Probability Theory

MTH 664

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Chapter 1.2 Exercises

1.2.2. Let χ have the standard normal distribution. Use Theorem 1.2.6 to get upper and lower bounds on $P(\chi \geq 4)$.

Proof

We are given that χ follows a standard normal distribution, meaning $\chi \sim N(0,1)$. We need to use Theorem 1.2.6 to get upper and lower bounds for $P(\chi \geq 4)$, where χ represents a standard normal random variable.

We know that Theorem 1.2.6 provides bounds for the tail probabilities of the standard normal distribution. The key inequality we will use is:

For $x > 0$:

$$\frac{1}{\sqrt{2\pi}} \frac{x}{x^2+1} e^{-x^2/2} \leq P(\chi \geq x) \leq \frac{1}{\sqrt{2\pi} x} e^{-x^2/2}.$$

We want to estimate $P(\chi \geq 4)$. Applying the upper and lower bounds from the theorem for $x = 4$:

Upper bound:

$$P(\chi \geq 4) \leq \frac{1}{\sqrt{2\pi} \cdot 4} e^{-4^2/2}.$$

Simplifying:

$$P(\chi \geq 4) \leq \frac{1}{4\sqrt{2\pi}} e^{-8}.$$

Using the approximation $e^{-8} \approx 0.000335$ and $\sqrt{2\pi} \approx 2.5066$:

$$P(\chi \geq 4) \leq \frac{1}{4 \cdot 2.5066} \cdot 0.000335 \approx 3.34 \times 10^{-5}.$$

Lower bound:

$$P(\chi \geq 4) \geq \frac{1}{\sqrt{2\pi}} \frac{4}{4^2+1} e^{-4^2/2}.$$

Simplifying:

$$P(\chi \geq 4) \geq \frac{1}{\sqrt{2\pi}} \frac{4}{17} e^{-8}.$$

Using $e^{-8} \approx 0.000335$ and $\sqrt{2\pi} \approx 2.5066$:

$$P(\chi \geq 4) \geq \frac{4}{17 \cdot 2.5066} \cdot 0.000335 \approx 3.13 \times 10^{-5}.$$