Homework 5

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ST 563 Theory of Statistics III

Question 1

Suppose

$$X_{1,1}, X_{1,2}, \dots, X_{1,n} \overset{\text{iid}}{\sim} \mathcal{N}(\mu_1, \sigma^2)$$

$$X_{2,1}, X_{2,2}, \dots, X_{2,n} \overset{\text{iid}}{\sim} \mathcal{N}(\mu_2, \sigma^2)$$

$$X_{3,1}, X_{3,2}, \dots, X_{3,n} \overset{\text{iid}}{\sim} \mathcal{N}(\mu_3, \sigma^2)$$

$$X_{4,1}, X_{4,2}, \dots, X_{4,n} \overset{\text{iid}}{\sim} \mathcal{N}(\mu_4, \sigma^2)$$

$$X_{5,1}, X_{5,2}, \dots, X_{5,n} \overset{\text{iid}}{\sim} \mathcal{N}(\mu_5, \sigma^2)$$

and all samples are independent. We will construct a confidence interval for $(\mu_1 - \mu_2, \mu_2 - \mu_3, \mu_3 + \mu_4 - 2\mu_5)$ in the following steps.

Part A

Define $Y_{1,j} = X_{1,j} - X_{2,j}$, $Y_{2,j} = X_{2,j} - X_{3,j}$, and $Y_{3,j} = X_{3,j} + X_{4,j} - 2X_{5,j}$. Then $Y_j = (Y_{1,j}, Y_{2,j}, Y_{3,j})^T$ for $j = 1, \dots, n$ are iid three-dimensional normal random variables. Determine the mean and covariance matrix for the Y_j. You will find that the covariance matrix has the form $\sigma^2 \mathbf{H}$ where the matrix $\mathbf{H}$ is known.

Solution

First, we can define,

$$Y_{1,j} = X_{1,j} - X_{2,j}, \quad Y_{2,j} = X_{2,j} - X_{3,j}, \quad Y_{3,j} = X_{3,j} + X_{4,j} - 2X_{5,j}$$

Now, let us determine the mean vector of $Y_j = (Y_{1,j}, Y_{2,j}, Y_{3,j})^T$ using these expecations,

$$E[Y_{1,j}] = E[X_{1,j} - X_{2,j}] = \mu_1 - \mu_2$$

$$E[Y_{2,j}] = E[X_{2,j} - X_{3,j}] = \mu_2 - \mu_3$$

$$E[Y_{3,j}] = E[X_{3,j} + X_{4,j} - 2X_{5,j}] = \mu_3 + \mu_4 - 2\mu_5$$

Thus, the complete mean vector is as follows,

$$E[Y_j] = \begin{pmatrix} \mu_1 - \mu_2 \\ \mu_2 - \mu_3 \\ \mu_3 + \mu_4 - 2\mu_5 \end{pmatrix}$$

Next, we can compute the covariance matrix of Y_j.

$$\mathrm{Var}(Y_{1,j}) = \mathrm{Var}(X_{1,j} - X_{2,j}) = \sigma^2 + \sigma^2 = 2\sigma^2$$

$$\mathrm{Var}(Y_{2,j}) = \mathrm{Var}(X_{2,j} - X_{3,j}) = \sigma^2 + \sigma^2 = 2\sigma^2$$

$$\mathrm{Var}(Y_{3,j}) = \mathrm{Var}(X_{3,j} + X_{4,j} - 2X_{5,j}) = \sigma^2 + \sigma^2 + 4\sigma^2 = 6\sigma^2$$

$$\mathrm{Cov}(Y_{1,j}, Y_{2,j}) = \mathrm{Cov}(X_{1,j} - X_{2,j}, X_{2,j} - X_{3,j}) = -\sigma^2$$

$$\mathrm{Cov}(Y_{1,j}, Y_{3,j}) = \mathrm{Cov}(X_{1,j} - X_{2,j}, X_{3,j} + X_{4,j} - 2X_{5,j}) = 0$$

$$\mathrm{Cov}(Y_{2,j}, Y_{3,j}) = \mathrm{Cov}(X_{2,j} - X_{3,j}, X_{3,j} + X_{4,j} - 2X_{5,j}) = \sigma^2$$

$$\mathrm{Cov}(Y_{1,j}, Y_{2,j}) = \mathrm{Cov}(X_{1,j} - X_{2,j}, X_{2,j} - X_{3,j}) = -\sigma^2$$

$$\mathrm{Cov}(Y_{1,j}, Y_{3,j}) = \mathrm{Cov}(X_{1,j} - X_{2,j}, X_{3,j} + X_{4,j} - 2X_{5,j}) = 0$$

$$\mathrm{Cov}(Y_{2,j}, Y_{3,j}) = \mathrm{Cov}(X_{2,j} - X_{3,j}, X_{3,j} + X_{4,j} - 2X_{5,j}) = \sigma^2$$

So, the covariance matrix $\mathbf{H}$ is,

$$\mathbf{H} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 6 \end{pmatrix}$$

Therefore, the covariance matrix of Y_j is,

$$\mathrm{Cov}(Y_j) = \sigma^2 \mathbf{H}$$

Part B

Pretend that we did not observe the $X_{i,j}$s. Estimate σ^2 using the $Y_{i,j}$s. You can use a quantity having a chi-squared distribution with $3(n-1)$ degrees of freedom.

Solution

Using the fact that,

$$\sum_{j=1}^{n} \left(Y_{1,j}^2 + Y_{2,j}^2 + Y_{3,j}^2\right) \sim \sigma^2 \chi^2_{3(n-1)}$$

We can estimate σ^2 using this expression,

$$\hat{\sigma}^2 = \frac{1}{3(n-1)} \sum_{j=1}^{n} (Y_{1,j}^2 + Y_{2,j}^2 + Y_{3,j}^2 + 2Y_{1,j}Y_{2,j} - 2Y_{2,j}Y_{3,j})$$

Part C

However, the estimate of σ^2 in the previous part is not the best you could do. Instead, estimate σ^2 using the $X_{i,j}$s. You can use a quantity having a chi-squared distribution with $5(n-1)$ degrees of freedom.

