

TD7

Exercice 1 :

i) $* g: \mathbb{R} \rightarrow \mathbb{R}$ est une intégrale à paramètre
 $x \mapsto \int_0^1 \frac{e^{-(t^2+1)x^2}}{t^2+1} dt$

La fonction $u: [0,1] \times \mathbb{R} \rightarrow \mathbb{R}$ admet une dérivée partielle par rapport à x qui est continue sur $[0,1] \times \mathbb{R}$. La

fonction g est \mathcal{C}^1 sur \mathbb{R} et

$$\begin{aligned} g'(x) &= \int_0^1 \frac{\partial u}{\partial x}(t, x) dt = \int_0^1 -2x(t^2+1)^{-1} \frac{e^{-(t^2+1)x^2}}{t^2+1} dt \\ &= -2x e^{-x^2} \int_0^1 e^{-(t^2+x^2)} dt. \end{aligned}$$

* $f: \mathbb{R} \rightarrow \mathbb{R}$ est dérivable (th fondamental de l'Analyse)
 $x \mapsto \int_x^n e^{-t^2} dt$ et $f'(x) = e^{-x^2}$.

* Démontre que $h = g + f^2$ est nulle. Comme h est dérivable

$$h'(x) = -2x e^{-x^2} \int_0^1 e^{-(t^2+x^2)} dt + 2 e^{-x^2} \int_0^n e^{-t^2} dt$$

$$\stackrel{T=t-x}{=} -2 e^{-x^2} \int_0^n e^{-t^2} dt + 2 e^{-x^2} \int_0^x e^{-t^2} dt.$$

$$= 0.$$

De plus

$$h(0) = \int_0^1 \frac{dt}{1+t^2} = \operatorname{Arctan}(1) = \frac{\pi}{4}.$$

2) On va montrer que $g(n) \rightarrow 0$ lorsque $n \rightarrow +\infty$.

on a pour tout $(t, n) \in [0, 1] \times \mathbb{R}$

$$0 \leq \frac{e^{-(t^2+1)n^2}}{t^2+1} \leq e^{-(t^2+1)n^2} \leq e^{-n^2}$$

et

$$0 \leq \underbrace{\int_0^1 \frac{e^{-(t^2+1)n^2}}{t^2+1} dt}_{= g(n)} \leq \int_0^1 e^{-n^2} dt = e^{-n^2} \xrightarrow{n \rightarrow \infty} 0$$

et $g(n) \rightarrow 0$ qd $n \rightarrow \infty$ par le th. des gendarmes

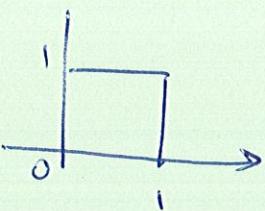
3) on a $\lim_{n \rightarrow \infty} f^2(n) = I^2 = \frac{\pi}{4}$.

Ainsi $I = \pm \sqrt{\frac{\pi}{4}}$ mais comme $e^{-t^2} \geq 0 \quad \forall t > 0$ on

a nécessairement $I = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$.

Exercise 2:

$$D = [0, 1] \times [0, 1]$$



$$\begin{aligned}
 \iint_D \frac{1}{(1+y+x)^2} dx dy &= \int_0^1 \int_0^1 \frac{1}{(1+y+x)^2} dx dy \\
 &= \int_0^1 \left[-(1+y+x)^{-1} \right]_0^1 dy \\
 &= \int_0^1 (1+y)^{-1} - (2+y)^{-1} dy \\
 &= \left[\ln(1+y) \right]_0^1 - \left[\ln(2+y) \right]_0^1 \\
 &= \ln 2 - (\ln 3 - \ln 2) = \ln(4) - \ln(3).
 \end{aligned}$$

Exercise 3:

$$D = \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0 \quad x+y \leq 1\}$$

$$I_1 = \iint_D (x+y) e^{-x} e^{-y} dx dy = \int_0^1 e^{-x} \underbrace{\left(\int_0^{1-x} (x+y) e^{-y} dy \right)}_{= \textcircled{1}} dx$$

$$\textcircled{1} = x \int_0^{1-x} e^{-y} dy + \int_0^{1-x} y e^{-y} dy \quad \begin{matrix} \text{IPP} \\ u=y \\ v=e^{-y} \end{matrix} \quad \begin{matrix} u'=1 \\ v'=e^{-y} \end{matrix}$$

$$= x \left[-e^{-y} \right]_0^{1-x} + \left[-ye^{-y} \right]_0^{1-x} + \int_0^{1-x} e^{-y} dy$$

$$= x \left(-e^{-1+x} + 1 \right) + \left(-(1-x)e^{-1+x} \right) + \left[-e^{-y} \right]_0^{1-x}$$

$$= -x/e^{-1+x} + x + 2e^{-1+x} - e^{-1+x} - e^{-1+x} + 1$$

$$= (x+1) - 2e^{-1+x}.$$

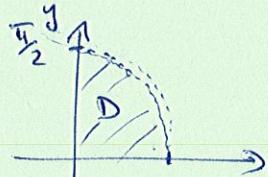
$$I_1 = \int_0^1 ((x+1)e^{-x} - 2e^{-1}) dx$$

