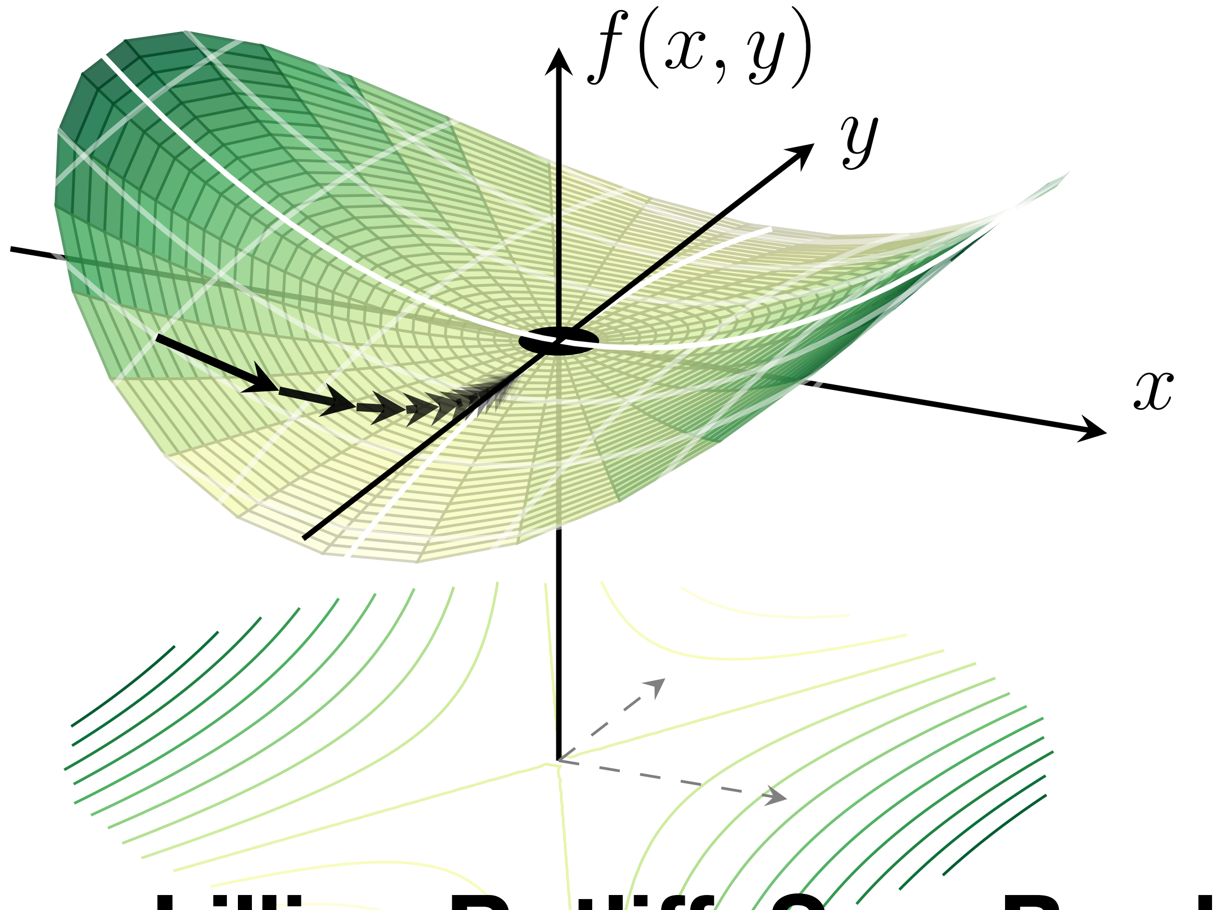