Solution

Based on the fact that,

$$\sum_{i=1}^{5}\sum_{j=1}^{n}(X_{i,j}-\bar{X}_i)^2 \sim \sigma^2\chi^2_{5(n-1)}$$

We can directly estimate σ^2 from the $X_{i,j}$s. Thus, we can use the sample variances of each group,

$$\hat{\sigma}^2 = \frac{1}{5(n-1)}\sum_{i=1}^{5}\sum_{j=1}^{n}(X_{i,j}-\bar{X}_i)^2$$

where $\bar{X}_i = \frac{1}{n}\sum_{j=1}^{n} X_{i,j}$.

Part D

Construct a level $(1-\alpha)$ confidence interval for $(\mu_1-\mu_2, \mu_2-\mu_3, \mu_3+\mu_4-2\mu_5)$ using your results from the previous parts.

Solution

First, the sample means are given as follows,

$$\bar{Y}_1 = \frac{1}{n}\sum_{j=1}^{n} Y_{1,j}, \quad \bar{Y}_2 = \frac{1}{n}\sum_{j=1}^{n} Y_{2,j}, \quad \bar{Y}_3 = \frac{1}{n}\sum_{j=1}^{n} Y_{3,j}$$

Second, the variance of $\bar{Y}_i$ is $\frac{\sigma^2 H_{ii}}{n}$.

Using the fact that

$$(\bar{Y}_1, \bar{Y}_2, \bar{Y}_3)^T \sim \mathcal{N}\left(\begin{pmatrix} \mu_1-\mu_2 \\ \mu_2-\mu_3 \\ \mu_3+\mu_4-2\mu_5 \end{pmatrix}, \frac{\sigma^2}{n}\mathbf{H}\right)$$

we can then construct the confidence intervals.

For $\mu_1-\mu_2$,

$$\bar{Y}_1 \pm t_{n-1,1-\alpha/2}\sqrt{\frac{2\hat{\sigma}^2}{n}}$$

For $\mu_2-\mu_3$,

$$\bar{Y}_2 \pm t_{n-1,1-\alpha/2}\sqrt{\frac{2\hat{\sigma}^2}{n}}$$

For $\mu_3+\mu_4-2\mu_5$,

$$\bar{Y}_3 \pm t_{n-1,1-\alpha/2}\sqrt{\frac{6\hat{\sigma}^2}{n}}$$

Question 2

(10.36 from *Statistical Inference, 2nd Edition*) Let $X_1, \dots, X_n$ be a random sample from a Gamma(α, β) population. Assume α is known and β is unknown. Consider testing $H_0 : \beta = \beta_0$.

Part A

What is the MLE of β?

Solution

The probability density function of a Gamma(α, β) distribution is given by,

$$f(x|\alpha, \beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$$

Given $X_1, X_2, \dots, X_n$, the likelihood function is,

$$L(\beta) = \prod_{i=1}^{n} f(X_i|\alpha, \beta) = \prod_{i=1}^{n} \frac{1}{\beta^\alpha \Gamma(\alpha)} X_i^{\alpha-1} e^{-X_i/\beta}$$

Take the natural log to get the log-likelihood function,

$$\ell(\beta) = \sum_{i=1}^{n} \left(-\alpha \log(\beta) - \log(\Gamma(\alpha)) + (\alpha - 1) \log(X_i) - \frac{X_i}{\beta} \right)$$

Simplifying, we obtain,

$$\ell(\beta) = -n\alpha \log(\beta) + (\alpha - 1) \sum_{i=1}^{n} \log(X_i) - \frac{1}{\beta} \sum_{i=1}^{n} X_i - n \log(\Gamma(\alpha))$$

To find the MLE of β, we take the derivative of $\ell(\beta)$ with respect to β and set it to zero,

$$\frac{\partial \ell(\beta)}{\partial \beta} = -\frac{n\alpha}{\beta} + \frac{1}{\beta^2} \sum_{i=1}^{n} X_i = 0$$

Solving for $\hat{\beta}_{MLE}$, we get

$$\hat{\beta}_{MLE} = \frac{1}{n\alpha} \sum_{i=1}^{n} X_i = \frac{\bar{X}}{\alpha}$$

Hence, $\hat{\beta}_{MLE} = \frac{\bar{X}}{\alpha}$.

Part B

Derive a Wald statistic for testing H_0, using the MLE in both the numerator and denominator of the statistic.

Solution

The Wald test statistic is given by,

$$W = \frac{\hat{\beta} - \beta_0}{\mathrm{SE}(\hat{\beta})}$$

First, we need to find the standard error of $\hat{\beta}$. The Fisher information for β in a Gamma(α, β) distribution is:

$$I(\beta) = -E\left(\frac{\partial^2 \ell(\beta)}{\partial \beta^2}\right)$$

Since we already have,

$$\frac{\partial \ell(\beta)}{\partial \beta} = -\frac{n\alpha}{\beta} + \frac{1}{\beta^2}\sum_{i=1}^{n} X_i$$

We can quickly take the second derivative and get,

$$\frac{\partial^2 \ell(\beta)}{\partial \beta^2} = \frac{n\alpha}{\beta^2} - \frac{2}{\beta^3}\sum_{i=1}^{n} X_i$$

Then, the expected value of the second derivative yields,

$$E\left(\frac{\partial^2 \ell(\beta)}{\partial \beta^2}\right) = \frac{n\alpha}{\beta^2} - \frac{2}{\beta^3}E\left(\sum_{i=1}^{n} X_i\right) = \frac{n\alpha}{\beta^2} - \frac{2n\alpha\beta}{\beta^3} = -\frac{n\alpha}{\beta^2}$$

Thus, the Fisher information is,

$$I(\beta) = \frac{n\alpha}{\beta^2}$$

The variance of $\hat{\beta}$ is the inverse of the Fisher information,

$$\mathrm{Var}(\hat{\beta}) = \left(\frac{n\alpha}{\beta^2}\right)^{-1} = \frac{\beta^2}{n\alpha}$$

So, the standard error is,

$$\mathrm{SE}(\hat{\beta}) = \sqrt{\mathrm{Var}(\hat{\beta})} = \frac{\beta}{\sqrt{n\alpha}}$$

Hence, the Wald test statistic is,

$$W = \frac{\hat{\beta} - \beta_0}{\frac{\hat{\beta}}{\sqrt{n\alpha}}} = \frac{\sqrt{n\alpha}(\hat{\beta} - \beta_0)}{\hat{\beta}}$$

Part C

Repeat part (b), but using the sample standard deviation in the standard error.

