The Local Stability of Equilibria in Two-Player Continuous Games

Benjamin J. Chasnov*, Daniel Calderone*, Behçet Açikmeşe, Samuel A. Burden, Lillian J. Ratliff

***Abstract*— We study the stability of learning dynamics in two-player games and introduce the quadratic numerical range as a tool to analyze the optimization landscape of games. In particular, we study the continuous-time dynamics of gradient play and determine the conditions under which their fixed points are differential Nash equilibria. Since the quadratic numerical range reduces to the analysis of a two-by-two matrix, we provide an in-depth characterization by decomposing the game derivative into interpretable coordinates. We characterize the stability regions for potential and zero-sum game, as well as the learning dynamics with non-uniform learning rates. In scalar games, there exists learning rates to destabilize an equilibrium if and only if the equilibrium is non-Nash. Our numerical experiments explore polynomial and vector-valued general-sum games.**

I. Introduction

The study of learning in games is experiencing a resurgence in the control theory [18], [21], [22], optimization [12], [14], [20], and machine learning [5]–[7], [9], [15], [25] communities. Partly driving this resurgence is the prospect for game theoretic analysis to yield machine learning algorithms that generalize better or converge more robustly. Towards understanding the optimization landscape in such game theoretic formulations, dynamical systems theory is emerging as a principal tool for analysis and ultimately synthesis [1], [2], [4], [12], [13].

A predominant learning paradigm used across these different domains is gradient-based learning. Updates in large decision spaces can be performed locally with minimal information, while still guaranteeing local convergence in many problems [6], [14].

One of the primary means to understand the optimization landscape of games is the eigenstructure and spectrum of the Jacobian of the learning dynamics in a neighborhood of a a stationary point, as illustrated in Fig. 1. In particular, for a zero-sum continuous game $(f, -f)$ for some continuously-differentiable f the Nash equilibria are saddle points of the function f. However, as the example in Fig. 1 demonstrates, not all saddle points are relevant. Loosely speaking, the equilibrium conditions for the game correspond to constraints on the curvature directions of the cost function and hence, on the eigenstructure of the Jacobian nearby equilibria.

The local stability of a hyperbolic fixed point in a nonlinear system can be assessed by examining the eigenstructure of the linearized dynamics [10], [19]. However, in a game context there are extra constraints coming from the underlying game—that is, players are constrained to move only along directions over which they have control (i.e., they can control their individual actions as opposed to the entire state of the dynamical system corresponding to the learning rules being applied by the agents). Indeed, it has been observed in earlier work that not all stable attractors of gradient play are local Nash equilibria and not all local Nash equilibria are stable attractors of gradient play [12]. Furthermore, changes in players learning rates—which corresponds to scaling rows of the Jacobian—can change an equilibrium from being stable to unstable and vice versa [6].

To summarize this observation, there is a subtle, but extremely important difference between game dynamics and traditional nonlinear dynamical systems: alignment conditions are important for distinguishing between equilibria that have game-theoretic meaning versus those which are simply stable attractors of learning rules, and features of learning dynamics such as learning rates can play an important role in shaping not only equilibria but also alignment properties. Motivated by this observation along with the recent resurgence of applications of learning in games in control, optimization, and machine learning, in this paper we provide a in depth analysis of the spectral properties of gradient-based learning in two player continuous games.

Contributions. Towards this end, we begin with an analysis of the spectral properties of block operator matrices using the numerical range and its close cousin, the quadratic numerical range [23]. We show that both are important tools for quantifying the spectrum of block operators; indeed, the

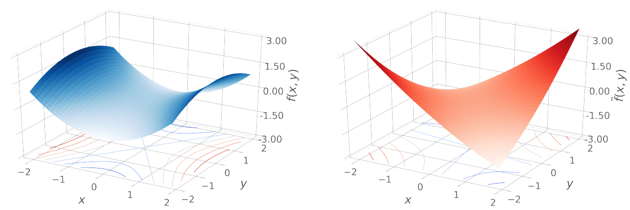


Fig. 1: *Cost landscape is crucial to understanding dynamics*: the zero-sum game defined by $f(x, y) = \frac{1}{2}x^2 - \frac{1}{4}y^2$ has a saddle point Nash equilibrium at $(0, 0)$ which is stable for gradient play. If the cost function is rotated to $\tilde{f}(x, y) = \frac{1}{8}x^2 + \frac{1}{8}y^2 + \frac{3}{4}xy$—i.e., a rotation by $\frac{3\pi}{4}$—then $(0, 0)$ is still a saddle point, yet it is no longer a stable point and, in fact, it is not even a Nash equilibrium. The directions of the saddle's curvature do not align with the axes corresponding to the players' individual actions.

B. Chasnov, S. Burden, and L. Ratliff are with the Department of Electrical and Computer Engineering, University of Washington, Seattle, WA 98115 {bchasnov, sburden, ratliffl}@uw.edu

D. Calderone and B. Açikmeşe are with the Department of Aeronautics and Astronautics, University of Washington, Seattle, WA 98115 {djcal, behcet}@uw.edu

* These authors contributed equally to this work.

(point) spectrum of a block operator matrix is contained in both the numerical and the quadratic numerical range. The quadratic numerical range for an arbitrary $n \times n$ block operator is completely characterized by two-by-two matrices.

With this observation, we provide a complete characterization of the spectral properties of two-by-two scalar continuous games, after which we return to 2-player continuous games. We leverage our insights about the scalar setting to characterize the quadratic numerical range for games on higher-dimensional strategy spaces. Our results include specialization to several important classes of games including zero-sum, potential, Hamiltonian, and *constant curvature* games, the latter of which we introduce in this paper.

II. Dynamics of Learning in Games

In multi-agent settings, an agent's objective is coupled to the decisions of others. We use a standard model of learning that captures the decision-making processes of agents.

An n-agent continuous game is a collection of costs $\mathcal{G} = (f_1, \dots, f_n)$ defined on $X = X_1 \times \cdots \times X_n$ where agent $i \in \mathcal{I} = \{1, \dots, n\}$ has cost $f_i \in C^r(X, \mathbb{R})$ with $r \geq 0$. Agent i's set of feasible strategies is the open subset $X_i \subseteq \mathbb{R}^{n_i}$ where n_i is its dimension.

