Homework 4

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ST

Question 1

Part A

Solution

We aim to derive the level α Likelihood Ratio Test (LRT) statistic for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$. The likelihood function for the iid Exponential(θ) sample is given by:

$$L(\theta; \mathbf{x}) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-x_i/\theta} = \theta^{-n} e^{-\sum_{i=1}^{n} x_i/\theta}.$$

The LRT statistic $\lambda(\mathbf{x})$ is defined as the ratio of the maximum likelihood under H_0 to the unrestricted maximum likelihood:

$$\lambda(\mathbf{x}) = \frac{L(\theta_0; \mathbf{x})}{\sup_{\theta \in \Theta} L(\theta; \mathbf{x})},$$

where Θ is the set of all possible values of θ. The maximum likelihood estimate (MLE) for θ is $\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} x_i$.

Substituting in the likelihood expressions, we get:

$$\lambda(\mathbf{x}) = \frac{\theta_0^{-n} e^{-\frac{1}{\theta_0} \sum_{i=1}^{n} x_i}}{\hat{\theta}^{-n} e^{-n}} = \left(\frac{\theta_0}{\hat{\theta}}\right)^{n} e^{-n(\frac{1}{\theta_0} - \frac{1}{\hat{\theta}}) \sum_{i=1}^{n} x_i}.$$

The critical function $\phi(\mathbf{x})$ is generally defined where $\lambda(\mathbf{x}) \leq c$, with c chosen to satisfy $E_{\theta_0}[\phi(\mathbf{X})] = \alpha$.

Part B

Solution

The plot of $\lambda(\mathbf{x})$ as a function of the minimal sufficient statistic for this family, $T(\mathbf{x}) = \sum_{i=1}^{n} x_i$, reflects that $\lambda(\mathbf{x})$ depends on $T(\mathbf{x})$. Generally, $\lambda(T)$ decreases as T deviates from $n\theta_0$, indicating both very small and very large values of T lead to rejection of H_0.

The rejection region is constructed such that $P(T < k_1 \text{ or } T > k_2) = \alpha$ for some constants k_1 and k_2. The critical value is typically determined using the chi-squared distribution or through simulation, ensuring that the overall type I error rate is controlled at α.

Part C

Solution

The reasoning mentioned in the textbook is incorrect because it fails to recognize that the distribution of $T(\mathbf{X})$ under the null hypothesis is not symmetric about its mean $n\theta_0$. Consequently, simply dividing the tails into equal areas does not correspond to the smallest values of $\lambda(\mathbf{x})$. Instead, the values of k_1 and k_2 should be chosen based on the distribution of $\lambda(\mathbf{x})$, not $T(\mathbf{x})$ directly, unless further symmetry or other properties are known.

The remaining questions require similarly detailed derivations. Each would elaborate on the theory of hypothesis testing, the use of likelihood ratio tests, and Bayesian estimation techniques tailored to the distributional assumptions and test hypotheses given. If you need continued solutions for the subsequent questions, let me know which specific parts you'd like me to solve!