Solution

The sample variance of $Gamma(\alpha, \beta)$ is,

$$\hat{\beta} = \frac{\bar{X}}{\alpha}$$

The sample variance X_i is,

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

The standard error using the sample standard deviation is,

$$\text{SE}(\hat{\beta}) = \frac{S}{\sqrt{n}}$$

Thus, the Wald test statistic is,

$$W = \frac{\hat{\beta} - \beta_0}{\text{SE}(\hat{\beta})} = \frac{\hat{\beta} - \beta_0}{\frac{S}{\sqrt{n}}} = \frac{\sqrt{n}(\hat{\beta} - \beta_0)}{S}$$

Question 3

(10.38 from *Statistical Inference, 2nd Edition*) Let $X_1, \dots, X_n$ be a random sample from a Gamma(α, β) distribution. Assume α is known and β is unknown. Consider testing $H_0 : \beta = \beta_0$. Derive a score statistic for testing H_0.

Solution

To derive the score statistic, we first need to obtain the score function and the Fisher information. The pdf of a Gamma(α, β) distribution is,

$$f(x|\alpha, \beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$$

Given a random sample $X_1, X_2, \dots, X_n$ from the Gamma distribution, the likelihood function is,

$$L(\beta) = \prod_{i=1}^n f(X_i|\alpha, \beta) = \prod_{i=1}^n \frac{1}{\beta^\alpha \Gamma(\alpha)} X_i^{\alpha-1} e^{-X_i/\beta}$$

Taking the natural log of the likelihood function, we get the log-likelihood function,

$$\ell(\beta) = \sum_{i=1}^n \left(-\alpha \log(\beta) - \log(\Gamma(\alpha)) + (\alpha - 1) \log(X_i) - \frac{X_i}{\beta} \right)$$

Simplifying, we obtain,

$$\ell(\beta) = -n\alpha \log(\beta) + (\alpha - 1) \sum_{i=1}^n \log(X_i) - \frac{1}{\beta} \sum_{i=1}^n X_i - n \log(\Gamma(\alpha))$$

The score function is the first derivative of the log-likelihood function with respect to β,

$$U(\beta) = \frac{\partial \ell(\beta)}{\partial \beta} = -\frac{n\alpha}{\beta} + \frac{1}{\beta^2} \sum_{i=1}^n X_i$$

Under the null hypothesis $H_0 : \beta = \beta_0$, the score function is,

$$U(\beta_0) = -\frac{n\alpha}{\beta_0} + \frac{1}{\beta_0^2} \sum_{i=1}^n X_i$$

Next, we need to calculate the Fisher information at β_0. The second derivative of the log-likelihood function with respect to β is,

$$\frac{\partial^2 \ell(\beta)}{\partial \beta^2} = \frac{n\alpha}{\beta^2} - \frac{2}{\beta^3}\sum_{i=1}^{n} X_i$$

The Fisher information $I(\beta)$ is the negative expected value of the second derivative,

$$I(\beta) = -E\left(\frac{\partial^2 \ell(\beta)}{\partial \beta^2}\right)$$

We already have,

$$\frac{\partial^2 \ell(\beta)}{\partial \beta^2} = \frac{n\alpha}{\beta^2} - \frac{2}{\beta^3}\sum_{i=1}^{n} X_i$$

Taking the expectation, we get,

$$E\left(\frac{\partial^2 \ell(\beta)}{\partial \beta^2}\right) = \frac{n\alpha}{\beta^2} - \frac{2}{\beta^3}E\left(\sum_{i=1}^{n} X_i\right) = \frac{n\alpha}{\beta^2} - \frac{2n\alpha\beta}{\beta^3} = -\frac{n\alpha}{\beta^2}$$

Thus, the Fisher information is,

$$I(\beta) = \frac{n\alpha}{\beta^2}$$

Under $H_0 : \beta = \beta_0$, the score statistic is,

$$S = \frac{U(\beta_0)}{\sqrt{I(\beta_0)}} = \frac{-\frac{n\alpha}{\beta_0} + \frac{1}{\beta_0^2}\sum_{i=1}^{n} X_i}{\sqrt{\frac{n\alpha}{\beta_0^2}}}$$

Simplifying, we get,

$$S = \frac{-n\alpha + \frac{1}{\beta_0}\sum_{i=1}^{n} X_i}{\sqrt{n\alpha}} = \frac{\frac{1}{\beta_0}\sum_{i=1}^{n} X_i - n\alpha}{\sqrt{n\alpha}}$$

This is the score statistic for testing $H_0 : \beta = \beta_0$.

Question 4

Let $X_1, \ldots, X_n$ be iid Weibull$(\lambda, 4)$, which has density function

$$f(x|\lambda) = \frac{4}{\lambda}\left(\frac{x}{\lambda}\right)^3 e^{-\left(\frac{x}{\lambda}\right)^4} \text{ for } x > 0$$

Part A

What is the MLE for λ?

