TD7

Exercice 1 :

1) * $g : \mathbb{R} \longrightarrow \mathbb{R}$ est une intégrale à paramètre

$$x \longmapsto \int_0^1 \frac{e^{-(t^2+1)x^2}}{t^2+1}\,dt$$

La fonction $u : [0,1]\times\mathbb{R} \longrightarrow \mathbb{R}$ admet une dérivée

$$(t,x) \longmapsto e^{-(t^2+1)x^2}/t^2+1$$

partielle par rapport à x qui est continue sur $[0,1]\times\mathbb{R}$. La fonction g est $\mathcal{C}^1$ sur $\mathbb{R}$ et

$$g'(x) = \int_0^1 \frac{\partial u}{\partial x}(t,x)\,dt = \int_0^1 -2x\,\cancel{(t^2+1)}\,\frac{e^{-(t^2+1)x}}{\cancel{t^2+1}}\,dt$$

$$= -2x e^{-x^2}\int_0^1 e^{-(tx)^2}\,dt.$$

* $f : \mathbb{R} \to \mathbb{R}$ est dérivable (th fondamental de l'Analyse)

$x \mapsto \int_0^x e^{-t^2}\,dt$ et $f'(x) = e^{-x^2}$.

* Montrons que $h = g + f^2$ est nulle. Comme h est dérivable

on a
$$h'(x) = -2x e^{-x^2}\int_0^1 e^{-(tx)^2}\,dt + 2e^{-x^2}\int_0^x e^{-t^2}\,dt$$

$$\overset{\tau = tx}{=} -2e^{-x^2}\int_0^x e^{-\tau^2}\,d\tau + 2e^{-x^2}\int_0^x e^{-t^2}\,dt.$$

$$= 0.$$

De plus

$$h(0) = \int_0^1 \frac{dt}{1+t^2} = \mathrm{Arctan}\,(1) = \frac{\pi}{4}.$$

2) On va montrer que $g(x) \longrightarrow 0$ lorsque $x \longmapsto +\infty$.

on a pour tout $(t,x) \in [0,1] \times \mathbb{R}$

$$0 \leq \frac{e^{-(t^2+1)x^2}}{t^2+1} \leq e^{-(t^2+1)x^2} \leq e^{-x^2}$$

et

$$0 \leq \underbrace{\int_0^1 \frac{e^{-(t^2+1)x^2}}{t^2+1}\,dt}_{=g(x)} \leq \int_0^1 e^{-x^2}\,dt = \underbrace{e^{-x^2}}_{\longrightarrow 0}$$

et $g(x) \to 0$ qd $x \to +\infty$ par le th. des gendarmes.

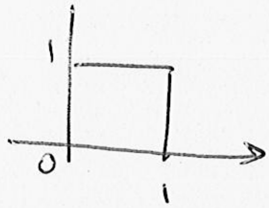
3) on a $\lim\limits_{x \to \infty} f^2(x) = I^2 = \frac{\pi}{4}$.

Ainsi $I = \pm\sqrt{\frac{\pi}{4}}$ mais comme $e^{-t^2} \geq 0$ $\forall t > 0$ on

a nécessairement $I = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$.

Exercice 2 :

$$D = [0,1] \times [0,1]$$



$$\iint_D \frac{1}{(1+y+x)^2}\,dx\,dy = \int_0^1 \int_0^1 \frac{1}{(1+y+x)^2}\,dx\,dy$$

$$= \int_0^1 \left[-(1+y+x)^{-1}\right]_0^1 dy$$

$$= \int_0^1 (1+y)^{-1} - (2+y)^{-1}\,dy$$

$$= \left[\ln(1+y)\right]_0^1 - \left[\ln(2+y)\right]_0^1$$

$$= \ln 2 - (\ln 3 - \ln 2) = \ln(4) - \ln(3).$$

Exercice 3 :

i) $D = \{ (x,y) \in \mathbb{R}^2 \mid x, y \geq 0 \quad x+y \leq 1 \}$

$$I_1 = \iint_D (x+y)e^{-x}e^{-y}\,dx\,dy = \int_0^1 e^{-x} \underbrace{\left(\int_0^{1-x} (x+y)e^{-y}\,dy \right)}_{=①} dx$$

$$① = x\int_0^{1-x} e^{-y}\,dy + \int_0^{1-x} y e^{-y}\,dy$$

IPP $\quad u = y \quad u' = 1$

$\quad v' = e^{-y} \quad v = e^{-y}$

$$= x\left[-e^{-y}\right]_0^{1-x} + \left[-ye^{-y}\right]_0^{1-x} + \int_0^{1-x} e^{-y}\,dy$$

$$= x\left(-e^{-1+x} + 1\right) + \left(-(1-x)e^{-1+x}\right) + \left[-e^{-y}\right]_0^{1-x}$$

$$= \cancel{-xe^{-1+x}} + x + \cancel{xe^{-1+x}} - e^{-1+x} - e^{-1+x} + 1$$

$$= (x+1) - 2e^{-1+x}.$$

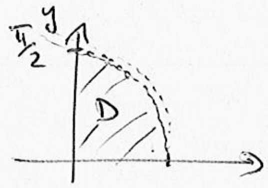
IPP : $u = (1+x)$ $u' = 1$
$v' = e^{-x}$ $v = -e^{-x}$

$$I_1 = \int_0^1 \left((x+1)e^{-x} - 2e^{-1}\right)dx$$

$$= \left[-(x+1)e^{-x}\right]_0^1 + \int_0^1 e^{-x}\,dx - 2e^{-1}$$

$$= -2e^{-x} + 1 + \underbrace{\left[-e^{-x}\right]_0^1}_{= -e^{-1}+1} - 2e^{-1} = 2 - 5e^{-1}$$

