Homework 1

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ST 543 Applied Stochastic Models

ST 563: Theory of Statistics III

Question 1

Let X be a *single* observation from the Geometric(p) distribution, where $X \in \{0, 1, 2, 3, ...\}$ is the number of Bernoulli(p) trials needed *before* seeing one success. The pmf for the Geometric(p) distribution is

$$p(x) = P(X = x) = (1 - p)^x p \text{ for } x = 0, 1, 2, 3, ...$$

We wish to test the null hypothesis $H_0 : p = 0.1$ vs. the alternative $H_1 : p = 0.5$.

(a) (2 points) What values of X are more likely under the null hypothesis $H_0 : p = 0.1$ than under the alternative hypothesis $H_1 : p = 0.5$?

$$H_0 : P(X = x) = (1 - 0.1)^x(0.1) = (0.9)^x(0.1)$$

$$H_1 : P(X = x) = (1 - 0.5)^x(0.5) = (0.5)^x(0.5)$$

$$H_0 > H_A = (0.9)^x(0.1) > (0.5)^x(0.5)$$

$$(0.9)^x > (0.5)^x \cdot 5 = xlog(0.9) > xlog(0.5) + log(5)$$

$$xlog(0.9) - xlog(0.5) > log(5) = x(log(\frac{0.9}{0.5})) > log(5)$$

$$x > \frac{log(5)}{log(\frac{0.9}{0.5})} \Rightarrow x > 2.738$$

Essentially, $X = x \geq 3$ given $X \in \mathbb{Z}$

(b) (2 points) Suppose we have the critical function, what is the Type I error probability for this test?

$$\psi(X) = \begin{cases} 1 & \text{for } X = 0 \\ 0 & \text{for } X > 0 \end{cases}$$

$$P_{p=0.1}(X = 0) = (0.9)^0(0.1)$$

$$\text{Type I error} = 10\%$$

(c) (3 points) How could we modify the above test to obtain a size $\alpha = 0.05$ test?

$$P_{p=0.1}(X \leq c) \leq 0.05 \Rightarrow \sum_{i=1}^{c} (0.9)^i (0.1) \leq 0.05$$

(d) (3 points) What is the Type II error probability for the test you suggested in the previous part?

Question 2

(8.13 from *Statistical Inference, 2nd Edition*) Let X_1, X_2 be iid Uniform$(\theta, \theta + 1)$. For testing $H_0 : \theta = 0$ versus $H_1 : \theta > 0$, we have two competing tests:

- Test 1: $\phi_1(X_1)$: Reject H_0 if $X_1 > 0.95$.
- Test 2: $\phi_2(X_1, X_2)$: Reject H_0 if $X_1 + X_2 > C$.

(a) Find the value of C so that ϕ_2 has the same size as ϕ_1.

(b) Calculate the power function of each test. Draw a well-labeled graph of each power function.

(c) Prove or disprove: ϕ_2 is a more powerful test than ϕ_1.

(d) Show how to get a test that has the same size but is more powerful than ϕ_2.

Question 3

(8.14 from *Statistical Inference, 2nd Edition*) For a random sample $X_1, \dots, X_n$ of Bernoulli(p) variables, it is desired to test

$$H_0 : p = 0.49 \qquad \text{versus} \qquad H_1 : p = 0.51$$

Use the Central Limit Theorem to determine, approximately, the sample size needed so that the two probabilities of error are both about 0.01. Use a test function that rejects H_0 if $\sum_{i=1}^{n} X_i$ is large.

Question 4

(8.15 from *Statistical Inference, 2nd Edition*) Show that for a random sample $X_1, \ldots, X_n$ from a Normal0, σ^2 population distribution, the most powerful test of $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma = \sigma_1$, where $\sigma_0 < \sigma_1$, is given by

$$\phi\left(\sum_{i=1}^{n} X_i^2\right) = \begin{cases} 1 & \text{if} \sum_{i=1}^{n} X_i^2 > c \\ 0 & \text{if} \sum_{i=1}^{n} X_i^2 \leq c \end{cases}$$

For a given value of α, the size of the Type I Error, show how the value of c is explicitly determined.

Question 5

(8.20 from *Statistical Inference, 2nd Edition*) Let X be a random variable whose pmf under H_0 and H_1 is given by

x	1	2	3	4	5	6	7
$f(x\|H_0)$	0.01	0.01	0.01	0.01	0.01	0.01	0.94
$f(x\|H_1)$	0.06	0.05	0.04	0.03	0.02	0.01	0.79

Use the Neyman-Pearson Lemma to find the most powerful test for H_0 versus H_1 with size $\alpha = 0.04$. Compute the probability of Type II Error for this test.

Question 6

(8.23 from *Statistical Inference, 2nd Edition*) Suppose X is one observation from a population with Beta$(\theta, 1)$ pdf.

(a) For testing $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$, find the size and sketch the power function of the test that rejects H_0

if $X > \frac{1}{2}$.

(b) Find the most powerful level α test of $H_0 : \theta = 1$ versus $H_1 : \theta = 2$.

Question 7

Make up your own question based on this week's material (hypothesis testing, Neyman-Pearson Lemma, Uniformly Most Powerful (UMP) tests), and provide a solution.