Lista 1 - Cálculo Pi pág 1

2. a) $\lim\limits_{n\to\infty}\left(1+\frac{1}{2n+3}\right)^{2n+3} \Rightarrow \lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e$

$x = 2n+3$

$n\to\infty \Rightarrow x\to\infty$

b) $\lim\limits_{n\to\infty}\left(1+\frac{1}{n^2}\right)^{n^2} \Rightarrow \lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e$

$x = n^2 \Rightarrow x\to\infty$

c) $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n} = \lim\limits_{n\to\infty}\underbrace{\left(1+\frac{1}{n}\right)^{n}}_{e}\cdot\underbrace{\left(1+\frac{1}{n}\right)^{n}}_{e} = e^2$

d) $\lim\limits_{n\to\infty}\left(1+\frac{1}{2n}\right)^{n} \Rightarrow \lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{x/2} = \sqrt{e}$

$x = 2n \Rightarrow x\to\infty$

3. 1) $0, \frac{1}{2}, \frac{2}{3}, \ldots \Rightarrow a_n = \frac{n}{n+1}$ converge p/ 1, $\lim\limits_{x\to\infty}\frac{x}{x+1} = \lim\limits_{x\to\infty}\frac{1}{1+\cancel{\frac{1}{x}}^{\to 0}} = 1$

2) $1, \frac{1}{2}, 1, \frac{1}{4}, \ldots \Rightarrow a_n = \begin{cases} 1, & \text{caso } n \text{ ímpar} \\ \frac{1}{2^{n/2}} & \text{caso } n \text{ par} \end{cases}$ $\Rightarrow$ converge p/ 1 / converge p/ 0

Portanto a sequência é divergente, pois ela se alterna entre n par e ímpar.

3) $\frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, -\frac{3}{4} \ldots$ diverge

4) converge, $\lim\limits_{x\to\infty}\left(4+\cancel{\frac{1}{x}}^{\to 0}\right)^{1/2} = 2$

5) $c_k = \frac{\sqrt{k+1}}{k-1}$, $k \geq 2$

$\lim\limits_{x\to\infty}\frac{\sqrt{x}+1}{x-1} \overset{L'H}{=} \lim\limits_{x\to\infty}\frac{1}{2}\cancel{\frac{1}{\sqrt{x}}}^{0} = 0$ //

6) $a_n = \frac{n^3+3n+1}{4n^3+2}$ $\lim\limits_{x\to\infty}\frac{1+\cancel{\frac{3}{n^2}}+\cancel{\frac{1}{n^3}}}{4+\cancel{\frac{2}{n^3}}} = \frac{1}{4}$ converge

7) $a_n = \sqrt{n+1}-\sqrt{n}$

$\lim\limits_{x\to\infty}\sqrt{n+1}-\sqrt{n}\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \lim\limits_{x\to\infty}\frac{\cancel{n}+1-\cancel{n}}{\sqrt{n+1}+\sqrt{n}} = \lim\limits_{x\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$ converge

8) $a_n = \frac{n+(-1)^n}{n-(-1)^n} \Rightarrow \begin{cases} a_{2k} = \frac{n+1}{n-1} \\ a_{2k+1} = \frac{n-1}{n+1} \end{cases}$ $\Rightarrow$ converge p/ 1

$\lim\limits_{x\to\infty}\frac{n+1}{n-1} = \frac{1+\cancel{\frac{1}{n}}}{1-\cancel{\frac{1}{n}}} = 1$

$\lim\limits_{x\to\infty}\frac{n-1}{n+1} = \frac{1-\cancel{\frac{1}{n}}}{1+\cancel{\frac{1}{n}}} = 1$

9) $a_n = \frac{2n}{n+1} - \frac{n+1}{2n} = \frac{4n^2-n^2-2n-1}{2n^2+2n}$ $\lim\limits_{x\to\infty}\frac{4n^2-n^2-2n-1}{2n^2+2n} = \lim\limits_{x\to\infty}\frac{4-1-\cancel{\frac{2}{n}}-\cancel{\frac{1}{n^2}}}{2+\cancel{\frac{2}{n}}}$ converge p/ $3/2$