2) $D = \{ (x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq \frac{\pi}{2} \text{ et } 0 \leq x \leq \cos y \}$



$$I_2 = \iint_D x \cos y \,dx\,dy = \int_0^{\pi/2} \left(\int_0^{\cos y} x\,dx \right) \cos y \,dy$$

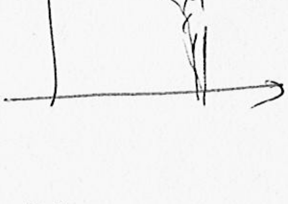
$$= \int_0^{\pi/2} \frac{\cos^2 y}{2} \cos y \,dy$$

$$= \frac{1}{2} \int_0^{\pi/2} \left(1 - \sin^2(y)\right) \cos y \,dy$$

$$= \frac{1}{2} \left(\left[\sin y\right]_0^{\pi/2} - \left[\frac{1}{3}\sin^3 y\right]_0^{\pi/2} \right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{3}\right) = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}.$$

3) $D = \{ (x,y) \in [0,1]^2 \mid x^2 + y^2 \geq 1 \}$



$$I_3 = \iint_D \frac{xy}{1+x^2+y^2}\,dx\,dy = \int_0^1 x \left(\int_{\sqrt{1-x^2}}^1 y(1+x^2+y^2)^{-1}\,dy \right) dx$$

$$I_3 = \frac{1}{2}\int_0^1 x\left[\ln(1+x^2+y^2)\right]_{\sqrt{1-x^2}}^{1}dx$$

$$= \frac{1}{2}\int_0^1 x\left(\ln(2+x^2) - \ln(1+\cancel{x^2}+1-\cancel{x^2})\right)dx$$

$$= \frac{1}{2}\left(\int_0^1 x\ln(2+x^2)\,dx - \frac{\ln 2}{2}\right)$$

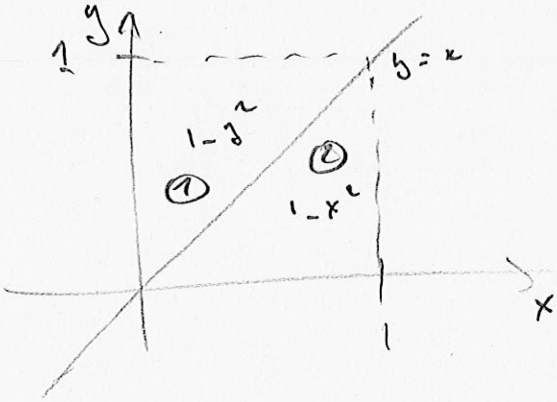
remarque $(u\ln u - u')' = u\ln u$

$$= \frac{1}{2}\left(\frac{1}{2}\left[(2+x^2)\ln(2+x^2) - (2+x^2)\right]_0^1 - \frac{\ln 2}{2}\right)$$

$$= \frac{1}{4}\left(3\ln(3) - 3 - 2\ln(2) + 1 - \ln 2\right)$$

$$= \frac{1}{4}\left(3\ln(3) - 3\ln(2) - 1\right)$$

4) $D = \{(x,y,z)\in\mathbb{R}^3 \mid x\geq 0,\ y\geq 0,\ z\geq 0$ et $z\leq \min\{1-x^2, 1-y^2\}\}$



si $|x|<|y|$ alors $1-x^2 \geq 1-y^2$ (1)

si $|y|<|x|$ alors $1-y^2 \geq 1-x^2$ (2)

$$\iiint_D z\,dx\,dy\,dz = \overbrace{\int_0^1\left(\int_0^y\left(\int_0^{1-y^2} z\,dz\right)dx\right)dy}^{(1)}$$

$$+ \overbrace{\int_0^1\left(\int_0^x\left(\int_0^{1-x^2} z\,dz\right)dy\right)dx}^{(2)}$$

$$= \int_0^1\int_0^y\left[\frac{z^2}{2}\right]_0^{1-y^2}dx\,dy + \int_0^1\int_0^x\left[\frac{z^2}{2}\right]_0^{1-x^2}dy\,dx$$

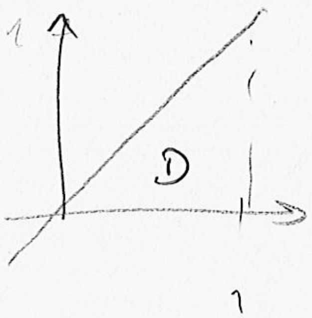
$$= \int_0^1 \int_0^y \frac{(1-y^2)^2}{2}\, dx\, dy \quad + \int_0^1 \int_0^x \frac{(1-x^2)^2}{2}\, dy\, dx$$

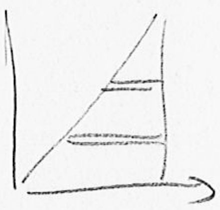
$$= \int_0^1 y\,(1-y^2)^2 \,\frac{dy}{2} \quad + \int_0^1 x\,(1-x^2)^2\, \frac{dx}{2}$$

$$= \frac{1}{2}\left[-\frac{1}{6}\left(1-y^2\right)^3\right]_0^1 + \frac{1}{2}\left[-\frac{1}{6}\left(1-x^2\right)\right]_0^1 = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}.$$

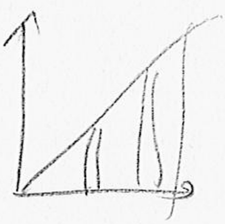
Exercice :

$$D = \left\{ (x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq x \leq 1 \right\}$$





①



②

× 2 "manières" de calculer cette intégrale.