IPP: $u = (1+x) \quad u' = 1$
 $u' = e^{-x} \quad v = -e^{-x}$

$$= \left[-(x+1)e^{-x} \right]_0^1 + \int_0^1 e^{-x} dx - 2e^{-1}$$

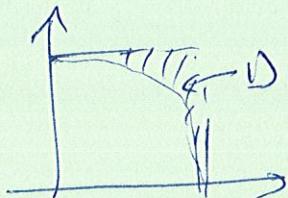
$$= -2e^{-1} + 1 + \underbrace{\left[-e^{-x} \right]_0^1}_{= -e^{-1} + 1} - 2e^{-1} = 2 - 5e^{-1}$$

2) $D = \{ (x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq \frac{\pi}{2} \text{ et } 0 \leq x \leq \cos y \}$



$$\begin{aligned} I_2 &= \iint_D x \cos y \, dx \, dy = \int_0^{\pi/2} \left(\int_0^{\cos y} x \, dx \right) \sin y \, dy \\ &= \int_0^{\pi/2} \frac{\cos^2 y}{2} \sin y \, dy \\ &= \frac{1}{2} \int_0^{\pi/2} (1 - \sin^2 y) \cos y \, dy \\ &\stackrel{u = \sin y}{=} \frac{1}{2} \left(\left[\cos y \right]_0^{\pi/2} - \left[\frac{1}{3} \sin^3 y \right]_0^{\pi/2} \right) \\ &= \frac{1}{2} \left(1 - \frac{1}{3} \right) = \frac{1}{2} \times \frac{2}{3} = \frac{1}{6}. \end{aligned}$$

3) $D = \{ (x,y) \in [0,1]^2 \mid x^2 + y^2 \geq 1 \}$



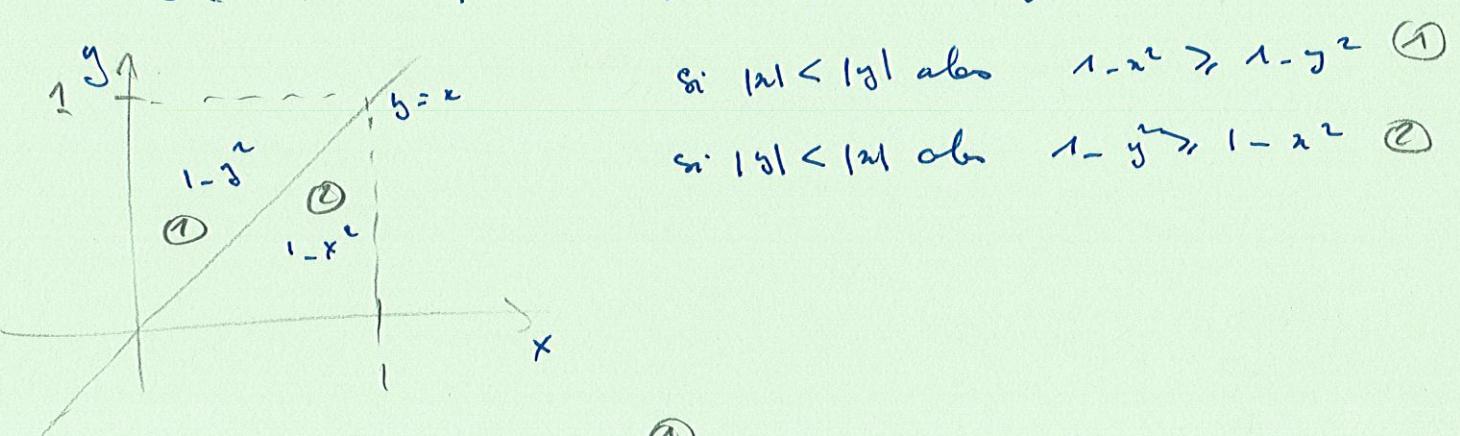
$$I_3 = \iint_D \frac{xy}{1+x^2+y^2} \, dx \, dy = \int_0^1 x \left(\int_{\sqrt{1-x^2}}^1 y (1+x^2+y^2)^{-1} dy \right) dx$$

$$\begin{aligned}
 I_3 &= \frac{1}{2} \int_0^1 x \left[\ln(1+x^2+y^2) \right]_{\sqrt{1-x^2}}^1 dx \\
 &= \frac{1}{2} \int_0^1 x \left(\ln(2+x^2) - \ln(1+x^2+1-x^2) \right) dx \\
 &= \frac{1}{2} \left(\int_0^1 x \ln(2+x^2) dx - \frac{\ln 2}{2} \right)
 \end{aligned}$$

remarque $(u \ln u - u')' = u \ln u$

$$\begin{aligned}
 &= \frac{1}{2} \left(\frac{1}{2} \left[(2+x^2) \ln(2+x^2) - (2+x^2) \right]_0^1 - \frac{\ln 2}{2} \right) \\
 &= \frac{1}{4} \left(3 \ln(3) - 3 - 2 \ln(2) + 1 - \ln 2 \right) \\
 &= \frac{1}{4} (3 \ln(3) - 3 \ln(2) - 1)
 \end{aligned}$$

4) $D = \{(x,y,z) \in \mathbb{R}^3 \mid x \geq 0, y \geq 0, z \geq 0 \text{ et } z \leq \min\{1-x^2, 1-y^2\}\}$



