Math 4610 Lecture Notes

Root Finding Problems for Real Values Function of One Variable *

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Root Finding Problem: Definition of a Root Finding Problem

Many problems can be cast or recast in the form of finding places where a function is zero. In a standard first semester calculus course finding extreme values of a function, g, of one variable can be recast into the problem of determining locations where the derivative, g' of the original function is zero. Since, a necessary condition for the existence of a local minimum or local maximum value at a point x^* is that the derivative is zero. That is, we want to compute a solution to the equation

$$g'(x^*) = 0.$$

The result is a root finding problem on the derivative.
Now, a general definition of the problem.

Definition 1 The General Root Finding Problem: *The general root finding problem can be written as follows: For a given real-valued function, f, of a single real variable find a real number, x^*, such that*

$$f(x^*) = 0$$

The problem seems like it should be easy to solve these types of problems. However, there are many sources of errors and difficulties in this problem.
There are all kinds of issues that arise in solving these types of problems. For example, the function may have multiple roots. In searching for a specific root, we may find other roots that are not of interest. To deal with all of the issues in this problem, we will develop a number of algorithms that can be used in a variety of root finding problems.
More often than not, we will need to locate roots that cannot be represented exactly due to finite precision in number representation. For example, finding the roots of

$$sin(x) = 0$$

is easy from an analytic point of view, the zeros are $x_n = n\ \pi$ where n is an arbitrary integer. If n is not equal to zero, the root is an irrational number and cannot be represented exactly. So, we will necessarily heve to settle for an approximation. It should be noted that an algebraic solution will be available only in cases where $f(x)$ has a simple definition, say a linear or quadratic polynomial. We might be able to guarantee a solution exists, but there may be no analytic means of finding a root or multiple roots.
One very complicated function is a part of one of the oldest problems in all of mathematics. The problem is the Riemann conjecture regarding the distribution of prime numbers in amongst all real numbers. The Riemann-Zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

where s represents an arbitrary complex number. This innocent looking formula is still not completely understood and the Riemann conjecture remains unsolved. One related to the distribution of primes involves encryption of data and cyber security.