Solution

The likelihood function for the Weibull distribution is given,

$$L(\lambda) = \prod_{i=1}^{n} \frac{4}{\lambda}\left(\frac{X_i}{\lambda}\right)^3 e^{-\left(\frac{X_i}{\lambda}\right)^4}$$

Next, the log-likelihood function is found by taking the log,

$$\ell(\lambda) = \sum_{i=1}^{n}\left[\log\left(\frac{4}{\lambda}\right) + 3\log\left(\frac{X_i}{\lambda}\right) - \left(\frac{X_i}{\lambda}\right)^4\right]$$

Simplifying, we now obtain,

$$\ell(\lambda) = \sum_{i=1}^{n}\left[\log(4) - \log(\lambda) + 3\log(X_i) - 3\log(\lambda) - \left(\frac{X_i}{\lambda}\right)^4\right]$$

$$\ell(\lambda) = n\log(4) - n\log(\lambda) + 3\sum_{i=1}^{n}\log(X_i) - 3n\log(\lambda) - \sum_{i=1}^{n}\left(\frac{X_i}{\lambda}\right)^4$$

$$\ell(\lambda) = n\log(4) + 3\sum_{i=1}^{n}\log(X_i) - 4n\log(\lambda) - \sum_{i=1}^{n}\left(\frac{X_i}{\lambda}\right)^4$$

Now, we solve for the MLE,

$$\frac{\partial \ell(\lambda)}{\partial \lambda} = -\frac{4n}{\lambda} + 4\sum_{i=1}^{n}\frac{X_i^4}{\lambda^5} = 0$$

Solving for λ, we get,

$$\frac{4n}{\lambda} = 4\sum_{i=1}^{n}\frac{X_i^4}{\lambda^5}$$

$$\lambda^5 = \frac{\sum_{i=1}^n X_i^4}{n}$$

$$\hat{\lambda} = \left(\frac{\sum_{i=1}^n X_i^4}{n} \right)^{1/5}$$

So the MLE of λ is $\hat{\lambda} = \left(\frac{\sum_{i=1}^n X_i^4}{n} \right)^{1/5}$.

Part B

What is the information $I_1(\lambda)$?

Solution

The Fisher information $I(\lambda)$ can be calculated using the negative expectation of the second derivative from the likelihood function,

$$I(\lambda) = -E\left(\frac{\partial^2 \ell(\lambda)}{\partial \lambda^2}\right)$$

Since we already calculated the first derivative,

$$\frac{\partial \ell(\lambda)}{\partial \lambda} = -\frac{4n}{\lambda} + 4\sum_{i=1}^{n} \frac{X_i^4}{\lambda^5}$$

We can solve for the second derivative, and get

$$\frac{\partial^2 \ell(\lambda)}{\partial \lambda^2} = \frac{4n}{\lambda^2} - 20\sum_{i=1}^{n} \frac{X_i^4}{\lambda^6}$$

Taking the expectation,

$$E\left(\frac{\partial^2 \ell(\lambda)}{\partial \lambda^2}\right) = \frac{4n}{\lambda^2} - 20\sum_{i=1}^{n} E\left(\frac{X_i^4}{\lambda^6}\right)$$

We know that for a Weibull$(\lambda, 4)$ distribution,

$$E\left(X_i^4\right) = \lambda^4 \Gamma\left(1 + \frac{4}{4}\right) = \lambda^4 \Gamma(2) = \lambda^4 \cdot 1 = \lambda^4$$

Thus,

$$E\left(\frac{X_i^4}{\lambda^6}\right) = \frac{\lambda^4}{\lambda^6} = \frac{1}{\lambda^2}$$

So,

$$E\left(\frac{\partial^2 \ell(\lambda)}{\partial \lambda^2}\right) = \frac{4n}{\lambda^2} - 20 \cdot \frac{n}{\lambda^2} = \frac{4n}{\lambda^2} - \frac{20n}{\lambda^2} = -\frac{16n}{\lambda^2}$$

Therefore, the Fisher information is:

$$I(\lambda) = \frac{16n}{\lambda^2}$$

Part C

Find the Likelihood Ratio test statistic for testing $H_0 : \lambda = 1$ vs. $H_1 : \lambda \neq 1$.

Solution

The likelihood ratio test statistic is given by,

$$\lambda_{LR} = -2\left(\ell(\lambda_0) - \ell(\hat{\lambda})\right)$$

Under $H_0 : \lambda = 1$, the log-likelihood function is,

$$\ell(1) = n\log(4) + 3\sum_{i=1}^{n}\log(X_i) - 4n\log(1) - \sum_{i=1}^{n}X_i^4$$

$$\ell(1) = n\log(4) + 3\sum_{i=1}^{n}\log(X_i) - \sum_{i=1}^{n}X_i^4$$

Under H_1, the log-likelihood function at $\hat{\lambda}$ is,

$$\ell(\hat{\lambda}) = n\log(4) + 3\sum_{i=1}^{n}\log(X_i) - 4n\log(\hat{\lambda}) - \sum_{i=1}^{n}\left(\frac{X_i}{\hat{\lambda}}\right)^4$$

$$\ell(\hat{\lambda}) = n\log(4) + 3\sum_{i=1}^{n}\log(X_i) - 4n\log\left(\left(\frac{\sum_{i=1}^{n}X_i^4}{n}\right)^{1/5}\right) - \sum_{i=1}^{n}\left(\frac{X_i^4}{\left(\frac{\sum_{i=1}^{n}X_i^4}{n}\right)^{4/5}}\right)$$

$$\ell(\hat{\lambda}) = n\log(4) + 3\sum_{i=1}^{n}\log(X_i) - \frac{4n}{5}\log\left(\sum_{i=1}^{n}X_i^4\right) + \frac{4n}{5}\log(n) - n$$

Hence, the likelihood ratio test statistic is,

$$\lambda_{LR} = -2\left(\ell(1) - \ell(\hat{\lambda})\right)$$

$$\lambda_{LR} = -2\left[n\log(4) + 3\sum_{i=1}^{n}\log(X_i) - \sum_{i=1}^{n}X_i^4 - \left(n\log(4) + 3\sum_{i=1}^{n}\log(X_i) - \frac{4n}{5}\log\left(\sum_{i=1}^{n}X_i^4\right) + \frac{4n}{5}\log(n) - n\right)\right]$$

$$\lambda_{LR} = -2\left[-\sum_{i=1}^{n}X_i^4 + \frac{4n}{5}\log\left(\sum_{i=1}^{n}X_i^4\right) - \frac{4n}{5}\log(n) + n\right]$$

$$\lambda_{LR} = 2\sum_{i=1}^{n}X_i^4 - \frac{8n}{5}\log\left(\frac{\sum_{i=1}^{n}X_i^4}{n}\right) - 2n$$

Part D

Find the Score test statistic for testing $H_0 : \lambda = 1$ vs. $H_1 : \lambda \neq 1$.

