Learning in Games: Solution Concepts

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An optimization problem with multiple 'players.'

Single objective:

- Cost: $f : X \to \mathbb{R}$
- Variable: $x \in X$

Optimality:
$f(x^*) \leq f(x), x \in X_{x^*}$ (local)

$$\boxed{\min_{x \in X} f(x)}$$

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Game (two players):

- Game: $\mathcal{G} = (f_1, f_2)$.
- Costs: $f_i : X_1 \times X_2 \to \mathbb{R}, \; i \in [1, 2]$
- Variables: $(x_1, x_2) \in X_1 \times X_2$.

Optimality:
$f_1(x_1^*, x_2^*) \leq f_1(x_1, x_2^*), \; x_1 \in X_1,$
$f_2(x_1^*, x_2^*) \leq f_2(x_1^*, x_2), \; x_2 \in X_2.$

$$\boxed{\min_{x\in X} f(x)}$$

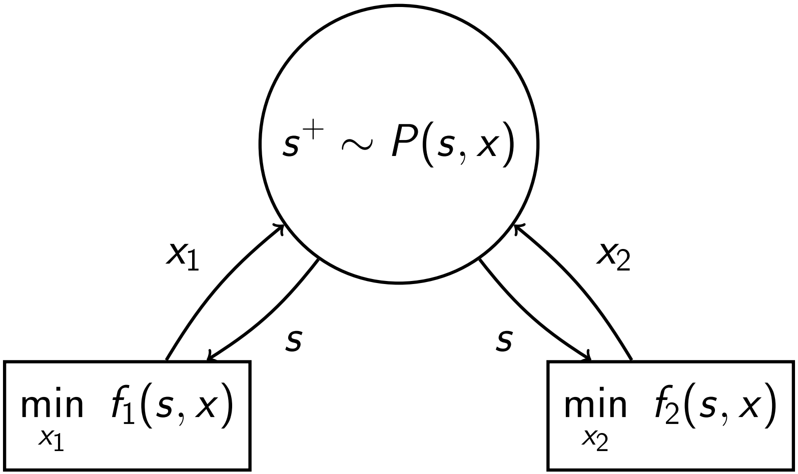
Single objective

$$\boxed{\min_{x_1\in X_1} \; f_1(x_1, x_2^*)} \underset{x_2^*}{\overset{x_1^*}{\rightleftarrows}} \boxed{\min_{x_2\in X_2} \; f_2(x_1^*, x_2)}$$

Player 1 Player 2

Learning in Dynamic Environments

Shared state s, individual costs and actions



where actions $x = (x_1, x_2)$ and shared state s that evolves via dynamics $\sim P$

- Dynamic Games
- Stochastic Games
- Multi-agent Reinforcement learning

Differential Notions of Equilibrium

Optimization (Unconstrained)

Twice-continuously differentiable objective $c \in \mathcal{C}^2$.
Local optimality of u^*:

$$Dc(u^*) = 0, \;\; D^2c(u^*) > 0.$$

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Continuous Games

Local optimality of $u^* = (x_1^*, x_2^*)$:

First player's condition

$$D_1 f_1(x_1^*, x_2^*) = 0, \; D_1^2 f_1(x_1^*, x_2^*) > 0$$

Second player's condition

$$D_2 f_2(x_1^*, x_2^*) = 0, \; D_2^2 f_1(x_1^*, x_2^*) > 0$$

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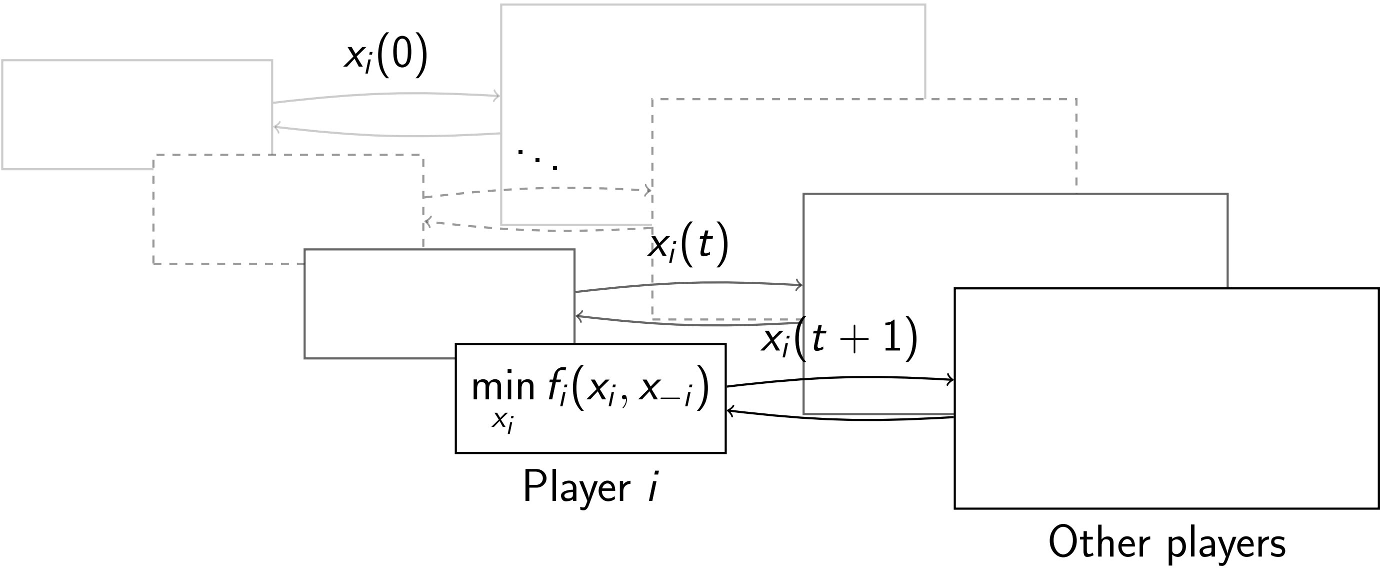
$$\text{“}D_i^2 f_i(x_i, x_{-j}) \equiv \frac{\partial}{\partial x_i^2} f_i(x)|_{x=(x_i, x_{-i})}\text{”}$$

Algorithms for Learning in Games

Develop a learning rule to generate sequence $x_i(0), \cdots, x_i(t)$ over time...



Algorithms for Learning in Games

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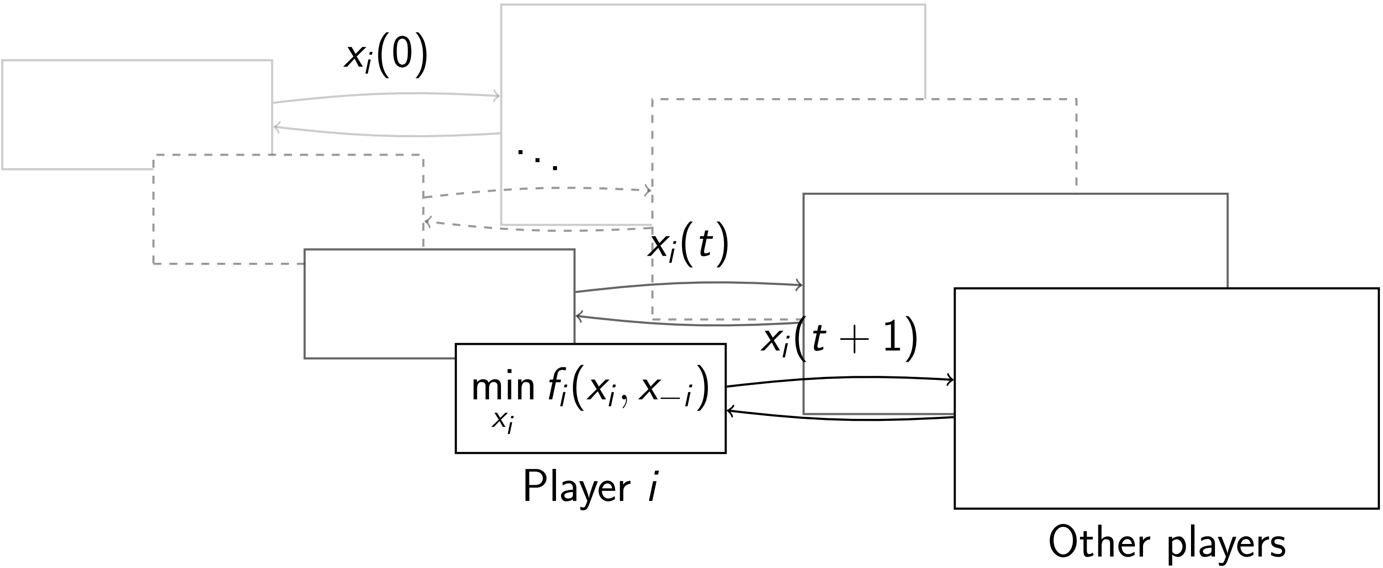
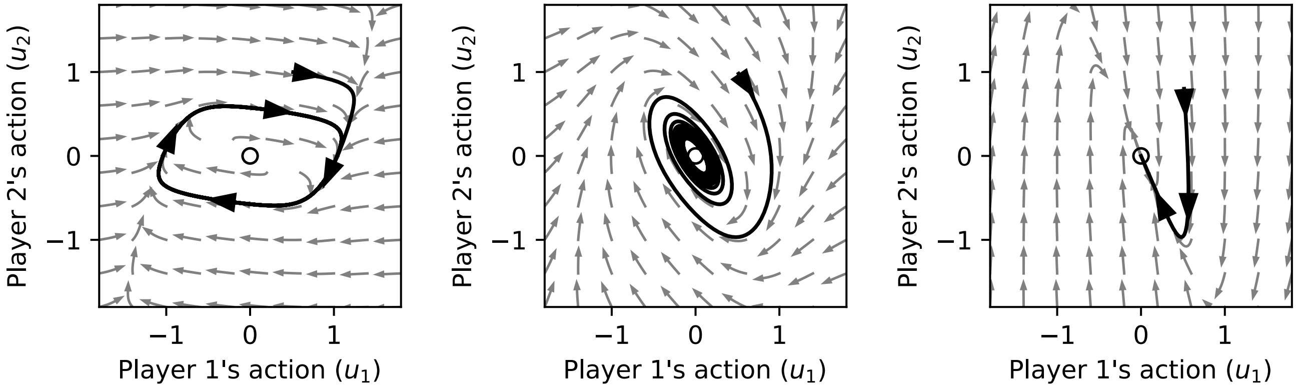