10) $a_n = n(\sqrt{n^2+1}-n)$ $\lim\limits_{n\to+\infty}\frac{(n\sqrt{n^2+1}-n^2)(n\sqrt{n^2+1}+n^2)}{n\sqrt{n^2+1}+n^2} = \lim\limits_{x\to\infty}\frac{n^2}{n\sqrt{n^2+1}+n^2} = \frac{1}{\sqrt{1+\cancel{\frac{1}{n^2}}}+1} = \frac{1}{2}$

converge p/ 1/2

11) $a_n = \frac{\operatorname{sen} n}{n}$ $\quad \lim_{x\to\infty} \frac{\operatorname{sen} x}{x}$ Pelo Teorema do Confronto, temos

$-1 \leq \operatorname{sen} x \leq 1 \Rightarrow -\frac{1}{x} \leq \frac{\operatorname{sen} x}{x} \leq \frac{1}{x}$, como $\lim_{x\to\infty} -\frac{1}{x} = 0$ e $\lim_{x\to\infty} \frac{1}{x} = 0$

a sequência irá convergir p/ 0.

12) $a_n = \operatorname{sen} n$; $b_n = \operatorname{sen}(n\pi)$; $c_n = \operatorname{sen}\left(\frac{n\pi}{2}\right)$
divergente $\quad$ converge p/ 0. $\quad$ divergente

13) $a_n = \frac{2n + \operatorname{sen} n}{5n+1}$ $\quad \lim_{n\to+\infty} \frac{2n+\operatorname{sen} n}{5n+1} = \lim_{n\to+\infty} \frac{2 + \frac{\operatorname{sen} n}{n} \to 0}{5 + \frac{1}{n} \to 0} = \frac{2}{5}$ ∴ converge.

14) $a_n = \frac{(n+3)! - n!}{(n+4)!} \Rightarrow a_n = \frac{((n+3)(n+2)(n+1) - 1) \cdot \cancel{n!}}{(n+4)(n+3)(n+2)(n+1) \cdot \cancel{n!}}$

$\lim_{n\to+\infty} \frac{(n+3)(n+2)(n+1) - 1}{(n+4)(n+3)(n+2)(n+1)} = 0$ ∴ converge

15) $a_n = \sqrt[n]{n^2+n} = \left(n^2\left(1+\frac{1}{n}\right)\right)^{1/n}$ $\quad \lim_{x\to\infty} \cancel{x}^{2/x} \cdot \left(1+\frac{1}{x}\right)^{1/x} = \lim_{x\to\infty} e^{\frac{2}{x}\ln x \to 0} \cdot e^{\frac{1}{x}\ln\left(1+\frac{1}{x}\right) \to 0} = 1$

converge p/ 1.

16) $a_n = \frac{n \operatorname{sen}(n!)}{n^2+1} \Rightarrow -1 \leq \operatorname{sen}(n!) \leq 1$
$-\frac{n}{n^2+1} \leq \frac{n \operatorname{sen}(n!)}{n^2+1} \leq \frac{n}{n^2+1}$

$\lim_{x\to\infty} -\frac{x}{x^2+1} = 0$ e $\lim_{x\to\infty} \frac{x}{x^2+1} = 0$ ∴ $\lim_{n\to\infty} \frac{n \operatorname{sen}(n!)}{n^2+1} = 0$

17) $a_n = \frac{3^n}{2^n + 10^n} = \frac{\left(\frac{3}{10}\right)^n}{\left(\frac{2}{10}\right)^n + 1}$ $\quad \lim_{x\to\infty} \frac{\left(\frac{3}{10}\right)^n \to 0}{\frac{1}{5^n} \to 0 + 1} = 0$

18) $a_n = \left(\frac{n+2}{n+1}\right)^n$ $\quad \lim_{n\to\infty} \frac{\left(1+\frac{2}{n}\right)^n \to e^2}{\left(1+\frac{1}{n}\right)^n \to e} = e$