Following [8], at time t, a myopic agent i optimizes its individual objective f_i over decision variable x_i given the decisions of others $x_{-i} \in X_1 \times X_{i-1} \times X_{i+1} \times X_n$. The synchronous adaptive process that arises is $x_i(t+1) \in \arg\min_{x_i} f_i(x_i, x_{-i}(t)),\ \forall i \in \mathcal{I}$, a discrete-time process. Gradient play is the continuous-time analog for continuously-differential costs with $r \geq 2$, where each agent steps infinitesimally in the direction of the steepest gradient of its objective,

$$\dot{x}_i = -D_i f_i(x_i, x_{-i}),\ \forall i \in \mathcal{I}.$$

A. Gradient-Based Learning for Two-Player Games

We present the local optimality condition of individual agents in two-player games $\mathcal{G} = (f_1, f_2)$.

Definition 1: A strategy $(x, y) \in X_1 \times X_2$ is a *strict local Nash equilibrium* for minimizing agents if there exists open sets $E_1 \subset X_1,\ E_2 \subset X_2$ such that $x \in E_1,\ y \in E_2$, and the following two conditions hold:

$$f_1(x, y) < f_1(x', y),\ \forall x' \in E_1 \backslash \{x\}$$
$$f_2(x, y) < f_2(x, y'),\ \forall y' \in E_2 \backslash \{y\}.$$

At a local Nash, no agent can decrease their cost through an infinitesimal variation of their decision variables.

Given an equilibrium, we locally analyze the vector of derivatives

$$g \equiv (D_1 f_1, D_2 f_2) \tag{1}$$

that captures the directions along which players can adjust their actions to minimize their respective costs. The definiteness of the derivatives of $D_1 f_1$ and $D_2 f_2$ at a fixed point of g provides a sufficient condition for the optimality of an individual agent [18].

Definition 2: A strategy $(\tilde{x}, \tilde{y}) \in X_1 \times X_2$ is a *differential Nash equilibrium* for minimizing agents if $g(\tilde{x}, \tilde{y}) = 0$ and

$$-D_1^2 f_1(\tilde{x}, \tilde{y}) \prec 0,\ -D_2^2 f_2(\tilde{x}, \tilde{y}) \prec 0.$$

A differential Nash equilibrium is a strict local Nash equilibrium [17, Thm. 1].

B. Linearizing the Game Dynamics

To analyze the convergence properties of gradient play, consider the stability of the following dynamical system. A strategy $(\tilde{x}, \tilde{y}) \in X_1 \times X_2$ is a *locally exponentially stable fixed point* of dynamical system $g \in C^1(X, \mathbb{R})$ in a neighbourhood of $(\tilde{x}, \tilde{y})$, where $g \equiv (g_1, g_2)$ is given by

$$\begin{aligned} \dot{x} &= -g_1(x, y), \\ \dot{y} &= -g_2(x, y), \end{aligned} \tag{2}$$

if and only if $g(\tilde{x}, \tilde{y}) = 0$ and all eigenvalues of $-Dg(\tilde{x}, \tilde{y})$ are in $\mathbb{C}_-^\circ$, the open-left half plane [10, Corr. 4.3]. We linearize the dynamics (2) about fixed point $(\tilde{x}, \tilde{y})$ and partition the Jacobian $J(\tilde{x}, \tilde{y})$ into blocks,

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} -D_1^2 f_1(\tilde{x}, \tilde{y}) & -D_{12} f_1(\tilde{x}, \tilde{y}) \\ -D_{21} f_2(\tilde{x}, \tilde{y}) & -D_2^2 f_2(\tilde{x}, \tilde{y}) \end{bmatrix}.$$

We drop the dependence on $(\tilde{x}, \tilde{y})$ when the chosen fixed point is clear from context.

Nash optimality is defined with respect to the definiteness of $J_{11} \prec 0$ and $J_{22} \prec 0$, whereas stability considers the spectrum of the whole J. Because the diagonal blocks are symmetric, J is similar to a block matrix with diagonal block-diagonals, illustrated in Figure 2. For the remainder

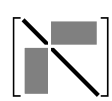
$J(\tilde{x}, \tilde{y}) \sim$ 

Fig. 2: The derivative of the game dynamics g is similar to a block matrix with diagonal block-diagonals.

of the paper, we will study the linearization at a fixed point of the dynamics (2).

While some of the stable attractors correspond to Nash optimal points, many do not. To illustrate this, we state the following. If agents are willing and able to fully cooperate and jointly minimize f, as in to play the cooperative game $\mathcal{G} = (f, f)$, then $(\tilde{x}, \tilde{y})$ is a strict local optimum if and only if dynamics (2) is stable around a neighbourhood of $(\tilde{x}, \tilde{y})$ Standard results from non-linear programming [3] and non-linear systems [19] apply.

However, in game-theoretic settings such as zero-sum games $(f, -f)$, potential games[1] (f_1, f_2) with potential function ϕ, or general-sum games with no additional restriction on the costs, not all stable attractors of gradient play are optimal. The coupling effects of J_{12} and J_{21} results in dynamics that do not correspond to a gradient field.

[1]$\mathcal{G} = (f_1, f_2)$ is a potential game if and only if there exists potential function $\phi : X \to \mathbb{R}$ such that $D_1 f_1 \equiv D_1 \phi$ and $D_2 f_2 \equiv D_2 \phi$ [16].

C. Block Numerical Range

The numerical range and quadratic numerical range of a block operator matrix are sets that contain the operator's spectrum [23]. The numerical range, defined by

$$W(J) = \{z \in \mathbb{C}^{n_1+n_2} : \ \langle Jz, z\rangle, \ \|z\| = 1\},$$

is a convex subset of $\mathbb{C}$. Given a block operator J, as in the preceding subsection, let

$$J_{v,w} = \begin{bmatrix} \langle J_{11}v, v\rangle & \langle J_{12}w, v\rangle \\ \langle J_{21}v, w\rangle & \langle J_{22}w, w\rangle \end{bmatrix}.$$

The quadratic numerical range of J is defined by

$$W^2(J) = \bigcup_{v\in\mathcal{S}_1, w\in\mathcal{S}_2} \sigma_p(J_{v,w})$$

where $\sigma_p(\cdot)$ denotes the (point) spectrum of its argument and $\mathcal{S}_i = \{z \in \mathbb{C}^{n_i} : \ \|z\| = 1\}$.

