Homework 6

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ST 543: Applied Stochastic Models

Problem 1

Solution

The chance that any male and female will mate in a small time interval h is given by $\lambda h + o(h)$. The total number of mating events depends on how many males $N_1(t)$ and females $N_2(t)$ there are because each male can mate with each female. So, the rate at which mating happens in the population is,

$$\nu(N_1, N_2) = \lambda N_1 N_2$$

When a mating event happens, one offspring is produced, which is equally likely to be a male or a female. This means that the transitions from state (N_1, N_2) can be described as,

$$(N_1, N_2) \to \begin{cases} (N_1 + 1, N_2) & \text{with probability } \frac{1}{2} \\ (N_1, N_2 + 1) & \text{with probability } \frac{1}{2} \end{cases}$$

Now, given the state (N_1, N_2), the next state is either $(N_1 + 1, N_2)$ or $(N_1, N_2 + 1)$ each with probability $\frac{1}{2}$:

$$P_{(N_1,N_2)\to(N_1+1,N_2)} = \frac{1}{2}$$

$$P_{(N_1,N_2)\to(N_1,N_2+1)} = \frac{1}{2}$$

Therefore, the rate at which mating events occur is,

$$\nu(N_1, N_2) = \lambda N_1 N_2$$

Problem 2

Solution

Part A

True: $N(t) < n$ means that the count of events by time t is less than n. This implies that the n^{th} event has not occurred by time t, which is equivalent to saying $S_n > t$. Here, S_n is the time of the n^{th} event.

Part B

True: $N(t) \leq n$ means that the count of events by time t is at most n. This implies that the n^{th} event occurs at or after time t, which is equivalent to saying $S_n \geq t$. Again, S_n is the time of the n^{th} event.

Part C

True: $N(t) > n$ means that the count of events by time t is greater than n. This implies that the n^{th} event has already occurred by time t, which is equivalent to saying $S_n < t$. Once again, S_n is the time of the n^{th} event.

Problem 3

Part A

Let us denote:

- T_{work}: the time the machine is working.
- T_{repair}: the total time the machine is being repaired, which is the sum of the times for each repair phase.
- P_{ij} : Proportion of time in phase i

The time for each phase i of the repair is exponential with rate μ_i. Therefore, the expected time for each phase i is $1/\mu_i$.

The total repair time T_{repair} is the sum of k independent exponential r.v.s,

$$T_{\text{repair}} = \sum_{i=1}^{k} T_i$$

where T_i is exponential with rate μ_i. The expected total repair time is,

$$E[T_{\text{repair}}] = \sum_{i=1}^{k} \frac{1}{\mu_i}$$

The proportion of time the machine is undergoing a phase i repair is given by the ratio of the expected time spent in phase i to the total expected time,

$$P_{ij} = \frac{E[T_i]}{E[T_{\text{work}}] + E[T_{\text{repair}}]}$$

Since $E[T_i] = \frac{1}{\mu_i}$ and $E[T_{\text{work}}] = \frac{1}{\lambda}$, we get the expression,

$$P_{ij} = \frac{\frac{1}{\mu_i}}{\frac{1}{\lambda} + \sum_{j=1}^{k} \frac{1}{\mu_j}}$$

Part B

Let us denote:

- W_i : Proportion of time working

The proportion of time the machine is working is given by the ratio of the expected working time to the total expected time,

$$W_i = \frac{E[T_{\text{work}}]}{E[T_{\text{work}}] + E[T_{\text{repair}}]}$$

Using $E[T_{\text{work}}] = \frac{1}{\lambda}$ and $E[T_{\text{repair}}] = \sum_{i=1}^{k} \frac{1}{\mu_i}$, the final expression is,

$$W_i = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + \sum_{i=1}^{k} \frac{1}{\mu_i}}$$

Problem 4

Part A

To find the distribution of S_n, we need to understand that $S_n = \sum_{i=1}^{n} X_i$, where each X_i is an independent Poisson random variable with mean μ. Plus, when you add up n independent Poisson random variables, each with mean μ, the result is also a Poisson random variable with mean $n\mu$.

Hence, the distribution of S_n follows,

$$S_n \sim \text{Poisson}(n\mu)$$

Part B

The number of arrivals $N(t)$ in a renewal process up to time t is Poisson distributed with parameter λt, where λ is the rate of the Poisson process. Since the time between arrivals X_n is Poisson distributed with mean μ, the rate λ is $\frac{1}{\mu}$.

Thus, the distribution of $N(t)$ follows,

$$N(t) \sim \text{Poisson}\left(\frac{t}{\mu}\right)$$

Therefore, the probability that $N(t) = n$ is:

$$P(N(t) = n) = \frac{\left(\frac{t}{\mu}\right)^n e^{-\frac{t}{\mu}}}{n!}, \quad n = 0, 1, 2, ...$$

Problem 5

Part A

The interarrival times of $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are independent by definition, since they are independent renewal processes. When combining these two processes into $\{N(t), t \geq 0\}$, the resulting interarrival times are sums of independent interarrival times from the two original processes. However, the interarrival times of $\{N(t), t \geq 0\}$ are not independent because they depend on the occurrences in both $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$.

Therefore, $\{N_1(t), t \geq 0\}$ are **NOT** independent

Part B

The interarrival times of $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are identically distributed within each process. However, the combined process $\{N(t), t \geq 0\}$ has interarrival times that are not identically distributed. The distribution of interarrival times for $\{N(t), t \geq 0\}$ depends on the sum of the distributions of the interarrival times of $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$, leading to a more complex distribution.

Hence, $\{N_1(t), t \geq 0\}$ are **NOT** identically distributed.

Part C

A renewal process requires that the interarrival times be independent and identically distributed.

By the logic of parts A and B, $\{N_1(t), t \geq 0\}$ is **NOT** a renewal process

Problem 6

Part A

The probability that an event occurs within a time d of the previous event is given by the exponential distribution, which is the interarrival time distribution of a Poisson process. So, the probability that the interarrival time X is less than or equal to d is given,

$$P(X \leq d) = 1 - e^{-\lambda d}$$

Since the events occur at rate λ, the rate of d-events becomes,

$$\lambda_d = \lambda \cdot P(X \leq d) = \lambda(1 - e^{-\lambda d})$$

Part B

The proportion of all events that are d-events is simply the probability that an event occurs within time d of the previous event, which is,

$$P(X \leq d) = 1 - e^{-\lambda d}$$

Therefore, the proportion of all events that are d-events can be given as,

$$1 - e^{-\lambda d}$$