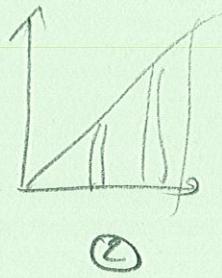
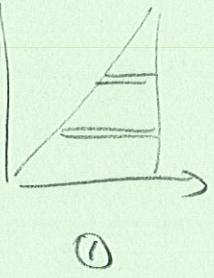
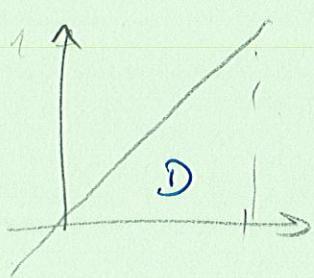
$$\begin{aligned}
 \iiint_D z \, dx \, dy \, dz &= \overbrace{\int_0^1 \left(\int_0^y \left(\int_0^{1-y^2} z \, dz \right) dx \right) dy}^{\textcircled{1}} \\
 &\quad + \overbrace{\int_0^1 \left(\int_0^x \left(\int_0^{1-x^2} z \, dz \right) dy \right) dx}^{\textcircled{2}}
 \end{aligned}$$

$$\begin{aligned}
 &= \int_0^1 \int_0^y \left[\frac{z^2}{2} \right]_0^{1-y^2} dx \, dy + \int_0^1 \int_0^x \left[\frac{z^2}{2} \right]_0^{1-x^2} dy \, dx
 \end{aligned}$$

$$\begin{aligned}
 &= \int_0^1 \int_0^y \frac{(1-y^2)^2}{2} dx dy + \int_0^1 \int_0^x \frac{(1-x^2)^2}{2} dy dx \\
 &= \int_0^1 y (1-y^2)^2 \frac{dy}{2} + \int_0^1 x (1-x^2)^2 \frac{dx}{2} \\
 &= \frac{1}{2} \left[-\frac{1}{6} (1-y^2)^3 \right]_0^1 + \frac{1}{2} \left[-\frac{1}{6} (1-x^2)^3 \right]_0^1 = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}.
 \end{aligned}$$

Exercice :

$$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq x \leq 1\}$$



2 manières de calculer cette intégrale.

$$\textcircled{1} \quad \iint_D e^{x^2} dx dy = \int_0^1 \left(\int_y^1 e^{x^2} dx \right) dy \quad \rightarrow \text{il faut partitionner } e^{x^2} \\ \text{cela n'aboutit pas.}$$

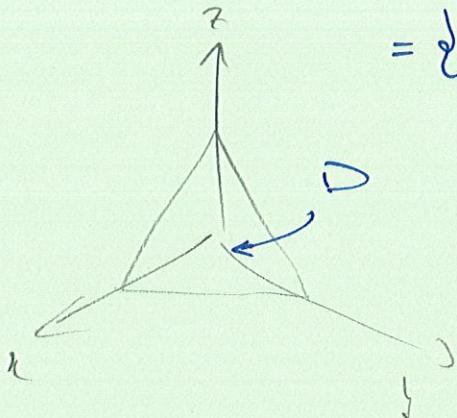
$$\textcircled{2} \quad \iint_D e^{x^2} dx dy = \int_0^1 \left(\int_0^x e^{y^2} dy \right) dx$$

$$= \int_0^1 x e^{x^2} dx = \left[\frac{1}{2} e^{x^2} \right]_0^1 = \frac{1}{2} e - \frac{1}{2} = \frac{(e-1)}{2}.$$

Exercise 5 :

$$D = \{ (x, y, z) \in \mathbb{R}^3 \mid x \geq 0, y \geq 0, z \geq 0, x+y+z \leq 1 \}$$

$$= \{ z = 1 - x - y \mid (x, y) \in T \}$$



$$\text{et } T = \{ (x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x+y \leq 1 \}$$

$$\iiint_D \frac{dx dy dz}{(1+x+y+z)^3} = \int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} (1+x+y+z)^{-3} dz \right) dy \right) dx$$

$$= \int_0^1 \left(\int_0^{1-x} \left[-\frac{1}{2} (1+x+y+z)^{-2} \right]_{z=0}^{1-x-y} dy \right) dx$$

$$= \int_0^1 \int_0^{1-x} -\frac{1}{8} + \frac{1}{2} (1+x+y)^{-2} dy dx$$

$$= \int_0^1 \left[-\frac{1}{8}y - \frac{1}{2} (1+x+y)^{-1} \right]_{y=0}^{1-x} dx$$

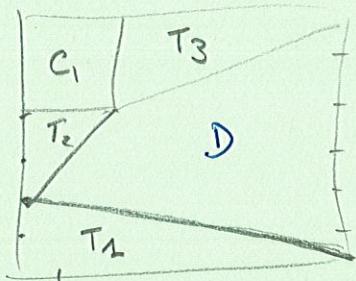
$$= \int_0^1 -\frac{1}{8}(1-x) - \frac{1}{4} + \frac{(1+x)^{-1}}{2} dx$$

$$= -\frac{1}{8} + \left[\frac{x^2}{16} \right]_0^1 - \frac{1}{4} + \frac{1}{2} \left[\ln(1+x) \right]_0^1$$

$$= -\frac{2}{16} + \frac{1}{16} - \frac{4}{16} + \frac{\ln 2}{2} = -\frac{5}{16} + \frac{\ln 2}{2}$$

Exercice 6.