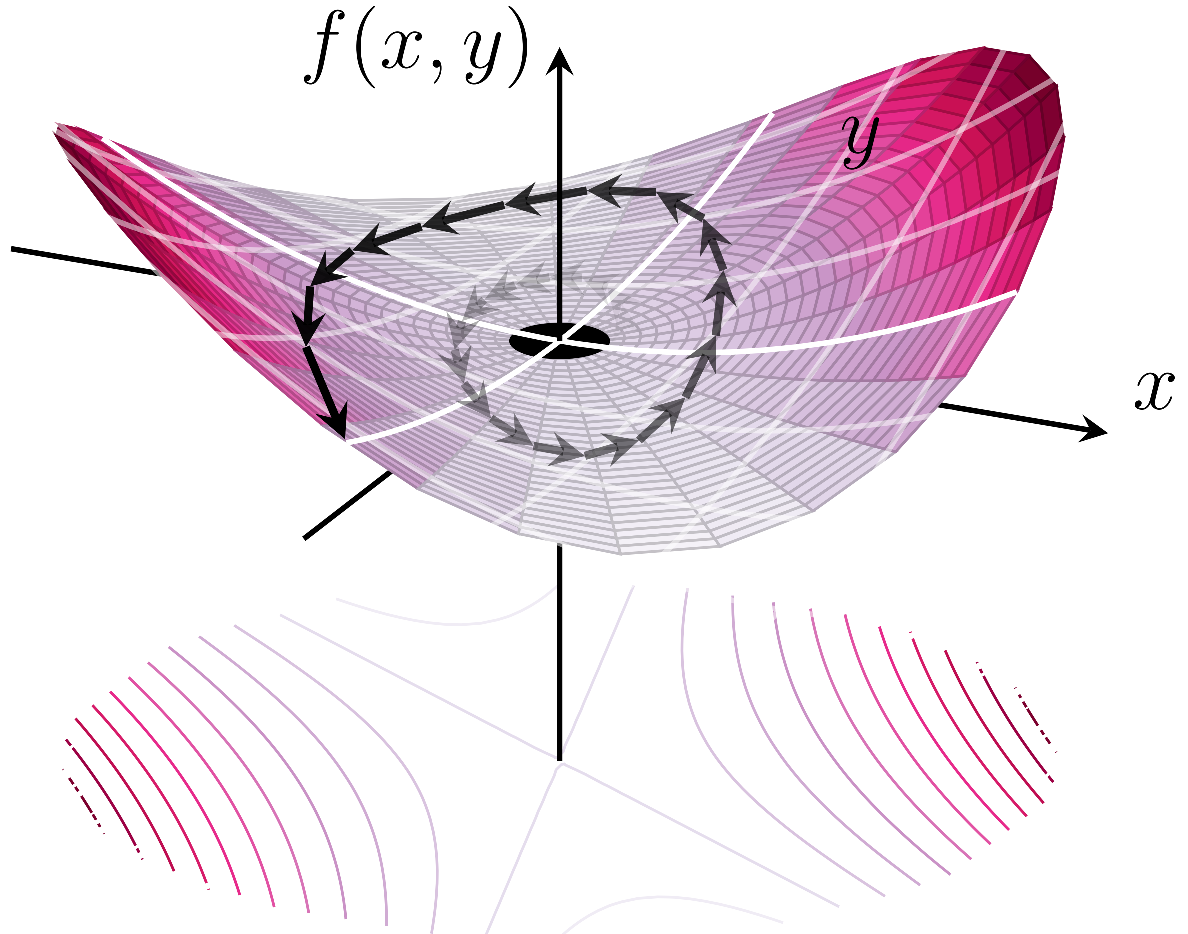
University of Washington
Seattle WA