19) $a_n = \frac{(n+1)^n}{n^{n+1}} = \lim_{n\to+\infty} \frac{1}{n} \cdot \frac{(n+1)^n}{n^n} = \lim_{n\to+\infty} \frac{1}{n} \to 0 \cdot \left(1+\frac{1}{n}\right)^n \to e = 0$

20) $a_n = n a^n$, $a \in \mathbb{R} \Rightarrow \begin{cases} a = 1 & \text{diverge } +\infty \\ |a| > 1 & \text{diverge } +\infty \\ |a| < 1 & \text{converge } 0 \end{cases}$

21) $a_n = \frac{n!}{n^n}$ $\quad$ Como $n^n >> n!$ converge p/ 0

22) $a_n = n - n^2 \operatorname{sen} 1/n$ $\quad \lim_{n\to\infty} n\left(1 - \frac{\operatorname{sen} 1/n}{1/n}\right)^{\to 1} = 0$

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23) $a_n = (-1)^n + \frac{(-1)^n}{n} \Rightarrow \begin{cases} a_{2k} = 1 + \frac{1}{n} \\ a_{2k+1} = -1 - \frac{1}{n} \end{cases}$ $\lim_{k\to\infty} a_{2k} = 1$, $\lim_{n\to\infty} a_{2k+1} = -1$ $\therefore$ diverge

24) $a_n = \sqrt[n]{a^n + b^n}$, $0 < a < b$ $\lim_{n\to\infty} \sqrt[n]{b^n\left(\frac{a^n}{b^n} + 1\right)} = \lim_{n\to\infty} b \sqrt[n]{\left(\frac{a}{b}\right)^{n\to\infty} + 1} = b$

25) $a_n = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{n}\right) = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{3}} \cdots \frac{\cancel{n-1}}{n} = \frac{1}{n}$ converge p/ 0

26) $a_n = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right) = \frac{2^2-1}{2^2} \cdot \frac{3^2-1}{3^2} \cdots \frac{n^2-1}{n^2} =$

$\Rightarrow \frac{\cancel{1.3}}{2^2} \cdot \frac{\cancel{2.4}}{\cancel{3^2}} \cdot \frac{\cancel{3.8}}{\cancel{4^2}} \cdot \frac{\cancel{4.6}}{5^2} \cdots \frac{\cancel{(n-1)}(n+1)}{n^2} = \frac{n+1}{2n}$ $\lim_{x\to\infty} \frac{x+1}{2x} = \frac{1}{2}$

27) $a_n = \frac{1}{n} \cdot \frac{1.3.5\ldots(2n-1)}{2.4.6.\,2n} = \frac{1}{n} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{(2n-1)}{2n}$

$\lim_{n\to\infty} \frac{\cancel{1}}{\cancel{n}}^{\to 0} \cdot \frac{1/n}{2/n} \cdot \frac{3/n}{4/n}^{\to 0} \cdots \frac{2 - 1/n^{\to 0}}{2} = 0$

28) $a_n = \sqrt[n]{n} = n^{1/n}$ $\lim_{x\to\infty} e^{\frac{1}{x}\ln x} = e^{\lim_{x\to\infty} \frac{1}{x}\ln x} = e^0 = 1$

29) $a_n = \frac{n^\alpha}{e^n}$, $\alpha \in \mathbb{R} \Rightarrow \begin{cases} \alpha = 1 & \lim_{n\to\infty} \frac{n}{e^n} \overset{L'H}{=} 0 \\ \alpha = 0 & \lim_{n\to\infty} \frac{1}{e^n} = 0 \\ \alpha < 0 & \lim_{n\to\infty} \frac{1}{n^\alpha e^n} = 0 \end{cases}$

30) $a_n = \frac{\ln(n)}{n^a}$, $a > 0$ converge p/ 0

31) $a_n = \sqrt[n]{n!} = \sqrt[n]{n.(n-1)\ldots 2.1} = \sqrt[n]{n^n(1.(1-1/n)\ldots 2/n . 1/n)} =$

$\lim_{n\to\infty} n = \infty$ diverge

32) $a_n = \sqrt[n]{a}$, $a > 0$ $a_n = a^{1/n}$ $\lim_{n\to\infty} e^{\frac{1}{n}\ln a} = 1$ $\therefore$ converge