① $\iint_D e^{x^2}\, dx\, dy = \int_0^1 \left(\int_y^1 e^{x^2}\, dx \right) dy$ ⇝ il faut primitiver e^{x^2}
cela n'aboutit pas.

② $\iint_D e^{x^2}\, dx\, dy = \int_0^1 \left(\int_1^x e^{x^2}\, dy \right) dx$

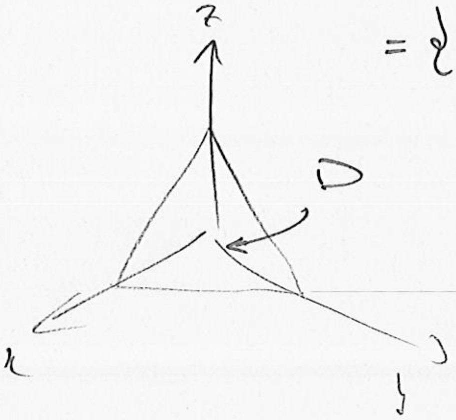
$$= \int_0^1 x e^{x^2}\, dx = \left[\frac{1}{2} e^{x^2} \right]_0^1 = \frac{1}{2} e - \frac{1}{2} 1 = \frac{(e-1)}{2}.$$

Exercice 5 :

$D = \{ (x,y,z) \in \mathbb{R}^3 \mid x \geq 0 ;\ y \geq 0 ;\ z \geq 0 \quad x+y+z \leq 1 \}.$

$= \{ \text{——} \mid \quad 0 \leq z \leq \quad \quad \text{ou } (x,y) \in T \}$

$\text{et } T = \{ (x,y) \in \mathbb{R}^2 \mid x \geq 0 ;\ y \geq 0 ;\ x+y \leq 1 \}$



$$\iiint_D \frac{dx\,dy\,dz}{(1+x+y+z)^3} = \int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} (1+x+y+z)^{-3}\, dz \right) dy \right) dx$$

$$= \int_0^1 \left(\int_0^{1-x} \left[-\frac{1}{2} (1+x+y+z)^{-2} \right]_{z=0}^{1-x-y} dy \right) dx$$

$$= \int_0^1 \int_0^{1-x} -\frac{1}{8} + \frac{1}{2} (1+x+y)^{-2}\, dy\, dx$$

$$= \int_0^1 \left[-\frac{1}{8} y - \frac{1}{2} (1+x+y)^{-1} \right]_{y=0}^{1-x} dx$$

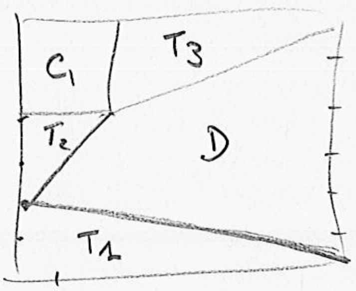
$$= \int_0^1 -\frac{1}{8}(1-x) - \frac{1}{4} + \frac{(1+x)^{-1}}{2}\, dx$$

$$= -\frac{1}{8} + \left[\frac{x^2}{16} \right)_0^1 - \frac{1}{4} + \frac{1}{2} \left[\ln(1+x) \right]_0^1$$

$$= -\frac{2}{16} + \frac{1}{16} - \frac{4}{16} + \frac{\ln 2}{2} = -\frac{5}{16} + \frac{\ln 2}{2}$$

Exercice 6.

1ère méthode : Remarque que : $\text{Aire}(D) = 36 - \text{Aire}(T_1) - \text{Aire}(T_2) - \text{Aire}(T_3) - \text{Aire}(C_1)$

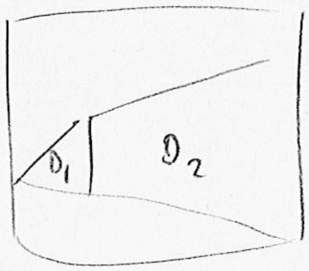


$$= 36 - 6 - 2 - 4 - 4$$

$$= 20.$$

2ème méthode : décomposer la région D en 2 parties :

$$D = D_1 \cup D_2$$



$$\iint_D dx\,dy = \int_{-3}^{-1} \left(\int_{-\frac{1}{3}x - 2}^{x+2} dy \right) dx + \int_{-1}^{3} \left(\int_{-\frac{1}{3}x - 2}^{\frac{x}{2}+\frac{3}{2}} dy \right) dx$$

$$= \int_{-3}^{-1} x + 2 + \frac{1}{3}x + 2 \, dx + \int_{-1}^{3} \frac{x}{2} + \frac{3}{2} + \frac{x}{3} + 2 \, dx$$

