# Introduction to Mobile Robotics

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# Differential Drive Model

The standard differential drive model:

$$\dot{x} = \frac{r}{2}(\dot{\phi}_1 + \dot{\phi}_2)\cos(\theta)$$

$$\dot{y} = \frac{r}{2}(\dot{\phi}_1 + \dot{\phi}_2)\sin(\theta)$$

$$\dot{\theta} = \frac{r}{2L}(\dot{\phi}_1 - \dot{\phi}_2)$$

# Another Example

Assume that you have a differential drive robot.

If the drive wheel is 20cm in diameter and turns at 10 rpm, what is the linear speed of the rolling wheel (with no slip or skid).

We see that distance covered $s = \theta r$ and so $v = ds/dt = rd\theta/dt$. Note that $d\theta/dt = 2\pi\omega$, where $\omega$ is the rpm. So

$$v = 2\pi r\omega = 2\pi * 10 * 10 = 200\pi.$$

# Another Example

Let the distance between the wheels be 30cm (axle length). If the right wheel is turning at 10 rpm and the left is turning at 10.5 rpm, find a formula for the resulting motion.

As stated earlier, the motion for this robot would be a circle. Thus the two wheels trace out two concentric circles. The two circles must be traced out in the same amount of time:

$$t = \frac{d_1}{v_1} = \frac{d_2}{v_2} \Rightarrow \frac{d_1}{10.5 * 20\pi} = \frac{d_2}{10 * 20\pi} \Rightarrow \frac{2\pi(R+30)}{210\pi} = \frac{2\pi R}{200\pi}$$

$$\frac{30}{105} = R\left(\frac{1}{100} - \frac{1}{105}\right) = \frac{5R}{100 * 105}$$

$$\Rightarrow R = \frac{100 * 105}{5}\frac{30}{105} = 600$$

Thus we have $x^2 + y^2 = 600^2$ as the basic formula for the curve of motion.

# Discrete Differential Drive Model

The discretization of the differential drive model:

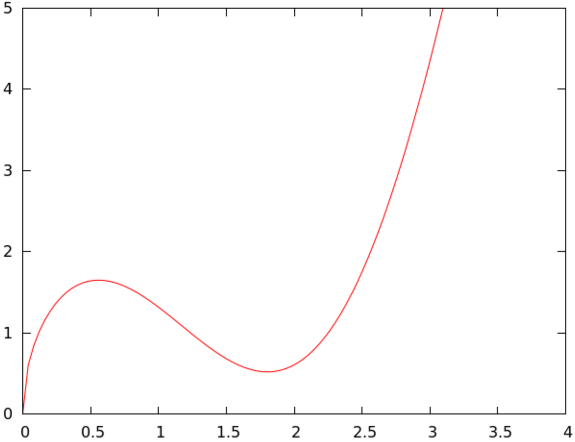
$$x_{k+1} = x_k + \frac{r\Delta t}{2}(\omega_{1,k} + \omega_{2,k})\cos(\theta_k)$$

$$y_{k+1} = y_k + \frac{r\Delta t}{2}(\omega_{1,k} + \omega_{2,k})\sin(\theta_k)$$

$$\theta_{k+1} = \theta_k + \frac{r\Delta t}{2L}(\omega_{1,k} - \omega_{2,k})$$

# Inverse Kinematics for the Differential Drive

Say you want to traverse a path: $x(t), y(t)$.



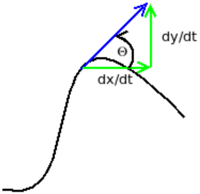
# Inverse Kinematics

You want to prescribe $x(t), y(t)$ and obtain $\dot{\phi}_1, \dot{\phi}_2$.

Clearly if you have $x(t), y(t)$, differentiation will yield $\dot{x}(t), \dot{y}(t)$, so we may assume that we know $\dot{x}(t), \dot{y}(t)$.

The direction of the robot is the direction of the curve:

$$\theta = \arctan \frac{\dot{y}}{\dot{x}} \ .$$




Differentiation gives

$$\dot{\theta} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2}$$

# Inverse Kinematics

Define $v = \sqrt{\dot{x}^2 + \dot{y}^2} = \frac{r}{2}|\dot{\phi}_1 + \dot{\phi}_2|$, so

$$|\dot{\phi}_1 + \dot{\phi}_2| = \frac{2v}{r}$$

Using the third differential equation, $\dot{\phi}_1 = \dot{\phi}_2 + \frac{2L\dot{\theta}}{r}$, we can solve for $\dot{\phi}_2$. We get,

$$|\dot{\phi}_2 + \frac{L\dot{\theta}}{r}| = \frac{v}{r}.$$

Solving for $\dot{\phi}_2$ and then plugging back in for $\dot{\phi}_1$, we have

$$\dot{\phi}_1 = \frac{L\dot{\theta}}{r} \pm \frac{v}{r}, \quad \dot{\phi}_2 = -\frac{L\dot{\theta}}{r} \pm \frac{v}{r}$$

# Inverse Kinematics

Plugging in we have

$$\dot{\phi}_1 = \frac{L}{r}\left(\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2}\right) \pm \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{r}$$

$$\dot{\phi}_2 = -\frac{L}{r}\left(\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2}\right) \pm \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{r}$$

Direction along the path is selected depending on the $\pm$.

