SOUTH DAKOTA SCHOOL OF MINES & TECHNOLOGY

RIAS Robotics & Intelligent Autonomous Systems

Introduction to Mobile Robotics

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Examples and Applications

- GSL
  - Simple GSL code
  - Compiling
  - Random numbers
  - Matrices and Linear Systems
- Octave
  - Matrix notion
  - Linear solves
- Fixed frame
- Current frame

Combined Transforms

Begin with a point x in space. An application of a transformation, T_1, with respect to the global frame carries this point to a new point x':

$$x' = T_1 x$$

In essense, the fixed frame views the point as in motion. Now apply another transformation T_2 to the new point x':

$$x" = T_2 x' = T_2(x') = T_2(T_1 x) = T_2 T_1 x$$

Note that each transform was done with respect to the fixed frame.

Combined Transforms

Again, begin with a point x in space. If we view the transformation, T from the perspective of the point (which will be fixed), then it appears that the "fixed" frame is moving

AND

that the motion is in the *opposite* direction of the fixed frame transformation. Opposite here would be the inverse transformation: T^{-1}.

Thus combined transformations from the point's "point of view":

$$T^{-1} = T_2^{-1} T_1^{-1}, \quad \text{or} \quad T = \left(T_2^{-1} T_1^{-1}\right)^{-1}$$

$$T = T_1 T_2$$

Reverse order.

Combined Transforms

Successive transformations relative to the global frame are left multiplied:

$$T = T_n T_{n-1} \dots T_1 T_0$$

For example, take a rotation about z of 30 degrees, R_1, followed by a rotation about x by 60 degrees, R_2:

$$R = R_2 R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 60 & -\sin 60 & 0 \\ 0 & \sin 60 & \cos 60 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos 30 & -\sin 30 & 0 & 0 \\ \sin 30 & \cos 30 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Combined Transforms

Successive transformations relative to the moving frame are right multiplied:

$$T = T_0 T_1 \dots T_{n-1} T_n$$

For example, take a rotation about x by 45 degrees, R, followed by a translation in z by 4 cm, T:

$$M = TR = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 60 & -\sin 60 & 0 \\ 0 & \sin 60 & \cos 60 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Inverse Transforms

Inverting transformation matrices ...

$$T^{-1} = (T_n T_{n-1} \dots T_1 T_0)^{-1} = T_0^{-1} T_1^{-1} \dots T_{n-1}^{-1} T_n^{-1}$$

How does one invert the transformations?

Inverses

Rotation matrices are orthogonal and so

$$R^{-1} = R^T$$

For example, the inverse of the 60 degree rotation mentioned above:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 60 & -\sin 60 & 0 \\ 0 & \sin 60 & \cos 60 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 60 & \sin 60 & 0 \\ 0 & -\sin 60 & \cos 60 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Inverses

Translation matrices are simple as well. One just negates the translation components.

Thus:

$$\begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & -a \\ 0 & 1 & 0 & -b \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus we can just undo the transformations individually.

RPY Angles

RPY angles provide the position and orientation of a craft by using a translation to body center and then three rotation matrices for craft pose.

- Rotation about a (z axis) - Roll
- Rotation about o (y axis) - Pitch
- Rotation about n (x axis) - Yaw

$$M = R_n R_o R_a T$$

Euler Angles

Euler angles provide the position and orientation of a craft by using a translation to body center and then three rotation matrices for craft pose. However - reference is with respect to the body, not the world coordinates.

- Rotation about a (z axis) - Roll
- Rotation about o - Pitch
- Rotation about a - Roll

$$M = R_a R_o R_a T$$

Forward and Inverse Kinematics

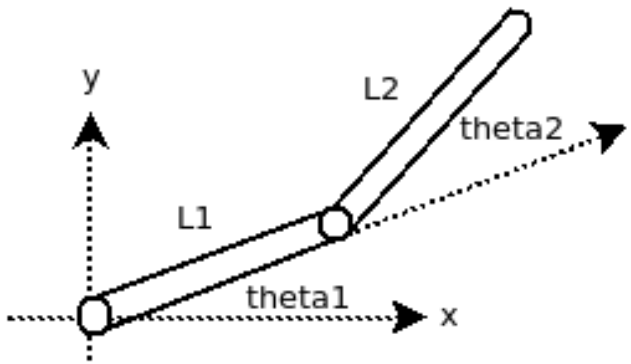
Given joint angles and actuator lengths it is straightforward to compute end effector position. Thus it is easy to find effector path as a function of rotations.