$$= \int_{-3}^{-1} \frac{4}{3}x + 4 \, dx + \int_{-1}^{3} \frac{5}{6}x + \frac{7}{2} \, dx = \left[\frac{4}{3}\frac{x^2}{2} + 4x \right]_{-3}^{-1} + \left[\frac{5}{6}\frac{x^2}{2} + \frac{7}{2}x \right]_{-1}^{3}$$

$$= \underbrace{\frac{4}{3} \times \frac{1}{2}}_{\frac{2}{3}} \underbrace{- 4 - \frac{4}{3} \times \frac{9}{2} + 12}_{= 2} + \underbrace{\frac{5 \times 9}{12}}_{\frac{15}{4}} + \frac{21}{2} - \frac{5}{12} + \frac{7}{2} = 20.$$

$$\frac{15}{4} + \frac{42}{4} - \frac{5}{12} + \frac{14}{4}$$

Exercice 7 :

1) Pour trouver ϕ et $\varphi : [-1,1] \longrightarrow \mathbb{R}$ on remarque que :

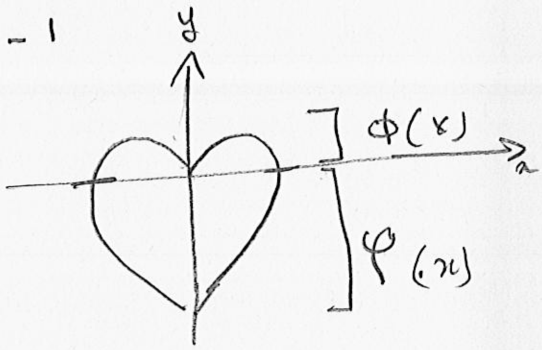
$$(y+1)^2 - 2|x|(y+1) + (2x^2-1) = 0$$

$$\Leftrightarrow \quad y+1 = \frac{1}{2}\left(2|x| \pm 2\sqrt{|x|^2-(2x^2-1)}\right)$$

$$\Leftrightarrow \quad y = |x| \pm \sqrt{1-x^2} - 1$$

on pose alors $\phi(x) = |x| - 1 + \sqrt{1-x^2}$

$\varphi(x) = |x| - 1 - \sqrt{1-x^2}$



2) Calcul d'une primitive de $x \mapsto \sqrt{1-x^2}$.

$$\int_0^x \sqrt{1-t^2}\, dt = \int_0^{\mathrm{Asin}\, x} \sqrt{1-\sin^2\theta}\, \cos\theta\, d\theta = \int_0^{\mathrm{Asin}\, x} \cos^2\theta\, d\theta.$$

1ère méthode : Pose $A = \int_0^{\mathrm{Asin}} \cos^2\theta\, d\theta$ et faire une IPP.

$$A = \int_0^{\mathrm{Asin}\, x} \cos^2\theta\, d\theta = \underbrace{\cos(\mathrm{Asin}\, x)}_{=\sqrt{1-x^2}}\underbrace{\sin(\mathrm{Asin}\, x)}_{=x} + \int_0^{\mathrm{Asin}\, x} \sin^2\theta\, d\theta.$$

et $2A = x\sqrt{1-x^2} + \int_0^{\mathrm{Asin}\, x} (\sin^2\theta + \cos^2\theta)\, d\theta = x\sqrt{1-x^2} + \mathrm{Asin}(x)$

$$\therefore A = \frac{1}{2}\left(x\sqrt{1-x^2} + \mathrm{Asin}(x)\right)$$

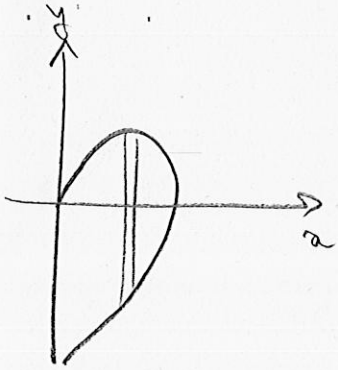
2ème méthode : linéariser le $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ et $\sin(2\theta) = 2\sin\theta\cos\theta$

$$\int_0^{\mathrm{Asin}\, x} \cos^2\theta\, d\theta = \frac{1}{2}\left(\mathrm{Asin}\, x + \left[\frac{1}{2}\sin 2\theta\right]_0^{\mathrm{Asin}\, x}\right)$$

$$= \frac{1}{2}\left(\mathrm{Asin}\, x + \frac{1}{2}\sin(2\,\mathrm{Asin}\, x)\right)$$

$$= \frac{1}{2}\left(\mathrm{Asin}\, x + \sin(\mathrm{Asin}\, x)\cos(\mathrm{Asin}\, x)\right)$$

$$= \frac{1}{2}\left(\mathrm{Asin}\, x + x\sqrt{1-x^2}\right).$$



3) Utiliser le fait que $\mathcal{D}$ est symétrique par rapport à l'axe Oy.