Solution

The score function is the first derivative of the log-likelihood function with respect to λ,

$$U(\lambda) = \frac{\partial \ell(\lambda)}{\partial \lambda} = -\frac{4n}{\lambda} + 4\sum_{i=1}^{n} \frac{X_i^4}{\lambda^5}$$

Under the null hypothesis $H_0 : \lambda = 1$, the score function is,

$$U(1) = -4n + 4\sum_{i=1}^{n} X_i^4$$

The Fisher information at $\lambda = 1$ is,

$$I(1) = \frac{16n}{1^2} = 16n$$

Thus, the score test statistic is,

$$S = \frac{U(1)}{\sqrt{I(1)}} = \frac{-4n + 4\sum_{i=1}^{n} X_i^4}{4\sqrt{n}} = \frac{\sum_{i=1}^{n} X_i^4 - n}{\sqrt{n}}$$

Part E

Find the Wald test statistic for testing $H_0 : \lambda = 1$ vs. $H_1 : \lambda \neq 1$.

Solution

The Wald test statistic is as follows,

$$W = \frac{\hat{\lambda} - \lambda_0}{\mathrm{SE}(\hat{\lambda})}$$

From part A, we have the MLE,

$$\hat{\lambda} = \left(\frac{\sum_{i=1}^{n} X_i^4}{n} \right)^{1/5}$$

So, the Fisher information is,

$$I(\lambda) = \frac{16n}{\lambda^2}$$

Next, the variance of $\hat{\lambda}$ is just the inverse of the Fisher information,

$$\mathrm{Var}(\hat{\lambda}) = \left(\frac{16n}{\lambda^2} \right)^{-1} = \frac{\lambda^2}{16n}$$

Therefore, the standard error is,

$$\mathrm{SE}(\hat{\lambda}) = \frac{\lambda}{4\sqrt{n}}$$

The Wald test statistic is,

$$W = \frac{\hat{\lambda} - 1}{\frac{\hat{\lambda}}{4\sqrt{n}}} = 4\sqrt{n}(\hat{\lambda} - 1)$$

Question 5

Let $X_1, \dots, X_n$ be iid Poisson(λ_X) and let $Y_1, \dots, Y_m$ be iid Poisson(λ_Y), with the two samples independent of each other.

Part A

Find the unconstrained MLE of (λ_X, λ_Y).

Solution

The pmf of a Poisson(λ) random variable is,

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

Given $X_1, X_2, \dots, X_n$ are iid Poisson(λ_X), the likelihood function becomes,

$$L(\lambda_X) = \prod_{i=1}^{n} \frac{\lambda_X^{X_i} e^{-\lambda_X}}{X_i!}$$

Next, the log-likelihood function is calculated as follows,

$$\ell(\lambda_X) = \sum_{i=1}^{n} (X_i \log(\lambda_X) - \lambda_X - \log(X_i!))$$

Simplifying, we obtain,

$$\ell(\lambda_X) = \sum_{i=1}^{n} X_i \log(\lambda_X) - n\lambda_X - \sum_{i=1}^{n} \log(X_i!)$$

Next, the MLE of λ_X is found by,

$$\frac{\partial \ell(\lambda_X)}{\partial \lambda_X} = \sum_{i=1}^{n} \frac{X_i}{\lambda_X} - n = 0$$

Solving for λ_X, we get:

$$\hat{\lambda}_X = \frac{1}{n} \sum_{i=1}^{n} X_i = \bar{X}$$

Similarly, for $Y_1, Y_2, \ldots, Y_m$ which are iid Poisson(λ_Y), we get:

$$\hat{\lambda}_Y = \frac{1}{m}\sum_{j=1}^{m} Y_j = \bar{Y}$$

Therefore, the unconstrained MLE of (λ_X, λ_Y) is $(\hat{\lambda}_X, \hat{\lambda}_Y) = (\bar{X}, \bar{Y})$.

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Part B

Find the constrained MLE of (λ_X, λ_Y) subject to the constraint $\lambda_X = \lambda_Y$.

Solution

Under the constraint $\lambda_X = \lambda_Y = \lambda$, the combined likelihood function is,

$$L(\lambda) = \prod_{i=1}^{n} \frac{\lambda^{X_i} e^{-\lambda}}{X_i!} \prod_{j=1}^{m} \frac{\lambda^{Y_j} e^{-\lambda}}{Y_j!}$$

Then, the log-likelihood function is found by,

$$\ell(\lambda) = \sum_{i=1}^{n} (X_i \log(\lambda) - \lambda - \log(X_i!)) + \sum_{j=1}^{m} (Y_j \log(\lambda) - \lambda - \log(Y_j!))$$

$$\ell(\lambda) = \left(\sum_{i=1}^{n} X_i + \sum_{j=1}^{m} Y_j \right) \log(\lambda) - (n+m)\lambda - \left(\sum_{i=1}^{n} \log(X_i!) + \sum_{j=1}^{m} \log(Y_j!) \right)$$

We solve for the MLE of λ,

$$\frac{\partial \ell(\lambda)}{\partial \lambda} = \frac{\sum_{i=1}^{n} X_i + \sum_{j=1}^{m} Y_j}{\lambda} - (n+m) = 0$$

$$\hat{\lambda} = \frac{1}{n+m} \left(\sum_{i=1}^{n} X_i + \sum_{j=1}^{m} Y_j \right)$$

$$\hat{\lambda} = \frac{n\bar{X} + m\bar{Y}}{n+m}$$

Hence, the constrained MLE of (λ_X, λ_Y) is $(\hat{\lambda}, \hat{\lambda}) = \left(\frac{n\bar{X}+m\bar{Y}}{n+m}, \frac{n\bar{X}+m\bar{Y}}{n+m} \right)$.

Part C

Find the Likelihood Ratio test statistic to test $H_0 : \lambda_X = \lambda_Y$ vs. $H_1 : \lambda_X \neq \lambda_Y$.

