# Flat Norm on Graphs

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**Abstract**

In this paper we implement and test a method for computing the multiscale flat norm signature for characteristic functions over irregular grids in $\mathbb{R}^2$ and $\mathbb{R}^3$.

## Contents



## 1 Multiscale Flat Norm

In 2005, Chan and Esedoglu introduced an edge preserving total variation regularization functional:

$$F_{CE}(u) = \int_\Omega |\nabla u dx + \lambda \int_\Omega |u - f| dx \tag{1}$$

Where $\Omega$ is a domain on which we have greyscale data $f : \Omega \to \mathbb{R}$ we would like to denoise. Solving the associated minimization problem above we obtain:

$$u^* = \underset{u}{\operatorname{argmin}}\, F_{CE}(u)$$

Which is the denoised greyscale approximation $u^* : \Omega \to \mathbb{R}$ to the data $f$. The strength of the denoising may be adjusted by the parameter $\lambda \in [0, \infty)$, a large value of $\lambda$ corresponds to enacting a stricter penalty for candidates $u$ that deviate too far from the original data and thus enforce less denoising. We henceforth refer to (1) as the $L^1$TV functional.

It was recognized in [2] by Simon Morgan and Kevin Vixie that the $L^1$TV functional was both a special case of and an extension of the flat norm in geometric measure theory (GMT). Work was done in [3] by Kevin Vixie Et al. to explore the implications of this.

# 2 Discrete Implementations

If $\chi_E$ is the characteristic function of $E$, with $\chi_E(x) = 1$ if $x \in E$ and 0 otherwise, and $u$ may be taken to be of this form, then the $L^1$TV functional in (1) reduces to:

$$F_{CE}(\Sigma) = \operatorname{Per}(\Sigma) + \lambda|\Sigma\Delta\Omega|$$

Where $\Sigma$ is the support of $u = \chi_\Sigma$, $\operatorname{Per}(\Sigma)$ is the perimeter of $\Sigma$, and $\Sigma\Delta\Omega$ is the symmetric difference between $\Sigma$ and the support $\Omega$ of the data $f = \chi_\Omega$.

The flat norm with scale $\lambda$ of an oriented 1-dimensional set $T$ is given by:

$$\mathbb{F}(T) = \min_S\{V_1(T - \partial S) + \lambda V_2(S)\} \tag{2}$$

Where $V_1$ is 1-dimensional volume (length), $V_2$ is 2-dimensional volume (area) and $S$ varies over 2-dimensional regions. We refer to the pair of the 1D and 2D sets $\{T, S\}$ as the flat norm decomposition. By [2] we have for fixed $\lambda$:

$$\mathbb{F}(\partial\Omega) = F_{CE}(\Sigma) \tag{3}$$

with flat norm decomposition $\{\partial\Omega, \Sigma\Delta\Omega\}$, with $\partial\Omega$ denoting the measure theoretic boundary of $\Omega$. In [3], Kevin Et al. thus represented the problem of computing the flat norm as minimizing the $L^1$TV functional and computing the minimzer by graph cuts as introduced by [1]. Applied to images, this is realized by representing each pixel as a node on a rectangular grid which forms the working space. A characteristic function $\chi_\Omega$ is defined on the nodes which represents a black and white thresholded image. Graph edges are added between each node using 16 nearest neighbors in image space. Each of these edges are weighted by minimizing gradient computation error on known functions. After, a virtual sink ($t$) and a virtual source node ($s$) are added. The source node is connected to every node in $\Omega$ and the sink to every node on the grid not in $\Omega$.

A particular cut of this graph corresponds has a capacity equal to the value of the flat norm $\mathbb{F}(S)$ for a set of nodes $S$ consisting of nodes $n$ for which either the edge $(n, s)$ or $(n, t)$ is in the cut. Any cut of the graph incurs a penalty proportional to the number of image nodes it cuts, with the penalty exactly equal to $V_1(T - \partial S)$ in (2) for parameter $\lambda$. Hence finding a cut with minimal capacity is the same as computing the flat norm.

The vector of weights $w^*$ calculated in [3] were chosen more specifically to approximate the function $g_\theta : \mathbb{R}^2 \to \mathbb{R}$ whose gradient is $\nabla g_\theta = (\cos\theta, \sin\theta)^T$ for all $\theta$:

$$w^* = \underset{w}{\operatorname{argmin}} \int_0^{2\pi} (h(w, \theta) - 1)^2 d\theta \tag{4}$$

With

$$h(w, \omega) = \sum_{j=1}^{4} w_1 |\nabla g_\theta \cdot v_j| + \sum_{j=5}^{8} w_2 |\nabla g_\theta \cdot v_j| + \sum_{j=9}^{16} w_3 |\nabla g_\theta \cdot v_j|$$

Where $v_j, j = 1, ..., 16$ are the vectors from a fixed point in the grid to its 16 nearest neighbors, where three types of neighbor groupings are identified as below.