$$\text{Aire}\,(\mathcal{D}) = 2\int_0^1 \int_{\varphi(x)}^{\phi(x)} dy\,dx$$

$$= 2\int_0^1 \phi(x) - \varphi(x)\,dx$$

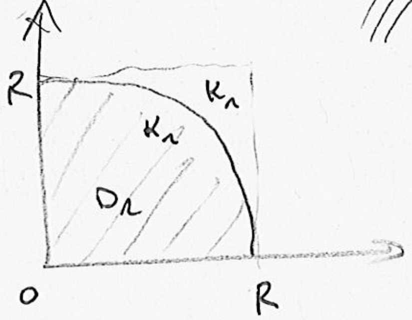
$$= 2\int_0^1 2\sqrt{1-x^2}\,dx = \frac{4}{2}\left[x\sqrt{1-x^2} + \text{Asin}\,(x)\right]_0^1$$

$$= 2\,\text{Asin}\,(1) = \pi$$

Exercice 2 : (Intégrale de Gauss)

cf le chapitre sur les espaces normés :

1)



$$\frac{1}{2}\|\cdot\|_2 \leq \|\cdot\|_\infty \leq \|\cdot\|_2$$

$$\Rightarrow \underbrace{B_2(0,2R)}_{=D_{2R}} \supset \underbrace{B_\infty(0,R)}_{=K_R} \supset \underbrace{B_2(0,R)}_{=D_R}$$

$$\Rightarrow \iint_{D_R} e^{-(x^2+y^2)}\,dx\,dy \leq \iint_{K_R} e^{-x^2-y^2}\,dx\,dy \leq \iint_{D_{2R}} e^{-x^2-y^2}\,dx\,dy$$

car $e^{-x^2-y^2} \geq 0 \quad \forall x,y \in \mathbb{R}^2$.

2) $$\iint_{D_R} e^{(-x^2-y^2)}\,dx\,dy = \int_0^{\pi/2} d\theta \int_0^R r e^{-r^2}\,dr = \frac{\pi}{2}\left[-\frac{1}{2}e^{-r^2}\right]_0^R = \frac{\pi}{4}\left(1-e^{-R^2}\right)$$

$$\iint_{D_{2R}} e^{-(x^2+y^2)}\,dx\,dy = \frac{\pi}{4}\left(1-e^{-2R^2}\right)$$

$$\iint_{K_R} e^{-x^2}e^{-y^2}\,dx\,dy = \left(\int_0^R e^{-x^2}\,dx\right)^2$$

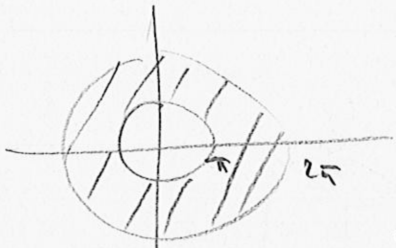
Théorème de Fubini $\Rightarrow$ $\lim_{R\to+\infty}\left(\int_0^R e^{-x^2}dx\right)^2$ existe et vaut $\frac{\pi}{4}$

et on a $\int_0^{+\infty} e^{-x^2}dx = \lim_{R\to\infty}\int_0^R e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$.

Exercice 8 :

1) $D = \{(x,y)\in\mathbb{R}^2 \mid \pi^2 \leq x^2+y^2 \leq 4\pi^2\}$

$$\iint_D \sin\sqrt{x^2+y^2}\,dx\,dy = \int_\pi^{2\pi} r\sin r\,dr \int_0^{2\pi} d\theta$$



$$= \left(\left[-r\cos r\right]_\pi^{2\pi} + \left[\sin r\right]_\pi^{2\pi}\right)2\pi$$

$u = r \quad u' = 1$

$v' = \sin r \quad v = -\cos r$

$$= (-2\pi - \pi)\,2\pi = -6\pi^2.$$

$\phi^{-1}(D) = \begin{cases}\pi < r \leq 2\pi \\ 0 \leq \theta \leq 2\pi.\end{cases}$

2) $D = \{(x,y)\in\mathbb{R}^2 \mid \frac{x^2}{a^2}+\frac{y^2}{b^2} \leq 1\}$ $\quad a, b > 0.$

Posons $\begin{cases} x = a\,r\cos\theta \\ y = b\,r\sin\theta\end{cases}$ $\quad \phi : \underbrace{]0,+\infty[\times]0,2\pi[}_{J} \longrightarrow \mathbb{R}^2$

$(r,\theta) \longrightarrow \begin{pmatrix}x\\y\end{pmatrix}$

* $|\mathrm{Jac}_\phi(r,\theta)| = |abr| = abr$
* $\phi^{-1}(D) = \{(r,\theta)\in J \mid 0 \leq r^2 \leq 1\}$

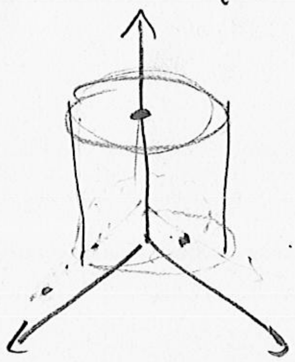
$$\iint_D x^2+y^2\,dx\,dy = ab\int_0^1\int_0^{2\pi}(a^2\cos^2\theta + b^2\sin^2\theta)\,r^3\,d\theta\,dr$$

$$= \left(\int_0^1 r^3 dr\right)\left(\int_0^{2\pi}\frac{a^2}{2}(\cos 2\theta + 1) + \frac{b^2}{2}(-\cos(2\theta)+1)\,d\theta\right)ab$$

$$= ab\left[\frac{r^4}{4}\right]_0^1\left(\left[\frac{a^2}{2}\left(\frac{1}{2}\sin(2\theta)+\theta\right)\right]_0^{2\pi} + \left[\frac{b^2}{2}\left(\frac{-\sin 2\theta}{2}+\theta\right)\right]_0^{2\pi}\right)$$

$$= \frac{ab}{4}\left(\frac{a^2}{2}(2\pi) + \frac{b^2}{2}(2\pi)\right) = \frac{ab}{4}\left(a^2\pi + b^2\pi\right)$$

