Acoustic time-reversal mirrors in the framework of one-way wave theories

Maarten V. de Hoop*, Louis Fishman†, and B. Lars G. Jonsson‡

* *Center for Wave Phenomena, Colorado School of Mines, Golden CO 80401-1887, USA.*
† *Department of Physics, University of New Orleans, New Orleans LA 70148.*
‡ *Department of Electromagnetic Theory, Royal Institute of Technology, Stockholm, Sweden.*

ABSTRACT
We investigate the implications of directional wavefield decomposition with a view to time reversibility. In particular, we discuss how wavefield decomposition preserves the reciprocity theorem of time-convolution type but looses the reciprocity theorem of time-correlation type. As a consequence, a perfect 'time-reversal mirror' in the framework of one-way wave theory does not exist: We find that on the wavefront set ('classical limit') a time-reversal mirror can retrofocus the wavefield to its originating source, but that non-perfectly retrofocusing lower-order distributions contribute to the process as well. These distributions can be attributed to 'evanescent' wave constituents but are not negligible; we will study them explicitly. As a peculiarity, we discuss how a Schrödinger-like equation can be obtained out of the (exact) frequency-domain one-way wave equation. This involves an approximation – known in ocean acoustics and exploration seismology as the 'parabolic equation' approximation – that restores time-reversibility.

1 INTRODUCTION

In time-reversal acoustics a signal is recorded by an array of transducers, time-reversed, and then re-transmitted into the configuration. The re-transmitted signal propagates back through the same medium and retrofocuses on the source that generated the signal. In this process the actual medium properties are not used. In a time-reversal cavity the array completely surrounds the source; if the time-reversal is carried out on a limited area (presumably flat) we speak of a time-reversal mirror. In this paper, we analyze these experiments theoretically in the framework of *one-way* wave theory.

We investigate the implications of directional wavefield decomposition (Weston, 1987; De Hoop, 1996) with a view to time reversibility. In particular, we discuss how wavefield decomposition preserves the reciprocity theorem of time-convolution type but looses the reciprocity theorem of time-correlation type. As a consequence, a perfect time-reversal mirror in the framework of one-way wave theory does not exist: We find that on the wavefront set a time-reversal mirror can retrofocus the wavefield to its originating source, but that non-perfectly retrofocusing lower-order distributions contribute to the process as well. These distributions can be attributed to 'evanescent' wave constituents but are not negligible; we will study them explicitly. (In terms of a plane-wave expansion, evanescent waves correspond with non-homogeneous constituents.)

As an intermediate result, we will briefly establish a connection between reciprocity theorems (Rayleigh, 1873; De Hoop and De Hoop, 2000), time-reversal cavities (TRCs) and mirrors (TRMs) (Jackson and Dowling, 1991; Fink, 1992; Fink, 1993; Fink, 1999), and boundary control theory (Lions, 1971): A reciprocity theorem describes the interaction of two states, a time-reversal mirror reflects one state into another, and boundary control theory aims to control the wavefield (in a desired state) at a particular point/region in space and time through its boundary values, the control being determined by a model state. The question whether the time-reversal mirror provides *optimal* boundary control is addressed in a separate paper (Gustafsson et al., 2001). Cheney et al. (2001) invoked a TRM to understand optimal acoustical measurements. The one-way wave TRM can be employed in *inverse scattering* theory as well, viz. using the generalized Bremmer coupling series (Bremmer, 1939; De Hoop, 1996) as the direct scattering model and a stripping procedure.

TRMs have been implemented experimentally (the medium is not known) and in data processing (a medium is assumed) in various fields of application. We mention ocean acoustics, guided wave optics, seismology, and medical imaging. DeRode et al. (1995) carried out the first experiment demonstrating time-reversibility of an acoustic wave propagating through a random collection of scatterers. The stochastic theory associated with such experiments has been developed by Blomgren et al. (2001) and led to the notion of *super-resolution*. Ultrasonic experiments in a waveguide showing the use of time-reversal to 'compensate' for multiples have been conducted by Roux et al. (1997). In fact, to compensate for such reverberation in underwater (ocean) acoustics, single-channel time reversal was introduced by Parvulescu and Clay (1965). (Their experiments were restricted to retrofocusing in time and did not encompass the spatial focusing resulting from the use of an array.) Jackson and Dowling (1991) developed further a formalism based on modal decomposition of the wavefield to describe phase conjugation, the frequency-domain (monochromatic) counterpart of time-reversal. Underwater acoustics TRM experiments have since been carried out by Kuperman et al. (1998) and others. (One ocean acoustics application is that of two-way underwater communication (Catipovic, 1997)). In seismic data processing, TRMs appeared in the form of 'controlled illumination' (Rietveld et al., 1992) leading to the generation of so-called 'common focal point' gathers (Thorbecke, 1997) for data analysis. In data processing, a TRM can be viewed to synthesize a desired 'source' array from original measurements, an idea that goes back to Taner (1976) and Schultz and Claerbout (1978). In medical applications, iterative use of TRMs has been proposed to aid the process of kidney stone destruction (Thomas et al., 1996). Iterative use of TRMs in inverse scattering is a current subject of research.

Time-reversal retrofocusing may degrade or be lost when the effective TRM aperture becomes small, in dynamic media (changing with time; see, for example, Khosla and Dowling (1998), and in noisy environments Khosla and Dowling (2001). However, for simulation and data processing, one-way wave rather than full-wave theories have been widely used in all fields of application mentioned above. Upon directional decomposition the retrofocusing of waves degrades also. Hence the subject of this paper.

As a peculiarity, we discuss how a Schrödinger-like equation can be obtained out of the (exact) frequency-domain one-way wave equation. This involves an approximation – known in ocean acoustics and exploration seismology as the 'parabolic equation' (PE) approximation – that restores time-reversibility.

2 THE HYPERBOLIC SYSTEM

2.1 The first-order system

First, we introduce the matrix form of the equations that govern acoustic wave motion. Let the *field matrix* $F_P = F_P(\boldsymbol{x}, t)$ of the wave motion be composed of the components of the two wavefield quantities whose inner product represents the area density of power flow (Poynting vector). Then, F_P satisfies a system of linear, first-order, partial differential equations of the form

$$(\mathsf{D}_{IP} + M_{IP}\partial_t)\, F_P = Q_I \ , \quad \mathsf{D}_{IP} = \mathsf{D}_{IP}(\nabla) \ , \ M_{IP} = M_{IP}(\boldsymbol{x}) \ , \tag{1}$$

where uppercase Latin subscripts are used to denote the pertaining matrix elements and the summation convention for repeated subscripts applies. We assume that $\boldsymbol{x} \in \Omega$ and $t \in [0, T]$. In equation (1), D_{IP} is a symmetric, block off-diagonal *spatial differentiation operator matrix* that contains the operator ∂_p in a homogeneous linear fashion, M_{IP} is the *medium matrix* that is representative for the properties of the media in which the waves propagate and $Q_I = Q_I(\boldsymbol{x}, t)$ is the *volume source density matrix* that is representative for the action of the volume sources that generate the wavefield.

Let $\mathcal{N}_{IP}$ be the *unit normal operator* that arises from replacing ∂_p in D_{IP} by n_p, where $\boldsymbol{n}$ is, on each of the two faces of the surface, the unit vector along the normal oriented away from the domain that surrounds that surface, $\mathcal{N}_{IP} = \mathsf{D}_{IP}(\boldsymbol{n})$. The medium parameters are assumed to be piecewise continuous. Across a surface of discontinuity in medium properties, the parameters may jump by finite amounts. On the assumption that the interface is passive (i.e., free from surface sources) and that the wavefield quantities must remain bounded on either side of the interface, the wavefield must satisfy the boundary condition of the continuity type

$$\mathcal{N}_{IP} F_P \quad \text{is continuous across source-free interface} \ . \tag{2}$$

Second, let us identify

$$F = (p, v_1, v_2, v_3)^T \ ,$$

where p = acoustic pressure, and $v_{1,2,3}$ = particle velocity, and

$$Q = (q, f_1, f_2, f_3)^T \ ,$$

where q = volume source density of injection rate and $f_{1,2,3}$ = volume source density of force. The differential operator matrix and the medium matrix are given by

$$\mathsf{D} = \begin{pmatrix} 0 & \partial_1 & \partial_2 & \partial_3 \\ \partial_1 & 0 & 0 & 0 \\ \partial_2 & 0 & 0 & 0 \\ \partial_3 & 0 & 0 & 0 \end{pmatrix} , \quad M = \begin{pmatrix} \kappa & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & \rho \end{pmatrix} , \tag{3}$$

respectively, where ρ is the volume density of mass, and κ is the compressibility. We thus recover the physical equations: the equation of motion and the constitutive relations.

