Math 4610 Fundamentals of Computational Mathematics - Topic 13.

In the previous section of these topics, a need for Taylor series expansions was identified in the evaluation of a natural exponential function. In this section of the notes, some examples of Taylor series expansions will be computed and/or stated for completeness. Recall that the definition of the Taylor series for a function, $f(x)$ is the following.

$$f(x) = f(x_0) + \frac{1}{1!}\ f'(x_0)(x - x_0) + \frac{1}{2!}\ f''(x_0)(x - x_0)^2 + \frac{1}{3!}\ f'''(x_0)(x - x_0)^3 + \cdots$$

where, x_0, is the center of the series expansion. More often than not we will use the following form of the series.

$$f(x) = f(x_0 + h) = f(x_0) + \frac{1}{1!}\ f'(x)h + \frac{1}{2!}\ f''(x)h^2 + \frac{1}{3!}\ f'''(x)h^3 + \cdots$$

Note that in many of the problems where we will need the Taylor series we will need to expand everything in an expression about a center, x_0 and look at the distance, h, from that point.
So, on to some examples.

Examples With Some Details:

The Natural Exponential Function

The first and maybe most important example is the natural exponential function. If $f(x) = e^x$, then the derivatives we need are simply

$$f'(x_0) = e^{x_0}, \quad f''(x_0) = e^{x_0}, \quad f'''(x_0) = e^{x_0}, \ldots$$

with the n^{th} derivative being

$$f^{(n)}(x_0) = e^{x_0}$$

Substituting all of this into the definition of the Taylor series gives the following expansion.

$$e^x = e^{x_0} + e^{x_0}(x - x_0) + \frac{1}{2}e^{x_0}(x - x_0)^2 + \frac{1}{6}e^{x_0}(x - x_0)^2 + \ldots = \sum_{k=0}^{\infty} \frac{e^{x_0}}{k!}\ (x - x_0)^k = e^{x_0} \sum_{k=0}^{\infty}\ \frac{(x - x_0)^k}{k!}$$

or if we let $h = x - x_0$

$$e^x = e^{x_0} \sum_{k=0}^{\infty} \frac{h^k}{k!}$$

This Taylor series is very easy to compute due to the fact that the derivative of the exponential function is itself. In most calculus courses students are introduced to the Maclarin series where the center of the series is $x_0 = 0$. Since $e^0 = 1$ the series becomes

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

This series is used in many examples in calculus and as a basis for modeling population problems and exponential growth and decay.

The Cosine Function

This example is a bit more difficult, but can be done with a little effort. We will determine the Taylor expansion of $f(x) = cos(x)$. The derivatives we need in this case are

$$f'(x) = -sin(x), \quad f''(x) = -cos(x), \quad f'''(x) = sin(x), \quad f''''(x) = cos(x) = f(x)$$

From this we see that the form of the derivative cycles after four derivatives. So, we see a pattern that helps us write the general Taylor series. The series with center x_0 is

$$cos(x) = cos(x_0) - \frac{sin(x_0)}{1!}(x - x_0) - \frac{cos(x_0)}{2!}(x - x_0)^2 + \frac{sin(x_0)}{3!}(x - x_0)^3 + \frac{cos(x_0)}{4!}(x - x_0)^4 + \dots$$

In most calculus textbooks, the general series is not presented. Instead, the Maclarin series with $x_0 = 0$ is used as an example. This gives the more commonly used series

$$cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

The Sine Function

In this section, the Maclarin series for $f(x) = sin(x)$ will be given without any details. The series is

$$sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{2k-1}}{(2k-1)!}$$

It is not too hard to change the center of the series in these cases. It is not that commonly done in mathematics courses.

Taylor Series For A Logarithm Function:

In this example, consider

$$f(x) = ln(1 + x)$$

This is a common example in second semester calculus courses. If we expand about $x_0 = 0$ we will obtain the following derivatives.

$$\begin{aligned}
f(0) &= ln(1) = 0 \\
f'(0) &= \frac{1}{1+x} \rightarrow f'(0) = 1, \\
f''(0) &= -\frac{1}{(1+x)^2} \rightarrow f'(0) = -1, \\
f'''(0) &= 2\,\frac{1}{(1+x)^3} \rightarrow f'(0) = 2, \\
f''''(0) &= -6\,\frac{1}{(1+x)^4} \rightarrow f'(0) = -6
\end{aligned}$$

This can be repeated as many times as needed until a pattern is found. Substituting these derivatives in gives

$$ln(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{x^k}{k}$$

Taylor Series Applied to Finite Difference Approximations:

In a couple of topics we have seen approximations of derivatives of functions using difference quotients. This can be stated as

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

We can expand the terms in the expression using the following at the arbitrary point, x. This gives

$$\begin{aligned}
\frac{1}{h}\left(f(x+h)-f(x)\right) &= \frac{1}{h}\left(f(x)+f'(x)(x+h-x)+\frac{1}{2}f''(x)(x+h-x)^2\cdots-f(x)\right)\\
&= \frac{1}{h}\left(f'(x)h+\frac{1}{2}f''(x)h^2+\cdots\right)\\
&= f'(x)+\frac{1}{2}f''(x)h+\cdots
\end{aligned}$$

The result of using a Taylor series approximation is that the difference quotient is equal to the derivative of the function plus extra terms from the Taylor series expansion.
We will take up this idea in the next topic in our list.