The quadratic numerical range can be described as the set of solutions of the characteristic polynomial

$$\begin{aligned} &\lambda^2 - \lambda(\langle J_{11}x, x\rangle + \langle J_{22}y, y\rangle) \\ &\quad + \langle J_{11}x, x\rangle\langle J_{22}y, y\rangle - \langle J_{12}x, y\rangle\langle J_{21}y, x\rangle = 0 \end{aligned}$$

for $x \in \mathcal{S}_1$ and $y \in \mathcal{S}_2$. We use the notation $\langle Jx, y\rangle = x^*Jy$ to denote the inner product. Note that $W^2(J)$ is a (potentially non-convex) subset of $W(J)$ and contains $\sigma_p(J)$.

The generalization of quadratic numerical range to $n \times n$ block matrices—i.e., $W^n(J)$—derived from an n-player game involves taking the union of the solutions to the characteristic polynomial of $n \times n$ matrices. The quadratic numerical range of J informs us of the regions in which the eigenvalues of J must lie. These regions consists of at most n connected, closed and bounded sets that in general are non-convex [23].

Example 1: Consider the Jacobian of the zero sum game defined by

$$f(x, y) = -\tfrac{1}{2}x_1^2 + \tfrac{5}{2}x_2^2 + 7y_1x_1 - 3y_2x_2 - 2y_1^2 - 6y_2^2.$$

The numerical range, quadratic numerical range, spectrum and diagonal entries of $J = -Dg(0, 0)$ derived from the dynamics of gradient play are plotted in Fig. 3. Note that $W(J)$ is convex and that $W^2(J)$ is a subset of $W(J)$. Furthermore, $\sigma_p(J) \subset W^2(J)$, and $\mathrm{diag}(J) \subset \mathbb{R}$. Albeit non-convex, $W^2(J)$ provides a tighter characterization of the spectrum[2].

Towards characterizing the quadratic numerical range of a two-player game derivative, we now study 2×2 real-valued scalar games. In Section IV, we return to analyze the higher dimensional case.

[2]There are numerous computational approaches for estimating the $W(\cdot)$ and $W^2(\cdot)$ (see, e.g., [11, Sec. 6]).

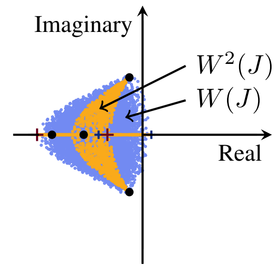


Fig. 3: The spectrum of J ($\bullet$) is contained in the numerical range (convex region) and in quadratic numerical range (non-convex region). The eigenvalues of J_{ii} are real (dark $+$'s).

III. Stability in Two-Player Scalar Games

In the scalar strategy case, we decompose the game Jacobian evaluated at a fixed point as

$$J(x, y) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} m & -z \\ z & m \end{bmatrix} + \begin{bmatrix} h & p \\ p & -h \end{bmatrix} \quad (3)$$

where $m = (a + d)/2$, $h = (a - d)/2$, $p = (b + c)/2$, $z = (c - b)/2$. We provide easily verifiable characterizations of several quantities associated with J.

Statement 1: Given a matrix $J \in \mathbb{R}^{2\times2}$ and its eigenvalues λ_1, λ_2, the decomposition (3) gives rise to the following:

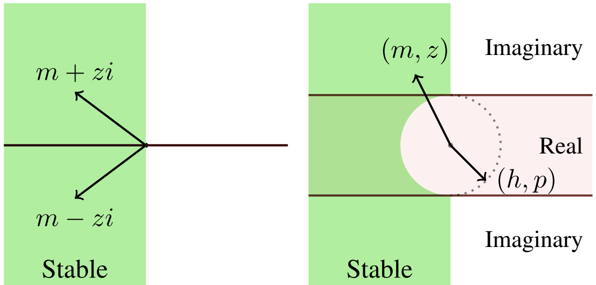
$$\begin{aligned} \tfrac{1}{2}\mathrm{Tr}(J) &= \tfrac{1}{2}\big(\lambda_1 + \lambda_2\big) = \tfrac{1}{2}\big(a + d\big) = m, \\ \det\big(J\big) &= \lambda_1\lambda_2 = ad - bc, \\ &= m^2 + z^2 - p^2 - h^2 = \left\|\begin{bmatrix} m \\ z \end{bmatrix}\right\|^2 - \left\|\begin{bmatrix} h \\ p \end{bmatrix}\right\|^2, \\ 4\mathrm{disc}\big(J\big) &= h^2 + p^2 - z^2, \\ \lambda_{1,2} &= m \pm \sqrt{h^2 + p^2 - z^2}. \end{aligned}$$

This decomposition provides several insights into linear 2×2 vector fields and, in particular, to games.

A. Vector Field Insights

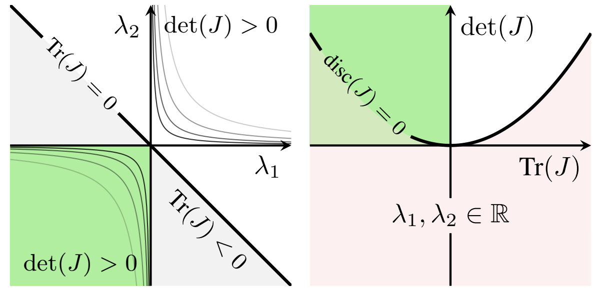
1) Relationship to Complex Plane: Fig. 4 plots the coordinates of m, z, h, p relative to each other to illustrate the decomposition in Statement 1. If $\begin{bmatrix} h & p \end{bmatrix}^T = 0$, then the eigenvalues of J are $\lambda_{1,2} = m \mp zi$. Figure 4a corresponds to a plot of eigenvalues in the complex plane where the $\mathtt{mh}$-axis corresponds to the real axis and the $\mathtt{pz}$-axis corresponds to the imaginary axis. Stability is given by the familiar open-left half plane condition: $\rho(J) \subset \mathbb{C}_-^\circ$. If $\begin{bmatrix} h & p \end{bmatrix}^T \neq 0$ a circular region in the center of the plane expands the values of m, z for which the eigenvalues of the matrix are purely real. Fig. 4b shows that, indeed, the eigenvalues are purely real if and only if $z^2 \leq h^2 + p^2$.