In view of the linearity of the wave motion, the principle of superposition ensures that the wavefield F_P that is generated by the volume source distribution Q_I (and any surface source distributions) can be written as the superposition of point-source contributions through the use of a Green's tensor. The latter is a solution of the system of differential equations

$$(\mathsf{D}_{IP} + M_{IP}\partial_t)\, G_{PI'} = \delta_{II'}\delta(. - \boldsymbol{x}')\delta(. - t') , \tag{4}$$

where $\delta_{II'}$ is the unit matrix and δ is the Dirac distribution. In view of the time invariance of the medium, the Green's tensor depends on t and t' only through the difference $t - t'$, i.e.,

$$G_{PI'} = G_{PI'}(\boldsymbol{x}, \boldsymbol{x}', t, t') = G_{PI'}(\boldsymbol{x}, \boldsymbol{x}', t - t') .$$

To arrive at the reciprocity relations, revealing the structure of the system of partial differential equations, we consider the ('Fourier real') bilinear form of the time-convolution type

$$\langle Q \,|\, F \rangle = Q_I \overset{(t)}{*} F_I ; \tag{5}$$

we introduce the Minkowski delta

$$\delta^- = \mathrm{diag}[1, -1, -1, -1]$$

to form

$$\langle Q \,|\, \delta^- F \rangle$$

with respect to which we have the algebraic symmetry property

$$\delta^- \mathsf{D} = -\mathsf{D}^T \delta^- . \tag{6}$$

This symmetry property implies the reciprocity theorem and relations of the time-convolution type (De Hoop and De Hoop, 2000, (8.11)). On the other hand, consider the ('Fourier complex') bilinear form of the time-correlation type

$$\langle Q \,|\, F \rangle_{\mathbb{C}} = Q_I \overset{(-t)}{*} F_I . \tag{7}$$

With respect to this form, which we can write also as $\langle Q \,|\, \delta\, F \rangle_{\mathbb{C}}$, we have the algebraic symmetry property

$$\mathsf{D}^T = \mathsf{D} . \tag{8}$$

This symmetry property implies the reciprocity theorem and relations of the time-correlation type (De Hoop and De Hoop, 2000, (10.7)).

2.2 The reduced system of equations

To develop the directional decomposition and the subsequent 'wave tracing', we should carry out our analysis in the time-Laplace domain. To show the notation, we give the expression for the acoustic pressure,

$$\hat{p}(\boldsymbol{x}, s) = \int_{t=0}^{\infty} \exp(-st) p(\boldsymbol{x}, t)\, \mathrm{d}t . \tag{9}$$

Under this transformation, assuming zero initial conditions, we have $\partial_t \to s$. However, for the purpose of the analysis to follow, we will invoke the limit, $\lim_{s \to \mathrm{i}\omega}$.

The decomposition procedure requires a separate handling of the horizontal components of the particle velocity. From equation (1) we obtain

$$\hat{v}_\mu = \mathrm{i}\rho^{-1}\omega^{-1}(\partial_\mu \hat{p} - \hat{f}_\mu) , \quad \mu = 1, 2 , \tag{10}$$

leaving, upon substitution, the matrix differential equation

$$(\delta_{IP}\,\partial_3 + \mathrm{i}\omega\hat{\mathsf{A}}_{IP})\hat{F}_P = \hat{N}_I \ , \quad \hat{\mathsf{A}}_{IP} = \hat{\mathsf{A}}_{IP}(\boldsymbol{x}, D_{1,2}) \ , \quad D_\nu \equiv \frac{\mathrm{i}}{\omega}\partial_\nu \ , \tag{11}$$

in which δ_{IP} is the Kronecker delta, and the elements of the acoustic field matrix are given by

$$\hat{F} = (\hat{p}, \hat{v}_3)^T \ , \tag{12}$$

the elements of the notional source matrix by

$$\hat{N} = (\hat{f}_3, D_1(\rho^{-1}\hat{f}_1) + D_2(\rho^{-1}\hat{f}_2) + \hat{q})^T \ , \tag{13}$$

and the elements of the acoustic system's operator matrix by

$$\hat{\mathsf{A}} = \begin{pmatrix} 0 & \rho \\ -D_1(\rho^{-1}D_1\,) - D_2(\rho^{-1}D_2\,) + \kappa & 0 \end{pmatrix} \ . \tag{14}$$

It is observed that the right-hand side of equation (10) and $\hat{\mathsf{A}}_{IP}$ contain the spatial derivatives D_ν with respect to the horizontal coordinates only. D_ν has the interpretation of *horizontal slowness* or horizontal momentum operator.

2.3 The coupled system of one-way wave equations

To distinguish up- and downgoing constituents in the wavefield, we shall construct an appropriate linear operator $\hat{L}_{IJ}$ with

$$\hat{F}_I = \hat{L}_{IJ}\hat{W}_J \ , \tag{15}$$

that, with the aid of the commutation relation $(\partial_3\hat{L}_{IJ}) = [\partial_3, \hat{L}_{IJ}]$, transforms equation (11) into

$$\hat{L}_{IJ}\,(\partial_3\delta_{JM} + \mathrm{i}\omega\hat{\Lambda}_{JM})\hat{W}_M = -(\partial_3\hat{L}_{IJ})\,\hat{W}_J + \hat{N}_I \ , \tag{16}$$

as to make $\hat{\Lambda}_{JM}$, satisfying

$$\hat{\mathsf{A}}_{IJ}\hat{L}_{JM} = \hat{L}_{IJ}\hat{\Lambda}_{JM} \ , \tag{17}$$

a diagonal matrix of operators. We denote $\hat{L}_{IJ}$ as the composition operator and $\hat{W}_M$ as the wave matrix. The matrix expression in parentheses on the left-hand side of equation (16) is diagonal and its diagonal entries are the two so-called *one-way* wave operators. The first term on the right-hand side of equation (16) is representative for the scattering due to variations of the medium properties in the vertical direction. The scattering due to variations of the medium properties in the horizontal directions is contained in $\hat{\Lambda}_{JM}$ and, implicitly, in $\hat{L}_{IJ}$ also.

To investigate whether solutions of equation (17) exist, we introduce the generalized eigenvector operators $\hat{L}_I^{(\pm)}$ according to

$$\hat{L}_I^{(+)} = \hat{L}_{I1} \ , \quad \hat{L}_I^{(-)} = \hat{L}_{I2} \ . \tag{18}$$

Upon writing

$$\hat{\Lambda} = \mathrm{diag}[\hat{\Gamma}^{(+)}, \hat{\Gamma}^{(-)}] \ , \tag{19}$$

the diagonal entries representing the generalized eigenvalue operators, equation (17) decomposes into the two systems of equations

$$\hat{\mathsf{A}}_{IJ}\hat{L}_J^{(\pm)} = \hat{L}_I^{(\pm)}\hat{\Gamma}^{(\pm)} \ . \tag{20}$$

By analogy with the case where the medium is translationally invariant in the horizontal directions, we shall denote $\hat{\Gamma}^{(\pm)}$ as the *vertical slowness* or vertical momentum operators. Notice that the operators $\hat{L}_1^{(\pm)}$ compose the acoustic pressure and that the operators $\hat{L}_2^{(\pm)}$ compose the vertical particle velocity from the elements of $\hat{W}_M$ associated with the up- and downgoing constituents.

In De Hoop (1996) an Ansatz procedure has been followed to solve the generalized eigenvalue-eigenvector problem (20) in operator sense: choosing the *vertical acoustic-power-flux normalization* analog, we satisfy the commutation rule

$$[\hat{\mathsf{A}}_{12}^{-1/2}\hat{L}_1^{(\pm)}, \hat{\mathsf{A}}_{12}^{1/2}\hat{\mathsf{A}}_{21}\hat{\mathsf{A}}_{12}^{1/2}] = 0 \ . \tag{21}$$

In this normalization, we find the vertical slowness operator or generalized eigenvalues to be

$$\hat{\Gamma}^{(+)} = -\hat{\Gamma}^{(-)} = \hat{\Gamma} = \hat{\mathrm{A}}^{1/2} \ , \quad \hat{\mathrm{A}} \equiv \hat{\mathsf{A}}_{12}^{1/2}\hat{\mathsf{A}}_{21}\hat{\mathsf{A}}_{12}^{1/2} \ ; \quad \hat{\Gamma}^2 = \hat{\mathrm{A}} \tag{22}$$

is the characteristic operator equation, while the generalized eigenvectors constitute the *composition* operator

$$\hat{L} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{\mathsf{A}}_{12}^{1/2}\hat{\Gamma}^{-1/2} & \hat{\mathsf{A}}_{12}^{1/2}\hat{\Gamma}^{-1/2} \\ \hat{\mathsf{A}}_{12}^{-1/2}\hat{\Gamma}^{1/2} & -\hat{\mathsf{A}}_{12}^{-1/2}\hat{\Gamma}^{1/2} \end{pmatrix} . \tag{23}$$

The *decomposition* operator then follows as

$$\hat{L}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{\Gamma}^{1/2}\hat{\mathsf{A}}_{12}^{-1/2} & \hat{\Gamma}^{-1/2}\hat{\mathsf{A}}_{12}^{1/2} \\ \hat{\Gamma}^{1/2}\hat{\mathsf{A}}_{12}^{-1/2} & -\hat{\Gamma}^{-1/2}\hat{\mathsf{A}}_{12}^{1/2} \end{pmatrix} . \tag{24}$$

The (de)composition operators account for the radiation patterns of the different source and receiver types.