3) $\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq 1 \; ; \; 0 \leq z \leq h\}$ $h > 0$.



$$\iiint_D z \, dx\, dy\, dz = \int_0^1 \int_0^{2\pi} \int_0^h z \, dz \, d\theta \, r\, dr$$

$$= \left(\int_0^1 r\, dr\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^h z\, dz\right)$$

Coordonné cylindrique.

$$= \frac{h^2}{2} \times 2\pi \times \frac{1}{2} = \frac{\pi}{2} h^2 .$$

$$\phi^{-1}(D) = \begin{cases} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq z \leq h . \end{cases}$$

$\mathrm{Jac}_\phi(r,\theta,z) = r$.

4) $D = \{(x,y,z) \; \mathbb{R}^3 \mid 1 \leq x^2+y^2+z^2 \leq 4\}$

coordonnées sphérique

$$\phi :]0,+\infty[\times]0, 2\pi[\times]-\tfrac{\pi}{2}, \tfrac{\pi}{2}[\longrightarrow \mathbb{R}^3$$

$$(r,\theta,\varphi) \longmapsto \begin{pmatrix} r\cos\theta\cos\varphi \\ r\sin\theta\cos\varphi \\ r\sin\varphi \end{pmatrix} .$$

* $|\mathrm{Jac}_\phi(r,\theta,\varphi)| = r^2 \cos\varphi$
* $\phi^{-1}(D) = \begin{cases} 1 \leq r \leq 2 \\ 0 \leq \theta \leq 2\pi \\ -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2} \end{cases}$.

$$\iiint_D (x^2+y^2+z^2)^\alpha \, dx\, dy\, dz = \iiint_{\phi^{-1}(D)} r^{2\alpha} \times r^2 \cos\varphi \, dr\, d\theta\, d\varphi$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_1^2 r^{2\alpha+2}\, dr\right)\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi \, d\varphi\right)$$

$$= 2\pi \left[\frac{r^{2\alpha+3}}{2\alpha+3}\right]_1^2 \left[\sin\varphi\right]_{-\pi/2}^{\pi/2}$$

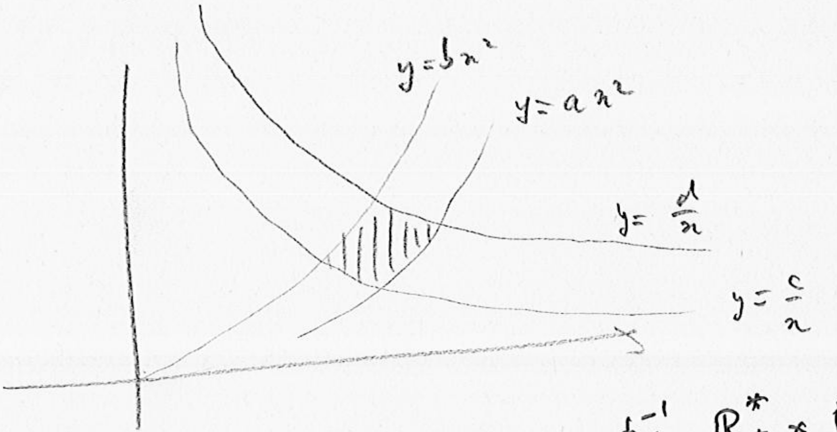
$$= 2\pi \left(\frac{2^{2\alpha+3}}{2\alpha+3} - \frac{1}{2\alpha+3}\right) \underbrace{\left(\sin\frac{\pi}{2} + \sin\frac{\pi}{2}\right)}_{=2}$$

$$= \frac{4\pi}{2\alpha+3}\left(2^{2\alpha+3} - 1\right).$$

Exercice 9 : $\begin{cases} 0 < a \leq b, \\ 0 < c \leq d \end{cases}$

$$D = \left\{ ax^2 \leq y \leq bx^2 \text{ et } \frac{c}{x} \leq y \leq \frac{d}{x} \ , (x,y) \in \mathbb{R}^2 \right\}$$

Poser $\begin{cases} u = y/x^2 \\ v = xy \end{cases}$



* changement de variable : $\phi^{-1} : \mathbb{R}_+^* \times \mathbb{R}_+^* \longrightarrow \mathbb{R}_+^* \times \mathbb{R}_+^*$.

$$(x, y) \longmapsto (yx^{-2}, xy)$$

$$\det \mathrm{Jac}_{\phi^{-1}}(x,y) = \det \begin{pmatrix} yx^{-3} \times (-2) & x^{-2} \\ y & x \end{pmatrix} = -2yx^{-2} - yx^{-2} = -3x^{-2}y^{-2}$$