2) Stability: It is well-known that a matrix $J \in \mathbb{R}^{2\times2}$ is stable if and only if $\mathrm{Tr}(J) = \lambda_1 + \lambda_2 < 0$ and $\det(J) = \lambda_1\lambda_2 > 0$. These conditions on the eigenvalues λ_1, λ_2 of J are shown as the intersection of a half-plane and a circular exclusion region in Fig. 4b. For real eigenvalues, these conditions parameterize the negative orthant in the λ_1-λ_2 space, illustrated in Fig. 5a. A familiar diagram relating the determinant, trace, and discriminant of J is illustrated in Fig. 5b.



(a) The complex plane. (b) A representation of the m, z, h, p coordinates.

Fig. 4: The stability of a 2×2 matrix J is governed by its trace and determinant conditions. Decomposing J into (m, z) and (h, p) defined in Equation 3 reveals its stability and draws an analogy to the complex plane. When h and p are zero, the eigenvalues of J are $\lambda_{1,2} = m \pm zi$. When h or p is non-zero, a circle with radius $\| \begin{bmatrix} h & p \end{bmatrix}^T \|_2$ centered around the origin is excluded from left-half stability region.



(a) Eigenvalues of J and level sets of $\det(J) = \lambda_1 \lambda_2$. (b) The trace, determinant and discriminant of J.

Fig. 5: The relationship between the trace, determinant and eigenvalues of two-by-two matrix J. The dynamics $\dot{x} = Jx$ is stable (green) if and only if $\det(J) > 0$ and $\mathrm{Tr}(J) < 0$.

As a consequence of the trace and determinant conditions, we state the following.

Proposition 1: A matrix $J \in \mathbb{R}^{2\times 2}$ is stable if and only if $m < 0$ and $m^2 + z^2 > h^2 + p^2$.

3) Potential vs. Rotational Vector Fields: The Helmholtz decomposition of the game form around an equilibrium gives the potential and rotational piece as

$$J = \begin{bmatrix} m+f & p \\ p & m-h \end{bmatrix} + \begin{bmatrix} 0 & -z \\ z & 0 \end{bmatrix}$$

For a potential vector field ($z = 0$), there is no rotational component and J is symmetric, so stability is equivalent to negative definiteness of J. Ignoring the uninteresting, unstable case where the diagonal elements are positive and assuming without loss of generality, that $a = m + h < 0$, we consider the Schur complement of J,

$$d - ca^{-1}b = d - p^2 a^{-1}.$$

From the Schur complement, it is clear that stability in the potential case is determined by a trade off between the interaction term p (scaled by a^{-1}) and the magnitude of d. If $d > 0$, then the system cannot be stable because p can only increase the Schur complement. If $d < 0$, then d still needs to be negative enough to outweigh $-p^2 a^{-1}$ which is positive. Adding rotation ($z \neq 0$) can only improve stability. If $d > 0$, the prerequisite assumption that $m < 0$ ensures that the stable player is converging faster than the unstable player is diverging. If z is large enough, whenever agents start to diverge because of the unstable mode of the potential field, the rotational component returns the agents to the stable and faster mode of the potential field.

B. Game Insights

For 2×2 game dynamics given by $\dot{x} = Jx$, the condition for a fixed point of the dynamics to be a differential Nash equilibrum is $m < -|h|$, illustrated as the left shaded region in Figure 6b. Note that since the Nash condition is based on the magnitude of h, it implies also that $m < 0$, ie. $\begin{bmatrix} m & z \end{bmatrix}^T$ must be in the left half plane. Therefore, if a Nash equilibrium has unstable dynamics it must be because $\begin{bmatrix} m & z \end{bmatrix}^T$ is inside the hp-disk. We have the following classification of fixed points in 2×2 games.

Proposition 2 (Fixed Points of 2×2 Game Dynamics): For a two-player scalar game, the following conditions certify whether a fixed point is a strict differential Nash equilibrium and/or a locally exponentially stable fixed point of gradient play, using the decomposition given by (3):

1) Nash & stable: $m < -|h|,\ m^2 + z^2 > h^2 + p^2$.
2) Nash & unstable: $m < -|h|,\ m^2 + z^2 \leq h^2 + p^2$.
3) Non-Nash & stable:
$$-|h| \leq m < 0,\ m^2 + z^2 > h^2 + p^2.$$
4) Non-Nash & unstable:
$$-|h| \leq m,\ m \geq 0, \quad \text{or} \quad m^2 + z^2 \leq h^2 + p^2.$$

These conditions are illustrated in Fig. 6b. We can then consider these conditions in the four types of games. Each of these situations is illustrated in Fig. 7.

Potential games ($z = 0$)**:** $\begin{bmatrix} m & z \end{bmatrix}^T$ must live on the horizontal axis, thus stable fixed points are a subset of Nash equilibria. The magnitude of the interaction term p as well as the Hamiltonian term Z can only hurt stability. For Nash equilibria this interaction is precisely quantified by the Schur compliment formula $a - d^{-1}p^2$.

Zero-sum games ($p = 0$)**:** $\begin{bmatrix} h & p \end{bmatrix}^T$ must live on the horizontal axis, thus all Nash equilibria are stable. There are, however stable fixed points that are not Nash equilibria. Here, the magnitude of the interaction term z helps stability and may make a fixed point stable even if it is not Nash. Intuitively, a strong enough interaction term can cause a Nash player with stronger negative curvature to pull another player with weaker positive curvature toward a fixed point even if that point is a local minimum for the weaker player.

Hamiltonian games ($m = 0$)**:** $\begin{bmatrix} m & z \end{bmatrix}^T$ must live on the vertical axis, thus no strict Nash equilibria can exist. At best these games are marginally stable if $|z|$ is large enough relative to the magnitude of $\begin{bmatrix} h & p] \end{bmatrix}^T$.

Constant-curvature games ($h = 0$): $\begin{bmatrix} h & p \end{bmatrix}^T$ must live on the vertical axis, so any stable point is also a Nash.