Using the decomposition operator, equation (16) transforms into

$$(\delta_{IP}\,\partial_3 + \mathrm{i}\omega\hat{\Lambda}_{IP})\hat{W}_P = -(\hat{L}^{-1})_{IM}(\partial_3\hat{L}_{MP})\hat{W}_P + (\hat{L}^{-1})_{IM}\hat{N}_M , \tag{25}$$

which can be interpreted as a coupled system of one-way wave equations. The coupling between the counter-propagating components, $\hat{W}_1$ and $\hat{W}_2$, is apparent in the first source-like term on the right-hand side, which can be written as

$$-\hat{L}^{-1}(\partial_3\hat{L}) = \begin{pmatrix} \hat{T} & \hat{R} \\ \hat{R} & \hat{T} \end{pmatrix} , \tag{26}$$

in which $\hat{T}$ and $\hat{R}$ represent the *transmission* and *reflection* operators, respectively. We introduce the shorthand notation

$$\hat{\mathsf{B}} \equiv \hat{\Lambda} + \hat{L}^{-1}((\mathrm{i}\omega)^{-1}\partial_3\hat{L}) \tag{27}$$

and

$$\hat{X} \equiv \hat{L}^{-1}\,\hat{N} \tag{28}$$

so that the system of one-way equations can be written in the simplified form (the counterpart of equation (1), where x_3 has taken the role of t)

$$(\delta_{IP}\,\partial_3 + \mathrm{i}\omega\hat{\mathsf{B}}_{IP})\hat{W}_P = \hat{X}_I , \quad \hat{\mathsf{B}}_{IP} = \hat{\mathsf{B}}_{IP}(\boldsymbol{x}, D_{1,2}) . \tag{29}$$

We now introduce the *one-way* Green's functions according to

$$(\partial_3 \pm \mathrm{i}\omega\hat{\Gamma})\hat{\mathcal{G}}^{(\pm)} = \delta(. - x'_{1,2})\,\delta(. - x'_3) , \tag{30}$$

supplemented with the condition of causality enforcing that $\hat{\mathcal{G}}^{(\pm)}$ decays as $x_3 \to \pm\infty$. With the aid of Duhamel's principle (Dieudonné, 1983, 23.66.10), we find the alternative formulation that $\hat{\mathcal{G}}^{(+)}$ satifies the homogeneous equation for $x_3 > 0$ and is subjected to (Hörmander, 1983, 23.1.2)

$$\begin{aligned} &\lim_{x_3 \downarrow x'_3} \hat{\mathcal{G}}^{(+)}(x_{1,2}, x_3, x'_{1,2}, x'_3) = \delta(x_{1,2} - x'_{1,2}) , \\ &\lim_{x_3 \uparrow x'_3} \hat{\mathcal{G}}^{(+)}(x_{1,2}, x_3, x'_{1,2}, x'_3) = 0 , \end{aligned} \tag{31}$$

for all $\omega > 0$, and similarly for $-\hat{\mathcal{G}}^{(-)}$ with the limits interchanged. Together they form

$$\hat{\mathcal{G}} = \mathrm{diag}[\hat{\mathcal{G}}^{(+)}, \hat{\mathcal{G}}^{(-)}] , \tag{32}$$

and satisfy system (29) where in $\hat{\mathsf{B}}$ the coupling term has been dropped (as if $\omega \to \infty$), i.e.

$$(\delta_{IP}\,\partial_3 + \mathrm{i}\omega\hat{\Lambda}_{IP})\hat{\mathcal{G}}_{PI'} = \delta_{II'}\delta(. - x'_{1,2})\delta(. - x'_3) \tag{33}$$

cf. equation (30).

Consider the real inner product (of time-convolution type)

$$\langle \hat{V} \,|\, \hat{W} \rangle = \int_{\mathbb{R}^2} \hat{V}_P\hat{W}_P \,\mathrm{d}x_1\mathrm{d}x_2 . \tag{34}$$

With respect to this product, it follows that the vertical slowness operator is symmetric,

$$\hat{\Gamma}^T = \hat{\Gamma} ,$$

understood in the following sense (see, e.g., Reed and Simon (1980, VIII): The operator $\hat{\Gamma} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is essentially self-adjoint ($(\hat{\Gamma}^T)^T = \hat{\Gamma}^T$) and has a unique self-adjoint extension that reduces to $\hat{\Gamma}$ on $H^1(\mathbb{R}^2)$ (see also Jonsson and De Hoop(2001)).

Here, $\mathbb{R}^2 \ni (x_1, x_2)$ and L^2 is the space of Lebesgue square integrable functions. Also (see De Hoop (1996, IV.3-IV.4))

$$J\,\hat{\mathsf{B}} = -\hat{\mathsf{B}}^T J\;, \tag{35}$$

the counterpart of equation (6), where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\;.$$

(These symmetries hold in the particular normalization only.) Note that the operators are *not* self-adjoint with respect to the complex inner product (of time-correlation type) $\langle \hat{V} \mid K\hat{W}\rangle_{\mathbb{C}}$ with

$$\langle \hat{V} \mid \hat{W}\rangle_{\mathbb{C}} = \int_{\mathbb{R}^2} \hat{V}_P \overline{\hat{W}_P}\,\mathrm{d}x_1\mathrm{d}x_2 \tag{36}$$

and

$$K = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

(see also De Hoop (1996, II.49)). This is at the basis of loosing perfect time reversibility.

3 TIME-REVERSAL CAVITIES: THE FULL-WAVE EQUATION

3.1 Reciprocity theorem of the time-correlation type

We consider two states, Y and Z say, each satisfying a system of equations of the form (1) on a common domain Ω. We will assume that the media in states Y and Z are each other's (time-reversed) adjoint. The reciprocity theorem of the time-correlation type can then be formulated as (De Hoop and De Hoop, 2000, 10.7)

$$\underbrace{\int_{\partial\Omega} \langle \mathcal{N}F^Y \mid \overbrace{F^Z}^{\text{control}}\rangle_{\mathbb{C}}\,\mathrm{d}A(\boldsymbol{x})}_{\text{TRC `experiment'}} = \underbrace{\int_{\Omega} \left(\langle Q^Y \mid F^Z\rangle_{\mathbb{C}} + \langle F^Y \mid Q^Z\rangle_{\mathbb{C}}\right)\,\mathrm{d}V(\boldsymbol{x})}_{\text{reciprocity}}\;, \tag{37}$$

where we made use of Gauss' theorem. Here, $\Omega \subset \mathbb{R}^3$ is compact with boundary $\partial\Omega$. In applications, not all of $\partial\Omega$ need be controllable.

3.2 Time-reversal cavities

Let us consider instantaneous point sources, i.e., let

$$Q_K^Y(\boldsymbol{x}, t) = a_K^Y \delta(\boldsymbol{x} - \boldsymbol{x}^Y)\,\delta(t - T) \tag{38}$$

be the source distribution of State Y, and

$$Q_L^Z(\boldsymbol{x}, t) \;= a_L^Z \delta(\boldsymbol{x} - \boldsymbol{x}^Z)\,\delta(t) \tag{39}$$

be the source distribution of State Z (the model state). Here, a^Y, a^Z are constant 4×1 matrices controlling the source types. In terms of Green's functions, we write the field matrices of the respective states as

$$F_P^Y(\boldsymbol{x}, t) = a_K^Y G_{PK}^Y(\boldsymbol{x}, \boldsymbol{x}^Y, t - T)\;, \tag{40}$$

and

$$F_Q^Z(\boldsymbol{x}, t) = a_L^Z G_{QL}^Z(\boldsymbol{x}, \boldsymbol{x}^Z, t)\;. \tag{41}$$

Substituting equations (40)-(41) into the left-hand side of equation (37) yields

$$\begin{aligned} &\int_{\partial D} \langle \mathcal{N}F^Y \mid F^Z\rangle_{\mathbb{C}}\,\mathrm{d}A(\boldsymbol{x}) \\ &= a_K^Y a_L^Z \int_{\partial D} \mathcal{N}_{QP} G_{PK}^Y(\boldsymbol{x}, \boldsymbol{x}^Y, . - T) \overset{(-t)}{*} G_{QL}^Z(\boldsymbol{x}, \boldsymbol{x}^Z, .)\,\mathrm{d}A(\boldsymbol{x})\;. \end{aligned} \tag{42}$$

We will investigate the surface integral

$$\mathcal{E}_{LK}(\boldsymbol{x}^Y,\boldsymbol{x}^Z,t) \equiv \int_{\partial D} \mathcal{N}_{QP} G^Y_{PK}(\boldsymbol{x},\boldsymbol{x}^Y,.-T) \overset{(-t)}{*} G^Z_{QL}(\boldsymbol{x},\boldsymbol{x}^Z,.)\,\mathrm{d}A(\boldsymbol{x}). \tag{43}$$