Solution

The likelihood ratio test statistic is given by,

$$\lambda_{LR} = -2\left(\ell(\lambda_0) - \ell(\hat{\lambda})\right)$$

The log-likelihood under $H_0 : \lambda_X = \lambda_Y = \lambda$ is,

$$\ell(\lambda) = \left(\sum_{i=1}^{n} X_i + \sum_{j=1}^{m} Y_j\right)\log(\lambda) - (n+m)\lambda - \left(\sum_{i=1}^{n}\log(X_i!) + \sum_{j=1}^{m}\log(Y_j!)\right)$$

$$\ell(\lambda) = \left(\sum_{i=1}^{n} X_i + \sum_{j=1}^{m} Y_j\right)\log\left(\frac{n\bar{X}+m\bar{Y}}{n+m}\right) - (n+m)\left(\frac{n\bar{X}+m\bar{Y}}{n+m}\right) - \left(\sum_{i=1}^{n}\log(X_i!) + \sum_{j=1}^{m}\log(Y_j!)\right)$$

$$\ell(\lambda) = (n\bar{X}+m\bar{Y})\log\left(\frac{n\bar{X}+m\bar{Y}}{n+m}\right) - (n\bar{X}+m\bar{Y}) - \left(\sum_{i=1}^{n}\log(X_i!) + \sum_{j=1}^{m}\log(Y_j!)\right)$$

The log-likelihood under the alternative hypothesis is,

$$\ell(\hat{\lambda}_X, \hat{\lambda}_Y) = \sum_{i=1}^{n}(X_i\log(\hat{\lambda}_X) - \hat{\lambda}_X - \log(X_i!)) + \sum_{j=1}^{m}(Y_j\log(\hat{\lambda}_Y) - \hat{\lambda}_Y - \log(Y_j!))$$

$$\ell(\hat{\lambda}_X, \hat{\lambda}_Y) = \sum_{i=1}^{n}(X_i\log(\bar{X}) - \bar{X} - \log(X_i!)) + \sum_{j=1}^{m}(Y_j\log(\bar{Y}) - \bar{Y} - \log(Y_j!))$$

$$\ell(\hat{\lambda}_X, \hat{\lambda}_Y) = \sum_{i=1}^{n} X_i\log(\bar{X}) - n\bar{X} + \sum_{j=1}^{m} Y_j\log(\bar{Y}) - m\bar{Y} - \left(\sum_{i=1}^{n}\log(X_i!) + \sum_{j=1}^{m}\log(Y_j!)\right)$$

$$\ell(\hat{\lambda}_X, \hat{\lambda}_Y) = n\bar{X}\log(\bar{X}) - n\bar{X} + m\bar{Y}\log(\bar{Y}) - m\bar{Y} - \left(\sum_{i=1}^{n}\log(X_i!) + \sum_{j=1}^{m}\log(Y_j!)\right)$$

Therefore, the likelihood ratio test statistic is,

$$\lambda_{LR} = -2\left(\ell(\lambda) - \ell(\hat{\lambda}_X, \hat{\lambda}_Y)\right)$$

$$\lambda_{LR} = -2\left[(n\bar{X}+m\bar{Y})\log\left(\frac{n\bar{X}+m\bar{Y}}{n+m}\right) - (n\bar{X}+m\bar{Y}) - \left(n\bar{X}\log(\bar{X}) - n\bar{X} + m\bar{Y}\log(\bar{Y}) - m\bar{Y}\right)\right]$$

$$\lambda_{LR} = 2\left[n\bar{X}\log(\bar{X}) + m\bar{Y}\log(\bar{Y}) - (n\bar{X} + m\bar{Y})\log\left(\frac{n\bar{X} + m\bar{Y}}{n+m}\right)\right]$$

This is the likelihood ratio test statistic for testing $H_0 : \lambda_X = \lambda_Y$ vs. $H_1 : \lambda_X \neq \lambda_Y$.

Question 6

Let $X_1, \dots, X_n \sim_{iid}$ Exponential(θ_X), and $Y_1, \dots, Y_n \sim_{iid}$ Exponential(θ_Y) be independent samples. The density function of an Exponential(θ) random variable is:

$$f(x|\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}} \text{ for } x > 0$$

Part A

Derive the unconstrained MLEs of θ_X and θ_Y.

Solution

The likelihood function for $X_1, \dots, X_n \sim$ Exponential(θ_X) is expressed as,

$$L(\theta_X) = \prod_{i=1}^{n} \frac{1}{\theta_X} e^{-\frac{X_i}{\theta_X}} = \frac{1}{\theta_X^n} e^{-\frac{1}{\theta_X} \sum_{i=1}^{n} X_i}$$

Now, the log-likelihood is formulated as,

$$\ell(\theta_X) = -n \log(\theta_X) - \frac{1}{\theta_X} \sum_{i=1}^{n} X_i$$

We solve for the MLE,

$$\frac{\partial \ell(\theta_X)}{\partial \theta_X} = -\frac{n}{\theta_X} + \frac{1}{\theta_X^2} \sum_{i=1}^{n} X_i = 0$$

$$\hat{\theta}_X = \frac{1}{n} \sum_{i=1}^{n} X_i = \bar{X}$$

Similarly, for $Y_1, \dots, Y_n \sim$ Exponential(θ_Y), we can get,

$$\hat{\theta}_Y = \frac{1}{n} \sum_{j=1}^{n} Y_j = \bar{Y}$$

So, the unconstrained MLEs of θ_X and θ_Y are $\hat{\theta}_X = \bar{X}$ and $\hat{\theta}_Y = \bar{Y}$.

Part B

Derive the constrained MLE of θ_X and θ_Y under the constraint $\theta_X = \theta_Y$.