C. Non-Uniform Learning Rates

We now consider how these results change if agents optimize with non-uniform learning rates. For varied learning rates, γ_1, γ_2, the game Jacobian becomes

$$J(\tilde{x}, \tilde{y}) = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{4}$$

We note that since $\gamma_1, \gamma_2 > 0$, modifying the learning rates does not change whether or not a fixed point is a Nash equilibrium. It also does not change the sign of $\det(J)$ as a result any change in stability will come from a sign change in $\text{Tr}(J)$. The trace of the modified J is given by

$$\text{Tr}(J) = \gamma_1 a + \gamma_2 d = (\gamma_1 + \gamma_2)m + (\gamma_1 - \gamma_2)h$$

so that $m = \frac{\gamma_2 - \gamma_1}{\gamma_1 + \gamma_2} h$ is the boundary at which its sign flips. Let $\beta = (\gamma_2 - \gamma_1)/(\gamma_1 + \gamma_2)$. Note that $-1 < \beta < 1$ since $\gamma_1, \gamma_2 > 0$. The change in the region of stability is shown in Figure 8.

Theorem 1: Consider a fixed point $(\tilde{x}, \tilde{y})$ of two-player scalar game (f_1, f_2). Suppose the rows of the game Jacobian $J(\tilde{x}, \tilde{y})$ are scaled by learning rates $\gamma_1, \gamma_2 > 0$ as in Equation 4. Then, the following are true:

1) If a Nash equilibrium is stable for some learning rates, then it is stable for all learning rates.
2) If a stable fixed-point is non-Nash, there exists learning rates that make it unstable.
3) For any fixed point, if $\det(J) > 0$ and if at least one player has negative curvature $a < 0$ or $d < 0$, then there exists learning rates that make it stable.

Proof: To prove 1), we observe that if $m < -|h|$, then $m \leq \beta f$ for all β such that $|\beta| < 1$. To prove 2), choose $\gamma_1 > \left|\frac{d}{a}\right| \gamma_2$. Without loss of generality, assume $a < 0$ and $d > 0$. Then, it directly follows that $\gamma_1 a + \gamma_2 d < 0$. To prove 3), choose $\gamma_1 > \left|\frac{d}{a}\right| \gamma_2$. Without loss of generality, assuming $a < 0$ ensures that $\gamma_1 a + \gamma_2 d < 0$. ∎

Stable Nash equilibria in scalar games are robust to variations in learning rates. Non-Nash fixed points are not.

IV. Stability of Nash in Two-Player Games

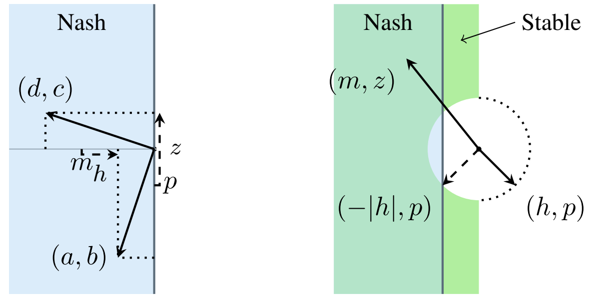
We consider some results on block 2×2 matrix games.

A. Potential Games

In potential games, $J = J^T$ and thus J is stable if and only if $J \prec 0$. Since $J_{11} \prec 0$ and $J_{22} \prec 0$ are necessary condition for $J \prec 0$. We have the following proposition.

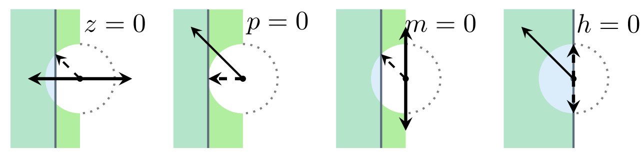
Proposition 3 (Block 2×2 Potential Games): Suppose the game Jacobian at fixed point $(\tilde{x}, \tilde{y})$ has block structure

$$J(x, y) = \begin{bmatrix} J_{11} & P \\ P^T & J_{22} \end{bmatrix}. \tag{5}$$



(a) The rows of a 2×2 matrix are plotted along the `ad`- and `bc`-axes. (b) The regions of stability and Nash in the `mh`- and `zp`-axes.

Fig. 6: We plot the elements of J for a two-player, scalar game, in both a, b, c, d and m, z, h, p coordinates from (3). Shaded blue regions represent Nash where $a, d < 0$. Green regions represent stability, determined by the trace condition intersected with determinant condition. Visible in (b) are regions of stable non-Nash or an unstable Nash.



(a) Potential: Stable $\subset$ Nash. (b) Zero-sum: Stable $\supset$ Nash (c) Hamiltonian: $\text{Re}(\lambda_{1,2}) = 0$. (d) Matching: Stable $\subset$ Nash.

Fig. 7: Stability and Nash of Nash different classes of games. (a) Potential games: symmetric interaction term only hurts stability. (b) Zero-sum games: rotation can compensate for unhappy player. (c) Hamiltonian games: players have zero total curvature, $a + d = 0$. (d) Constant curvature $a = d$: there are no stable non-Nash equilibria.

Then (i) stable fixed points are Nash equilibria, (ii) non-Nash fixed points are unstable, and (iii) Nash equilibria are stable if and only if $J_{11} - P J_{22}^{-1} P^T \prec 0$.

The proof follows from properties of negative definite, symmetric matrices. These results are consistent from our intuition in the scalar case that the interaction terms in potential games only discourage stability.

We certify the instability of the following non-Nash equilibrium of a potential game. A direct application of [23, Thm. 1.2.1] gives us the following result.

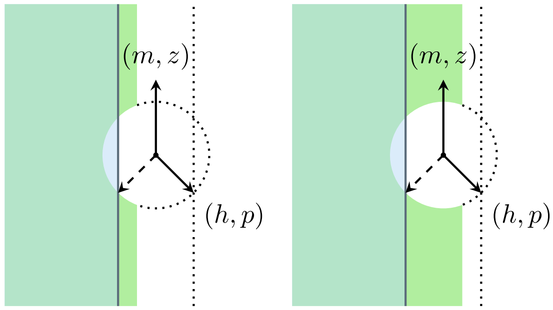
Proposition 4: Suppose without loss of generality $J_{11} \prec 0, J_{22} \succ 0$ and J has block structure (5). Then $W^2(J) \subset \{z \leq \lambda_{\max}(J_{11})\} \cup \{z \geq \lambda_{\min}(J_{22})\}$ consists of two disjoint connected regions of real eigenvalues on both sides of the complex plane.

B. Zero-Sum Games

In the zero-sum case, we have the following propositions.