Invoking the reciprocity relation (of the time-convolution type) for the Green's function,

$$\delta^-_{PM} G^Y_{PK}(\boldsymbol{x},\boldsymbol{x}^Y,t) = \delta^-_{PK} G^Y_{PM}(\boldsymbol{x}^Y,\boldsymbol{x},t)\ ,$$

leads to

$$\begin{aligned}&\mathcal{E}_{LK}(\boldsymbol{x}^Y,\boldsymbol{x}^Z,t)\\ &= \int_{\partial D} \mathcal{N}_{QP} \delta^-_{PR} G^Y_{SR}(\boldsymbol{x}^Y,\boldsymbol{x},.-T)\delta^-_{SK} \overset{(-t)}{*} G^Z_{QL}(\boldsymbol{x},\boldsymbol{x}^Z,.)\,\mathrm{d}A(\boldsymbol{x})\end{aligned}$$

and with the symmetry property of $\mathcal{N}$ inherited from equation (6),

$$\begin{aligned}&\mathcal{E}_{LK}(\boldsymbol{x}^Y,\boldsymbol{x}^Z,t)\\ &= -\delta^-_{SK} \int_{\partial D} G^Y_{SR}(\boldsymbol{x}^Y,\boldsymbol{x},.) \overset{(t)}{*} \delta^-_{QP} \mathcal{N}_{PR} G^Z_{QL}(\boldsymbol{x},\boldsymbol{x}^Z,T-.)\mathrm{d}A(\boldsymbol{x})\ .\end{aligned} \tag{44}$$

$\mathcal{E}$ represents the following experiment: a field G^Z is emitted from a point source at $\boldsymbol{x}^Z$ and time 0, and its components are recorded over a surface ∂D – the *mirror* – in a time interval $[0,T]$. These are then *time reversed*, properly transformed according the surface normal, and re-emitted through State Y (indentifyable as a Love integral representation). Observations are made at $\boldsymbol{x}^Y$ as a function of t.

To predict the outcome of this experiment, we evaluate the surface integral with the aid of the volume integral representation in the right-hand side of equation (37). Since then

$$\int_D \langle Q^Y \mid F^Z\rangle_{\mathbb{C}}\,\mathrm{d}V(\boldsymbol{x}) = a^Y_K a^Z_L G^Z_{KL}(\boldsymbol{x}^Y,\boldsymbol{x}^Z,T-t)\ ,$$

while

$$\int_D \langle F^Y \mid Q^Z\rangle_{\mathbb{C}}\,\mathrm{d}V(\boldsymbol{x}) = a^Y_K a^Z_L G^Y_{LK}(\boldsymbol{x}^Z,\boldsymbol{x}^Y,t-T)\ ,$$

we find that

$$\begin{aligned}&a^Y_K a^Z_L\ \mathcal{E}_{LK}(\boldsymbol{x}^Y,\boldsymbol{x}^Z,t)\\ &= a^Y_K a^Z_L\ \underbrace{G^Y_{LK}(\boldsymbol{x}^Z,\boldsymbol{x}^Y,t-T)}_{\delta^-_{KR} G^Y_{RS}(\boldsymbol{x}^Y,\boldsymbol{x}^Z,t-T)\,\delta^-_{SL}}\ + a^Y_K a^Z_L G^Z_{KL}(\boldsymbol{x}^Y,\boldsymbol{x}^Z,T-t)\ .\end{aligned}$$

The first term on the right-hand side represents an outgoing wave, while the second term represents an imploding (incoming) wave. The imploding wave collapses where $\boldsymbol{x}^Y = \boldsymbol{x}^Z$, the original source location; it is followed by the outgoing wave, the 'switch' taking place at instant T. Through the superposition of these two constituents the field remains finite in Ω.

We observe that, in general, the field (on the boundary) does not decay for finite times T. Local energy decay (Ávila and Costa, 1980), as T becomes large, allows one to focus with the time-reversal cavity the acoustic energy to an arbitrary subregion of the cavity.

4 TIME-REVERSAL MIRRORS: THE ONE-WAY WAVE EQUATION

4.1 Extraction of the self-adjoint state

We extract the *self-adjoint (real) part* of the vertical slowness operator,

$$\hat{\mathring{\Gamma}} = \tfrac{1}{2}(\hat{\Gamma} + \hat{\Gamma}^*)\ , \tag{45}$$

which, on the wavefront set (classical limit) coincides with the original vertical slowness operator ⋆. We now replace the vertical slowness operator by its real part, and we indicate the induced state with $\breve{\ }$. In this state, we then have the symmetry property

$$K\,\hat{\breve{\mathsf{B}}} = \hat{\breve{\mathsf{B}}}^{*} K\ , \tag{46}$$

the counterpart of equation (8).

4.2 Reciprocity theorem of the time-correlation type

Consider the interaction of states, Y and Z say,

$$\langle \hat{\breve{W}}^{Y} \mid K\,\hat{\breve{W}}^{Z} \rangle_{\mathbb{C}}$$

which represents a time correlation nested in a transverse space integration. By virtue of symmetries (46), we find that

$$\partial_3 \langle \hat{\breve{W}}^{Y} \mid K\,\hat{\breve{W}}^{Z} \rangle_{\mathbb{C}} = \langle \hat{X}^{Y} \mid K\,\hat{\breve{W}}^{Z} \rangle_{\mathbb{C}} + \langle \hat{\breve{W}}^{Y} \mid K\,\hat{X}^{Z} \rangle_{\mathbb{C}}. \tag{47}$$

States Y and Z each satisfy a system of equations of the form (25). We have assumed that the media in states Y and Z are each other's (time-reversed) adjoint.

The reciprocity theorem of the time-correlation type can then be formulated as

$$\underbrace{\langle \hat{\breve{W}}^{Y} \mid \overbrace{K\,\hat{\breve{W}}^{Z}}^{\text{control}} \rangle_{\mathbb{C}} \Big|_{x_3'}^{x_3}}_{\text{TRM `experiment'}} = \underbrace{\int_{[x_3',x_3]} \left(\langle \hat{X}^{Y} \mid K\,\hat{\breve{W}}^{Z} \rangle_{\mathbb{C}} + \langle \hat{\breve{W}}^{Y} \mid K\,\hat{X}^{Z} \rangle_{\mathbb{C}} \right) \mathrm{d}x_3}_{\text{one-way reciprocity}}\ , \tag{48}$$

where

$$\langle \hat{\breve{W}}^{Y} \mid K\,\hat{\breve{W}}^{Z} \rangle_{\mathbb{C}} \Big|_{x_3'}^{x_3} = \langle \hat{\breve{W}}^{Y} \mid K\,\hat{\breve{W}}^{Z} \rangle_{\mathbb{C}} \Big|_{x_3} - \langle \hat{\breve{W}}^{Y} \mid K\,\hat{\breve{W}}^{Z} \rangle_{\mathbb{C}} \Big|_{x_3'}$$

and $[x_3, x_3'] \subset \mathbb{R}$ is a closed and bounded interval.

4.3 Time-reversal mirrors in the self-adjoint-operator state

Let us consider instantaneous point sources, i.e., let

$$\hat{X}^{Y}_{I}(\boldsymbol{x}, t) = A^{Y}_{I}\,\delta(\boldsymbol{x} - \boldsymbol{x}^{Y})\exp(-\mathrm{i}\omega T) \tag{49}$$

be the source distribution of State Y, and

$$\hat{X}^{Z}_{J}(\boldsymbol{x}, t) = A^{Z}_{J}\,\delta(\boldsymbol{x} - \boldsymbol{x}^{Z}) \tag{50}$$

be the source distribution of State Z. Here, A^{Y}, A^{Z} are constant 2×1 matrices controling the directional source types through equation (28). We let $x^{Y}_{3}, x^{Z}_{3} \in [x_3', x_3]$. Consider the boundary contributions on the left-hand side of equation (48),

$$\hat{\breve{\mathcal{E}}}(\boldsymbol{x}^{Y}, \boldsymbol{x}^{Z}, \omega) \equiv \langle \hat{\breve{W}}^{Y} \mid K\,\hat{\breve{W}}^{Z} \rangle_{\mathbb{C}} \Big|_{x_3'}^{x_3} = \int_{\mathbb{R}^2} \hat{\breve{W}}^{Y}_{P} K_{PQ} \overline{\hat{\breve{W}}^{Z}_{Q}}\,\mathrm{d}x_1 \mathrm{d}x_2 \Big|_{x_3'}^{x_3}. \tag{51}$$

The conjugation here represents the monochromatic wave counterpart of time reversal. If we now drop the coupling or interaction term in $\hat{\mathsf{B}}$, we have

$$\hat{\breve{W}}^{Y}_{P}(x_{1,2}, x_3) = A^{Y}_{K}\hat{\breve{\mathcal{G}}}^{Y}_{PK}(x_{1,2}, x_3, x^{Y}_{1,2}, x^{Y}_{3})\exp(-\mathrm{i}\omega T)\ , \tag{52}$$

while

$$\hat{\breve{W}}^{Z}_{Q}(x_{1,2}, x_3) = A^{Z}_{L}\hat{\breve{\mathcal{G}}}^{Z}_{QL}(x_{1,2}, x_3, x^{Z}_{1,2}, x^{Z}_{3})\ . \tag{53}$$

⋆ In terms of operator symbols, $\hat{\Gamma}\exp[-\mathrm{i}(\alpha_1 x_1 + \alpha_2 x_2)] = \hat{\gamma}(x_{1,2}, x_3, \alpha_{1,2})\ \exp[-\mathrm{i}(\alpha_1 x_1 + \alpha_2 x_2)]$, we have $\hat{\breve{\gamma}}^{\mathrm{prin}} = \mathrm{Re}\{\hat{\gamma}^{\mathrm{prin}}\}$.

