Homework 1

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ST 543 Applied Stochastic Models

Problem 1: Marbles Drawn with Replacement

(a) Sample Space

$$S = \{(R,R),(R,G),(R,B),(G,R),(G,G),(G,B),(B,R),(B,G),(B,B)\}$$

(b) Probability of Each Point in the Sample Space

$$P((x,y)) = \frac{1}{9}$$

Problem 2: Marbles Drawn without Replacement

(a) Sample Space

$$S = \{(R, G), (R, B), (G, R), (G, B), (B, R), (B, G)\}$$

(b) Probability of Each Point in the Sample Space

$$P((x, y)) = \frac{1}{6}$$

Problem 3: Probability of Drawing a Black Marble

$$P(\text{Black}|\text{Box1}) = \frac{1}{2}$$
$$P(\text{Black}|\text{Box2}) = \frac{2}{3}$$
$$P(\text{Box1}) = P(\text{Box2}) = \frac{1}{2}$$

$$P(\text{Black}) = P(\text{Black}|\text{Box1}) \cdot P(\text{Box1}) + P(\text{Black}|\text{Box2}) \cdot P(\text{Box2})$$
$$P(\text{Black}) = \frac{1}{2} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$$

Problem 4: Two Balls Selection

(a) Sample Space

$$S = \{\text{All combinations of two balls from 5 red, 3 orange, and 2 blue}\}$$

$$S = \binom{10}{2} = 45$$

(b) Support of R.V. X

$$\text{Support}(X) = \{0, 1, 2\}$$

(c) P(X = 0)

$$P(X = 0) = \frac{\binom{5}{2} + \binom{2}{2}}{\binom{10}{2}} = \frac{10 + 1}{45} = \frac{11}{45} = 0.24\bar{4} \approx 24.4\%$$

Problem 5: Five Fair Coins Tossed

(a) What outcomes in the sample space does $I_E = 1$ equal 1?

$$\text{Outcomes} = \{(H, H, H, H, H)\}$$

(b) What is $P(I_E = 1)$?

$$P(I_E = 1) = \left(\frac{1}{2}\right)^5 = \frac{1}{32} = 0.03125$$

Problem 6: Random Variable with a Biased Coin

(a) Random Variable X Equally Likely to be 0 or 1

The probability that O_1 and O_2 are the same is $(1-p)p + p(1-p) = 2p(1-p)$, and the probability that O_1 and O_2 are different is $2p(1-p)$. Since the procedure alternates until O_1 and O_2 are different, the probability of X being 0 or 1 is equal.

Hence,

$$P(X = 0) = P(X = 1) = \frac{1}{2}$$

(b) Simpler Method Validation

Yes, we could use a simpler procedure that continues flipping the coin until the last two flips are different. This method maintains the equal likelihood of $X = 0$ or $X = 1$.

Problem 7: Drawing Black Given Red

$$P(\text{First ball is Black} \mid \text{Second ball is Red}) = \frac{P(\text{First ball is Black} \cap \text{Second ball is Red})}{P(\text{Second ball is Red})}$$

$$P(\text{First ball is Black} \cap \text{Second ball is Red}) = \frac{b}{b+r+c} \cdot \frac{r}{b+r+c} = \frac{br}{(b+r+c)^2}$$

$$P(\text{Second ball is Red}) = \frac{r}{b+r+c}$$

$$P(\text{First ball is Black} \mid \text{Second ball is Red}) = \frac{\frac{br}{(b+r+c)^2}}{\frac{r}{b+r+c}} = \frac{br}{(b+r+c)^2} \cdot \frac{b+r+c}{r} = \frac{b}{b+r+c}$$

So, we find that

$$P(\text{First ball is Black} \mid \text{Second ball is Red}) = \frac{b}{b+r+c}$$

Problem 8: Expectation of Nonnegative Integer R.V.

$E(X) = \sum_{n=1}^{\infty} P(X \geq n) = \sum_{n=0}^{\infty} P(X > n),$ where X is a nonnegative integer valued random variable.

$I_n = \begin{cases} 1 & \text{if } X \geq n \\ 0 & \text{if } X < n \end{cases}$, for $n \geq 1$.

Express X using I_n : $X = \sum_{n=1}^{\infty} I_n$.

$E(X)$ can be rewritten as:

$E(X) = E\left(\sum_{n=1}^{\infty} I_n\right) = \sum_{n=1}^{\infty} E(I_n),$ by the linearity of expectation.

Since $E(I_n) = P(X \geq n),$ we get:

$E(X) = \sum_{n=1}^{\infty} P(X \geq n),$ which is the sum of probabilities that X is at least n.

Thus,

$$E(X) = \sum_{n=0}^{\infty} P(X > n)$$