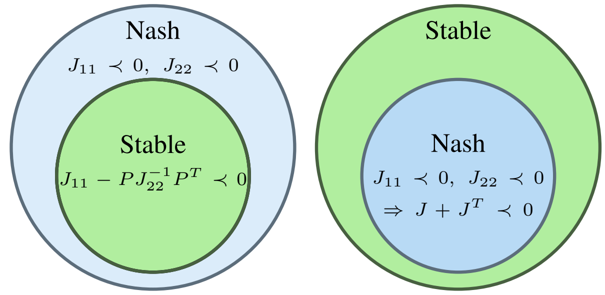
Proposition 5 (Block 2×2 Zero-Sum Games): Suppose the game Jacobian at a fixed point (x, y) has block structure

$$J(x, y) = \begin{bmatrix} J_{11} & Z \\ -Z^T & J_{22} \end{bmatrix}. \tag{6}$$



(a) $\gamma_1 > \gamma_2$ (b) $\gamma_1 < \gamma_2$

Fig. 8: Effect of positive learning rates $\gamma_1,\ \gamma_2 > 0$. The factor $\beta = \frac{\gamma_2-\gamma_1}{\gamma_1+\gamma_2}$, which has magnitude no greater than one, expands or shrinks the region for stability. For $\beta \to \pm 1$, the vertical boundary for stability approaches $\pm h$.



(a) Potential games (b) Zero-sum games

Fig. 9: Nash and stability of fixed points in potential and zero-sum games. All stable equilibria of potential games are Nash because the Schur complement of J is negative definite. All Nash equilibria of zero-sum games are stable because the symmetric component of J is negative definite.

Then (i) Nash equilibria are stable fixed points and (ii) unstable fixed points are not-Nash equilibria.

Proof: The proof follows directly from Lyapunov theory using Lyapunov function $\|x\|^2$. Indeed, since $J_{12} = -J_{21}^T$, if $J_{11} \prec 0$ and $J_{22} \prec 0$, then $J + J^T \prec 0$. ■

We characterize the real and imaginary components of spectrum of a zero-sum game using the quadratic numerical range. For a zero sum game $(f_1, f_2) = (f, -f)$, define the following quantities: for $i = 1, 2$,

$$\lambda_i^- = \min \sigma_p(-D_i^2 f_i(x)),\ \ \lambda_i^+ = \max \sigma_p(-D_i^2 f_i(x)).$$

and let $\underline{\lambda} = \frac{1}{2}(\lambda_1^- + \lambda_2^-)$ and $\overline{\lambda} = \frac{1}{2}(\lambda_1^+ + \lambda_2^+)$.

Proposition 6: Consider a zero-sum game $\mathcal{G} = (f, -f)$. The Jacobian $J(x)$ of the dynamics $\dot{x} = -g(x)$ at fixed points x is such that

$$\sigma_p(J(x)) \cap \mathbb{R} \subset [\min\{\lambda_1^-, \lambda_2^-\}, \max\{\lambda_1^+, \lambda_2^+\}] \qquad (7)$$

and $\sigma_p(J(x)) \backslash \mathbb{R}$ is contained in

$$\{z \in \mathbb{C}:\ \mathrm{Re}(z) \in [\underline{\lambda}, \overline{\lambda}],\ |\mathrm{Im}(z)| \le \|D_{12}f(x)\|\}. \qquad (8)$$

Furthermore, if $\lambda_1^- - \lambda_2^+ > 0$ or $\lambda_2^- - \lambda_1^+ > 0$ then the following two implications hold for $\delta = \lambda_1^- - \lambda_2^+$ or $\delta = \lambda_2^- - \lambda_1^+$, respectively: (i) $\|D_{12}f(x)\| \le \delta/2 \implies \sigma_p(J(x)) \subset \mathbb{R}$; (ii) $\|D_{12}f(x)\| > \delta/2 \implies \sigma_p(J(x)) \backslash \mathbb{R} \subset \{z \in \mathbb{C}|\ |\mathrm{Im}(z)| \le \sqrt{\|D_{12}f(x)\|^2 - \delta^2/4}\}$.

Proof: First, observe that $\overline{\det(J_{v,w}(x) - \lambda I)} = \det(J_{v,w}(x) - \bar{\lambda} I)$ for $v, w \in \mathbb{C}^{d_1+d_2}$ such that $\|v\| = \|w\| = 1$, since $D_1^2 f(x)$ and $-D_2^2 f(x)$ are symmetric, which in turn implies that $W^2(J(x)) = W^2(J(x))^*$. Since $-w^* D_{12} f(x)^T v v^* D_{12} f(x) w \le 0$, (i) and (ii) follow from [24, Lemma 5.1-(ii)] and (7) and (8) follow from [23, Prop. 1.2.6]. ■

The effects of non-uniform learning rates for zero-sum games can be stated as follows. Let $\rho(\cdot)$ be the spectral radius of its argument. Consider the discretized dynamics

$$x_{i,k+1} = x_{i,k} - \gamma_i D_i f_i(x_k) \qquad (9)$$

with $\gamma_1, \gamma_2 > 0$. Without loss of generality, let $\tau = \gamma_2/\gamma_1$ be the learning rate ratio and let

$$J_\tau = \begin{bmatrix} J_{11} & J_{12} \\ -\tau J_{12}^T & \tau J_{22} \end{bmatrix}.$$

Proposition 7: Consider a zero-sum game $(f_1, f_2) = (f, -f)$ with game Jacobian J. Suppose $(\tilde{x}, \tilde{y})$ such that $\sigma_p(J(\tilde{x}, \tilde{y})) \subset \mathbb{C}_-^\circ$. If $\gamma_1 > 0$ is selected so that $\rho(I - \gamma_1 J_\gamma) < 1$, then the discrete time update (9) is stable for any learning rate ratio $\tau > 0$.

Proof: First, observe that $W^2(J(\tilde{x}, \tilde{y})) \subset \mathbb{C}_-^\circ$ by the following reasoning. Let $a = \langle -J_{11} v, v\rangle$, $d = \langle J_{22} w, w\rangle$ and $b = \langle -J_{12} w, v\rangle$. Then the eigenvalues of $-J_{w,v}$ are

$$\lambda = \tfrac{a-d}{2} \pm \tfrac{1}{2}\sqrt{(a+d)^2 - 4b^2}$$

where $d < 0$ and $a > 0$ for all $(v, w) \in S_1 \times S_2$ since $(\tilde{x}, \tilde{y})$ is stable. Then, $\mathrm{Re}(\lambda) > 0$ when $(a+d)^2 \le 4b^2$. On the other hand, when $(a+d)^2 > 4b^2$, $\mathrm{Re}(\lambda) > 0$ if

$$\tfrac{a-d}{2} - \tfrac{1}{2}\sqrt{(a+d)^2 - 4b^2} > 0 \iff ad < b^2.$$

Recall that $a > 0$ and $d < 0$ everywhere on S_1, S_2 so this always holds since $ad < 0 < b^2$. Hence, $\sigma(-W^2(J)) \subset \mathbb{C}_+^\circ$ which shows the inclusion.