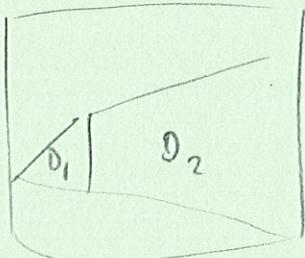
1^{ère} méthode: Remarque que: $\text{Aire}(D) = 36 - \text{Aire}(T_1) - \text{Aire}(T_2)$
 $- \text{Aire}(T_3) - \text{Aire}(C_1)$



$$= 36 - 6 - 2 - 4 - 4 \\ = 20.$$

2^{ème} méthode: décompose la région D en 2 parties:

$$D = D_1 \cup D_2$$



$$\iint_D dx dy = \int_{-3}^{-1} \left(\int_{-\frac{1}{3}x+2}^{x+2} dy \right) dx \\ + \int_{-1}^3 \left(\int_{-\frac{1}{3}x+2}^{\frac{x}{2}+\frac{3}{2}} dy \right) dx$$

$$= \int_{-3}^{-1} x+2 + \frac{1}{3}x+2 dx + \int_{-1}^3 \frac{x}{2} + \frac{3}{2} + \frac{x}{3} + 2 dx$$

$$= \int_{-3}^{-1} \frac{4}{3}x+4 dx + \int_{-1}^3 \frac{5}{6}x+\frac{7}{2} dx = \left[\frac{4}{3}\frac{x^2}{2} + 4x \right]_{-3}^{-1} + \left[\frac{5}{6}\frac{x^2}{2} + \frac{7}{2}x \right]_{-1}^3$$

$$= \underbrace{\frac{4}{3} \times \frac{1}{2}}_{\frac{2}{3}} - 4 - \underbrace{\frac{4}{3} \times \frac{9}{2}}_{= 2} + 12 + \underbrace{\frac{5x^9}{12}}_{\frac{15}{4}} + \frac{21}{2} - \frac{5}{12} + \frac{7}{2} = 20.$$

$$+ \frac{42}{4} - \frac{5}{12} + \frac{16}{4}$$

Exercise 2:

i) Pour trouver ϕ et $\psi : [-1, 1] \rightarrow \mathbb{R}$ on remarque que :

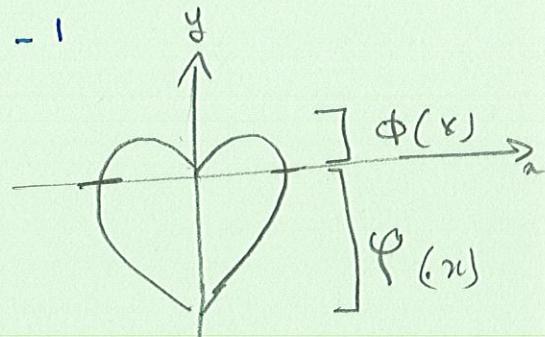
$$(y+1)^2 - 2|z| |y+1| + (2z^2 - 1) = 0$$

$$\Leftrightarrow y+1 = \frac{1}{2} (2|z| \pm \sqrt{4|z|^2 - (2z^2 - 1)})$$

$$\Leftrightarrow y = |z| \pm \sqrt{1-z^2} - 1$$

on pose alors $\phi(z) = |z| - 1 + \sqrt{1-z^2}$

$$\psi(z) = |z| - 1 - \sqrt{1-z^2}$$



c) Calcul d'une primitive de $x \mapsto \sqrt{1-x^2}$.

$$\int_0^{\pi} \sqrt{1-\cos^2 \theta} d\theta = \int_0^{\pi} \sqrt{1-\sin^2 \theta} \cos \theta d\theta = \int_0^{\pi} \cos^2 \theta d\theta.$$

1^{ere} méthode : Poser $A = \int_0^{\pi} \cos^2 \theta d\theta$ et faire un IPP.

$$A = \int_0^{\pi} \cos^2 \theta d\theta = \underbrace{\cos(\pi \sin \theta)}_{= \sqrt{1-\cos^2 \theta}} \underbrace{\sin(\pi \sin \theta)}_{= \theta} + \int_0^{\pi} \sin^2 \theta d\theta.$$

$$\text{et } 2A = x \sqrt{1-x^2} + \int_0^{\pi} (\sin^2 \theta + \cos^2 \theta) d\theta = x \sqrt{1-x^2} + A \sin(\pi)$$

$$\therefore A = \frac{1}{2} (x \sqrt{1-x^2} + A \sin(\pi))$$

2^{eme} méthode : Linéariser le $\cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta)$ et $\sin(2\theta) = 2 \sin \theta \cos \theta$

