Probability Theory

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Chapter 1.1 Exercises

1.1.1. Let $\Omega = \mathbb{R}$, $\mathcal{F}$ be all subsets so that A or A^c is countable, $P(A) = 0$ in the first case and $= 1$ in the second. Show that $(\Omega, \mathcal{F}, P)$ is a probability space.

1.1.2. Recall the definition of S_d from Example 1.1.5. Show that $\sigma(S_d) = \mathcal{R}^d$, the Borel subsets of $\mathbb{R}^d$.

1.1.3. A σ-field $\mathcal{F}$ is said to be *countably generated* if there is a countable collection $C \subset \mathcal{F}$ so that $\sigma(C) = \mathcal{F}$. Show that $\mathbb{R}^d$ is countably generated.

1.1.4.

(i) Show that if $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots$ are σ-algebras, then $\cup_i \mathcal{F}_i$ is an algebra.

(ii) Give an example to show that $\cup_i \mathcal{F}_i$ need not be a σ-algebra.

1.1.5. A set $A \subset \{1, 2, \dots\}$ is said to have *asymptotic density* θ if

$$\lim_{n \to \infty} \frac{|A \cap \{1, 2, \dots, n\}|}{n} = \theta.$$

Let $\mathcal{A}$ be the collection of sets for which the asymptotic density exists. Is $\mathcal{A}$ a σ-algebra? An algebra?

Chapter 1.2 Exercises

1.2.1. Suppose X and Y are random variables on $(\Omega, \mathcal{F}, P)$ and let $A \in \mathcal{F}$. Show that if we let $Z(\omega) = X(\omega)$ for $\omega \in A$ and $Z(\omega) = Y(\omega)$ for $\omega \in A^c$, then Z is a random variable.

1.2.2. Let χ have the standard normal distribution. Use Theorem 1.2.6 to get upper and lower bounds on $P(\chi \geq 4)$.

1.2.3. Show that a distribution function has at most countably many discontinuities.

1.2.4. Show that if $F(x) = P(X \leq x)$ is continuous, then $Y = F(X)$ has a uniform distribution on $(0,1)$, that is, if $y \in [0,1]$, $P(Y \leq y) = y$.

1.2.5. Suppose X has continuous density f, $P(\alpha \leq X \leq \beta) = 1$ and g is a function that is strictly increasing and differentiable on (α, β). Then $g(X)$ has density $\frac{f(g^{-1}(y))}{g'(g^{-1}(y))}$ for $y \in (g(\alpha), g(\beta))$ and 0 otherwise. When $g(x) = ax + b$ with $a > 0$, $g^{-1}(y) = \frac{y-b}{a}$, so the answer is $\frac{1}{a} f\left(\frac{y-b}{a}\right)$.

1.2.6. Suppose X has a normal distribution. Use the previous exercise to compute the density of $\exp(X)$. (The answer is called the *lognormal distribution*.)

1.2.7.

(i) Suppose X has a density function f. Compute the distribution function of X^2 and then differentiate to find its density function.

(ii) Work out the answer when X has a standard normal distribution to find the density of the *chi-square distribution*.

Chapter 1.3 Exercises

1.3.1. Show that if $\mathcal{A}$ generates $\mathcal{S}$, then $X^{-1}(\mathcal{A}) \equiv \{\{X \in A\} : A \in \mathcal{A}\}$ generates $\sigma(X) = \{\{X \in B\} : B \in \mathcal{S}\}$.

1.3.2. Prove Theorem 1.3.6 when $n = 2$ by checking $\{X_1 + X_2 \le x\} \in \mathcal{F}$.

1.3.3. Show that if f is continuous and $X_n \to X$ almost surely then $f(X_n) \to f(X)$ almost surely.

1.3.4.

(i) Show that a continuous function from $\mathbb{R}^d \to \mathbb{R}$ is a measurable map from $(\mathbb{R}^d, \mathcal{R}^d)$ to $(\mathbb{R}, \mathcal{R})$.

(ii) Show that $\mathcal{R}^d$ is the smallest σ-field that makes all the continuous functions measurable.

1.3.5. A function f is said to be *lower semicontinuous* or l.s.c. if

$$\liminf_{y \to x} f(y) \ge f(x)$$

and *upper semicontinuous* (u.s.c.) if $-f$ is l.s.c. Show that f is l.s.c. if and only if $\{x : f(x) \le a\}$ is closed for each $a \in \mathbb{R}$ and conclude that semicontinuous functions are measurable.

1.3.6. Let $f : \mathbb{R}^d \to \mathbb{R}$ be an arbitrary function and let $f^\delta(x) = \sup\{f(y) : |y - x| < \delta\}$ and $f_\delta(x) = \inf\{f(y) : |y - x| < \delta\}$ where $|z| = (z_1^2 + \cdots + z_d^2)^{1/2}$. Show that f^δ is l.s.c. and f_δ is u.s.c. Let $f^0 = \lim_{\delta \to 0} f^\delta$, $f_0 = \lim_{\delta \to 0} f_\delta$, and conclude that the set of points at which f is discontinuous is $\{f^0 \ne f_0\}$ and that $\{f^0 \ne f_0\}$ is measurable.

