Homework 2

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ST 543 Applied Stochastic Models

Question 1

There wasn't much to discuss here. The problem was pretty straight forward. (Update, problem 5 torn me to pieces)

Part A

$$X = \sum_{i=1}^{N} T_i$$

Part B

$$E[N] = \frac{1}{p} = \frac{1}{1/3} = 3$$

Part C

$$E[T_N] = \frac{1}{3}(2) + \frac{1}{3}(3) + \frac{1}{3}(5) = \frac{10}{3}$$

Part D

$$E[\sum_{i=1}^{N} T_i | N = n] = n \cdot E[T_N] = \frac{10}{3}n$$

Part E

$$E[X] = E[N] \cdot E[T_N] = (3)(\frac{10}{3}) = 10$$

Question 2

Part A

The probability of getting no heads in ten flips, calculated as a mixture of the probabilities for each coin.

$$P(N=0) = \frac{1}{3}(1-0.3)^{10} + \frac{1}{3}(1-0.5)^{10} + \frac{1}{3}(1-0.7)^{10} \approx 0.00974$$

Part B

Generalizing $P(N=0)$, where instead of 0 heads, we have n heads in ten flips and the given mixture of different coin probabilities. This results the following equation:

$$P(N=n) = \frac{1}{3}\left(\binom{10}{n}0.3^n(1-0.3)^{10-n} + \binom{10}{n}0.5^n(1-0.5)^{10-n} + \binom{10}{n}0.7^n(1-0.7)^{10-n}\right)$$

Here's all the probabilities for $n = 0, 1, 2, ..., 10$

- $P(N=0) \approx 0.00974$
- $P(N=1) \approx 0.04365$
- $P(N=2) \approx 0.09296$
- $P(N=3) \approx 0.13101$
- $P(N=4) \approx 0.14732$
- $P(N=5) \approx 0.15064$
- $P(N=6) \approx 0.14732$
- $P(N=7) \approx 0.13101$
- $P(N=8) \approx 0.09296$
- $P(N=9) \approx 0.04365$
- $P(N=10) \approx 0.00974$

Part C

N does not strictly follow a binomial distribution since it is derived from a mixture of binomial distributions due to different head probabilities per selected coin.

Part D

$$E[\text{Heads per flip}] = \frac{1}{3}(0.3) + \frac{1}{3}(0.5) + \frac{1}{3}(0.7) = 0.5$$

The expected outcome of the game per flip is 0, which points towards a fair game in the long run. In other words, if you continue to play this game for over n runs, your expected win should be \$0. Now, that's a game theory!

Question 3

Part A

Let Y_i be the amount of money spent by the i^{th} customer, which is uniformly distributed over (0, 100). Then,

$$E[Y_i] = \frac{0 + 100}{2} = 50$$

Given that mean number of customers entering the store is poisson distributed, we have:

$$E[N] = \lambda = 10$$

Now, X is the sum of the individual purchases of N customers:

$$X = Y_1 + Y_2 + \cdots + Y_N$$

Therefore, we can calculate the mean of the amount of money X that the store takes in on a given day by:

$$E[X] = E[N] \cdot E[Y_i] = 10 \cdot 50 = 500$$

Hence, the mean amount of money that the store takes in on a given day is $500.

Part B

To find the variance of the amount of money X that the store takes in on a given day, we do the following algebraic manipulations:

$$\text{Var}(N) = \lambda = 10$$

$$\text{Var}(Y_i) = \frac{(100-0)^2}{12} = \frac{10000}{12} \approx 833.33$$

$$\text{Var}(X) = E[N] \cdot \text{Var}(Y_i) + \text{Var}(N) \cdot (E[Y_i])^2$$

Thus, substituting these into the equation about yields:

$$\text{Var}(X) = 10 \cdot 833.33 + 10 \cdot 50^2 = 8333.3 + 25000 = 33333.3$$

Therefore, the variance of the amount of money X that the store takes in on a given day is approximately \$33,333.30. Scary.

Question 4

Part A

Since $E[Z_k] = 0$, we have:

$$E[X_{k+1}] = E[X_k + Z_k] = E[X_k] + E[Z_k] = E[X_k]$$

Given that $X_0 = x_0$, the expected position after the first step is:

$$E[X_1] = E[X_0] = x_0$$

Applying the same reasoning iteratively for each step:

$$E[X_2] = E[X_1] = x_0$$

$$\vdots$$

$$E[X_n] = x_0$$

Hence, the expected position $E[X_n]$ after n steps remains x_0, the initial position.