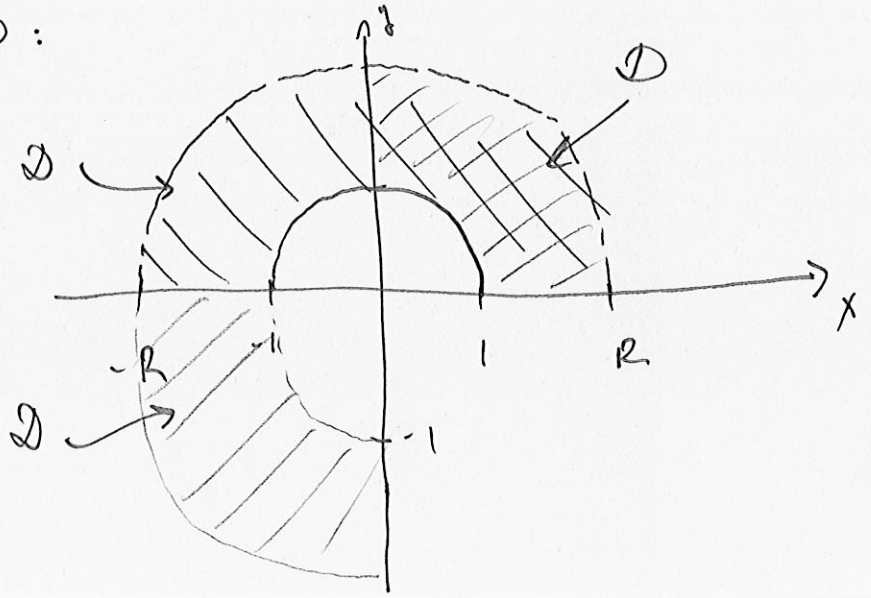
$$\Leftrightarrow \det \mathrm{Jac}_\phi(u,v) = \frac{1}{\det J_{\phi^{-1}}(\phi(u))} = -\frac{1}{3u}.$$

* $\phi^{-1}(D) = \{ (u,v) \mid a \leq u \leq b \text{ et } c \leq v \leq d \}$

$$\iint_D dx\,dy = \iint_{\phi^{-1}(D)} \frac{du\,dv}{3|u|} = \left(\int_a^b \frac{du}{3u} \right) \left(\int_c^d dv \right)$$

$$= \frac{1}{3} \left[\ln u \right]_a^b (d - c) = \frac{1}{3} \ln\left(\frac{b}{a}\right)(d-c)$$

Exercice 10 :



2) On passe en coordonnées polaires :

$$\mathcal{D} = \left\{ (r,\theta) \mid 1 \leq r \leq R \text{ et } 0 \leq \theta \leq \frac{3\pi}{2} \right\}$$

* Calcul du centre de gravité : $\frac{1}{\text{Aire}(\mathcal{D})} \left(\iint_{\mathcal{D}} x\,dx\,dy, \iint_{\mathcal{D}} y\,dx\,dy \right)$

* $\text{Aire}(\mathcal{D}) = \iint_{\mathcal{D}} dx\,dy = \int_1^R \int_0^{3\pi/2} d\theta\, r\,dr = \left(\int_1^R r\,dr \right) \left(\int_0^{3\pi/2} d\theta \right) = \frac{3\pi}{4}(R^2-1)$

* $\iint_{\mathcal{D}} x\,dx\,dy = \int_1^R \left(\int_0^{3\pi/2} r^2 \cos\theta\, d\theta \right) dr = \left(\int_1^R r^2\,dr \right) \left(\int_0^{3\pi/2} \cos\theta\, d\theta \right) = -\frac{(R^3-1)}{3}$

* $\iint_{\mathcal{D}} y\,dx\,dy = \left(\int_1^R r^2\,dr \right) \left(\int_0^{3\pi/2} \sin\theta\, d\theta \right) = \frac{R^3-1}{3}$.

On a alors :

$$(x_G, y_G) = \frac{4}{3\pi(R^2-1)} \left(-\frac{R^3-1}{3}, \frac{R^3-1}{3} \right)$$

$$= \frac{4(R^2+R+1)}{9\pi(R+1)} (-1, 1).$$

3) Le centre de gravité est situé sur $\Delta = \{y = -x\}$ et appartient à $\mathcal{D}$ ssi $\frac{4(R^2+R+1)}{9\pi(R+1)} \sqrt{2} \geq 1$

ssi $4\sqrt{2}R^2 + 4\sqrt{2}R + 4\sqrt{2} \geq 9\pi R + 9\pi$

ssi $4\sqrt{2}R^2 + R(4\sqrt{2} - 9\pi) + (4\sqrt{2} - 9\pi) \geq 0$

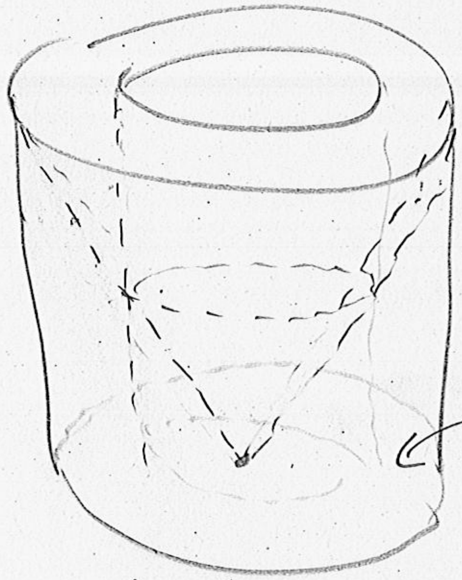
ssi $4\sqrt{2}\left[R^2 + R\left(1 - \frac{9\pi}{4\sqrt{2}}\right) + \left(1 - \frac{9}{4\sqrt{2}}\right) \right] \geq 0$