1.3.7. A function $\varphi : \Omega \to \mathbb{R}$ is said to be simple if

$$\varphi(\omega) = \sum_{m=1}^{n} c_m 1_{A_m}(\omega).$$

Chapter 1.4 Exercises

1.4.1. Show that if $f \geq 0$ and $\int f \, d\mu = 0$ then $f = 0$ a.e.

1.4.2. Let $f \geq 0$ and $E_{n,m} = \{x : \frac{m}{2^n} \leq f(x) < \frac{m+1}{2^n}\}$. As $n \to \infty$,

$$\sum_{m=1}^{\infty} \frac{m}{2^n} \mu(E_{n,m}) \uparrow \int f \, d\mu.$$

1.4.3. Let g be an integrable function on $\mathbb{R}$ and $\epsilon > 0$.

(i) Use the definition of the integral to conclude there is a simple function $\varphi = \sum_k b_k 1_{A_k}$ with $\int |g - \varphi| \, dx < \epsilon$.

(ii) Use Exercise A.2.1 to approximate the A_k by finite unions of intervals to get a step function

$$q = \sum_{j=1}^{k} c_j 1_{(a_{j-1}, a_j)}.$$

with $a_0 < a_1 < \cdots < a_k$, so that $\int |\varphi - q| \, dx < \epsilon$. (iii) Round the corners of q to get a continuous function r so that $\int |q - r| \, dx < \epsilon$.

(iii) To make a continuous function, replace each $c_j 1_{(a_{j-1}, a_j)}$ by a function that is 0 on $(a_{j-1}, a_j)^c$, c_j on $[a_{j-1} + \delta, a_j - \delta]$, and linear otherwise. If the δ_j are small enough and we let $r(x) = \sum_{k=j}^{k} r_j(x)$, then

$$\int |q(x) - r(x)| \, d\mu = \sum_{j=1}^{k} \delta_j c_j < \epsilon.$$

1.4.4. Prove the Riemann-Lebesgue lemma. If g is integrable then

$$\lim_{n \to \infty} \int g(x) \cos(nx) \, dx = 0.$$

Hint: If g is a step function, this is easy. Now use the previous exercise.

Chapter 1.5 Exercises

1.5.1. Let $\|f\|_\infty = \inf\{M : \mu(\{x : |f(x)| > M\}) = 0\}$. Prove that

$$\int |fg| d\mu \le \|f\|_1 \|g\|_\infty.$$

1.5.2. Show that if μ is a probability measure then

$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p.$$

1.5.3. Minkowski's inequality.

(i) Suppose $p \in (1,\infty)$. The inequality $\|f+g\|_p \le 2^p(\|f\|_p^p + \|g\|_p^p)$ shows that if $\|f\|_p$ and $\|g\|_p$ are $< \infty$ then $\|f+g\|_p < \infty$. Apply Hölder's inequality to $|f|^{p-1}$ and $|g||f+g|^{p-1}$ to show $\|f+g\|_p \le \|f\|_p + \|g\|_p$.

(ii) Show that the last result remains true when $p = 1$ or $p = \infty$.

1.5.4. If f is integrable and E_m are disjoint sets with union E then

$$\sum_{m=0}^{\infty} \int_{E_m} f d\mu = \int_E f d\mu.$$

So if $f \ge 0$, then $\nu(E) = \int_E f d\mu$ defines a measure.

1.5.5. If $g_n \uparrow g$ and $\int g_1^- d\mu < \infty$ then $\int g_n d\mu \uparrow \int g d\mu$.

1.5.6. If $g_m \ge 0$ then $\int \sum_{m=0}^\infty g_m d\mu = \sum_{m=0}^\infty \int g_m d\mu$.

1.5.7. Let $f \ge 0$. (i) Show that $\int f \wedge n d\mu \uparrow \int f d\mu$ as $n \to \infty$. (ii) Use (i) to conclude that if g is integrable and $\epsilon > 0$ then we can pick $\delta > 0$ so that $\mu(A) < \delta$ implies $\int_A |g| d\mu < \epsilon$.

1.5.8. Show that if f is integrable on $[a,b]$, $g(x) = \int_{[a,x]} f(y) dy$ is continuous on (a,b).

1.5.9. Show that if f has $\|f\|_p = (\int |f|^p d\mu)^{1/p} < \infty$, then there are simple functions φ_n so that $\|\varphi_n - f\|_p \to 0$.

1.5.10. Show that if $\sum_n \int |f_n| d\mu < \infty$ then $\int \sum_n f_n d\mu = \sum_n \int f_n d\mu$.