Stability of Gradient Learning Dynamics in Continuous Games

i) Scalar action spaces
ii) Vector action spaces





Ben Chasnov*, Dan Calderone*, Behcet Acikmese, Lillian Ratliff, Sam Burden

Department of Aeronautics and Astronautics
Department of Electrical and Computer Engineering
*Equal contribution

Learning in games

Motivation

- Explain how players grapple with each other in seeking an equilibrium
- Towards characterizing the optimization landscape of games

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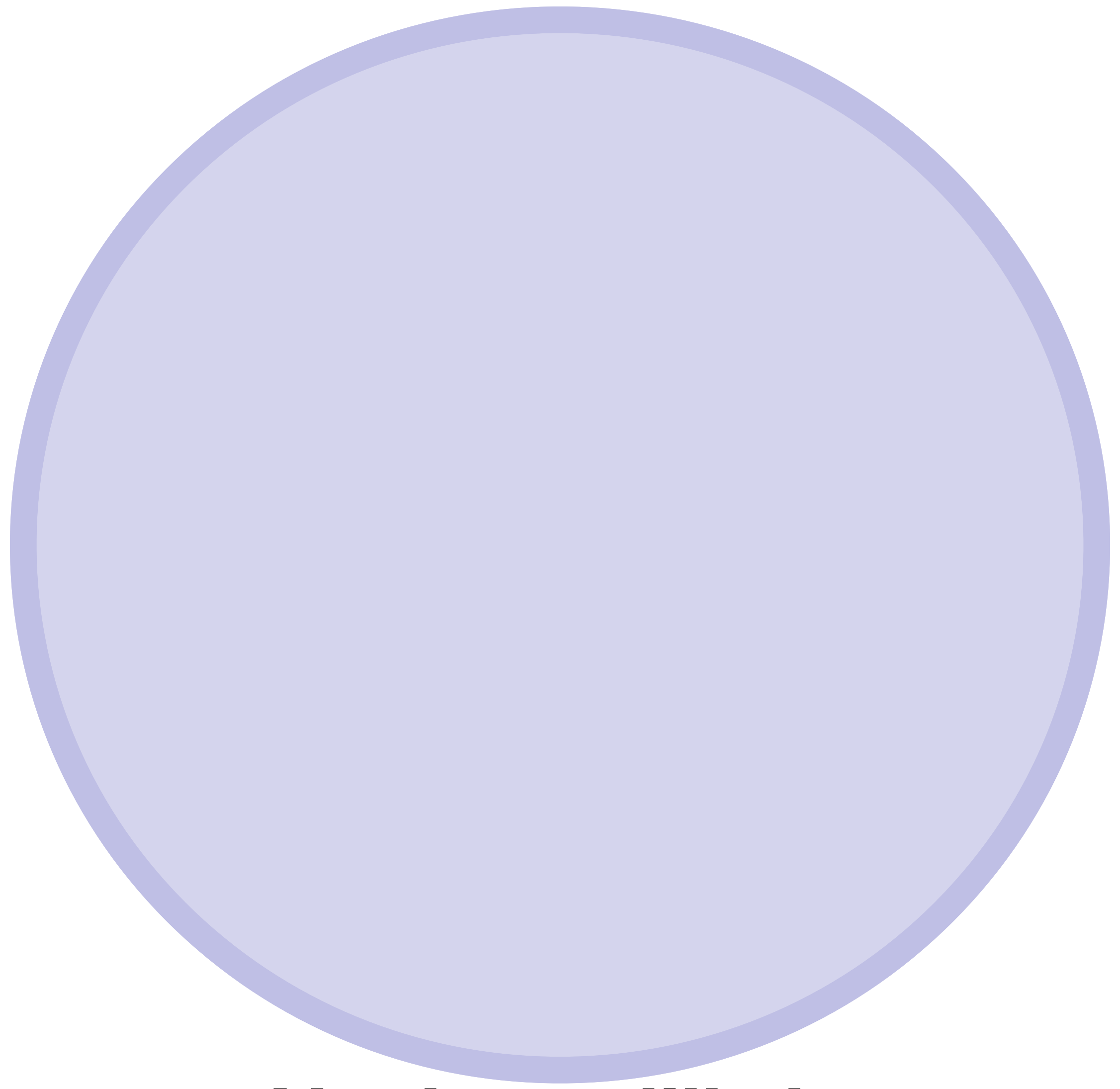

