ST561: HOMEWORK 1

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October 9, 2023

Question 1

1. Textbook, Page 37, Exercise 1.1

   (a) We know that a coin has 2 sides, H or T. For each flip, there are 2 possible options. Hence, we can describe each flip using this formula: $S = 2^n$. Given that we want to do 4 coin flips, our $n = 4$ and then $S = 2^4$. Therefore, our sample space will be,

   $$S = 16$$

   (b) Varies! Depends how many leaves are insect damaged! If I observed 3 damaged leaves out of 100, our sample space, S, would be 3. There can exist 0 undamaged leaves but no negative leaves. Hence, we can say that the observable sample space will be

   $$S = [0, \infty)$$

   (c) A 'lifetime' depends on how well manufactured the light bulb is. It can expire as soon as 0 hours or be immortal and continue to ∞. Hence, similar to part (b), our sample space will be

   $$S = [0, \infty)$$

   (d) Right off the bat, we cannot have a weight equal 0. And given that there is no definite maximum for the weights of the rats, we can assume that the weight can approach ∞. Therefore, our sample space will be

   $$S = (0, \infty)$$

   (e) Alright, since we are looking at a proportion, it can only mean that the sample space will be from 0. So, the sample space will be

   $$S = (0, 1)$$

Question 2

2. Suppose that you own 6 different Spanish textbooks, 4 different History textbooks, 4 different Geology textbooks, and 5 different English textbooks. You have reserved a shelf on a bookcase on which you place all these textbooks. How many different ways are there to place all 19 textbooks on the shelf so that all the textbooks in a given subject are grouped together?

   Given that we have $n = 19$ number of books, we can apply the formula

   $$\left(\frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!}\right)$$

   We know the following information: $n = 16, n_1 = 6, n_2 = 4, n_3 = 4, n_4 = 5$. We can substitute those values into the formula and get

   $$\frac{19!}{6! \cdot 4! \cdot 4! \cdot 5!} = \frac{121645100408832000}{49766420} = 2444321820$$

   . Therefore, there are a total of 2444321820 different ways you can combine the textbooks

Question 3

3. Four cards are drawn from a deck of 52 ordinary playing cards, without replacement.

   (a) What is the probability that all 4 cards are aces?

   - Given $P(A) = \frac{|A|}{|S|}$, let's determine the sample space $|S|$. Here is how we find the sample space,

   $$|S| = \binom{52}{4} = \frac{52!}{4!(52-4)!} = \frac{6497400}{24} = 270725$$

   Now, we know that there are only 4 aces in a ordinary deck of 52 playing cards. The probability of $|A|$ is as follows

   $$|A| = \binom{4}{4} = \frac{4!}{4!(4-4)!} = \frac{4!}{4! \cdot 1} = 1$$

   . Therefore,

   $$P(A) = \frac{1}{270725} = 0.0000037 \approx 0.00037\%$$

   (b) What is the probability of drawing, in order, the aces of clubs, diamonds, hearts, and spades?

   - The probability that of drawing the aces of clubs $(\frac{4}{52})$, diamonds $(\frac{3}{52})$, hearts $(\frac{2}{52})$, and spades $(\frac{1}{52})$ is as follows:

   $$(\frac{4}{52}) \cdot (\frac{3}{52}) \cdot (\frac{2}{52}) \cdot (\frac{1}{52}) = \frac{1}{270725} = 0.0000037 \approx 0.00037\%$$

   (c) Answer (a) and (b) above if each card is replaced (and the deck shuffled) after it is drawn.

   - [Part (a)] Since each card is drawn independently and it is a fair deck, the probability changes to

   $$(\frac{4}{52})^4 = \frac{256}{7311696} = 0.0000349797 \approx 0.0035\%$$

   - [Part (b)] It will the follow the same logic as [Part (a)], where the aces of clubs $(\frac{4}{52})$, diamonds $(\frac{4}{52})$, hearts $(\frac{4}{52})$, and spades $(\frac{4}{52})$ yield the same probability:

   $$(\frac{4}{52})^4 = \frac{256}{7311696} = 0.0000349797 \approx 0.0035\%$$

Question 4

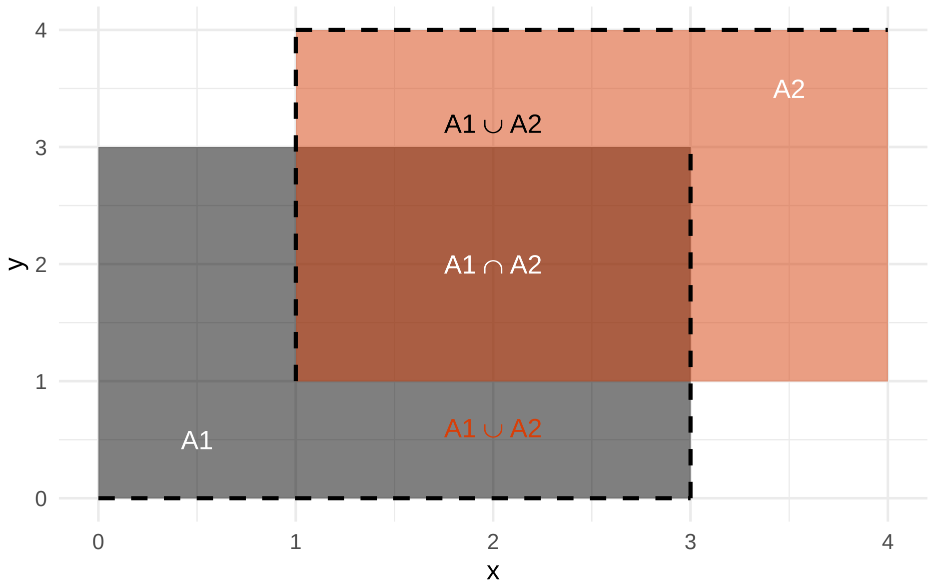
4. Find the union $A_1 \cup A_2$ and the intersection $A_1 \cap A_2$ of the two sets A_1 and A_2 where

   (a) $A_1 = \{x : 0 < x < 1\}, A_2 = \{x : 1 \leq x < 3\}$;

   - $A_1 \cup A_2 = \{x : 0 < x < 3\}$
   - $A_1 \cap A_2 = \emptyset$

   (b) $A_1 = \{(x, y) : 0 \leq x < 3, 0 < y \leq 3\}, A_2 = \{(x, y) : 1 < x \leq 4, 1 \leq y < 4\}$;
   (Draw a picture to show the set.)

   - $A_1 \cup A_2 = \{(x, y)\ :\ 0 \leq x \leq 4,\ 0 < y \leq 3\}$
   - $A_1 \cap A_2 = \{(x, y)\ :\ 1 < x \leq 3,\ 1 < y \leq 3\}$



   (c) $A_1 = \{(x, y) : x + y \leq 1, x \geq 0, y \geq 0\}, A_2 = \{(x, y) : x^2 + y^2 \leq \frac{1}{2}, x \geq 0, y \geq 0\}$.

   - $A_1 \cup A_2 = \{(x, y) : x + y \leq 1, x \geq 0, y \geq 0\}$
   - $A_1 \cap A_2 = \{(x, y) : x + y \leq 1$ and $x^2 + y^2 \leq \frac{1}{2}, x \geq 0, y \geq 0\}$

You are not required to show your work for this problem.

Question 5

5. (Urn Model) r different balls $B_1, \dots, B_r$ are to be randomly put into n different urns $U_1, \dots, U_n$ ($r \leq n$). What is the probability of the following events?

   (a) Urns $U_1, \dots, U_r$ each contains exactly one ball;

   - We want to distribute r balls into n urns in such a way that each of the first r urns contains exactly one ball. We have n choices for each of the r balls, giving us a total of n^r possible ways.
   - To count the possible outcomes where the first r urns have exactly one ball each, we can arrange these r balls in $r!$ different ways.
   - Hence, we get

$$P(A) = \frac{r!}{n^r}$$

   (b) No urn contains more than one ball;

   - To distribute r balls into n urns such that no urn contains more than one ball, you can think of it this way:
     - For the first ball, you have n choices of which urn to place it in.
     - For the second ball, there are $(n-1)$ urns left to choose from since you can't place it in the same urn as the first ball.
     - Similarly, for the third ball, you have $(n-2)$ choices, and so on, until you have $(n-r+1)$ choices for the rth ball.
   - To find the total number of ways to distribute the r balls into n urns, you multiply these choices together:
     - $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1) = \frac{n!}{(n-r)!}$
   - Now, to calculate the probability, we need to consider the total number of outcomes for distributing the r balls into the n urns, which is n^r.
   - Hence,

$$P(B) = \frac{n!}{(n-r)!} \cdot \frac{1}{n^r} = \frac{n!}{n^r(n-r)!}$$

   (c) Urn U_1 contains exactly m balls ($m \leq r$).