33) $a_n = \left(\frac{n-1}{n}\right)^n$ converge p/ $1/e$

34) $a_n = \left(\frac{n+1}{n}\right)^{n^2}$ $\lim_{x\to\infty} e^{x^2 \ln\left(\frac{x+1}{x}\right)} = +\infty$ $\therefore$ diverge

$\lim_{x\to+\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{1/x^2} \overset{L'H\ \frac{0}{0}}{=} \lim_{x\to+\infty} \frac{\frac{1}{1+1/x} \cdot -\frac{1}{x^2}}{\frac{-2}{x^3}} = \lim_{x\to+\infty} \frac{x^2}{2x+1} \overset{L'H}{=} \lim_{x\to+\infty} \frac{2x}{2} = +\infty$

35) $a_n = \left(\frac{n+1}{n}\right)^{\sqrt{n}}$ $\lim\limits_{x\to\infty} e^{\sqrt{x}\cdot\ln\left(1+\frac{1}{x}\right)} = e^0 = 1$

$$\lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{1/\sqrt{x}} \underset{(0/0)}{\overset{L'H}{=}} \lim_{x\to\infty} \frac{\frac{x}{x+1}\cdot -\frac{1}{x^2}}{-\frac{1}{2}x^{3/2}} = \lim_{x\to\infty} \frac{2x^{3/2}}{(x+1)\,x} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x+1} = 0$$

36) $a_n = \left(\frac{3n+5}{5n+11}\right)^n$

$$\lim_{n\to\infty} \left(\frac{3}{5}\right)^n \xrightarrow{} 0 \cdot \frac{\left(1+\frac{1}{3n/5}\right)^n \to e^{5/3}}{\left(1+\frac{1}{5n/11}\right)^n \to e^{11/5}} = 0 \therefore \text{converge}$$

37) $a_n = \left(\frac{3n+5}{5n+1}\right)^n \left(\frac{5}{3}\right)^n$ converge p/ $e^{22/15}$

38) $a_n = \left(1+\frac{1}{n^2}\right)^n$ $u = n^2$ $\lim\limits_{u\to\infty}\left(1+\frac{1}{u}\right)^{\sqrt{u}} = 1$ (ex 35)

1. a) Sendo $a,b \in \mathbb{R}$, $0 \leq a < b$,

$$\frac{b^{n+1}-a^{n+1}}{b-a} = \frac{\cancel{(b-a)}(b^n + b^{n-1}a + \ldots + ba^{n-1} + a^n)}{\cancel{(b-a)}} = b^n\Big(1 + \underbrace{\underset{<0}{\frac{a}{b}} + \underset{<0}{\frac{a^2}{b^2}} \cdots \underset{<0}{\frac{a^{n-1}}{b^{n-1}}} + \underset{<0}{\frac{a^n}{b^n}}}_{n}\Big)$$

$$< b^n(1+n)$$

b) $b^n[(n+1)a - nb] < a^{n+1}$ $0 < n \in \mathbb{N}$ $a,b \in \mathbb{R}$ $0 \leq a < b$

$$\frac{b^{n+1}-a^{n+1}}{b-a} < (n+1)b^n \Rightarrow b^{n+1} - a^{n+1} < b^n(n+1)(b-a)$$

$$\Rightarrow a^{n+1} > b^n(n+1)(a-b) + b^n b$$

$$\Rightarrow a^{n+1} > b^n[(n+1)a - nb \cancel{-b} \cancel{+b}]$$

$$\Rightarrow b^n[(n+1)a - nb] < a^{n+1}$$

c) $a = 1 + \frac{1}{(n+1)}$, $b = 1 + \frac{1}{n}$

$$\left(1+\frac{1}{n}\right)^n \left[(n+1)\cdot\left(1+\frac{1}{(n+1)}\right) - n\cdot\left(1+\frac{1}{n}\right)\right] = \left(1+\frac{1}{n}\right)^n \left[\cancel{n+1} + 1 \cancel{-n} \cancel{-1}\right]$$

$$= \left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{(n+1)}\right)^{n+1} \Rightarrow a_n < a_{n+1}$$