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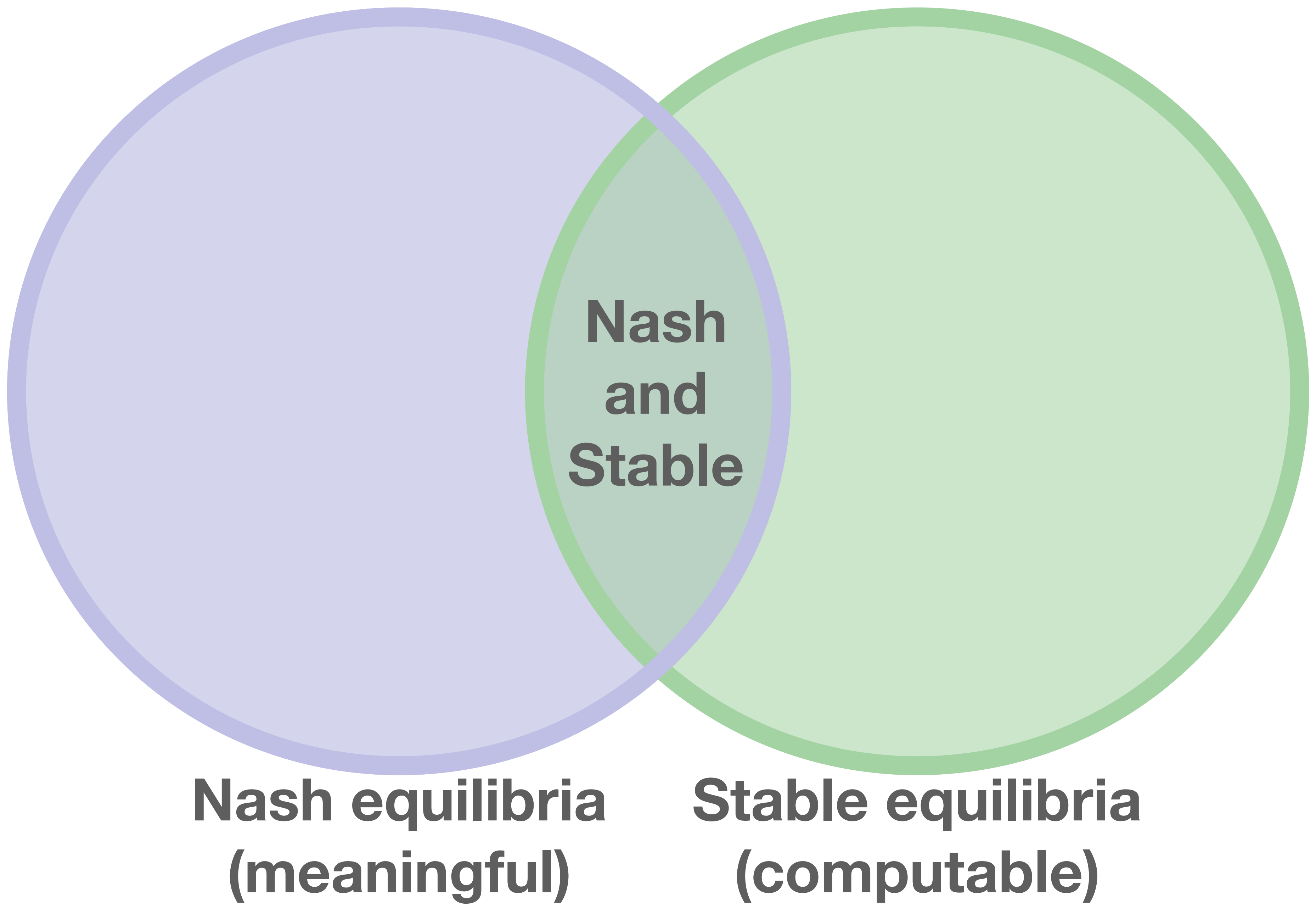
- Perspective 1: Meaningful equilibrium
- Perspective 2: Computable equilibrium

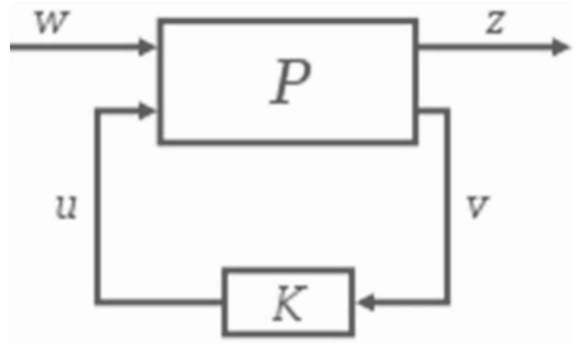


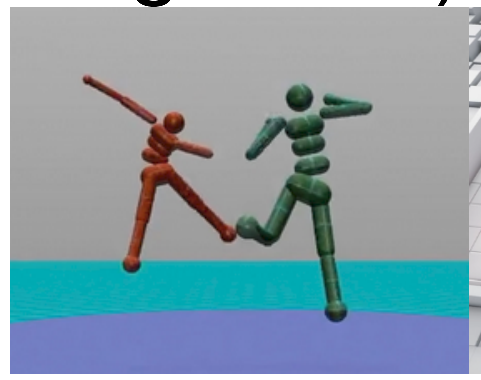
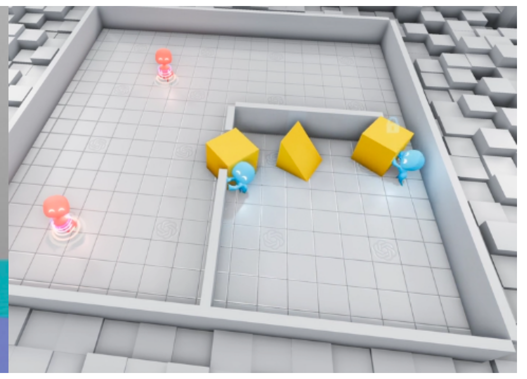
Learning in games
Applications