Then

$$\langle \hat{\check{W}}^{Y} \mid K\,\hat{\check{W}}^{Z}\rangle_{\mathbb{C}}\Big|_{x_3'}^{x_3} \tag{54}$$
$$= A_1^Y\overline{A_1^Z}\int_{\mathbb{R}^2}\hat{\check{\mathcal{G}}}^{(+)}(x_{1,2},.,x_{1,2}^Y,x_3^Y)\exp(-\mathrm{i}\omega T)\overline{\hat{\check{\mathcal{G}}}^{(+)}(x_{1,2},.,x_{1,2}^Z,x_3^Z)}\,\mathrm{d}x_1\mathrm{d}x_2\Big|_{x_3'}^{x_3}$$
$$-A_2^Y\overline{A_2^Z}\int_{\mathbb{R}^2}\hat{\check{\mathcal{G}}}^{(-)}(x_{1,2},.,x_{1,2}^Y,x_3^Y)\exp(-\mathrm{i}\omega T)\overline{\hat{\check{\mathcal{G}}}^{(-)}(x_{1,2},.,x_{1,2}^Z,x_3^Z)}\,\mathrm{d}x_1\mathrm{d}x_2\Big|_{x_3'}^{x_3}$$
$$= A_1^Y\overline{A_1^Z}\int_{\mathbb{R}^2}\hat{\check{\mathcal{G}}}^{(+)}(x_{1,2},x_3,x_{1,2}^Y,x_3^Y)\overline{\hat{\check{\mathcal{G}}}^{(+)}(x_{1,2},x_3,x_{1,2}^Z,x_3^Z)\exp(\mathrm{i}\omega T)}\,\mathrm{d}x_1\mathrm{d}x_2$$
$$-A_2^Y\overline{A_2^Z}\int_{\mathbb{R}^2}\hat{\check{\mathcal{G}}}^{(-)}(x_{1,2},x_3',x_{1,2}^Y,x_3^Y)\overline{\hat{\check{\mathcal{G}}}^{(-)}(x_{1,2},x_3',x_{1,2}^Z,x_3^Z)\exp(\mathrm{i}\omega T)}\,\mathrm{d}x_1\mathrm{d}x_2\ ,$$

using the boundary conditions. Substituting the reciprocity relation (of time-convolution type)

$$\hat{\mathcal{G}}^{(\pm)}(x_{1,2},.,x_{1,2}^Y,x_3^Y) = -\hat{\mathcal{G}}^{(\mp)}(x_{1,2}^Y,x_3^Y,x_{1,2},.)\ ,$$

we then obtain

$$\hat{\check{\mathcal{E}}}(\boldsymbol{x}^Y,\boldsymbol{x}^Z,\omega) = \tag{55}$$
$$A_1^Y\overline{A_1^Z}\int_{\mathbb{R}^2}\hat{\check{\mathcal{G}}}^{(-)}(x_{1,2}^Y,x_3^Y,x_{1,2},x_3)\overline{\hat{\check{\mathcal{G}}}^{(+)}(x_{1,2},x_3,x_{1,2}^Z,x_3^Z)\exp(\mathrm{i}\omega T)}\,\mathrm{d}x_1\mathrm{d}x_2$$
$$-A_2^Y\overline{A_2^Z}\int_{\mathbb{R}^2}\hat{\check{\mathcal{G}}}^{(+)}(x_{1,2}^Y,x_3^Y,x_{1,2},x_3')\overline{\hat{\check{\mathcal{G}}}^{(-)}(x_{1,2},x_3',x_{1,2}^Z,x_3^Z)\exp(\mathrm{i}\omega T)}\,\mathrm{d}x_1\mathrm{d}x_2\ ,$$

the counterpart of equation (44). Typically, we choose either $A_2^Z = 0$ or $A_1^Z = 0$ yielding a *one-sided* mirror †. With either choice, experiment $\hat{\check{\mathcal{E}}}$ yields the outcome

$$\hat{\check{\mathcal{E}}}(\boldsymbol{x}^Y,\boldsymbol{x}^Z,\omega)$$
$$= A_1^Y\overline{A_1^Z}\,\overline{\hat{\check{\mathcal{G}}}^{(+)}(\boldsymbol{x}^Y,\boldsymbol{x}^Z)\exp(\mathrm{i}\omega T)} + A_1^Y\overline{A_1^Z}\,\hat{\check{\mathcal{G}}}^{(+)}(\boldsymbol{x}^Z,\boldsymbol{x}^Y)\exp(-\mathrm{i}\omega T)$$
$$-A_2^Y\overline{A_2^Z}\,\overline{\hat{\check{\mathcal{G}}}^{(-)}(\boldsymbol{x}^Y,\boldsymbol{x}^Z)\exp(\mathrm{i}\omega T)} - A_2^Y\overline{A_2^Z}\,\hat{\check{\mathcal{G}}}^{(-)}(\boldsymbol{x}^Z,\boldsymbol{x}^Y)\exp(-\mathrm{i}\omega T)\ , \tag{56}$$

which, with the aid of the reciprocity properties of the one-way Green's functions, can be written in the form

$$\hat{\check{\mathcal{E}}}(\boldsymbol{x}^Y,\boldsymbol{x}^Z,\omega)$$
$$= A_1^Y\overline{A_1^Z}\,\overline{\hat{\check{\mathcal{G}}}^{(+)}(\boldsymbol{x}^Y,\boldsymbol{x}^Z)\exp(\mathrm{i}\omega T)} - A_1^Y\overline{A_1^Z}\,\hat{\check{\mathcal{G}}}^{(-)}(\boldsymbol{x}^Y,\boldsymbol{x}^Z)\exp(-\mathrm{i}\omega T)$$
$$-A_2^Y\overline{A_2^Z}\,\overline{\hat{\check{\mathcal{G}}}^{(-)}(\boldsymbol{x}^Y,\boldsymbol{x}^Z)\exp(\mathrm{i}\omega T)} + A_2^Y\overline{A_2^Z}\,\hat{\check{\mathcal{G}}}^{(+)}(\boldsymbol{x}^Y,\boldsymbol{x}^Z)\exp(-\mathrm{i}\omega T)\ . \tag{57}$$

As for the cavity, the second/fourth term on the right-hand side represents an outgoing wave, while the first/third term represents an imploding (incoming) wave. The imploding wave collapses where $\boldsymbol{x}^Y = \boldsymbol{x}^Z$, the original source location; it is followed by the outgoing wave, the 'switch' taking place at instant T when $x_3^Y = x_3^Z$. Invoking Duhamel's principle, for the example $A_2^Y = A_2^Z = 0$, yields

$$\lim_{x_3^Y \downarrow x_3^Z}\hat{\check{\mathcal{E}}}(\boldsymbol{x}^Y,\boldsymbol{x}^Z,\omega) = A_1^Y\overline{A_1^Z}\,\delta(x_{1,2}^Y - x_{1,2}^Z)\exp(-\mathrm{i}\omega T)\ . \tag{58}$$

We remark that one key difference between the TRC and the TRM is that the latter requires a *single* component (sensor) measurement only whereas the first requires dual-component sensors on the cavity boundary.

† We can interpret the one-sided mirror experiment, with source strength set to 1, as follows. If the one-way Green's function is considered to be the kernel of an operator, the experiment yields the composition of this operator with its adjoint, resulting in the so-called normal operator. This operator plays a central role in inverse scattering.

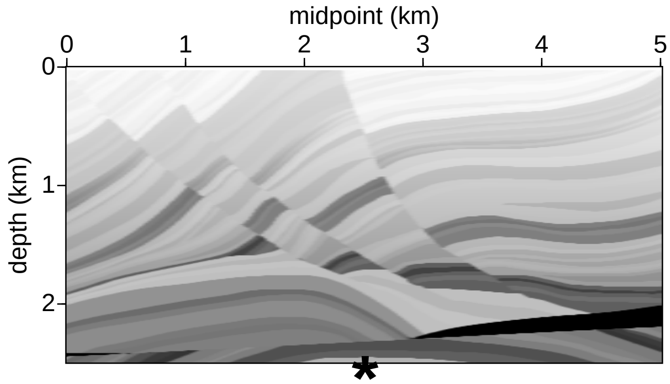

Figure 1. The Marmoussi medium velocity model (constant density); the source position $\boldsymbol{x}^Z$ is indicated by an asterisk.