Part B

Recall, $\text{Var}(X+Y) = \text{Var}(X)+\text{Var}(Y)$. While X_k and Z_k are not independent, we can still write:

$$\text{Var}(X_{k+1}) = \text{Var}(X_k + Z_k)$$

Given $\text{Var}(Z_k) = \beta X_k^2$, we can apply the law of total variance:

$$\text{Var}(X_{k+1}) = \text{Var}(E[X_{k+1}|X_k]) + E[\text{Var}(X_{k+1}|X_k)] = \text{Var}(X_k) + E[\beta X_k^2]$$

Fortunately, we can use the fact that $E[X_{k+1}|X_k] = X_k$, and simplify above:

$$\text{Var}(X_{k+1}) = \text{Var}(X_k) + \beta E[X_k^2]$$

Next, $E[X_k^2] = \text{Var}(X_k) + E[X_k]^2$ and $E[X_k] = x_0$ so,

$$\text{Var}(X_{k+1}) = \text{Var}(X_k) + \beta(\text{Var}(X_k) + x_0^2)$$

Thus, each step increments the variance by $\beta\text{Var}(X_k) + \beta x_0^2$. Starting from $\text{Var}(X_0) = 0$:

$$\text{Var}(X_1) = 0 + \beta x_0^2$$

$$\text{Var}(X_2) = \text{Var}(X_1) + \beta(x_0^2 + \text{Var}(X_1))$$

By iteratively applying this logic, we formulate the variance at n steps.

Hence, $\text{Var}(X_n)$, after n steps is given by the formula:

$$\text{Var}(X_n) = x_0^2\left((\beta+1)^n - 1\right)$$

Question 5

Part A

Let T denote the number of distinct types collected before collecting type i for the first time, where T takes values in $\{0, 1, \ldots, n-1\}$, with n being the total number of types. Each type is equally likely to appear. The probability $P(T = k)$ is $\frac{1}{n}$ for all k.

Proof $P(T = k)$:

$$P(T = k) = \binom{n-1}{k} \times k! \times \left(\frac{1}{n}\right)^k \times \left(\frac{n-1}{n}\right)^k \times \frac{1}{n}$$

Simplifying this, we find:

$$P(T = k) = \frac{(n-1)!}{(n-1-k)!n^k} \times \left(\frac{n-1}{n}\right)^k \times \frac{1}{n}$$

This results in:

$$P(T = k) = \frac{1}{n}$$

for all k from 0 to $n-1$. This proves that each k is equally probable under the assumption that type i appears after exactly k other types have been collected, establishing $P(T = k) = \frac{1}{n}$.

Define E_i as the event that type i appears. The probability $P(E_i = 1)$ is computed using:

$$P(E_i = 1) = \sum_{k=0}^{n-1} P(E_i = 1 | T = k) P(T = k)$$

where $P(E_i = 1 | T = k)$ is the conditional probability of collecting type i given k other types have been collected, which is $\frac{1}{n-k}$.

Thus, the total probability $P(E_i = 1)$ is:

$$P(E_i = 1) = \sum_{k=0}^{n-1} \frac{1}{n-k} \cdot \frac{1}{n}$$

Therefore, the probability of each type i being equally likely to be the last collected type.

Part B

Define X_j as the indicator random variable for the event that exactly one type j appears in the collection. We want to find the expected number of coupon types that appear exactly once, denoted by $X = \sum_{j=1}^{n} X_j$. The probability that a specific type j appears exactly once in n trials, given by $P(X_j = 1)$, can be calculated by considering that the type appears once and does not appear in the other $n-1$ trials.

