Suppose that the function, f, is a function with $n+1$ continuous derivatives at a point, a, in the domain of the function. Then, the Taylor series with remainder is*

$$f(x) \sim \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}\ (x-a)^k + R_f(a,n)$$

where

$$R_f(a,n) = \frac{f^{n+1}(\xi)}{(n+1)!}\ (x-a)^{n+1}$$

and ξ is a point between x and a.

The remainder in this definition can be used to establish the truncation error and upper bounds for the truncation error as will be seen in some examples in this topic.

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h **Form of the Taylor Series:**

Computational mathematicians should be able to use Taylor series with ease in the analysis of numerical methods. There are several different, but equivalent forms of the Taylor series. For the purposes in this course, we will use the h or increment form for Taylor series expansions. To get to the appropriate form, letting $h = x - a$ doing some simplification gives

$$f(a+h) \sim f(a) + f'(a)h + \frac{1}{2}f''(a)h^2 + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}\ h^k$$

or using the Taylor series with remainder

$$f(a+h) \sim \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}\ h^k + R_f(a,n)$$

where

$$R_f(a,n) = \frac{f^{n+1}(\xi)}{(n+1)!}\ h^{n+1}$$

It should

To create an account on GitHub, go to the Github site on any browser. I you already have an account, you can skip this step.

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https://github.com
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This site will display a place to create an account or sign in to an existin account.

GitHub Primer for Math 4610 at USU: Setting up a Repository

Once you log in, you will need to build a repository for use in the class and to turn in homework and completed tasks and projects. For the course, you will create a repository named the following:

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math4610
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Use only the characters above and using the following rules:

1. Use only lower case characters - github is case senesitive.

2. Do not put any blanks in the name of the repository.

Note that the instructor will use only this repository name in looking for your work.

Github Primer for Math 4610 at USU: List the Contents of the Home Directory

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If you have an account on GitHub, you will already know a lot about these things. However, when you are logged in you will see the main screen with any repositories you may already have created. We will go through the steps to build and name repositories in the next few pages.