- Asymptotic convergence: as $t \to \infty$, does $u(t) \to u^*$?
- Non-asymptotic bounds: for $t \geq T$, what is $\|u(t) - u^*\|$?
- Regret/no-regret learning: "best action *in hindsight*."

Non-uniform Learning Rates

- Vector field of $\omega \equiv (D_1 f_1, D_2 f_2)$ with costs
- $f_1(u) = \frac{1}{4}x_1^4 - \frac{1}{2}x_1^2 - x_1 x_2$
- $f_2(u) = \frac{1}{2}x_2^2 + 2x_1 x_2$



Extensions of the Nash equilibrium

Hierarchy of play

Leader

$$\min_{x_1} \; f_1(x_1, \overline{x_2})$$

$$\overline{x_2} \in \arg\min_{x_2} f_2(x_1, x_2)$$

$$\xrightarrow{x_1} \qquad \xleftarrow{x_2}$$

Follower

$$\min_{x_2} f_2(x_1, x_2)$$

Bilevel optimization, Stackelberg games, one-sided conjectural equilibrium.

Extensions of the Nash equilibrium

Hierarchy of play

$$\begin{tikzcd}[column sep=large] \text{Leader} & \text{Follower} \\ \boxed{\begin{array}{l} \min\limits_{x_1} \; f_1(x_1, \overline{x_2}) \\ \qquad \overline{x_2} \in \arg\min\limits_{x_2} f_2(x_1, x_2) \end{array}} \arrow[r, bend left, "x_1"] & \boxed{\min\limits_{x_2} f_2(x_1, x_2)} \arrow[l, bend left, "x_2"] \end{tikzcd}$$

Bilevel optimization, Stackelberg games, one-sided conjectural equilibrium.

Using models of others

$$\begin{tikzcd}[column sep=large] \text{Player 1} & \text{Player 2} \\ \boxed{\begin{array}{l} \min\limits_{x_1} \; f_1(x_1, \overline{x_2}) \\ \qquad \overline{x_2} \in \xi_1(x_1) \end{array}} \arrow[r, bend left, "x_1"] & \boxed{\begin{array}{l} \min\limits_{x_2} \; f_2(\overline{x_1}, x_2) \\ \qquad \overline{x_1} \in \xi_2(x_2) \end{array}} \arrow[l, bend left, "x_2"] \end{tikzcd}$$

Conjectural variations equilibrium, opponent models

Conjectural Variations Equilibria

$$
\begin{array}{ccc}
\text{Player 1} & & \text{Player 2} \\
\boxed{\begin{array}{r} \min\limits_{x_1} \; f_1(x_1, \overline{x_2}) \\ \overline{x_2} \in \xi_1(x_1) \end{array}} & \begin{array}{c} \xrightarrow{x_1} \\ \xleftarrow[x_2]{} \end{array} & \boxed{\begin{array}{r} \min\limits_{x_2} \; f_2(\overline{x_1}, x_2) \\ \overline{x_1} \in \xi_2(x_2) \end{array}}
\end{array}
$$

Conjectural Variations Equilibria

$$\begin{tikzcd} \text{Player 1} & \text{Player 2} \\ \boxed{\begin{array}{l} \min\limits_{x_1} \; f_1(x_1, \overline{x_2}) \\ \qquad\quad \overline{x_2} \in \xi_1(x_1) \end{array}} \arrow[r, bend left, "x_1"] & \boxed{\begin{array}{l} \min\limits_{x_2} \; f_2(\overline{x_1}, x_2) \\ \qquad\quad \overline{x_1} \in \xi_2(x_2) \end{array}} \arrow[l, bend left, "x_2"] \end{tikzcd}$$