d) $a = 1$ $b = 1 + \frac{1}{2n}$; $a_{2n} < 4, \forall n \in \mathbb{N}$

$$\left(1+\frac{1}{2n}\right)^n (n+1 \cancel{-n} - 1/2) < 1 \Rightarrow \left(1+\frac{1}{2n}\right)^n \cdot \frac{1}{2} < 1 \Rightarrow \left(1+\frac{1}{2n}\right)^n < 2$$

$$\Rightarrow \left(1+\frac{1}{2n}\right)^n \cdot \left(1+\frac{1}{2n}\right)^n < 2\cdot 2 \Rightarrow \left(1+\frac{1}{2n}\right)^{2n} < 4 \Rightarrow a_{2n} < 4.$$

1. e) $a_n < a_{n+1}$ e $a_{2n} < 4 \Rightarrow a_n < 4 \quad \forall n \in \mathbb{N}$

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1} < \left(1+\frac{1}{2n}\right)^{2n} < 4 \Rightarrow a_n < 4$$

Sendo a sequencia limitada superiormente por 4 e, sendo crescente, temos que o $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ existe.

4. a) $\lim_{n\to\infty} a_n = a \Rightarrow \lim_{n\to\infty} |a_n| = |a|$

- Caso $a_n > 0$, pela hipótese, temos que $\lim_{n\to\infty} a_n = a$
- Caso $a_n \leq 0$, como $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ tq $a - \varepsilon < a_n < a + \varepsilon$

$$\Rightarrow -a + \varepsilon > -a_n > -a - \varepsilon \Rightarrow -a - \varepsilon < -a_n < -a + \varepsilon$$

$$\therefore \lim_{n\to\infty} -a_n = -a$$

Logo, dada a hipótese, temos que $\lim_{n\to\infty} |a_n| = |a|$. Verdadeira.

b) Supondo que $\forall \varepsilon > 0, \exists n > n_0$ tq $||a_n| - |a|| < \varepsilon$ e pela desigualdade triangular reversa, temos que $||a_n| - |a|| \leq |a_n - a|$. Assim, não é possível afirmar que $|a_n - a| < \varepsilon$. Portanto, a afirmação é Falsa.

c) Sendo $a_n \leq 0$ e $\lim_{n\to\infty} a_n = a$, supondo que $a > 0$, temos que a_n é limitado superiormente por a, mas como $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ tq $n > n_0 \Rightarrow |a_n - a| < \varepsilon$, o que é um absurdo pois $a_n \leq 0$.

Portanto, $a \leq 0$ e a afirmação é Verdadeira.

d) $\lim_{n\to\infty} a_n = a$ e $a_n > 0 \Rightarrow a > 0$

Sendo $a_n = \frac{1}{n}$, temos que $\lim_{n\to\infty} \frac{1}{n} = 0$, logo pelo contraexemplo anterior, a afirmação é Falsa.

e) $\lim_{n\to\infty} a_n = a$ e b_n não converge, pelas propriedades de limites, temos que

$$\lim_{n\to\infty}(a_n + b_n) = \underbrace{\lim_{n\to\infty} a_n}_{a} + \lim_{n\to\infty} b_n = a + \lim_{n\to\infty} b_n$$

Logo, $a_n + b_n$ não converge $\therefore$ a afirmação é verdadeira

f) a_n e b_n divergem

Sendo $a_n = n$ e $b_n = -n$, $\lim_{n\to\infty} a_n + b_n = \lim_{n\to\infty} n - n = 0$;

logo, pelo contraexemplo a afirmação é Falsa.

g) Tomando $a_n = \frac{1}{n}$ e $b_n = \text{sen}\, n$, temos que, a_n converge p/ 0

e b_n diverge, mas $\lim_{n\to\infty} a_n \cdot b_n = \lim_{n\to\infty} \frac{\text{sen}\, n}{n} = 0$ pelo Teorema do

Confronto, logo a afirmação é Falsa.

h) Sendo $\lim_{n\to\infty} a_n \cdot b_n = 0$, pelas propriedades dos limites temos

$\lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n = 0$, logo $\lim_{n\to\infty} a_n = 0$ ou $\lim_{n\to\infty} b_n = 0$ $\therefore$ Portanto,

a afirmação é Verdadeira.