Now, we show $(\tilde{x}, \tilde{y})$ is stable for any $\tau > 0$. First, denoting $J_\tau = \mathrm{blkdiag}(I_{n_1}, \tau I_{n_2}) J(\tilde{x}, \tilde{y})$, the elements of $W^2(-J_\tau)$ are of the form

$$\lambda_\gamma = \tfrac{a-\tau d}{2} \pm \tfrac{1}{2}\sqrt{(a+\tau d)^2 - 4\tau b^2}.$$

Suppose $\tau > 0$ is such that $(a + \tau d)^2 \le 4b^2 \tau$. Then $\mathrm{Re}(\lambda_\tau) = (a - \tau d)/2 > 0$ trivially since $a - d > 0$. Now consider any $\tau > 0$ such that $(a+\tau d)^2 > 4\tau b^2$. In this case, we want to ensure

$$\mathrm{Re}(\lambda_\tau) > \tfrac{a-\tau d}{2} - \tfrac{1}{2}\sqrt{(a+\tau d)^2 - 4\tau b^2} > 0.$$

Combining to two conditions that define the case, we have that $(a - \tau d)^2 > (a + \tau d)^2 - 4\tau b^2$ is equivalent to

$$-2\tau ad > 2\tau(ad - 4b^2) \iff ad < b^2$$

and the right-hand side always holds since $a > 0$ and $d < 0$. ■

C. General-Sum Games

We characterize general two player continuous games by providing sufficient conditions for stability and instability.

Proposition 8 (Block 2×2 Stability): If $J + J^T \prec 0$, then the fixed point is stable.

Proof: Choose $x^T x + y^T y$ as a Lyapunov function. ■

The condition above is necessary and sufficient for stability in the potential game case as indicated in Proposition 3.

We now provide a sufficient condition for instability of fixed points in general sum games. Note we can write $J = 0.5(J + J^T) + 0.5(J - J^T)$. Let R be a rotation that diagonalizes $0.5(J + J^T)$ and sorts the eigenvalues so that

$$RJR^T = \begin{bmatrix} M_+ & 0 \\ 0 & M_- \end{bmatrix} + \begin{bmatrix} Z_1 & Z_2 \\ -Z_2^T & Z_3 \end{bmatrix} \tag{10}$$

where $M_+ \succ 0$, $M_- \preceq 0$ are diagonal and Z_1 and Z_3 are skew-symmetric. Let $\lambda_{\min}(M_+) > 0$ be the minimum eigenvalue of M_+ and $\lambda_{\max}(M_-) \leq 0$ be the maximum eigenvalue of M_-.

Theorem 2 (Block 2×2 Instability): Consider general-sum game (f_1, f_2). A fixed point $(\tilde{x}, \tilde{y})$ is unstable under gradient play (2) if

$$||Z_2|| < \tfrac{1}{2}\Big(\big|\lambda_{\max}(M_-)\big| + \big|\lambda_{\min}(M_+)\big|\Big) < |\lambda_{\min}(M_+)|. \tag{11}$$

with M_+, M_- and Z_2 defined in Equation 10.

Proof: Since Z_1 and Z_3 are skew-symmetric we have that $\mathrm{Re}\big(M_- + Z_3)\big) \leq \lambda_{\max}(M_-) \leq 0$ and $0 \leq \lambda_{\min}(M_+) \leq \mathrm{Re}\big(W(M_+ + Z_1)\big)$ [24, Prop. 1.1.12]. ■

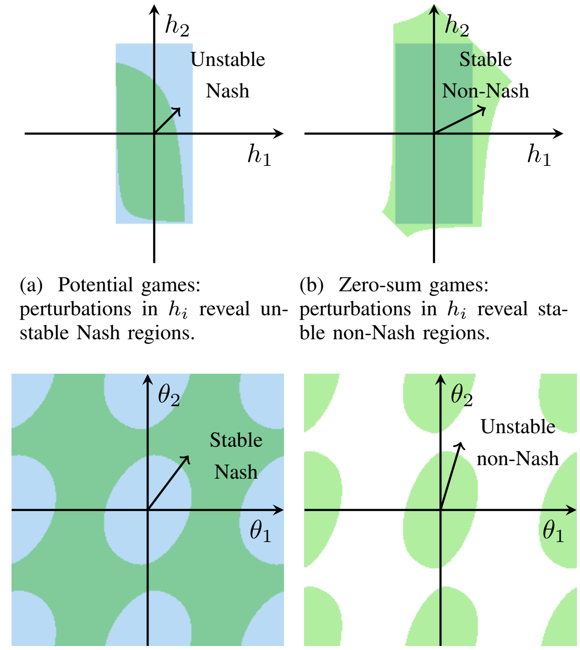
Intuitively, this result works by bounding some non-empty subset of the eigenvalues of J in $\mathbb{C}_+^\circ$ to guarantee instability. The inequalities guarantee that the interaction terms between the positive and negative eigenvalues of the symmetric piece of J are not strong enough to pull the positive eigenvalues into $\mathbb{C}_-^\circ$. Theorem 2 is the block matrix equivalent of being inside the circle of radius $\| \begin{bmatrix} h & p \end{bmatrix}^T \|_2$ in the scalar case. The first inequality is analogous to being inside the circle in the vertical direction. The second inequality is being far enough right along the horizontal axis.

V. Examples: Stability and Nash Boundaries

We explore the stability and Nash regions in two illustrative examples. First, we perturb the objectives of a polynomial game. Depending on their parameters or learning rate, players converge to a non-Nash fixed point or a stable limit cycle. Second, we perturb the scale and rotation of a block matrix to gain insight on the boundaries of stability and Nash for games with vector-valued strategies.