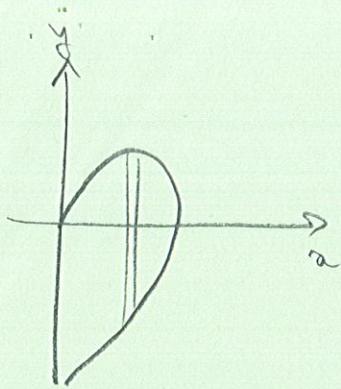
$$\int_0^{\pi} \cos^2 \theta d\theta = \frac{1}{2} \left(A \sin \pi + \left[\frac{1}{2} \sin 2\theta \right]_0^{\pi} \right)$$

$$= \frac{1}{2} \left(A \sin \pi + \frac{1}{2} \sin(2\pi) \right)$$

$$= \frac{1}{2} (A \sin \pi + \sin(\pi) \cos(\pi))$$

$$= \frac{1}{2} (A \sin \pi + x \sqrt{1-x^2}).$$

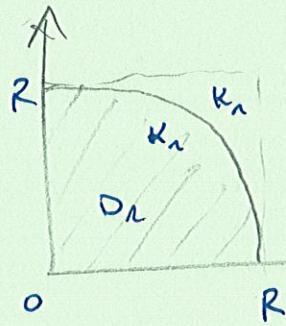
3) Utiliser le fait que \mathcal{D} est symétrique par rapport à l'axe Oy .



$$\begin{aligned} \text{Aire } (\mathcal{D}) &= 2 \int_0^a \int_{\phi(x)}^1 dy dx \\ &= 2 \int_0^a \phi(x) - \psi(x) dx \\ &= 2 \int_0^1 2 \sqrt{1-x^2} dx = \frac{4}{2} \left[x \sqrt{1-x^2} + A_{\sin}(x) \right]_0^1 \\ &= 2 A_{\sin}(1) = \pi \end{aligned}$$

Exercice 2 : (Intégrale de Gauss) cf le chapitre sur les espaces normés :

1)



$$\frac{1}{2} \|u\|_2 \leq \|u\|_\infty \leq \|u\|_2$$

$$\Rightarrow B_2(0, 2R) \supset B_\infty(0, R) \supset B_2(0, R)$$

" " "
D_{2R} K_n D_n

$$\Rightarrow \iint_{D_n} e^{-(x^2+y^2)} dx dy \leq \iint_{K_n} e^{-x^2-y^2} dx dy \leq \iint_{D_{2R}} e^{-x^2-y^2} dx dy$$

$$\text{car } e^{-(x^2+y^2)} \geq 0 \quad \forall x, y \in \mathbb{R}^2.$$

$$2) \iint_{D_R} e^{-(x^2+y^2)} dx dy = \int_0^{2\pi} d\theta \int_0^R r e^{-r^2} dr = \frac{\pi}{2} \left[-\frac{1}{2} e^{-r^2} \right]_0^R = \frac{\pi}{4} (1 - e^{-R^2})$$

$$\iint_{D_{2R}} e^{-(x^2+y^2)} dx dy = \frac{\pi}{4} (1 - e^{-4R^2})$$

$$\iint_{K_R} e^{-x^2} e^{-y^2} dx dy = \left(\int_0^R e^{-r^2} dr \right)^2$$

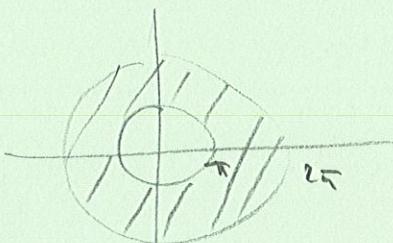
Théorème de Jordan $\Rightarrow \lim_{R \rightarrow \infty} \left(\int_0^R e^{-x^2} dx \right)^2$ existe et

$$\text{vaut } \frac{\pi}{4}$$

$$\text{et on a } \int_0^{+\infty} e^{-x^2} dx = \lim_{R \rightarrow \infty} \int_0^R e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

Exercice 8 :

$$1) D = \{(x,y) \in \mathbb{R}^2 \mid \pi^2 \leq x^2 + y^2 \leq 4\pi^2\}$$



$$\iint_D x \sin(r^2) dy dx = \int_{\pi}^{2\pi} r \sin r dr \int_0^{2\pi} d\theta$$

$$= \left[-r \cos r \right]_0^{2\pi} + \left[\sin r \right]_0^{2\pi} \cdot 2\pi$$

$$= (-2\pi - \pi) 2\pi = -6\pi^2.$$

$$\phi^{-1}(D) = \begin{cases} \pi \leq r \leq 2\pi \\ 0 \leq \theta \leq 2\pi. \end{cases}$$

$$\begin{matrix} u = r & u' = 1 \\ v = \sin \theta & v' = \cos \theta \end{matrix}$$

$$2) D = \{(x,y) \in \mathbb{R}^2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}$$

$a, b > 0$.

$$\text{Poser } \begin{cases} x = ar \cos \theta \\ y = br \sin \theta \end{cases}$$

$$\phi: \overbrace{[0, +\infty[\times]0, 2\pi[}^J \longrightarrow \mathbb{R}^2$$

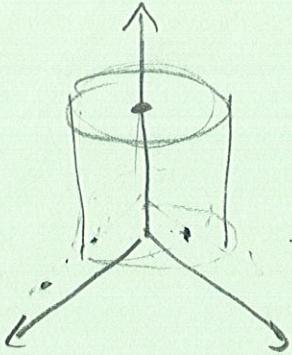
$$(r, \theta) \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\star |\text{Jac}_{\phi}(r, \theta)| = |abr| = abr$$