5a) $A \subset \mathbb{R}$, $f: A \to A \in C^0$, $a \in A$

a_n definida por $a_0 \in A$ e $a_{n+1} = f(a_n)$, $\forall n \geq 0$, $a_n \to a$

Sendo f uma função contínua, temos que, $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$

$= f(a)$, portanto, sendo $a_{n+1} = f(a_n)$ e a o limitante de a_n,

temos que, $f(a) = a$.

b) Considerando a sequência $a_1 = \sqrt{2}$, $a_2 = \sqrt{2\sqrt{2}}$, $a_3 = \sqrt{2\sqrt{2\sqrt{2}}}$ ..., podemos

defini-la indutivamente do seguinte forma: $a_1 = 2^{1/2}$, $a_2 = 2^{1/2}(2^{1/2})^{1/2} = 2^{3/4}$,

$a_3 = 2^{1/2}(a_2)^{1/2} = 2^{1/2}\cdot(2^{3/4})^{1/2} = 2^{1/2}\cdot 2^{3/8} = 2^{7/8}$, ..., $a_n = 2^{1/2}\cdot(a_{n-1})^{1/2}$

$\Rightarrow a_n = 2^{\frac{2^n-1}{2^n}}$, $\forall n \geq 1$.

Portanto, para $n \geq 1$, temos $2^n \geq 2 \Rightarrow \frac{1}{2^n} \leq \frac{1}{2} \Rightarrow 1 - \frac{1}{2^n} \geq 1 - \frac{1}{2}$

$\Rightarrow \frac{2^n-1}{2^n} \geq \frac{1}{2}$, logo, $2^{\frac{2^n-1}{2^n}} \geq 2^{1/2}$, sendo assim, $a_n \geq \sqrt{2}$ e

como $\lim_{n\to\infty} \frac{2^n-1}{2^n} \overset{L'H}{\underset{\frac{\infty}{\infty}}{=}} \lim_{n\to\infty} \frac{2^n \cdot \ln 2}{2^n \cdot \ln 2} = 1$, temos que, $\lim_{n\to\infty} a_n = 2^1 = 2$,

Disto, concluimos que a sequência é crescente e limitada superiormente por 2.

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5. c) $x_1 = \sqrt{2}$, $x_{n+1} = \sqrt{2 + x_n}$, $n \in \mathbb{N}$, $n \geqslant 1$

Do item a, seja $\lim\limits_{n\to\infty} a_n = a$, temos que,

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{2 + x_n} \Rightarrow a = \sqrt{2 + a} \overset{a > -2}{\Rightarrow} a^2 = 2 + a \Rightarrow a^2 - a - 2 = 0$$

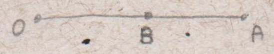
$\Rightarrow a = -1$ ou $a = 2$

Como $a_n > \sqrt{2}$, temos que, $\lim\limits_{n\to\infty} a_n = 2$.

d) Definindo a seguinte sequência, $a_1 = \sqrt{2}$, $a_2 = \sqrt{2}^{\sqrt{2}}$, ..., $a_n = \sqrt{2}^{a_{n-1}}$, do item (a) e, sendo $\lim\limits_{n\to\infty} a_n = a$, temos que

$$a = \sqrt{2}^{a} \Rightarrow a = 2$$

Logo, $\lim\limits_{n\to\infty} a_n = 2$

6. a) 

razão áurea

$$\frac{OA}{OB} = \frac{OB}{BA} \qquad \varphi = \frac{OB}{BA} \quad \text{raiz positiva de } x^2 - x - 1 = 0$$

$$OB^2 = (OB + BA)\,BA \Rightarrow OB^2 = OB.BA + BA^2 \Rightarrow \frac{OB^2}{BA^2} = \frac{OB.BA}{BA^2} + \frac{BA^2}{BA^2}$$

$$\Rightarrow \varphi^2 = \varphi + 1 \Rightarrow \varphi^2 - \varphi - 1 = 0 \Rightarrow \varphi = \frac{1 \pm \sqrt{5}}{2},$$

Mas OB e BA são segmentos de reta positivos, logo, $\boxed{\varphi = \frac{1 + \sqrt{5}}{2}}$

b) $f_0 = f_1 = 1$ $f_n = f_{n-2} + f_{n-1}$, $n \geqslant 2$ $x_n = \frac{f_{n+1}}{f_n} \longrightarrow \varphi$

1, 1, 2, 3, 5, 8, 13, 21, 34 ...