A. Polynomial Game

Consider the game $\mathcal{G} = (f_1, f_2)$ with costs $f_1(x, y) = \frac{ax^2}{2} - xy + \frac{x^4}{4}$, $f_2(x, y) = \frac{dy^2}{2} + 2xy$. The dynamics that arises from minimizing agents are $\dot{x} = -g_1(x, y)$, $\dot{y} = -g_2(x, y)$ where $g_1(x, y) = ax - y + x^3$, $g_2(x, y) = dy + 2x$. Let $a = -1 - \varepsilon_1 < 0$ and $d = 1 + \varepsilon_2 > 0$: agent 1 is locally repelled from $x = 0$ whereas agent 2 is locally attracted to $y = 0$, and ε_i encode the strength of these preferences.



(a) Potential games: perturbations in h_i reveal unstable Nash regions.
(b) Zero-sum games: perturbations in h_i reveal stable non-Nash regions.
(c) Rotate to converge to a Nash.
(d) Rotate to escape a non-Nash.

Fig. 10: Stable (green) and Nash (blue) regions of a 4×4 game derivative of a zero-sum game potential game. The effect of Scaling and rotation are described.

For generic $\varepsilon_1, \varepsilon_2 > 0$, the Jacobian linearized about the fixed point at the origin,

$$J(0, 0) = \begin{bmatrix} 1 + \varepsilon_1 & 1 \\ -2 & -1 - \varepsilon_2 \end{bmatrix},$$

has non-zero and non-repeating eigenvalues. Consider $\varepsilon_1, \varepsilon_2$ small enough such that the determinant of J is $1 - (\varepsilon_1\varepsilon_2 + \varepsilon_1 + \varepsilon_2) > 0$. If agent 2 is attracted to the origin more than agent 1 is repelled by it, then $\varepsilon_2 > \varepsilon_1$ and the origin is a stable attractor of the dynamics since $\mathrm{Tr}(J) = \varepsilon_1 - \varepsilon_2 < 0$.

The costs of the game are general-sum and the dynamics admit a stable limit cycle. Pick $\varepsilon_1 = 0.1, \varepsilon_2 = 0.2$. Suppose agents learn with non-uniform learning rates $\gamma_1, \gamma_2 > 0$ for dynamics $\dot{x} = -\gamma_1 g_1(x, y)$, $\dot{y} = -\gamma_2 g_2(x, y)$. Figure 8 illustrates the effects of non-uniform learning rates. For agent 1 learning faster $\gamma_1 > \frac{1.2}{1.1}\gamma_2$, agents converge to a stable limit cycle centered around the origin. These two scenarios are illustrated in Figure 11a and Figure 11b.

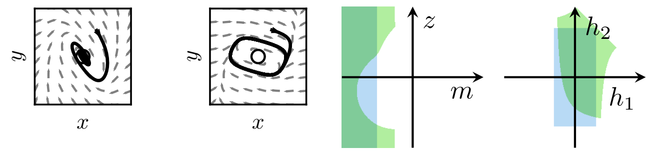
B. Block Matrix

Consider the block decomposition of a $n_1 \times n_2$ game,

$$\begin{bmatrix} M_1 & -Z \\ Z & M_2 \end{bmatrix} + \begin{bmatrix} H_1 & P \\ P & -H_2 \end{bmatrix},$$

where M_1, M_2 are symmetric and $\mathrm{Tr}(H_1) = \mathrm{Tr}(H_2) = 0$. We scale and rotate the blocks via the following parameterization for games with vector-valued strategies of dimensions $n_1 = n_2 = 2$.

The diagonal matrices are represented by scalars m_1, m_2 and h_1, h_2 via $M_i = m_i I$ and $H_i = h_i L$, where the



(a) (b) (c) (d)

Fig. 11: General-sum games may admit limit cycles. We visualize the stability and Nash regions for a 4×4 general-sum game to study the regions of interest.

identity I scales the average and $L = \operatorname{diag}(1, -1)$ scales the difference. The off-diagonal matrices are represented as congruent transformations $R_1^z \Sigma_z R_2^{zT}$ and $R_1^p \Sigma_p R_2^{pT}$ where Σ_z, Σ_p are diagonal matrices and $R_1^z, R_1^p, R_2^z, R_2^p$ are rotation matrices each defined relative to two angles,

$$R_i^{\{p,z\}} = \begin{bmatrix} \cos(\theta_i - \theta_{\{p,z\}}) & -\sin(\theta_i - \theta_{\{p,z\}}) \\ \sin(\theta_i - \theta_{\{p,z\}}) & \cos(\theta_i - \theta_{\{p,z\}}) \end{bmatrix},$$

where θ_1, θ_2 are associated with agents and θ_z, θ_p are associated with zero-sum and potential components.

We finely sample the region of Nash and stability in slices of two parameters, shown in Figure 10. We plot a plane $[-1, 1]^2$ for scalars and a torus $[-\pi, \pi]^2$ for angles. Consider the potential game with parameters $m_1 = -0.2, m_2 = -0.8$, $h_1 = 0.1, h_2 = -0.4$, $\Sigma_p = \operatorname{diag}(-0.7, 0.3)$ and rotations $\theta_1 = -\theta_2 = \frac{\pi}{4}$ in Figures 10a,c. Consider the zero-sum game with parameters $m_1 = -0.2, m_2 = -0.8$, $h_1 = 0.3, h_2 = -0.4$, $\Sigma_z = \operatorname{diag}(0.7, -0.3)$ and identity rotation in Figure 10b,d.

Towards the goal of characterizing general-sum games, we show slices of their landscape shown in Figures 11c-11d. One can further explore the stability and optimality regions of 4×4 matrices with our software tool[3]. We encourage the intrigued reader to experiment.

VI. Conclusion

We provide a complete understanding of the local stability and Nash optimality for fixed points of general-sum two-player scalar games. Leveraging the quadratic numerical range of a linearized game Jacobian, we provide sufficient conditions for stability and instability of higher dimensional general-sum games. Using coordinates meaningful to the game, we provide insights into synthesis of algorithms that are most accurately modeled as games, for instance, when agents cannot be trusted or reliably communicated with.

Truly strategic agents may learn to avoid undesirable equilibria by adapting their learning rates. How best to evaluate the quality of equilibria in general-sum games, however, remains an open problem.

Acknowledgements

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[3]Run Python script `demos/stability4x4.py` in repository `https://github.com/bchasnov/interact`

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