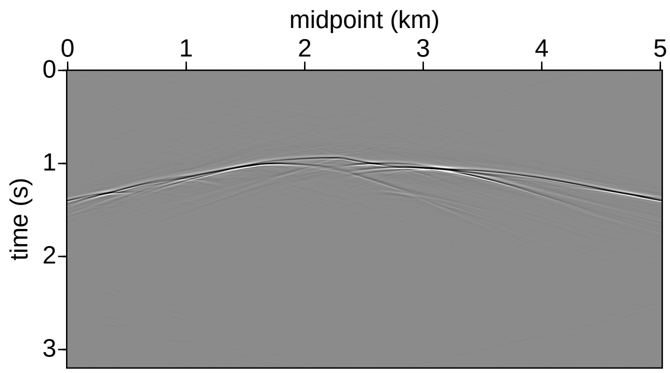

Figure 2. The boundary upgoing field measurement (State Z).

4.4 Numerical example from exploration seismology

We illustrate the above developed TRM by generating a self-adjoint-operator state. For this purpose, we employ the generalized screen expansion of the one-way Green's function (De Hoop et al., 2000; Le Rousseau and De Hoop, 2001). As the model, we use Marmoussi (figure 1; the source location is indicated by an asterisk) developed to mimic the geology offshore West Africa. In figure 2 we show the hydrophone recordings on the Earth's surface (the mirror in this case) for a time interval with $T = 3200$ms. These are then time reversed (figure 3) and re-emitted into the subsurface. The result is shown in figure 4 with horizontal section at the orginal source depth shown in figure 5. Retrofocusing has clearly been accomplished. Below the main figure is shown a sinc function corresponding to the equivalent numerical experiment in a homogeneous medium; the scattering in the heterogeneous medium leads to improved retrofocusing.

5 INCOMPLETE RETROFOCUSING WITH THE ONE-WAY WAVE EQUATION: HOMOGENEOUS MEDIUM

We will investigate the outcome of experiment $\mathcal{E}$ in the case of the original vertical slowness operator. The time-reversal mirror experiment, in a homogeneous medium, reduces to the composition (of Lagrangian distributions)

$$\int_{\mathrm{I\!R}^2} \hat{\mathcal{G}}^{(-)}(x^Y_{1,2}, x^Y_3, x_{1,2}, x_3)\overline{\hat{\mathcal{G}}^{(+)}(x_{1,2}, x_3, x^Z_{1,2}, x^Z_3)}\, \mathrm{d}x_1 \mathrm{d}x_2$$

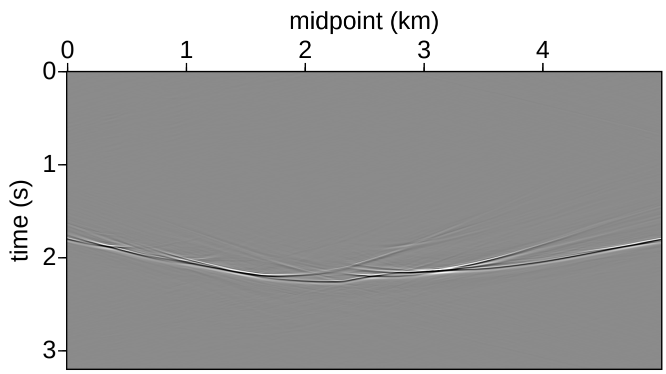

Figure 3. The boundary control (plotted through the mapping $t \to T - t$).

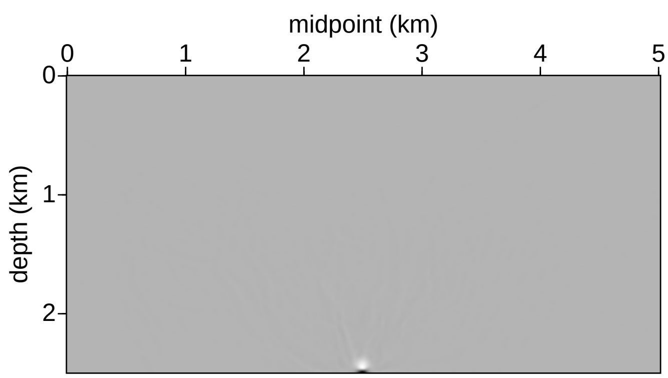

Figure 4. Retrofocusing at time T.

$$
\begin{aligned}
&= (2\pi)^{-4} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \exp[-\mathrm{i}K_3(k'_{1,2},\omega)(x_3^Y - x_3)] \exp[\mathrm{i}k'_\sigma(x_\sigma - x_\sigma^Y)] \\
&\overline{\int_{\mathbb{R}^2} \exp[\mathrm{i}K_3(k_{1,2},\omega)(x_3 - x_3^Z)] \exp[\mathrm{i}k_\sigma(x_\sigma - x_\sigma^Z)] \mathrm{d}k_1 \mathrm{d}k_2}\ \mathrm{d}x_1 \mathrm{d}x_2\, \mathrm{d}k'_1 \mathrm{d}k'_2 \\
&= (2\pi)^{-2} \int_{\mathbb{R}^2} \exp[-\mathrm{i}(K_3(k_{1,2},\omega)(x_3^Y - x_3) + \overline{K_3(k_{1,2},\omega)}(x_3 - x_3^Z)] \\
&\qquad \exp[-\mathrm{i}k_\sigma(x_\sigma^Y - x_\sigma^Z)] \mathrm{d}k_1 \mathrm{d}k_2\ . \qquad (59)
\end{aligned}
$$

In this expression,

$K_3(k_{1,2},\omega) = \sqrt{(\omega/c)^2 - k_1^2 - k_2^2}$

is deduced from the symbol $\omega\hat{\gamma}$, with $k_{1,2} = \omega\alpha_{1,2}$, of operator $\hat{\Gamma}$. The phase function in expression (59) has the appearance

$$
\begin{aligned}
&K_3(k_{1,2},\omega)(x_3^Y - x_3) + \overline{K_3(k_{1,2},\omega)}(x_3 - x_3^Z) \qquad (60) \\
&\quad = \begin{cases} K_3(k_{1,2},\omega)(x_3^Y - x_3^Z) & \text{if} \quad \sqrt{k_1^2 + k_2^2} < \omega/c \\ -\kappa_3(k_{1,2},\omega)\zeta & \text{if} \quad \sqrt{k_1^2 + k_2^2} > \omega/c \end{cases}
\end{aligned}
$$

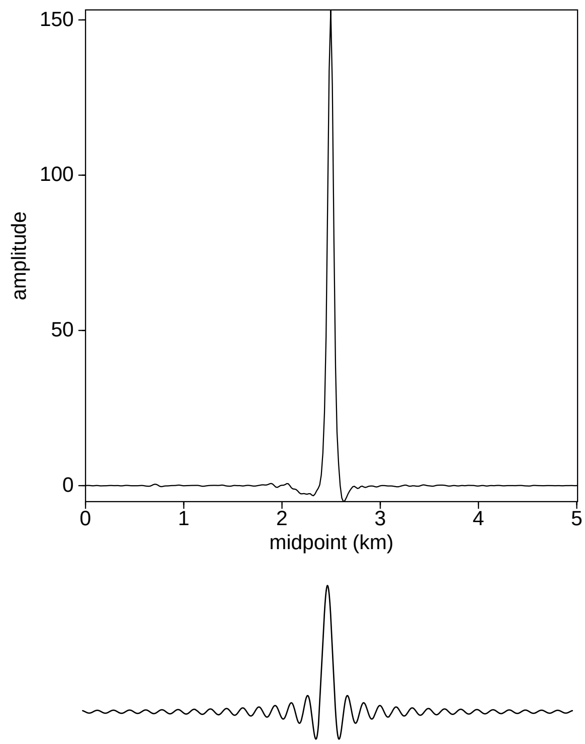

Figure 5. Section at x_3^Z through figure 4.

with

$$\zeta \equiv 2x_3 - x_3^Z - x_3^Y \ , \quad \kappa_3(k_{1,2},\omega) = -\mathrm{i}K_3(k_{1,2},\omega) \ .$$

We restrict $\omega > 0$. With this appearance we can introduce a parametrization in terms of scattering angle θ and azimuth ϕ:

$$\begin{aligned} & K_3(k_{1,2},\omega)(x_3^Y - x_3) + \overline{K_3(k_{1,2},\omega)}(x_3 - x_3^Z) \\ & \quad = \begin{cases} \omega c^{-1}\cos\theta\,(x_3^Y - x_3^Z) & \text{with} \quad \theta \in [0, \tfrac{1}{2}\pi] \\ \omega c^{-1}\sinh\tau\,\zeta & \text{with} \quad \tau \in [0,\infty) \end{cases} \end{aligned} \tag{61}$$

while

$$\begin{cases} k_1 \\ k_2 \end{cases} = \begin{cases} \begin{cases} \omega c^{-1}\cos\phi\sin\theta \\ \omega c^{-1}\sin\phi\sin\theta \end{cases} & \text{with} \quad \theta \in [0, \tfrac{1}{2}\pi] \\ \begin{cases} \omega c^{-1}\cos\phi\cosh\tau \\ \omega c^{-1}\sin\phi\cosh\tau \end{cases} & \text{with} \quad \tau \in [0,\infty) \end{cases} \tag{62}$$

with $\phi \in [0, 2\pi)$.