The probability that type j appears exactly once can be expressed as:

$$P(X_j = 1) = \binom{n}{1} \left(\frac{1}{n}\right)^1 \left(\frac{n-1}{n}\right)^{n-1}$$

This simplifies to:

$$P(X_j = 1) = n \cdot \frac{1}{n} \cdot \left(\frac{n-1}{n}\right)^{n-1} = \left(\frac{n-1}{n}\right)^{n-1}$$

The expectation of X, the total number of types that appear exactly once, is given by:

$$E(X) = E\left(\sum_{j=1}^{n} X_j\right) = \sum_{j=1}^{n} E(X_j) = \sum_{j=1}^{n} P(X_j = 1)$$

Since $P(X_j = 1)$ is the same for all j, this results in:

$$E(X) = n \cdot \left(\frac{n-1}{n}\right)^{n-1}$$

Question 6

Part A

By definition, a Markov chain is a stochastic process where future state depends only on the current state. By this logic:

- X_n represents the white balls in the first urn after n^{th} step.
- X_{n+1} depends solely on X_n and random outcome, independent of previous steps.

Thus, X_n is a Markov Chain.

Part B

The transition probabilities can be calculated based on the combinations of balls being switched between the urns. Let's denote the transition probability matrix as A, where A_{ij} is the probability of moving from state i to state j.

$$A_{ij} = \begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} \\ a_{1,0} & a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,0} & a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,0} & a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$$

Possible states are $0, 1, 2, 3$:

- **State 0**: 0 white, 3 black in the first urn.
- **State 1**: 1 white, 2 black in the first urn.
- **State 2**: 2 white, 1 black in the first urn.
- **State 3**: 3 white, 0 black in the first urn.

The state transition probabilities are:

- From **State 0**, the next state must be 1 because we are switching one black ball from the first urn with one white ball from the second urn.
- From **State 1**, transitions depend on which ball is drawn from each urn:
  - Transition to **State 0** if we draw a white ball from the first urn (1 white) and a black ball from the second urn (probability $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$).
  - Transition to **State 2** if we draw a black ball from the first urn (2 blacks) and a white ball from the second urn (probability $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$).
  - Stay in **State 1** otherwise (probability $\frac{1}{9} + \frac{4}{9} = \frac{5}{9}$).
- From **State 2**, transitions are the reverse of those from state 1:
  - Transition to **State 3** if we draw a black ball from the first urn (1 black) and a white ball from the second urn (probability $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$).
  - Transition to **State 1** if we draw a white ball from the first urn (2 whites) and a black ball from the second urn (probability $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$).
  - Stay in **State 2** otherwise (probability $\frac{1}{9} + \frac{4}{9} = \frac{5}{9}$).

- From **State 3**, the next state must be 1 because we are switching one white ball from the first urn with one black ball from the second urn.

Thus, the transition probability matrix A is:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{9} & \frac{5}{9} & \frac{4}{9} & 0 \\ 0 & \frac{4}{9} & \frac{5}{9} & \frac{1}{9} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Each row sums to 1, confirming the probabilities are correctly distributed. Google really likes Markov Chains (My source: Google).

Question 7

Part A

Given that $T_i = T_{i-1} + (T_i - T_{i-1})$, and $T_i - T_{i-1}$ given T_{i-1} has a geometric distribution, we can compute the expected value as follows:

$$E[T_i|T_{i-1}] = E[T_{i-1} + (T_i - T_{i-1})|T_{i-1}] = E[T_{i-1}|T_{i-1}] + E[T_i - T_{i-1}|T_{i-1}] = T_{i-1} + \frac{1}{p}$$

Part B

Noice that T_i and $T_i - T_{i-1}$ are independent of T_{i-1}:

$$\text{Var}(T_i|T_{i-1}) = \text{Var}(T_{i-1} + (T_i - T_{i-1})|T_{i-1}) = \text{Var}(T_{i-1}|T_{i-1}) + \text{Var}(T_i - T_{i-1}|T_{i-1}) = 0 + \frac{1-p}{p^2}$$

Here, $\text{Var}(T_{i-1}|T_{i-1}) = 0$ because T_{i-1} is known given T_{i-1}.

Part C

Considering $T_j = T_i + (T_j - T_i)$ and assuming $i < j$:

$$\text{Cov}(T_i, T_j) = \text{Cov}(T_i, T_i + (T_j - T_i)) = \text{Cov}(T_i, T_i) + \text{Cov}(T_i, T_j - T_i) = \text{Var}(T_i) + 0 = \text{Var}(T_i)$$

Next,

$$\text{Var}(T_i) = \text{Var}(T_{i-1}) + \frac{1-p}{p^2}$$

Thus, we get:

$$\text{Cov}(T_i, T_j) = \text{Var}(T_i) = i \cdot \frac{1-p}{p^2}$$