Solution

Under the constraint $\theta_X = \theta_Y = \theta$, the combined likelihood function is found by,

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{X_i}{\theta}} \prod_{j=1}^{n} \frac{1}{\theta} e^{-\frac{Y_j}{\theta}}$$

To continue, the log-likelihood is,

$$\ell(\theta) = -2n \log(\theta) - \frac{1}{\theta} \left(\sum_{i=1}^{n} X_i + \sum_{j=1}^{n} Y_j \right)$$

Then, the MLE is expressed as follows,

$$\frac{\partial \ell(\theta)}{\partial \theta} = -\frac{2n}{\theta} + \frac{1}{\theta^2} \left(\sum_{i=1}^{n} X_i + \sum_{j=1}^{n} Y_j \right) = 0$$

$$\hat{\theta} = \frac{1}{2n} \left(\sum_{i=1}^{n} X_i + \sum_{j=1}^{n} Y_j \right)$$

Hence, the constrained MLE of θ_X and θ_Y under the constraint $\theta_X = \theta_Y$ is:

$$\hat{\theta} = \frac{\sum_{i=1}^{n} X_i + \sum_{j=1}^{n} Y_j}{2n}$$

Part C

Derive an exact level α test for $H_0 : \theta_X = \theta_Y$ vs. $H_1 : \theta_X \neq \theta_Y$. (*Hint: find an implementable form for the Likelihood Ratio test.*)

Solution

Let's use our favorite likelihood ratio test statistic for the 1000th time!!! Here it is,

$$\lambda_{LR} = -2\left(\ell(\theta_0) - \ell(\hat{\theta})\right)$$

The log-likelihood under $H_0 : \theta_X = \theta_Y = \theta$ is,

$$\ell(\theta) = -2n\log(\theta) - \frac{1}{\theta}\left(\sum_{i=1}^{n} X_i + \sum_{j=1}^{n} Y_j\right)$$

$$\ell(\theta) = -2n\log\left(\frac{\sum_{i=1}^{n} X_i + \sum_{j=1}^{n} Y_j}{2n}\right) - 2n$$

The log-likelihood under H_1 is,

$$\ell(\hat{\theta}_X, \hat{\theta}_Y) = -n\log(\hat{\theta}_X) - n\log(\hat{\theta}_Y) - n$$

$$\ell(\hat{\theta}_X, \hat{\theta}_Y) = -n\log(\bar{X}) - n\log(\bar{Y}) - n$$

Therefore, the likelihood ratio test statistic is,

$$\lambda_{LR} = -2\left(\ell(\theta) - \ell(\hat{\theta}_X, \hat{\theta}_Y)\right)$$

$$\lambda_{LR} = -2\left[-2n\log\left(\frac{\sum_{i=1}^{n} X_i + \sum_{j=1}^{n} Y_j}{2n}\right) - 2n - \left(-n\log(\bar{X}) - n\log(\bar{Y}) - n\right)\right]$$

$$\lambda_{LR} = 2\left[n\log(\bar{X}) + n\log(\bar{Y}) - 2n\log\left(\frac{\sum_{i=1}^{n} X_i + \sum_{j=1}^{n} Y_j}{2n}\right)\right]$$

Therefore, we reject H_0 if $\lambda_{LR} > \chi^2_{1,\alpha}$, where $\chi^2_{1,\alpha}$ is the critical value from the chi-squared distribution with 1 degree of freedom at level α.

Part D

Compute the asymptotic variance-covariance matrix of the unconstrained MLE vector $\theta = (\hat{\theta}_X, \hat{\theta}_Y)$.

Solution

For the Exponential(θ) distribution, the Fisher information is,

$$I(\theta) = \frac{n}{\theta^2}$$

The variance of $\hat{\theta}$ is,

$$\mathrm{Var}(\hat{\theta}) = \frac{\theta^2}{n}$$

For $\theta = (\hat{\theta}_X, \hat{\theta}_Y)$, the asymptotic variance-covariance matrix is diagonal,

$$\mathbf{I}^{-1} = \begin{pmatrix} \mathrm{Var}(\hat{\theta}_X) & 0 \\ 0 & \mathrm{Var}(\hat{\theta}_Y) \end{pmatrix} = \begin{pmatrix} \frac{\theta_X^2}{n} & 0 \\ 0 & \frac{\theta_Y^2}{n} \end{pmatrix}$$

Hence, by the MLEs, we can get

$$\mathbf{I}^{-1} = \begin{pmatrix} \frac{\bar{X}^2}{n} & 0 \\ 0 & \frac{\bar{Y}^2}{n} \end{pmatrix}$$

Part E

Derive a large-sample Wald test for $H_0 : \theta_X - 2\theta_Y = 0$ vs. $H_1 : \theta_X - 2\theta_Y \neq 0$. (*Hint: Use the Mann-Wald Theorem/Delta Method to find the distribution of* $g(\theta) = \theta_X - 2\theta_Y$ *under the null hypothesis*).

Solution

Let $g(\theta) = \theta_X - 2\theta_Y$. Under H_0, $g(\theta) = 0$.

The variance of $g(\theta)$ is,

$$\text{Var}(g(\theta)) = \nabla g(\theta)^T \mathbf{I}^{-1} \nabla g(\theta)$$

Where

$$\nabla g(\theta) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

.

Thus,

$$\text{Var}(g(\theta)) = \begin{pmatrix} 1 & -2 \end{pmatrix} \begin{pmatrix} \frac{\bar{X}^2}{n} & 0 \\ 0 & \frac{\bar{Y}^2}{n} \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\text{Var}(g(\theta)) = \frac{\bar{X}^2}{n} \cdot 1^2 + \frac{\bar{Y}^2}{n} \cdot (-2)^2 = \frac{\bar{X}^2}{n} + \frac{4\bar{Y}^2}{n}$$

The standard error is calculated as,

$$\text{SE}(g(\theta)) = \sqrt{\frac{\bar{X}^2}{n} + \frac{4\bar{Y}^2}{n}}$$

Next, the Wald test statistic is formulated as,

$$W = \frac{g(\hat{\theta}) - 0}{\text{SE}(g(\hat{\theta}))} = \frac{\hat{\theta}_X - 2\hat{\theta}_Y}{\sqrt{\frac{\bar{X}^2}{n} + \frac{4\bar{Y}^2}{n}}}$$

Under H_0, $W \sim N(0, 1)$. Therefore, this Wald test statistic can be used for testing $H_0 : \theta_X - 2\theta_Y = 0$ vs. $H_1 : \theta_X - 2\theta_Y \neq 0$.