Differential Conjectural Variations Equilibrium

Local optimality of $u^* = (x_1^*, x_2^*)$:

First player's condition

$$Df_1(x_1^*, \xi_1(x_1^*)) = 0, \;\; D^2 f_1(x_1^*, \xi_1(x_1^*)) > 0$$

Second player's condition

$$Df_2(\xi_2(x_2^*), x_2^*) = 0, \;\; D^2 f_1(\xi_2(x_2^*), x_2^*) > 0$$

- Consistency: $x_2^* = \xi_1(x_1^*)$, $x_1^* = \xi_2(x_2^*)$

Conjectural learning dynamics

Ways we can construct learning dynamics to reflect this idea:

1. Descend the gradient of $Df_1(x, \xi_i(x))$ and $Df_2(\xi_2(y), y)$.

$$\begin{aligned} x_{k+1} &= x_k - \gamma(D_1 f_1(x_k, y) + D\xi_1(x_k)^\top D_2 f_1(x_k, y)) \\ y_{k+1} &= y_k - \gamma(D_2 f_2(x, y_k) + D\xi_2(y_k)^\top D_1 f_2(x, y_k)) \end{aligned} \tag{1}$$

   where

$$D\xi_1(x) = -(D_2^2 f_2(x, y_k))^{-1} D_{21} f_2(x, y_k).$$

2. Approximate the conjecture via taylor expansion of $\xi(x) \simeq y_k$

$$\begin{aligned} x_{k+1} &= x_k - \gamma g(x_k, h(x_k)), \\ y_k &= \arg\min_y f_2(x_k, y), \end{aligned} \tag{2}$$

   where

$$h(x) = y_k + D\xi(x_k)(x - x_k) + (x - x_k)^\top D^2 \xi(x_k)(x - x_k) + \mathcal{O}(x^3). \tag{3}$$

Block structure of zero-sum games Approximate discrete update $x_{k+1} = x_k - \gamma g(x)$ as $\dot{x} = -g(x)$. Linearization of g at x^* where $g(x^*) = 0$:

1. Simulteanous play games

$$Dg(x^*) = \begin{bmatrix} A & B \\ -B^\top & D \end{bmatrix}$$

2. Hierarchical (Stackelberg) game

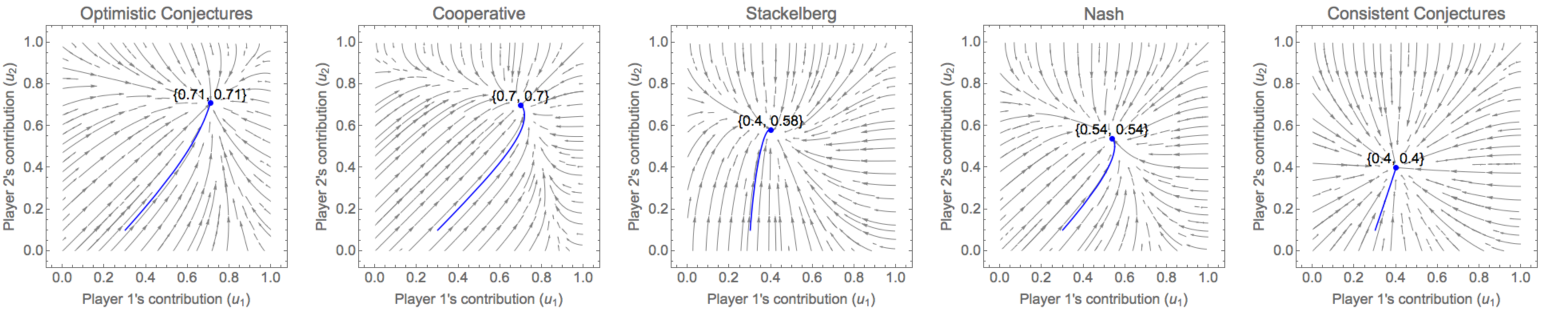
$$Dg(x^*) = \begin{bmatrix} A & 0 \\ -B^\top & D \end{bmatrix}$$

3. Conjectural (Stackelberg) game

$$Dg(x^*) = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$$

Example: Individual Contribution to Public Goods

- Models the tragedy of the commons
- Utilities: $-f_i(x_i, x_{-i}) = (I_i - x_i)^{\alpha_i}(x_i + x_{-i})^{1-\alpha_i}$
- Study of hierarchy of play and effects of various agent "conjectures."



Future Work in Learning in Games

...

- Characterize Conjectural Variations Equilibria: (stability, "performance metric")
- Devise learning rules
- Verify agent conjecture models behaviorally
- Lots more!