$$\star \phi^{-1}(D) = \{(r, \theta) \in J \mid 0 \leq r \leq 1\}$$

$$\begin{aligned} \iint_D x^2 + y^2 dy dx &= ab \int_0^1 \int_0^{2\pi} (a^2 \cos^2 \theta + b^2 \sin^2 \theta) r^3 dr d\theta \\ &= \left(\int_0^1 r^3 dr \right) \left(\int_0^{2\pi} \frac{a^2}{2} (\cos 2\theta + 1) + \frac{b^2}{2} (-\sin 2\theta + 1) d\theta \right) ab \\ &= ab \left[\frac{r^4}{4} \right]_0^1 \left(\left[\frac{a^2}{2} \left(\frac{1}{2} \sin(2\theta) + \theta \right) \right]_0^{2\pi} + \left[\frac{b^2}{2} \left(\frac{1}{2} \sin(2\theta) + \theta \right) \right]_0^{2\pi} \right) \\ &= \frac{ab}{4} \left(\frac{a^2}{2} (2\pi) + \frac{b^2}{2} (2\pi) \right) = \frac{ab}{4} \left(a^2 \pi + b^2 \pi \right) \end{aligned}$$

$$3) \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1, 0 \leq z \leq h\} \quad h > 0.$$



$$\iiint_D z \, dx \, dy \, dz = \int_0^1 \int_0^{2\pi} \int_0^h z \, dz \, d\theta \, dr$$

$$= \left(\int_0^1 r \, dr \right) \left(\int_0^{2\pi} d\theta \right) \left(\int_0^h z \, dz \right)$$

Coordonnées cylindriques.

$$\psi^{-1}(D) = \begin{cases} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq z \leq h \end{cases}$$

$$|\text{Jac}_\psi(r, \theta, z)| = r.$$

$$4) D = \{(x, y, z) \in \mathbb{R}^3 \mid 1 \leq x^2 + y^2 + z^2 \leq 4\}$$

Coordonnées sphériques

$$\phi: [0, \pi] \times [0, 2\pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \rightarrow \mathbb{R}^3$$

$$(r, \theta, \varphi) \mapsto \begin{pmatrix} r \cos \theta \cos \varphi \\ r \sin \theta \cos \varphi \\ r \sin \theta \sin \varphi \end{pmatrix}.$$

$$* |\text{Jac}_\phi(r, \theta, \varphi)| = r^2 \cos \varphi$$

$$* \psi^{-1}(D) = \begin{cases} 1 \leq r \leq 2 \\ 0 \leq \theta \leq 2\pi \\ -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2} \end{cases}$$

$$\iiint_D (x^2 + y^2 + z^2)^{\alpha} \, dx \, dy \, dz = \iiint_{\psi^{-1}(D)} r^{2\alpha} \cdot r^2 \cos \varphi \, dr \, d\theta \, d\varphi$$

$$= \left(\int_0^{2\pi} d\theta \right) \left(\int_1^2 r^{2\alpha+2} \, dr \right) \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \varphi \, d\varphi \right)$$

$$= 2\pi \left[\frac{r^{2\alpha+3}}{2\alpha+3} \right]_1^2 \left[\sin \varphi \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

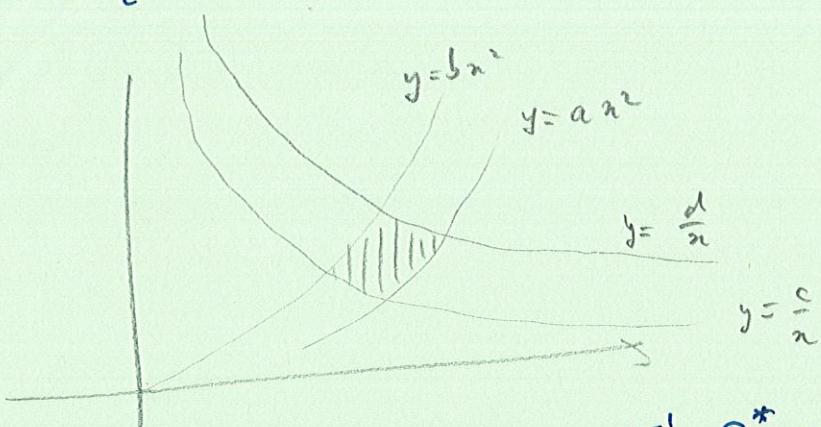
$$= 2\pi \left(\frac{2^{2\alpha+3}}{2\alpha+3} - \frac{1}{2\alpha+3} \right) \underbrace{\left(\sin \frac{\pi}{2} + \sin -\frac{\pi}{2} \right)}_{=2}$$

$$= \frac{4\pi}{2\alpha+3} \left(2^{2\alpha+3} - 1 \right).$$

Exercice 9 :

$$\begin{cases} 0 < a \leq b, \\ 0 < c \leq d \end{cases}$$

$$D = \left\{ ax^2 \leq y \leq bx^2 \text{ et } \frac{c}{x} \leq y \leq \frac{d}{x}, (x,y) \in \mathbb{R}^2 \right\}$$



$$\text{Poser } \begin{cases} u = \frac{y}{x^2} \\ v = xy \end{cases}$$

* changeert de variable : $\phi^{-1}: \mathbb{R}_+^* \times \mathbb{R}_+^* \rightarrow \mathbb{R}_+^* \times \mathbb{R}_+^*$
 $(x,y) \mapsto (yx^{-2}, xy)$