   - Choose m out of the r balls to place into urn U_1. Thus, we get $\binom{r}{m}$
   - Once we place the m balls in U_1, we will have $(r-m)$ balls remaining to distribute among the remaining $(n-1)$ urns.
   - For each of the remaining $(n-1)$ urns, we have $(n-1)$ choices for where to place each of the $(r-m)$ remaining balls.
   - Therefore,

$$U_1 = \binom{r}{m} \cdot (n-1)^{r-m}$$

Question 6

6. Suppose $n > 0, 0 \leq k \leq n$. Show that

   (a) $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

   - This is the identity, so let's break it down. We have $\binom{n}{k}$, which represents the number of ways to choose k items from a set of n distinct items, where order does not matter. Next, we have $n!$ which is the total number of possible arrangements of n unique items. Then, $k!$ is the total number of permutations of the k selected items. Lastly, we have $(n-k)!$ which is the total number of permutations of the remaining $(n-k)$ items. Hence:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

   (b) $\binom{n}{0} = \binom{n}{n} = 1$

   - Here is how both are equal to 1

$$\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{1 \cdot n!} = \frac{n!}{n!} = 1$$

$$\binom{n}{0} = \frac{n!}{n!(n-n)!} = \frac{n!}{n! \cdot 1} = \frac{n!}{n!} = 1$$

   (c) $\binom{n}{1} = n$

   - This simpifies down to n by this logic:

$$\binom{n}{0} = \frac{n!}{1!(n-1)!} = \frac{n!}{(n-1)!} = \frac{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1}{(n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1} = n$$

   (d) If $k > 1$, $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

   - Here is how we get back to $\binom{n}{k}$

$$\frac{(n-1)!}{(k-1)! \cdot (n-k)!} + \frac{(n-1)!}{k! \cdot (n-k-1)!} = \frac{n! \cdot (n-k)}{k! \cdot (n-k)! \cdot (n-k)} + \frac{n! \cdot k}{k! \cdot (n-k)! \cdot k}$$

$$= \frac{n! \cdot (n-k) + n! \cdot k}{k! \cdot (n-k)! \cdot (n-k)} = \frac{n! \cdot n}{k! \cdot (n-k)!} = \frac{n!}{k! \cdot (n-k)!} = \binom{n}{k}$$

Can you give intuitive explanations for the results above? (You do not have to turn in your explanations.)

Question 7

7. Consider the experiment of dealing a five-card poker hand at random from a deck of 52 ordinary playing cards. Let A be the event that you obtain a full house (3 cards of one rank and 2 cards of a second rank, e.g., 3 kings and 2 eights). Let B be the event that you obtain two pairs (2 cards of one rank, 2 cards of a second rank, and a 5th card of a third rank, e.g., 2 tens, 2 threes, and the Jack of diamonds). One of the following two probabilities is correct, and the other probability is incorrect.

   (a) Determine which probability is correct.

   - $P(A)$ is correct while $P(B)$ is incorrect

   $$P(A) = \frac{(13)(12)\binom{4}{3}\binom{4}{2}}{\binom{52}{5}} = \frac{3744}{2598960} = 0.1440576 \approx 0.14405762304922\%$$

   $$P(B) = \frac{(13)(12)\binom{4}{2}\binom{4}{2}(44)}{\binom{52}{5}} = \frac{247104}{2598960} = 0.09507803 \approx 9.5078031212485\%$$

   (b) Correct the incorrect probability.

   $$P(B) = \frac{\binom{4}{2}\cdot\binom{4}{2}\cdot\binom{44}{1}\cdot\binom{13}{2}}{\binom{52}{5}} = \frac{123552}{2598960} = 0.0475390156 \approx 4.575390156\%$$

   (c) Explain clearly how the correct probability is obtained and why the incorrect probability is wrongly calculated.

   - The correct probability for a full house (3 cards of one rank and 2 of another) in a five-card poker hand is calculated accurately using combinatorics. In contrast, the incorrect probability for two pairs fails to consider the possibility of one pair being a subset of the other, leading to an inaccurate calculation. Double counting leads to common miscalcuations such as this example.