Probability Theory

MTH 664

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Chapter 1.1 Exercises

1.1.1. Let $\Omega = \mathbb{R}$, $\mathcal{F}$ be all subsets so that A or A^c is countable, $P(A) = 0$ in the first case and $= 1$ in the second. Show that $(\Omega, \mathcal{F}, P)$ is a probability space.

Proof:

1. **Sample Space**: $\Omega = \mathbb{R}$ is non-empty.
2. **Sigma-Algebra** $\mathcal{F}$:
    - **Contains** Ω: Since $\Omega^c = \emptyset$ (countable), $\Omega \in \mathcal{F}$.
    - **Closed under Complements**: If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$ because either A or A^c is countable.
    - **Closed under Countable Unions**: For $A_n \in \mathcal{F}$:
        - If all A_n are countable, $\bigcup A_n$ is countable ($\in \mathcal{F}$).
        - If finitely many A_n^c are countable, $\bigcup A_n^c$ is countable; thus, $(\bigcup A_n)^c$ is countable ($\in \mathcal{F}$).
3. **Probability Measure** P:
    - **Defined by**:

$$P(A) = \begin{cases} 0, & \text{if } A \text{ is countable,} \\ 1, & \text{if } A^c \text{ is countable.} \end{cases}$$

    - **Non-negativity**: $P(A) \geq 0$.
    - **Normalization**: $P(\Omega) = 1$ (since Ω^c is countable).

- **Countable Additivity**: For disjoint $A_n \in \mathcal{F}$:
  - If any $P(A_n) = 1$, then $P\left(\bigcup A_n\right) = 1 = \sum P(A_n)$.
  - If all $P(A_n) = 0$, then $\bigcup A_n$ is countable, so $P\left(\bigcup A_n\right) = 0 = \sum P(A_n)$.

Conclusion: $(\Omega, \mathcal{F}, P)$ is a probability space.