Differential Geometry: A Cheat Sheet

MATH 583

Contents

Chapter 1

Background Theory

1.1 Manifolds and their properties

A topological manifold is a second countable hausdorff space. There is a lot of structure that this immediately implies. Here are some useful ones in no particular order

- Locally connected
- Locally path connected
- Locally compact
- has a countable basis of coordinate balls (neighborhood U homeomorphic to a ball in $\mathbb{R}^n$)
- Every open subset of an n manifold is an n manifold
- Every point on a manifold with boundary is either on the boundary or interior
- The interior of a n manifold with boundary is a manifold with boundary n

1.2 Theorems: Topological

Handy theorems that are of a topological flavor, useful for proving things about manifolds or proving something is a manifold

1.2.1 Continuous Functions

Theorem 1.2.1. *Let $S \subset M$ then the inclusion map $S \hookrightarrow M$ is continuous.*

Theorem 1.2.2. *If $X_1, .., X_n$ are topological spaces then the projection $\pi_i : X_1 \times ... \times X_n \to X_i$ is continuous*

Theorem 1.2.3. *The map $f : X_i \to X_1 \times ... \times X_n$ such that $f(x) = (x_1, .., x_{i-1}, x, x_{i+1}, ..., x_n)$ is a topological embedding.*

Theorem 1.2.4. *Let $f_i : X_i \to Y_i$ be maps then the product map is*

$$f_1 \times ... \times f_n : X_1 \times ... \times X_n \to Y_1 \times ... \times Y_n f_1 \times ... \times f_n(x) = (f_1(x), ..., f_n(x))$$

If the maps are all continuous the product is too, if they are homeomorphisms so is the product.

Theorem 1.2.5. *If each X_i is hausdorff or second countable then $X_1 \times ... \times X_n$ are too.*

Theorem 1.2.6. *Subspaces of any hausdorff or second countable space are too.*

1.2.2 Embeddings

An embedding is an injective continuous map that is a homeomorphism onto its image some useful facts

Theorem 1.2.7. *Let S be a subspace of M then $S \hookrightarrow M$ is an embedding*

Theorem 1.2.8. *A continuous injective map that is either open or closed is a topological embedding.*

Theorem 1.2.9. *Let $U \subset \mathbb{R}^N$ be open and $f : U \to \mathbb{R}^k$ any continuous map, then defint the graph*

$$\Gamma(f) = \{(x, y) = (x_1, ..., x_n, y_1, ..., y_n) : x \in U, y = f(x)\}$$

with the subspace topology of $\mathbb{R}^{n+k}$. Then the graph is a n manifold and more specifically is homeomorphic to U

Theorem 1.2.10. *Suppose $f : X \to Y$ is open or closed and injective then it is an embedding.*

1.2.3 Quotient Spaces

Theorem 1.2.11. *If P is second countable and M is a locally euclidean hausdorff space that is a quotient of P then M is a manifold.*

Theorem 1.2.12. *If $q : X \to Y$ is a surjective continuous map that is also an open or closed map, then it is a quotient map.*

Theorem 1.2.13. *Suppose $f : X \to Y$ is open or closed, and surjective then it is a quotient map.*

Theorem 1.2.14. *Suppose $q_1 : X \to Y_1$ and $q_2 : X \to Y_2$ are quotient maps that make the same identifications $q_1(x) = q_1(x')$ if and only if $q_2(x) = q_2(x')$. Then there is a unique homeomorphism between Y_1, Y_2.*

Theorem 1.2.15. *Suppose $\pi : X \to X/\sim$ is open, then $X/\sim$ is hausdorff if and only if $R = \{(x, y) | y \sim x\}$ is closed in $X \times X$*

1.3 Calculus for maps from $\mathbb{R}^n \to \mathbb{R}^m$

Theorem 1.3.1. *Implicit function theorem: Let $f : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be a continuously differentiable function with coordinates $(\mathbf{x}, \mathbf{y}) = (x_1, ..., x_n, y_1, ..., y_m)$and fix a point $(\mathbf{a}, \mathbf{b}) = (a_1, ..., a_n, b_1, ..., b_m)$ with $f(\mathbf{a}, \mathbf{b}) = \mathbf{0}$. Define the jacobian*

$$J_{f,y}(a, b) = \begin{bmatrix} \frac{\partial f_1}{\partial y_1}(\mathbf{a}, \mathbf{b}) & .. & \frac{\partial f_1}{y_n}(\mathbf{a}, \mathbf{b}) \\ \cdots & ... & \cdots \\ \frac{\partial f_m}{\partial y_1}(\mathbf{a}, \mathbf{b}) & .. & \frac{\partial f_m}{y_n}(\mathbf{a}, \mathbf{b}) \end{bmatrix}$$

Then if $det(J_{f,y}(\mathbf{a}, \mathbf{b})) \neq 0$ there exists an open set $U \subset \mathbb{R}^n$ containing $\mathbf{a}$ such that there is a function $g : U \to \mathbb{R}^m$ that sends $g(\mathbf{a}) = \mathbf{b}$ and $f(x, g(x)) = \mathbf{0}$ for all $x \in U$

Theorem 1.3.2. *Inverse Function theorem: Let U be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ a C^r mapping. If $a \in U$ and $DF(a) =\neq 0$ then there exists a neighborhood V of a such that*

1. *$f : U \to V$ is invertible and C^r*
2. *$f^{-1} : V \to U$ is C^r*
3. *$D(f^{-1})(x) = (DF(x))^{-1}$*

Theorem 1.3.3. *Let $U \subseteq \mathbb{R}^n$ and $f : U \to f(U)$ be C^∞. Then f is a diffeomorphism if and only if f is injective and DF is nonzero at every point in U.*

1.4 Linear Algebruh

SVD, change of coordinates goes here ...

Chapter 2

Smooth Manifolds

Throughout our study some proof techniques and ideas recur, he we collect them attempting to state the abstract idea with the minimal example of how to use it.

2.1 Recipes and Topoi

In the case of (smooth) manifolds we have access to coordinates, it is often easier to show than a set A is open or closed by constructing a continuous function f who's inverse image of some open or closed set is A. This also gives really easy proofs of something being a smooth manifold if A is an open subset of a space already known to be a smooth manifold.

Example 2.1.1: Proving GL_n is Open

$GL_n(F)$ is an open subset of $M_n(\mathbb{F})$ because $\det : M_n(\mathbb{F}) \to \mathbb{R}$ is a polynomial in the entries of a matrix and hence continuous. A matrix is in GL_n if and only if $\det \neq 0$ so

$$det^{-1}(\mathbb{R} - \{0\}) = GL_n$$

and the real line with a point removed is open, so GL_n is an open subset of $\mathbb{R}^{n^2}$ and hence a smooth manifold.