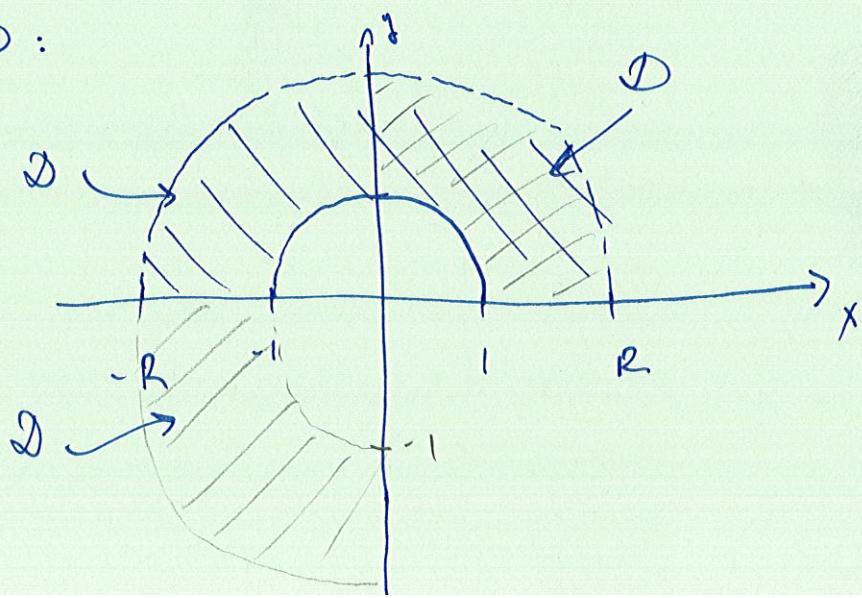
$$\det \text{Jac}_{\phi^{-1}}(x,y) = \det \begin{pmatrix} yx^{-3} + 2 & x^{-2} \\ y & x \end{pmatrix} = -2yx^{-2} - yx^{-3} = -3x^2y^{-2}$$

$$\Leftrightarrow \det \text{Jac}_{\phi}(u,v) = \frac{1}{\det \text{Jac}_{\phi^{-1}}(\phi(u))} = -\frac{1}{3u}$$

* $\phi^{-1}(D) = \{(u,v) \mid a \leq u \leq b \text{ et } c \leq v \leq d\}$

$$\iint_D dx dy = \iint_{\phi^{-1}(D)} \frac{du dv}{3|u|} = \left(\int_a^b \frac{du}{3u} \right) \left(\int_c^d dv \right) = \frac{1}{3} \left[\ln u \right]_a^b (d - c) = \frac{1}{3} \ln \left(\frac{b}{a} \right) (d - c)$$

Exercice 10 :



2) on passe en coordonnées polaires:

$$\mathcal{D} = \{(r, \theta) \mid 1 \leq r \leq R \text{ et } 0 \leq \theta \leq \frac{3\pi}{2}\}$$

* calcul du centre de gravité: $\frac{1}{Aire(\mathcal{D})} \left(\iint_{\mathcal{D}} x \, dx \, dy, \iint_{\mathcal{D}} y \, dx \, dy \right)$

$$* Aire(\mathcal{D}) = \iint_{\mathcal{D}} dx \, dy = \int_1^R \int_0^{\frac{3\pi}{2}} r \, dr \, d\theta = \left(\int_1^R r^2 \, dr \right) \left(\int_0^{\frac{3\pi}{2}} d\theta \right) = \frac{3\pi}{4} (R^2 - 1)$$

$$* \iint_{\mathcal{D}} x \, dx \, dy = \int_1^R \left(\int_0^{\frac{3\pi}{2}} r^2 \cos \theta \, d\theta \right) dr = \left(\int_1^R r^3 \, dr \right) \left(\int_0^{\frac{3\pi}{2}} \cos \theta \, d\theta \right) = -\frac{(R^3 - 1)}{3}$$

$$* \iint_{\mathcal{D}} y \, dx \, dy = \left(\int_1^R r^2 \, dr \right) \left(\int_0^{\frac{3\pi}{2}} r \sin \theta \, d\theta \right) = \frac{R^3 - 1}{3}.$$

on a alors:

$$(x_G, y_G) = \frac{4}{3\pi(R^2 - 1)} \left(-\frac{R^3 - 1}{3}, \frac{R^3 - 1}{3} \right)$$

$$= \frac{4(R^2 + R + 1)}{9\pi(R + 1)} (-1, 1).$$

3) Le centre de gravité se situe sur $\Delta = \{y = -x\}$ et appartient à \mathcal{D} si et

$$\frac{4(R^2 + R + 1)}{9\pi(R + 1)} \sqrt{2} \geq 1$$

$$\text{ssi } 4\sqrt{2}R^2 + 4\sqrt{2}R + 4\sqrt{2} \geq 9\pi R + 9\pi$$

$$\text{ssi } 4\sqrt{2}R^2 + R(4\sqrt{2} - 9\pi) + (4\sqrt{2} - 9\pi) > 0$$

$$\text{ssi } 4\sqrt{2} \left[R^2 + R \left(1 - \frac{9\pi}{4\sqrt{2}} \right) + \left(1 - \frac{9\pi}{4\sqrt{2}} \right) \right] > 0$$