In the TRM, in accordance with the symmetries invoked in the previous section, we introduce the cut-off function $\chi = \chi(k_{1,2},\omega)$ with cut-off criterion at ω/c. We thus define

$$\begin{aligned} & \hat{\mathcal{I}}_p = (2\pi)^{-2} \\ & \times \int_{\mathbb{R}^2} \chi(k_{1,2},\omega)\exp[-\mathrm{i}K_3(k_{1,2},\omega)(x_3^Y - x_3^Z)]\exp[-\mathrm{i}k_\sigma(x_\sigma^Y - x_\sigma^Z)]\mathrm{d}k_1\mathrm{d}k_2 \ , \end{aligned} \tag{63}$$

and similarly $\mathcal{I}_e$ with χ replaced by $1-\chi$. With the cut-off function, $\mathcal{I}_p$ becomes an oscillatory integral with real phase function. We have

$$\mathcal{I}_p = \mathcal{I}_p(\boldsymbol{x}^Y, \boldsymbol{x}^Z, t, T) = \pi^{-1}\mathrm{Re}\int_0^\infty \hat{\mathcal{I}}_p \exp(-\mathrm{i}\omega T)\ \exp(\mathrm{i}\omega t)\,\mathrm{d}\omega\ .$$

To simplify the further analysis, we introduce I_p according to

$$\mathcal{I}_p = \partial_{x_3}\ I_p\ , \tag{64}$$

where the derivative is taken in distributional sense.

5.1 Gel'fand's plane-wave expansion

We observe that I_p, with the aid of parametrization in scattering angle and azimuth, can be written in the form

$$\begin{aligned}&\tfrac{1}{2}\partial_t\ I_p = -\frac{1}{8\pi^2 c}\int_0^{\pi/2}\sin\theta\,\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi \\ &\delta''\left[(t-T) - c^{-1}((x_1^Y - x_1^Z)\cos\phi + (x_2^Y - x_2^Z)\sin\phi)\ \sin\theta - c^{-1}(x_3^Y - x_3^Z)\cos\theta\right]\ ,\end{aligned} \tag{65}$$

with $z = x_3^Y - x_3^Z$. To complete the right-hand side to Gel'fand's plane-wave expansion of the Dirac distribution (see, for example, Poritsky (1951)), we need to add the integral representation

$$\begin{aligned}&\tfrac{1}{2}\partial_t\ I_e = -\frac{1}{8\pi^3 c}\partial_t^2\int_0^\infty\cosh\tau\,\mathrm{d}\tau\int_0^{2\pi}\mathrm{d}\phi \\ &\frac{(t-T) - c^{-1}((x_1^Y - x_1^Z)\cos\phi + (x_2^Y - x_2^Z)\sin\phi)\ \cosh\tau}{[(t-T) - c^{-1}((x_1^Y - x_1^Z)\cos\phi + (x_2^Y - x_2^Z)\sin\phi)\cosh\tau]^2 + [c^{-1}\zeta\sinh\tau]^2}\,.\end{aligned} \tag{66}$$

This (missing) integral precisely represents the incomplete retrofocusing. It has a *cylindrical* singular support oriented along the vertical (ζ) axis.

5.2 The TRM generated distribution

For the sake of simplicity, we set $c = 1$. We take the cut-off function to be sharp. Changing to polar coordinates, we then obtain the oscillatory integral

$$I_p = 4\pi\int_0^\infty\int_0^\omega \sin(\omega t + \sqrt{\omega^2 - k^2}z)\frac{k\,J_0(kr)}{\sqrt{\omega^2 - k^2}}\,\mathrm{d}k\mathrm{d}\omega\ , \tag{67}$$

with

$$k^2 = k_1^2 + k_2^2\ ,\quad r^2 = (x_1^Z - x_1^Y)^2 + (x_2^Z - x_2^Y)^2\ ,\quad z = x_3^Z - x_3^Y$$

(read $t - T$ for t). We relate the inner integral to the Fourier transform of the generalized function $(t^2 - r^2)_+^{-1/2}$ with 'parameter' $|t|$ (Gel'fand and Shilov, 1964, p.185). To this end, we change variables according to $\omega^2 = \eta^2 + k^2$,

$$I_p = 4\pi\int_0^\infty kJ_0(kr)\int_0^\infty\frac{\sin(t\sqrt{\eta^2 + k^2} + \eta z)}{\sqrt{\eta^2 + k^2}}\mathrm{d}\eta\mathrm{d}k\ . \tag{68}$$

Then

$$I_p = 4\pi\int_0^\infty\left[\mathrm{sgn}(t)\ \cos(\eta z)\,\imath_p^s(k,\eta) + \sin(\eta z)\,\imath_p^c(k,\eta)\right]\mathrm{d}\eta\ , \tag{69}$$

with

$$\imath_p^s(k,\eta) = \int_0^\infty J_0(kr)\sin(|t|\sqrt{\eta^2 + k^2})\frac{k}{\sqrt{\eta^2 + k^2}}\mathrm{d}k\ , \tag{70}$$

$$\imath_p^c(k,\eta) = \int_0^\infty J_0(kr)\cos(|t|\sqrt{\eta^2 + k^2})\frac{k}{\sqrt{\eta^2 + k^2}}\mathrm{d}k\ . \tag{71}$$

Through the Fourier transform, these integrals represent the distributions,

$$\begin{aligned}\imath_p^s(k,\eta) &= (t^2 - r^2)_+^{-1/2}\cos(\eta\sqrt{t^2 - r^2})\ , \\ \imath_p^c(k,\eta) &= -(t^2 - r^2)_+^{-1/2}\sin(\eta\sqrt{t^2 - r^2})\end{aligned} \tag{72}$$

$$+(r^2-t^2)_+^{-1/2}\exp(-\eta\sqrt{r^2-t^2})\ . \tag{73}$$

Substituting equations (72)-(73) into equation (69), using the standard identities ($t^2-r^2\geq 0$)

$$\begin{aligned}(2/\pi)\int_0^\infty \cos(\eta z)\cos(\eta\sqrt{t^2-r^2})\,\mathrm{d}\eta &= \delta(\sqrt{t^2-r^2}-z)\\ &\quad+\delta(\sqrt{t^2-r^2}+z)\ ,\end{aligned} \tag{74}$$

$$\begin{aligned}(2/\pi)\int_0^\infty \sin(\eta z)\sin(\eta\sqrt{t^2-r^2})\,\mathrm{d}\eta &= \delta(\sqrt{t^2-r^2}-z)\\ &\quad-\delta(\sqrt{t^2-r^2}+z)\ ,\end{aligned} \tag{75}$$

and the classical integral

$$\int_0^\infty \sin(\eta z)\exp(-\eta\sqrt{r^2-t^2})\,\mathrm{d}\eta = \frac{z}{r^2+z^2-t^2}\ , \tag{76}$$

we arrive at

$$\begin{aligned}I_p = 4\pi^2[\mathrm{sgn}(t)-\mathrm{sgn}(z)]\,\overbrace{H(t^2-r^2)}^{r<t}\,\overbrace{\delta(t^2-(r^2+z^2))}^{\text{spherical}}\\ +4\pi\,\underbrace{(r^2-t^2)_+^{-1/2}\frac{z}{r^2+z^2-t^2}}_{r>t,\text{ cylindrical}}\ .\end{aligned} \tag{77}$$

We used the relations

$$\begin{aligned}&(t^2-r^2)_+^{-1/2}[\delta(\sqrt{t^2-r^2}-z)-\delta(\sqrt{t^2-r^2}+z)]\\ &\qquad=\mathrm{sgn}(z)\,H(t^2-r^2)\delta(t^2-r^2-z^2)\ ,\end{aligned}$$

$$\begin{aligned}&(t^2-r^2)_+^{-1/2}[\delta(\sqrt{t^2-r^2}-z)+\delta(\sqrt{t^2-r^2}+z)]\\ &\qquad=H(t^2-r^2)\delta(t^2-r^2-z^2)\ .\end{aligned}$$

Here, I_p can be considered as the kernel of an operator (normal operator) consisting of two parts. Note, as in the plane-wave expansion, that the second term in I_p brings in a cylindrical singular support oriented along the z-axis.

