Gaussian Quadrature

Seeks to obtain the best numerical estimate of an integral by picking optimal abscissas x_i at which to evaluate the function $f(x)$. The fundamental theorem of Gaussian quadrature states that the optimal abscissas of the m-point Gaussian quadrature formulas are precisely the roots of the orthogonal polynomial for the same interval and weighting function. Gaussian quadrature is optimal because it fits all polynomials up to degree $2m-1$ exactly. Slightly less optimal fits are obtained from Radau quadrature and Laguerre-Gauss quadrature.

$W(x)$	interval	x_i are roots of
1	$(-1, 1)$	$P_n(x)$
e^{-t}	$(0, \infty)$	$L_n(x)$
e^{-t^2}	$(-\infty, \infty)$	$H_n(x)$
$\left(1-t^2\right)^{-1/2}$	$(-1, 1)$	$T_n(x)$
$\left(1-t^2\right)^{1/2}$	$(-1, 1)$	$U_n(x)$
$x^{1/2}$	$(0, 1)$	$x^{-1/2} P_{2n+1}\left(\sqrt{x}\right)$
$x^{-1/2}$	$(0, 1)$	$P_{2n}\left(\sqrt{x}\right)$

To determine the weights corresponding to the Gaussian abscissas x_i, compute a Lagrange interpolating polynomial for $f(x)$ by letting

$$\pi(x) \equiv \prod_{j=1}^{m}\left(x-x_j\right) \tag{1}$$

Then fitting a Lagrange interpolating polynomial through the m points gives

$$\phi(x) = \sum_{j=1}^{m} \frac{\pi(x)}{(x - x_j)\,\pi'(x_j)} f(x_j) \tag{3}$$

for arbitrary points x. We are therefore looking for a set of points x_j and weights w_j such that for a weighting function $W(x)$,

$$\int_a^b \phi(x)\,W(x)\,dx = \int_a^b \sum_{j=1}^{m} \frac{\pi(x)\,W(x)}{(x - x_j)\,\pi'(x_j)}\,dx\,f(x_j) \tag{4}$$

$$\equiv \sum_{j=1}^{m} w_j f(x_j), \tag{5}$$

with weight

$$w_j = \frac{1}{\pi'(x_j)} \int_a^b \frac{\pi(x)\,W(x)}{x - x_j}\,dx. \tag{6}$$

The weights w_j are sometimes also called the Christoffel numbers (Chandrasekhar 1967). For orthogonal polynomials $\phi_j(x)$ with $j = 1, ..., n$,

$$\phi_j(x) = A_j\,\pi(x) \tag{7}$$

(Hildebrand 1956, p. 322), where A_n is the coefficient of x^n in $\phi_n(x)$, then

$$w_j = \frac{1}{\phi_n'(x_j)} \int_a^b W(x)\,\frac{\phi(x)}{x - x_j}\,dx \tag{8}$$

$$= -\frac{A_{n+1}\,\gamma_n}{A_n\,\phi_n'(x_j)\,\phi_{n+1}(x)}, \tag{9}$$

where

$$\gamma_m \equiv \int [\phi_m(x)]^2\,W(x)\,dx. \tag{10}$$

Using the relationship

$$\phi_{n+1}(x_i) = -\frac{A_{n+1}\,A_{n-1}}{A_n^2}\,\frac{\gamma_n}{\gamma_{n-1}}\,\phi_{n-1}(x_i) \tag{11}$$

(Note that Press et al. 1992 omit the factor A_n/A_{n-1}.) In Gaussian quadrature, the weights are all positive. The error is given by

$$E_n = \frac{f^{(2n)}(\xi)}{(2n)!}\int_a^b W(x)[\pi(x)]^2\,dx \tag{13}$$

$$= \frac{\gamma_n}{A_n^2}\frac{f^{(2n)}(\xi)}{(2n)!}, \tag{14}$$

where $a < \xi < b$ (Hildebrand 1956, pp. 320-321).

Other curious identities are

$$\sum_{k=0}^{n}\frac{[\phi_k(x)]^2}{\gamma_k} = \frac{A_n}{A_{n+1}\gamma_n}[\phi_{n+1}'(x)\phi_n(x)-\phi_n'(x)\phi_{n+1}(x)] \tag{15}$$

and

$$\sum_{k=0}^{n}\frac{[\phi_k(x_j)]^2}{\gamma_k} = -\frac{A_n\phi_n'(x_j)\phi_{n+1}(x_j)}{A_{n+1}\gamma_n} \tag{16}$$

$$= \frac{1}{w_j} \tag{17}$$

(Hildebrand 1956, p. 323).

In the notation of Szegö (1975), let $x_{1n} < \ldots < x_{nn}$ be an ordered set of points in $[a, b]$, and let $\lambda_{1n}, \ldots, \lambda_{nn}$ be a set of real numbers. If $f(x)$ is an arbitrary function on the closed interval $[a, b]$, write the Gaussian quadrature as

$$Q_n(f) = \sum_{\nu=1}^{n}\lambda_{\nu n} f(x_{\nu n}). \tag{18}$$

Here $x_{\nu n}$ are the abscissas and $\lambda_{\nu n}$ are the Cotes numbers.

SEE ALSO

Chebyshev Quadrature, Chebyshev-Gauss Quadrature, Chebyshev-Radau Quadrature, Fundamental Theorem of Gaussian Quadrature, Hermite-Gauss Quadrature, Jacobi-Gauss Quadrature, Laguerre-Gauss Quadrature, Legendre-Gauss Quadrature, Lobatto Quadrature, Radau Quadrature

= get a total greater than 45 with 5 12-sided dice

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CITE THIS AS:

Weisstein, Eric W. "Gaussian Quadrature." From *MathWorld*--A Wolfram Resource. https://mathworld.wolfram.com/GaussianQuadrature.html

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Applied Mathematics › Numerical Methods › Numerical Integration ›