# Example

Assume that you want to follow the path

$$x(t) = t^2, \quad y(t) = t$$

Then

$$\dot{x} = 2t, \quad \ddot{x} = 2, \quad \dot{y} = 1, \quad \ddot{y} = 0$$

Plug into the equations

$$\dot{\phi}_1 = \frac{L}{r}\left(\frac{(2t)(0) - (1)(2)}{(2t)^2 + (1)^2}\right) + \frac{\sqrt{(2t)^2 + (1)^2}}{r}$$

$$\dot{\phi}_2 = -\frac{L}{r}\left(\frac{(2t)(0) - (1)(2)}{(2t)^2 + (1)^2}\right) + \frac{\sqrt{(2t)^2 + (1)^2}}{r}$$

# Example

Working out the algebra, we obtain:

$$\dot{\phi}_1 = \frac{L}{r}\left(\frac{2}{4t^2+1}\right) + \frac{\sqrt{4t^2+1}}{r}$$

$$\dot{\phi}_2 = -\frac{L}{r}\left(\frac{2}{4t^2+1}\right) + \frac{\sqrt{4t^2+1}}{r}$$

The velocity along the curve is given by

$$v = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{4t^2+1}$$

# Example 2

How does one follow a curve given in function form: $y = f(x)$. First, you need to write in parametric form: $x(t)$, $y(t)$. This is easily done by

$$\text{Let } x = t \quad \rightarrow \quad y = f(x) = f(t)$$

Example: convert $y = x^2$ to parametric

$$\text{Let } x = t \quad \rightarrow \quad y = x^2 = t^2$$

Note that there are an infinite number of choices:

$$\text{Let } x = ct \quad c \text{ nonzero} \quad \rightarrow \quad y = x^2 = c^2 t^2$$

$$\text{Let } x = ce^t \quad c \text{ nonzero} \quad \rightarrow \quad y = x^2 = e^{2t}$$

etc ...

# Functional vs Parametric form

All the parametric forms provide the same curve, same shape, same geometry. They vary in the speed.

Think of the function form tells you the shape - like the shape of a road.

The parametric form gives you shape and velocity - includes how fast you drive the road.

# Setting the speed

Say you want to drive the curve $y = f(x)$ but also setting the speed. Is this possible?

Yes. It may be hard to compute however.

Start with an example to get the idea.. Change the speed traveling on $y = x^2$

$$\text{Let } x = ct \quad c \text{ nonzero} \quad \rightarrow \quad y = x^2 = c^2 t^2$$

Speed is

$$v = \sqrt{1 + 4c^2 t^2}$$

As you change $c$ you can change $v$. It clear that you a constant value for $c$ changes the speed, but does allow you to set the speed to anything you want.

# Setting the speed

In general you need to find the parametric form

$$x(t) = \phi(t) \quad y(t) = f(\phi(t))$$

such that

$$v(t) = \sqrt{\dot\phi^2(t) + [(df/dx)(\phi(t))]^2\dot\phi^2(t)}$$
$$= |\dot\phi(t)|\sqrt{1 + [(df/dx)(\phi(t))]^2}$$

where $v(t)$ is the given (or desired) speed function.

Assume that you don't reverse directions, so you can assume that $\dot\phi(t)$ is only one sign. We can even assume it is positive.

# Setting the speed

Thus we get

$$\dot{\phi}(t) = \frac{v(t)}{\sqrt{1 + [(df/dx)(\phi(t))]^2}}.$$

This is a nonlinear differential equation. It can be solved via separation of variables ... if the algebra works out.

In most cases the algebra is not nice and the integrals will not have analytic antiderivative.

A numerical approach will be necessary.

# Motion review

To model the robot, you need to be able to model general motion. Let $x(t)$ be position as a function of time. Then velocity as a function of time is $v(t) = \dot{x}(t)$ and acceleration as a function of time $a(t) = \dot{v}(t) = \ddot{x}(t)$.

Assume that you are given acceleration $a(t) = 1 - t$ and you know the starting position $x(0) = -1$ and the starting velocity $v(0) = 2$.

So

$$v(t) = t - t^2/2 + C, \quad v(0) = c = 2,$$

and

$$x(t) = t^2/2 - t^3/6 + 2t + c, \quad x(0) = c = -1.$$

The result is $x(t) = t^2/2 - t^3/6 + 2t - 1$.

# Euler review

Assume that you need to model motion numerically when you are given acceleration (for example if you could not antidifferentiate on the last slide).

So the problem is given $\ddot{x} = a(t)$, find $x$ using Euler's method.

Euler's method only applies to first order derivatives, so you must convert the second order equation to a first order system.

This is done introducing a new variable, say $v$ and setting $v = \dot{x}$. Then

$$\dot{x} = v, \quad \dot{v} = \ddot{x} = a(t).$$

# Euler review

You can convert $\dot{x} = v$ by

$$\frac{x(t+\Delta t) - x(t)}{\Delta t} \approx v(t) \quad \rightarrow \quad x(t+\Delta t) = x(t) + v(t)\Delta t$$

or

$$x_{k+1} = x_k + v_k \Delta t$$

The same is done for $v$

$$\frac{v(t+\Delta t) - v(t)}{\Delta t} \approx a(t) \quad \rightarrow \quad v(t+\Delta t) = v(t) + a(t)\Delta t$$

or

$$v_{k+1} = v_k + a_k \Delta t$$

# Euler review

So one algorithm is

Set $x_0 = x(0)$, $v_0 = v(0)$

for $k = 1$ to $k = n$:

$$v_{k+1} = v_k + a_k \Delta t$$

$$x_{k+1} = x_k + v_k \Delta t$$

You can plug the first equation (drop the index by one) into the second one:

Set $x_0 = x(0)$, $v_0 = v(0)$

for $k = 1$ to $k = n$:

$$x_{k+1} = x_k + (v_{k-1} + a_{k-1} \Delta t) \Delta t$$

# Summary

There are many different ways to do the numerical approximations and this subject is discussed in the numerical methods course from the math department.