Question 7

Let $X_1, \dots, X_n \sim_{iid} \mathcal{N}(\mu_X, \sigma_X^2)$, and $Y_1, \dots, Y_m \sim_{iid} \mathcal{N}(\mu_Y, \sigma_Y^2)$ be independent samples.

Part A

Find the unconstrained MLE of $(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2)$.

Solution

For $X_1, \dots, X_n \sim \mathcal{N}(\mu_X, \sigma_X^2)$, the likelihood function is,

$$L(\mu_X, \sigma_X^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left(-\frac{(X_i - \mu_X)^2}{2\sigma_X^2}\right)$$

Take the log and we can get the log-likelihood function,

$$\ell(\mu_X, \sigma_X^2) = -\frac{n}{2}\log(2\pi\sigma_X^2) - \frac{1}{2\sigma_X^2}\sum_{i=1}^{n}(X_i - \mu_X)^2$$

Solve for the MLES,

$$\frac{\partial \ell(\mu_X, \sigma_X^2)}{\partial \mu_X} = \frac{1}{\sigma_X^2}\sum_{i=1}^{n}(X_i - \mu_X) = 0 \quad \Rightarrow \quad \hat{\mu}_X = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$$

$$\frac{\partial \ell(\mu_X, \sigma_X^2)}{\partial \sigma_X^2} = -\frac{n}{2\sigma_X^2} + \frac{1}{2(\sigma_X^2)^2}\sum_{i=1}^{n}(X_i - \mu_X)^2 = 0 \quad \Rightarrow \quad \hat{\sigma}_X^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

Use similar steps to solve for $Y_1, \dots, Y_m \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$. After some algebra we get,

$$\hat{\mu}_Y = \frac{1}{m}\sum_{j=1}^{m} Y_j = \bar{Y}$$

$$\hat{\sigma}_Y^2 = \frac{1}{m}\sum_{j=1}^{m}(Y_j - \bar{Y})^2$$

Hence, the unconstrained MLEs of $(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2)$ are $(\hat{\mu}_X, \hat{\mu}_Y, \hat{\sigma}_X^2, \hat{\sigma}_Y^2) = (\bar{X}, \bar{Y}, \hat{\sigma}_X^2, \hat{\sigma}_Y^2)$.

Part B

Find the constrained MLE of $(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2)$ under the constraint $\mu_X = \mu_Y$.

Solution

Under the constraint $\mu_X = \mu_Y = \mu$, the combined likelihood function is,

$$L(\mu, \sigma_X^2, \sigma_Y^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left(-\frac{(X_i - \mu)^2}{2\sigma_X^2}\right) \prod_{j=1}^{m} \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left(-\frac{(Y_j - \mu)^2}{2\sigma_Y^2}\right)$$

Solving for the log-likelihood function,

$$\ell(\mu, \sigma_X^2, \sigma_Y^2) = -\frac{n}{2}\log(2\pi\sigma_X^2) - \frac{1}{2\sigma_X^2}\sum_{i=1}^{n}(X_i - \mu)^2 - \frac{m}{2}\log(2\pi\sigma_Y^2) - \frac{1}{2\sigma_Y^2}\sum_{j=1}^{m}(Y_j - \mu)^2$$

Find the MLE,

$$\frac{\partial \ell(\mu, \sigma_X^2, \sigma_Y^2)}{\partial \mu} = \frac{1}{\sigma_X^2}\sum_{i=1}^{n}(X_i - \mu) + \frac{1}{\sigma_Y^2}\sum_{j=1}^{m}(Y_j - \mu) = 0$$

Solving for μ, we can get,

$$\hat{\mu} = \frac{\frac{1}{\sigma_X^2}\sum_{i=1}^{n} X_i + \frac{1}{\sigma_Y^2}\sum_{j=1}^{m} Y_j}{\frac{n}{\sigma_X^2} + \frac{m}{\sigma_Y^2}}$$

Since we don't know σ_X^2 and σ_Y^2 yet, we can use their MLEs from the unconstrained case,

$$\hat{\mu} = \frac{\frac{1}{\hat{\sigma}_X^2}\sum_{i=1}^{n} X_i + \frac{1}{\hat{\sigma}_Y^2}\sum_{j=1}^{m} Y_j}{\frac{n}{\hat{\sigma}_X^2} + \frac{m}{\hat{\sigma}_Y^2}}$$

Now, substituting this $\hat{\mu}$ back, we re-estimate σ_X^2 and σ_Y^2,

$$\hat{\sigma}_X^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{\mu})^2$$

$$\hat{\sigma}_Y^2 = \frac{1}{m}\sum_{j=1}^{m}(Y_j - \hat{\mu})^2$$

Part C

Find the score function for (μ_X, σ_X^2).

Solution

The score function is the first derivative of the log-likelihood function with respect to the parameter of interest. For (μ_X, σ_X^2), the log-likelihood function is,

$$\ell(\mu_X, \sigma_X^2) = -\frac{n}{2}\log(2\pi\sigma_X^2) - \frac{1}{2\sigma_X^2}\sum_{i=1}^{n}(X_i - \mu_X)^2$$

The score functions are calculated as follows,

$$U_{\mu_X} = \frac{\partial \ell(\mu_X, \sigma_X^2)}{\partial \mu_X} = \frac{1}{\sigma_X^2}\sum_{i=1}^{n}(X_i - \mu_X)$$

$$U_{\sigma_X^2} = \frac{\partial \ell(\mu_X, \sigma_X^2)}{\partial \sigma_X^2} = -\frac{n}{2\sigma_X^2} + \frac{1}{2(\sigma_X^2)^2}\sum_{i=1}^{n}(X_i - \mu_X)^2$$

Therefore, the score functions for (μ_X, σ_X^2) are,

$$U_{\mu_X} = \frac{1}{\sigma_X^2}\sum_{i=1}^{n}(X_i - \mu_X)$$

$$U_{\sigma_X^2} = -\frac{n}{2\sigma_X^2} + \frac{1}{2(\sigma_X^2)^2}\sum_{i=1}^{n}(X_i - \mu_X)^2$$