$$\big(\theta_1(t), ..., \theta_n(t)\big) \rightarrow p(t)$$

It is MUCH harder to find the angle functions if you are given the end effector path:

$$p(t) \rightarrow \big(\theta_1(t), ..., \theta_n(t)\big)$$

Two link manipulator



Let the base of the first link be the origin of the fixed frame shown in the figure. The location of the second joint is $(\tilde{x}, \tilde{y})$ given by

$$\tilde{x} = L_1 \cos\theta_1, \quad \tilde{y} = L_1 \sin\theta_1$$

The location of the end effector is given by (x, y)

$$x = L_2 \cos(\theta_1 + \theta_2) + \tilde{x}, \quad y = L_2 \sin(\theta_1 + \theta_2) + \tilde{y}$$

Thus one obtains

$$x = L_2 \cos(\theta_1 + \theta_2) + L_1 \cos\theta_1, \quad y = L_2 \sin(\theta_1 + \theta_2) + L_1 \sin\theta_1$$

Review

Represent points by $\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$ and vectors by $\begin{pmatrix} v_x \\ v_y \\ v_z \\ 0 \end{pmatrix}$

Recall a rotation about x by 30 degrees

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos 30 & -\sin 30 & 0 & 0 \\ \sin 30 & \cos 30 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Denavit-Hartenberg Parameters

Provides a standard way to build kinematic models for a robot.

Simple concept.

Follow out the links of the manipulator, and see them as rotations and translations of the coordinate system:

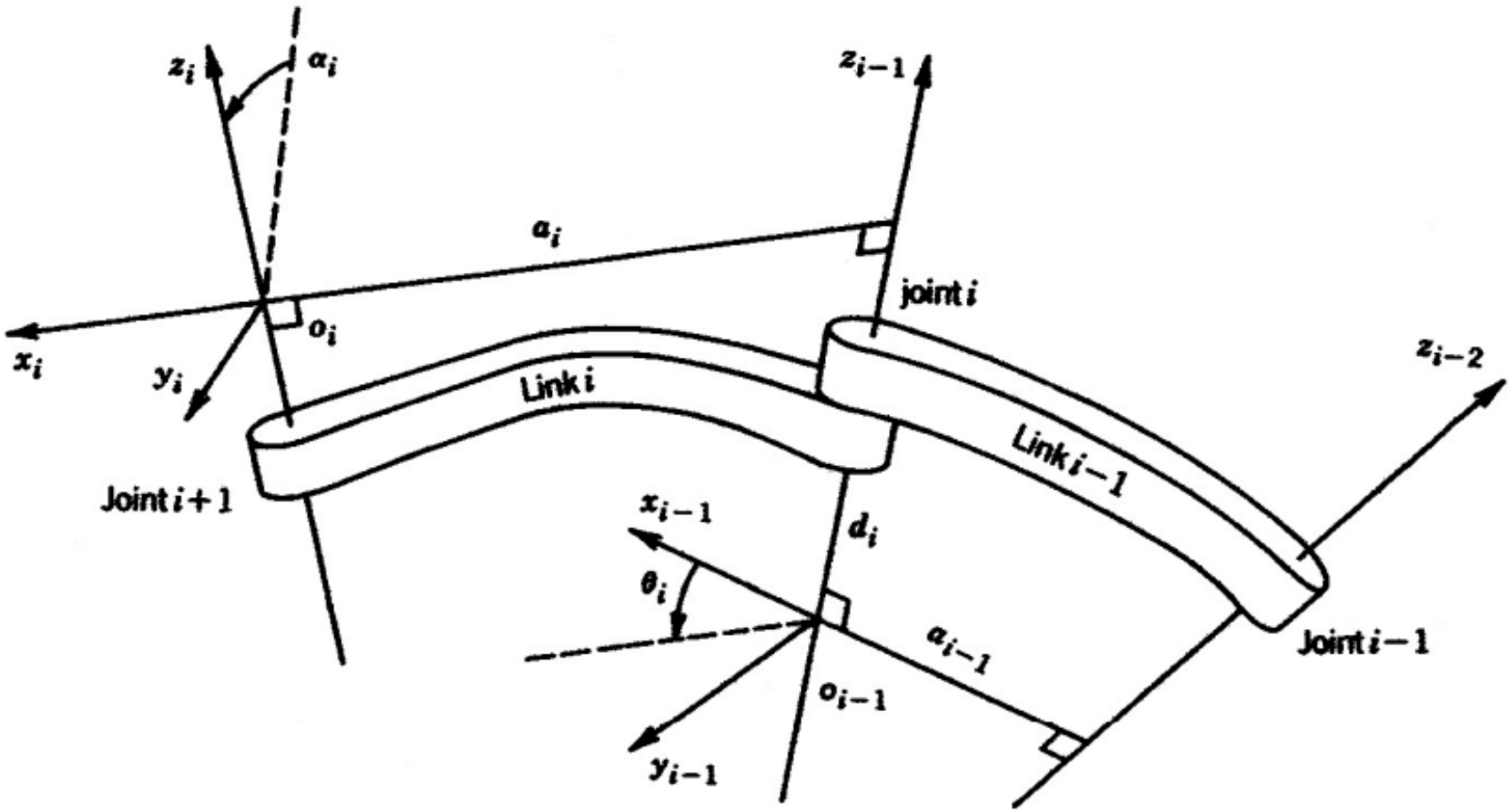
$$P = P_0 P_1 ... P_{n-1} P_n$$

where $P_k = R_z T_z T_x R_x$

Denavit-Hartenberg Parameters

- Each link is assigned a number. Normally start with the base and work towards the effector.
- All joints are represented by the z axis, z_i where the z axis is the axis of revolution (right hand rule for orientation).
- θ_i will represent the rotation about the joint.
- The x axis, x_i is in the direction that connects the links. [Well, connects the z axes of each joint.]
- a_i is link length.
- α_i will be the angles between z axes (if they are not parallel).
- d_i will represent the offset along the z axis.

Denavit-Hartenberg Parameters



Denavit-Hartenberg Parameters

Thus, the translation from one joint to the next involves a rotation, translation, translation and a rotation:

- Rotate about the local z axis angle θ.
- Translate along the z axis amount d.
- Translate along x amount a.
- Rotate about the new x axis (the joint twist) amount α.

This set of transformations will then change the coordinate system to the next link in the serial chain.

Denavit-Hartenberg Parameters

$$A_{n+1} =$$

$$\begin{pmatrix} \cos\theta_{n+1} & -\sin\theta_{n+1} & 0 & 0 \\ \sin\theta_{n+1} & \cos\theta_{n+1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_{n+1} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & a_{n+1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_{n+1} & -\sin\alpha_{n+1} & 0 \\ 0 & \sin\alpha_{n+1} & \cos\alpha_{n+1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Denavit-Hartenberg Parameters

$$A_{n+1} = \begin{pmatrix} \cos\theta_{n+1} & -\sin\theta_{n+1}\cos\alpha_{n+1} & \sin\theta_{n+1}\sin\alpha_{n+1} & a_{n+1}\cos\theta_{n+1} \\ \sin\theta_{n+1} & \cos\theta_{n+1}\cos\alpha_{n+1} & -\cos\theta_{n+1}\sin\theta_{n+1} & a_{n+1}\sin\theta_{n+1} \\ 0 & \sin\alpha_{n+1} & \cos\alpha_{n+1} & d_{n+1} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A parameter table keeps track for each link, the values of θ, d, a and α.

Starting from the base of the robot, we can built the transformation that defines the kinematics:

$$A = A_1 A_2 \ldots A_n$$

D-H Two Link Example

Link	θ	d	a	α
1	θ_1	0	a_1	0
2	θ_2	0	a_2	0

$$A_1 = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & a_1\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & a_1\sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & a_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & a_2\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Example

So,

$$A = A_1 A_2 = \begin{pmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) & 0 & a_2 \cos(\theta_1 + \theta_2) + a_1 \cos\theta_1 \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) & 0 & a_2 \sin(\theta_1 + \theta_2) + a_1 \sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Inverse Kinematics

How can we use this technology to solve the inverse kinematics problem?

$$T^{-1} = T_0^{-1} T_1^{-1} \dots T_{n-1}^{-1} T_n^{-1}$$

In each matrix one can solve algebraically for θ_i in terms of the orientation and displacement vectors.

What does this look like for the two link manipulator?