1, 2, 3/2, 5/3, 8/5, 13/8, 21/13, 34/21 ... $a_n = \frac{x}{y}$, $a_{n+1} = \frac{x+y}{x}$

$a_{n+1} = 1 + \frac{1}{a_n}$

$$a = 1 + \frac{1}{a} \Rightarrow a^2 = a + 1 \Rightarrow a^2 - a - 1 = 0$$

$\Rightarrow a = \varphi$ pois $a_n \geqslant 1$

Logo, temos que $\lim\limits_{n\to\infty} x_n = \varphi$

$$0 < \frac{1}{a_n} \leqslant 1$$

$$\therefore 1 < 1 + \frac{1}{a_n} \leqslant 2$$

$$1 < a_{n+1} \leqslant 2$$

logo é limitada e

7. $\left(\dfrac{p_n}{q_n}\right)_{n\in\mathbb{N}}$ $\quad p_1 = q_1 = 1$, p/ $n \geq 2$ $\begin{cases} p_n = p_{n-1} + 2q_{n-1} \\ q_n = p_{n-1} + q_{n-1} \end{cases}$

$$1,\ \frac{3}{2},\ \frac{7}{5},\ \frac{17}{13},\ \dots,\ a_n = \frac{x}{y},\quad a_{n+1} = \frac{x+2y}{x+y} = 1 + \frac{y}{x+y} = 1 + \frac{1}{a_n+1}$$

Sendo $x, y \in \mathbb{N}$, com $x > y > 0$, temos então

$$x+y > y > 0 \Rightarrow 0 < \frac{y}{x+y} < 1 \Rightarrow 1 < 1 + \frac{y}{x+y} < 2$$

Logo, a sequência converge p/ algum $\alpha \in \mathbb{R}$ tq $1 < \alpha < 2$, e, portanto, como $a_{n+1} = 1 + \dfrac{1}{a_n+1}$ e se a_{n+1} converge p/ a então a_n também convergirá p/ a, logo

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} 1 + \frac{1}{a_n+1} \Rightarrow a = 1 + \frac{1}{a+1} \Rightarrow a = \frac{a+2}{a+1}$$

$$\Rightarrow a^2 + \cancel{a} = \cancel{a} + 2 \Rightarrow a = \pm\sqrt{2},\ \text{mas } 1 < a < 2,\ \text{logo}$$

$$\boxed{a = \sqrt{2}}$$

8. $\dfrac{n+1}{n^2} + \dfrac{(n+1)^2}{n^3} + \dots + \dfrac{(n+1)^n}{n^{n+1}} \longrightarrow e - 1$

$$\left(\frac{n+1}{n}\right)\cdot\frac{1}{n} + \left(\frac{n+1}{n}\right)^2\cdot\frac{1}{n} + \dots + \left(\frac{n+1}{n}\right)^n\cdot\frac{1}{n}$$

$$= \frac{1}{n}\left[\left(1+\frac{1}{n}\right) + \left(1+\frac{1}{n}\right)^2 + \dots + \left(1+\frac{1}{n}\right)^n\right] = \frac{1}{\cancel{n}}\left[\frac{\left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n}\right)}{\frac{1}{\cancel{n}}}\right]$$

$$\therefore \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n}\right)\right] = \lim_{n\to\infty}\underbrace{\left(1+\frac{1}{n}\right)^n}_{e}\cdot\underbrace{\left(1+\cancel{\frac{1}{n}}\right)}_{1}^{0} - \underbrace{\lim_{n\to\infty}\left(1+\cancel{\frac{1}{n}}\right)^{0}}_{1}$$

$$= \boxed{e - 1}$$