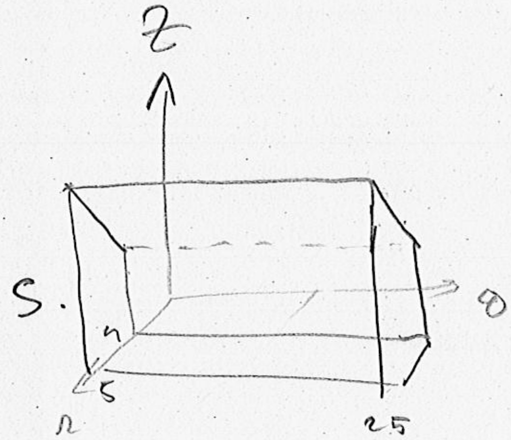
on pose $k = 1 - \frac{9\pi}{4\sqrt{2}}$ et

soit $R > \frac{k + \sqrt{k^2 + 4k}}{2} \simeq 4,826617132 \ldots$

Exercice 11 :

1)





$$S = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{cases} 4 \leq x^2 + y^2 \leq 25 \\ 0 \leq z \leq \sqrt{x^2 + y^2} \end{cases} \right\}$$

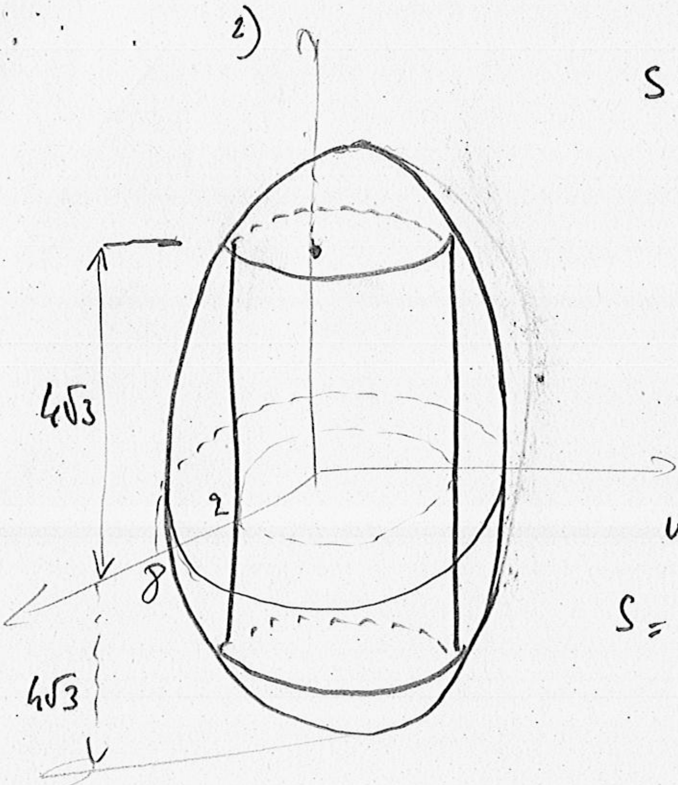
$$= \left\{ (r, \theta, z) \;\middle|\; \begin{cases} 2 \leq r \leq 5 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq z \leq r \end{cases} \right\}$$

$$\iiint_S dx\,dy\,dz = \int_2^5 \left(\int_0^{2\pi} \left(\int_0^r r\,dz \right) d\theta \right) dr$$

$$= \left(\int_0^{2\pi} d\theta \right) \int_2^5 r^2\,dr = 2\pi \left(\frac{5^3}{3} - \frac{2^3}{3} \right)$$

$$= 78\pi.$$

2)



$$S = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{cases} x^2+y^2 \leq 4 \\ \text{et} \\ 4(x^2+y^2)+z^2 \leq 64 \end{cases} \right\}$$

L'intersection du cylindre et de l'ellipsoïde

$$\begin{cases} x^2+y^2=4 \\ 16+z^2=64 \end{cases} \Rightarrow \begin{cases} x^2+y^2=4 \\ z=4\sqrt{3} \end{cases}$$

utilise la "formule de sommation par pile"

$$S = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \sqrt{-4(x^2+y^2)+64} \geq z \geq -\sqrt{64-4(x^2+y^2)},\ x,y \in \mathcal{D} \right\}$$

où $\mathcal{D} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 4\}$.

$$\iiint_S dx\,dy\,dz = \iint_{\mathcal{D}} \int_{-\sqrt{64-4(x^2+y^2)}}^{\sqrt{64-4(x^2+y^2)}} dz \; dx\,dy$$

$$= 2 \iint_{\mathcal{D}} \sqrt{64-4\cdot(x^2+y^2)}\; dx\,dy$$

$$= 4 \int_0^{2\pi}\int_0^2 \sqrt{16-r^2}\; r\,dr\,d\theta$$

$$= 4 \times 2\pi \times \left[-\frac{1}{\cancel{2}} \frac{\cancel{2}}{3} (16-r^2)^{3/2} \right]_0^2 = -\frac{8\pi}{3}\left((16-4)^{3/2} - 16^{3/2} \right)$$

$$= 8\pi \left(\frac{64}{3} - 8\sqrt{3} \right)$$