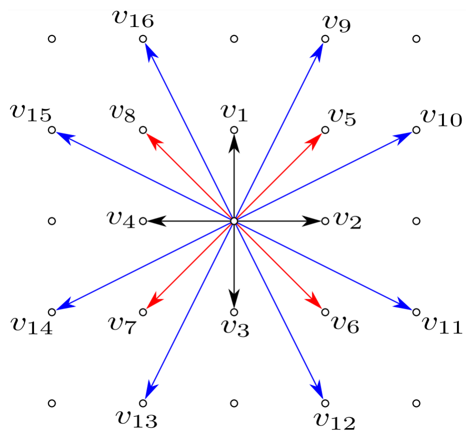


Figure 1: 16 vector neighborhood from [3].

Equation (4) was solved analytically to obtain weights $(w_1, w_2, w_3) \approx (0.1221, 0.0476, 0.0454)$.

# 3 Flat Norm on Arbitrary 2D and 3D Graphs

In the present work, we extend the multiscale flat norm computation to arbitrary graphs embedded in $\mathbb{R}^n$ using the framework above, and provide code to calculate explicitly in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Let $V = \{v_1, ..., v_N\}$ be a set of vertices in $\mathbb{R}^n$ with a set symmetric edges $E$ on $V$. Fix a particular vertex $v \in V$ with degree $D$ and associated connected $\{u_i, i = 1, ..., D\}$. We will provide a scheme for calculating the edge weights $\{w_i, i = 1, ..., D\}$ that both recover the weights obtained in (4) in the case of a regular unit grid but extend to arbitrary irregular grids and connections. Our weights will be chosen to minimize the distance (in the $L^2$ sense) between the linear approximation of a weighted sum of the edges and the total variation over $\partial B(0, 1)$, the unit sphere at the origin:

$$\mathcal{F}(w_1, w_2, ..., w_D) := \int_{\partial B(0,1)} \left| \sum_{i=1}^{D} w_i |\langle \nu, u_i \rangle| - \|\nu\| \right|^2 d\nu$$

Which expands into the quadratic:

$$\mathcal{F}(w_1, w_2, ..., w_D) = \sum_{i=1}^{D} w_i^2 C_1^i + 2\sum_{i=1}^{D}\sum_{j=1}^{i-1} w_i w_j C_3^{ij} - 2\sum_{i=1}^{D} w_i C_2^i + \alpha(n)$$

Where $\alpha(n) = \int_{\partial B(0,1)} d\nu$ is the $n$-dimensional measure of the unit $n$ sphere, with:

$$C_1^i := \int_{\partial B(0,1)} |\langle \nu, u_i\rangle|^2 d\nu$$
$$C_2^i := \int_{\partial B(0,1)} |\langle \nu, u_i\rangle| d\nu$$
$$C_3^{ij} := \int_{\partial B(0,1)} |\langle \nu, u_i\rangle\langle \nu, u_j\rangle| d\nu$$

The function above is convex, and thus we minimize it by calculating the gradient:

$$\frac{\partial \mathcal{F}}{\partial \omega_k} = 2\omega_k C_1^k + 2\sum_{m=1}^{D} \omega_m C_3^{km} - 2\omega_k C_3^{kk} - 2C_2^k$$

Which forms a linear system of $D$ equations in $D$ unknowns. WLOG this system may be presumed nonsingular, as if two edges $u_n$ and $u_m$ are linearly dependent, by scaling and symmetry they will receive the same weight and thus one may be safely removed from the calculation. Thus we can simply solve our system for the stationary point:

$$2\omega_1 C_1^1 + 2\sum_{m=2}^{D} \omega_m C_3^{1m} - 2C_2^1 = 0$$
$$\vdots$$
$$2\omega_D C_1^D + 2\sum_{m=1}^{D-1} \omega_m C_3^{Dm} - 2C_2^D = 0$$

If needed, the Hessian may also be quickly calculated:

$$\frac{\partial^2 \mathcal{F}}{\partial \omega_k \partial \omega_\ell} = \begin{cases} 2C_1^k & \text{if } \ell = k \\ 2C_3^{k\ell} & \text{otherwise} \end{cases}$$

Thus the problem reduces to calculation of $C_1^i$, $C_2^i$ and $C_3^{ij}$. We provide the following table of values:

| | 2D | 3D |
|---|---|---|
| $C_1^i$ | $\pi\|u_i\|^2$ | $\frac{4\pi}{3}\|u_i\|^2$ |
| $C_2^i$ | $4\|u_i\|$ | $2\pi\|u_i\|$ |
| $C_3^{ij}$ | Numerical | Numerical |

# 4 Tests

# 5 Edge Cases

# References


[1] Vladimir Kolmogorov and Ramin Zabih. What energy functions can be minimized via graph cuts? In Anders Heyden, Gunnar Sparr, Mads Nielsen, and Peter Johansen, editors, *Computer Vision — ECCV 2002*, pages 65–81, Berlin, Heidelberg, 2002. Springer Berlin Heidelberg.

[2] Simon P. Morgan and Kevin R. Vixie. L1tv computes the flat norm for boundaries. *Abstract and Applied Analysis*, 2007:1–14, 2007.

[3] Kevin Vixie, Keith Clawson, Thomas Asaki, Gary Sandine, Simon Morgan, and Brandon Price. Multiscale flat norm signatures for shapes and images. *Applied Mathematical Sciences*, 4:667–680, 01 2010.