6 TIME-REVERSAL MIRRORS: THE ‘PARABOLIC EQUATION’ APPROXIMATION

The parabolic equation approximation of the one-way wave equation is obtained upon expanding equation (22) in the horizontal ‘Laplacian’. We assume constant density of mass. In the one-way wave equation (25; $(-)$) the constituent operator $\mathrm{i}\omega\hat{\Gamma}$ is then replaced by

$$\mathrm{i}\omega\hat{\Gamma}_p = \frac{\mathrm{i}\omega}{c}-\frac{\nabla\cdot(c\,\nabla\ )}{2\mathrm{i}\omega}\ . \tag{78}$$

(For a detailed analysis of the consquences of such expansions, see De Hoop and De Hoop (1992). The equation describes narrow-angle propagation and small-angle scattering.) We can then write the homogeneous one-way wave equation for W_2 (or $\mathcal{G}^{(-)}$) in the form

$$\frac{\mathrm{i}}{\omega}\,\partial_{x_3}W_2 = -\frac{1}{2\omega^2}\,\nabla\cdot(c\,\nabla W_2)\,\underbrace{-\frac{1}{c}}_{`V'}\,W_2\ , \tag{79}$$

which resembles the Schrödinger equation in two dimensions and ‘time’ x_3 (Merzbacher, 1970) with potential V.

Clearly, $\hat{\Gamma}_p$ is $\mathbb{R}$ and $\mathbb{C}$ self-adjoint and hence time reversibility is restored. The TRM has perfect retrofocusing properties. (It is interesting to note a remark by Khosla and Dowling (2001) referring to PE approximations: ‘TRA retrofocusing .. *even* when approximate forms for the Green’s function of environment are used’.)

7 DISCUSSION

We have investigated the implications of directional wavefield decomposition or splitting with a view to time reversibility. In particular, we have discussed how wavefield decomposition preserves the reciprocity theorem of time-convolution type but looses the reciprocity theorem of time-correlation type. As a consequence, a perfect time-reversal mirror in the framework of one-way wave theory does not exist: We find that on the wavefront set a time-reversal mirror can retrofocus the wavefield to its originating source, but that non-perfectly retrofocusing lower-order distributions contribute to the process as well. We have given explicitly such distribution and its extended singular support in the case of a homogeneous half space.

We have also established a connection between reciprocity theorems, time-reversal cavities (TRCs) and mirrors (TRMs) and boundary control theory. TRMs appear in inverse scattering theories through the notion of *backpropagation* (continuation); reciprocity theorems appear in the *optimization* approach to inverse scattering; the boundary control method aims at determining the controls (through a response operator) producing 'standard waves' in space at a given time (*wave shaping*), these controls then being subjected to an inversion procedure (Belishev, 1987; Belishev, 1997). The latter procedure transfers the subsurface medium information to the acquisition surface controls; the ray-geometric solution represention is employed to relate the 'standard waves' to the boundary controls, while from the 'standard waves' the medium properties can be deduced.

The analysis presented here aims at providing the theoretical foundation for the application of TRMs in inverse scattering, a subject of further research.

As a final remark, we assumed throughout, and made implicit use of, the unique continuation properties of wavefields across hypersurfaces and mirrors in particular. Such properties are inherently connected to the controllability of (solutions to) the system of wave equations. (For example, problems arise if the mirror is oriented along a characteristic surface.) Such assumption can be justified by Holmgren's and Hörmander's uniqueness theorems extended by Tataru (1995).

ACKNOWLEDGMENT

The authors would like to thank J.H. Le Rousseau for many valuable discussions and for carrying out the TRM simulations and generating the figures.

References

Ávila G.S.S. and Costa D.G., 1980, Asymptotic properties of general symmetric hyperbolic systems, *Journal of Functional Analysis* **35**, 49-63.

Belishev M.I., 1987, On an approach to multidimensional inverse problems for the wave equation, *Dokl. Akad. Nauk SSSR* **297 (3)**, 524-527.

Belishev M.I., 1997, Boundary control and inverse problems: One-dimensional variant of the BC-method, *St. Petersburg Department of the Steklov Mathematical Institute Preprint*.

Blomgren
P., Papanicolaou G. and Zhao H., 2001, Super-resolution in time-reversal acoustics, *J. Acoust. Soc. Am.* submitted (http://www.georgep.stanford.edu/papanico/pubs.html).

Bremmer H., 1939, Geometrisch optische benadering van de golfvegelijking, *Handel. Ned. Nat. en Geneeskd. Congr.*, 88-91 (in Dutch).

Catipovic J.A., 1997, Acoustic telemetry, in *Encyclopedia of Acoustics* Vol.1, Pt.4, Ch.51, pp.591-596 (New York: Wiley).

Cheney M., Isaacson D. and Lassas M., 2001, Optimal acoustical measurements, *SIAM J. Appl. Math.* submitted (http://www.rpi.edu/ chenem/downloads.html)

De Hoop M.V., 1996, Generalization of the Bremmer coupling series, *J. Math. Phys.* **37**, 3246-3282.

De Hoop M.V. and De Hoop A.T., 1992, Scalar space-time waves in their spectral-domain first- and second-order Thiele approximations, *Wave Motion* **15**, 229-265.

De Hoop M.V. and De Hoop A.T., 2000, Wavefield reciprocity and optimization in remote sensing, *Proc. R. Soc. Lond. A (Mathematical, Physical and Engineering Sciences)* **456**, 641-682.

De Hoop M.V., Le Rousseau J.-H. and Wu R.-S., 2000, Generalization of the phase-screen approximation for the scattering of acoustic waves, *Wave Motion* **31**, 43-70.

Derode A., Roux P. and Fink M., 1995, Robust acoustic time reversal with high-order multiple scattering, *Phys. Rev. Lett.* **75**, 4206-4209.

Dieudonné J., 1983, *Grundzüge der modernen Analysis* (Braunsweig: Friedr. Vieweg & Sohn).

Fink M., 1992, *IEEE Trans. Ultrason. Ferroelectr. Freq. Control* **39**, 555-566.

Fink M., 1993, Time-reversal mirrors, *J. Phys. D: Applied Phys.* **26**, 1333-1350.

Fink M., 1999, Time-reversed acoustics, *Scientific American* **5**, 91-97.

Gel'fand M.I. and Shilov G., 1964, *Generalized functions* Vol.1 (New York: Academic Press).

Gustafsson M., Jonsson B.L.G. and De Hoop M.V., 2001, On the boundary control of acoustic wave fields by time reversal, *Preprint*.

Hörmander L., 1983, *The analysis of linear partial differential operators IV* (Berlin: Springer-Verlag).

Jackson D.R. and Dowling D.R., 1991, Phase conjugation in underwater acoustics, *J. Acoust. Soc. Am.* **89**, 171-181.

Jonsson B.L.G. and De Hoop M.V., 2001, Wave field decomposition in anisotropic fluids: A spectral theory approach, *Acta Appl. Math.* in print.

Khosla S.R. and Dowling D.R., 1998, Time-reversing array retrofocusing in simple dynamic underwater environments, *J. Acoust. Soc. Am.* **104**, 3339-3350.

Khosla S.R. and Dowling D.R., 2001, Time-reversing array retrofocusing in noisy environments, *J. Acoust. Soc. Am.* **109**, 538-546.

Kuperman W.A., Hodgkiss W.S., Song H.C., Akal T., Ferla T. and Jackson D.R., 1998, Phase-conjugation in the ocean: Experimental demonstration of an acoustic time reversal mirror, *J. Acoust. Soc. Am.* **103**, 25-40.

Le Rousseau J.-H. and De Hoop M.V., 2001, Modeling and imaging with the scalar generalized-screen algorithms in isotropic media, *Geoph.* in print.

Lions J.L., 1971, *Optimal control of systems governed by partial differential equations* (Berlin: Springer-Verlag).

Merzbacher E., 1970, *Quantum mechanics* (New York: Wiley).

Parvulescu A. and Clay C.S., 1965, *Radio Electron. Eng.* **29**, 233.

Poritsky H., 1951, Extension of Weyl's integral for harmonic spherical waves to arbitrary shapes, *Comm. Pure Appl. Math.* **4**, 33-42.

Rayleigh Lord, 1873, Some general theorems relating to vibrations, *Proc. Lond. Math. Soc.* **4**, 357-368.

Reed M. and Simon B., 1980, *Methods of modern mathematical physics, Vol. I: Functional analysis* (San Diego: Academic Press).

Rietveld W.E.A., Berkhout A.J. and Wapenaar C.P.A., 1992, Optimum seismic illumination of hydrocarbon reservoirs, *Geoph.* **57**, 1334-1345.

Roux P., Roman B. and Fink M., 1997, *Appl. Phys. Lett.* **79**, 1811.

Schultz P.S. and Claerbout J.F., 1978, Velocity estimation and downward continuation by wavefront synthesis, *Geoph.* **43**, 691-714.

Taner M.T., 1976, Simplan: simulated plane-wave exploration, *46th Ann. Mtg. Soc. Expl. Geophys., Expanded Abstracts*, 186-187.

Tataru D., 1995, Unique continuation for solutions to PDE's: Between Hörmander's theorem and Holmgren's theorem, *Comm. Partial Differential Equations* **20**, 855-884.

Thomas J.-L., Wu F. and Fink M., 1996, *Ultrason. Imaging* **18**, 106-121.

Thorbecke J., 1997, *Common focus point technology* (Delft University of Technology: PhD thesis).

Weston V.H., 1987, Factorization of the wave equation in higher dimensions, *J. Math. Phys.* **28**, 1061-1068.