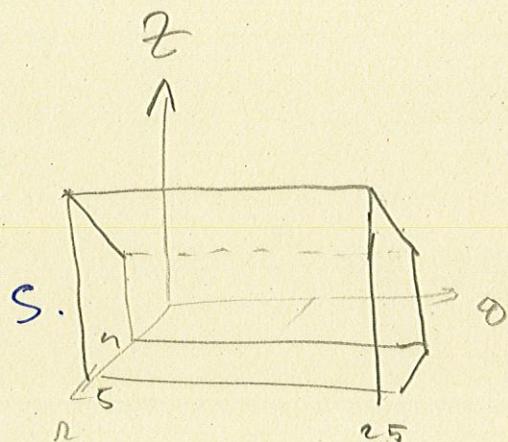
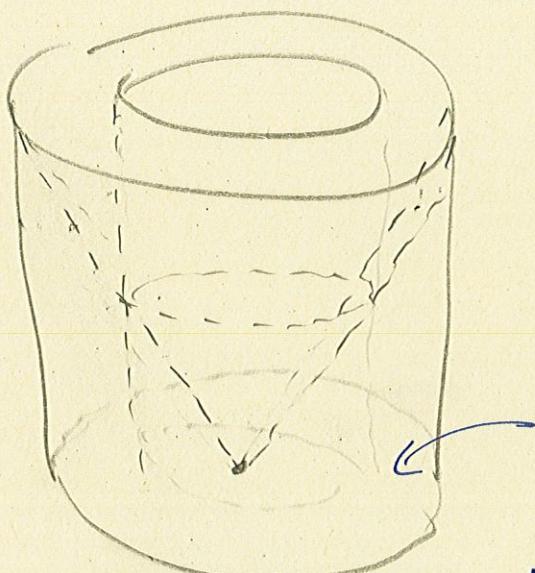
$$\text{pose } k = 1 - \frac{9\pi}{4\sqrt{2}} \text{ et}$$

$$\text{so } R > \frac{-k + \sqrt{k^2 + 4k}}{2} \simeq 4,826617132 \dots$$

✓

Exercise 11 :

1)



$$S = \left\{ (x, y, z) \in \mathbb{R}^3 \mid \begin{cases} 4(x^2 + y^2) \leq 25 \\ 0 \leq z \leq \sqrt{x^2 + y^2} \end{cases} \right\}$$

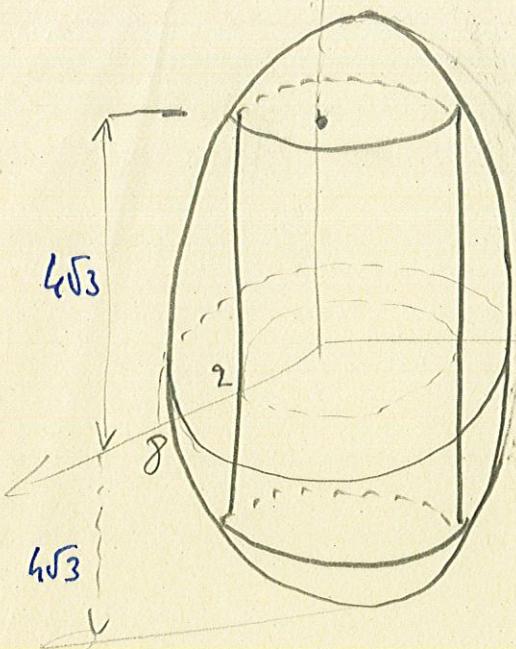
$$= \left\{ (r, \theta, z) \mid \begin{cases} 2 \leq r \leq 5 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq z \leq r \end{cases} \right.$$

$$\iiint_S dx dy dz = \int_2^5 \left(\int_0^{2\pi} \left(\int_0^r r dz \right) d\theta \right) dr$$

$$= \left(\int_0^{2\pi} d\theta \right) \int_2^5 r^2 dr = 2\pi \left(\frac{5^3}{3} - \frac{2^3}{3} \right) = 78\pi$$

2)

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid \begin{cases} x^2 + y^2 \leq 4 \\ \text{et} \\ 4(x^2 + y^2) + z^2 \leq 64 \end{cases}\}$$



L'intersection du cylindre et de l'ellipsoïde

$$\begin{cases} x^2 + y^2 = 4 \\ 16 + z^2 = 64 \end{cases} \Rightarrow \begin{cases} x^2 + y^2 = 4 \\ z = 4\sqrt{3} \end{cases}$$

utilise la "faute de Symétrie par paire"

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid \sqrt{-4(x^2 + y^2) + 64} \geq 3 \geq -\sqrt{64 - 4(x^2 + y^2)}\}$$

$$x, y \in \mathbb{R}$$

$$\text{et } D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}.$$

$$\iiint_S dx dy dz = \iint_D \int_{-\sqrt{64 - 4(x^2 + y^2)}}^{\sqrt{64 - 4(x^2 + y^2)}} dz \, dx \, dy$$

$$= 2 \iint_D \sqrt{64 - 4(x^2 + y^2)} \, dx \, dy$$

$$= 4 \iint_D \sqrt{16 - r^2} \, r dr \, d\theta$$

$$= 4 \times 2\pi \times \left[-\frac{1}{2} \frac{r^2}{3} (16 - r^2)^{\frac{3}{2}} \right]_0^2 = -\frac{8\pi}{3} \left((16 - 4)^{\frac{3}{2}} - 16^{\frac{3}{2}} \right)$$

$$= 8\pi \left(\frac{64}{3} - 8\sqrt{3} \right)$$