Learning in games

Applications

- **Control theory/reinforcement learning** (e.g. robust control, multiagent RL)



(OpenAI, 2017-2019)

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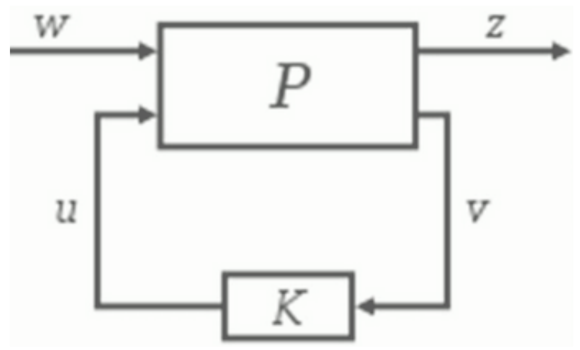


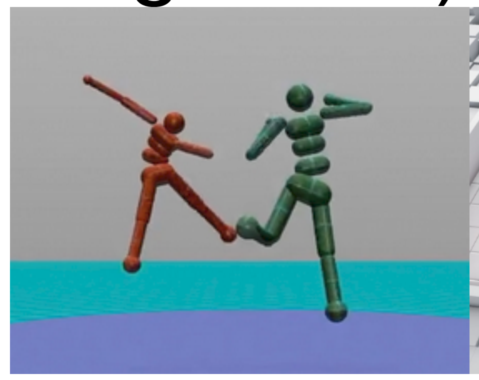
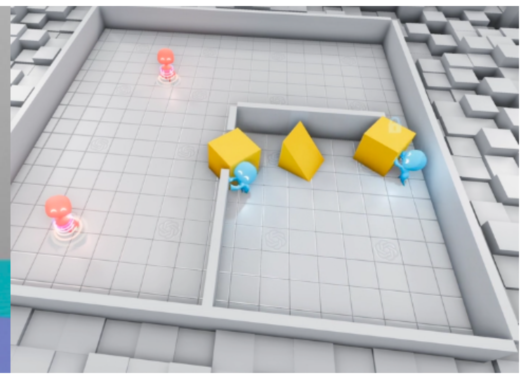

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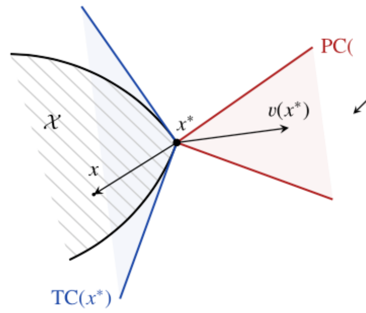
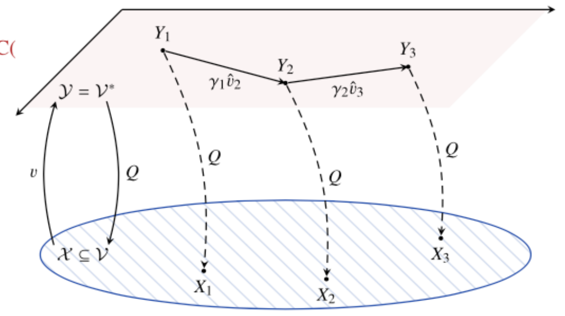
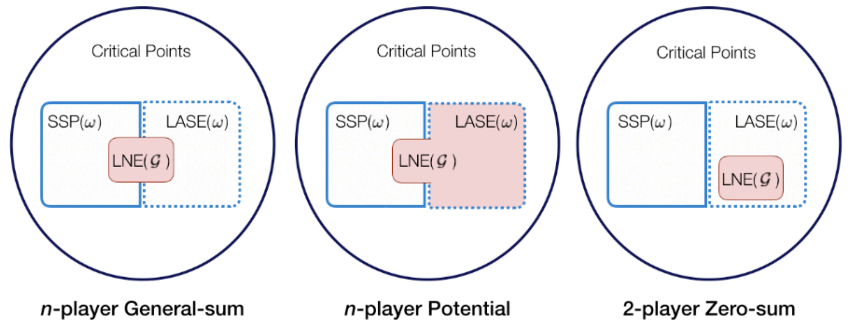



(Mertikopoulos, Zhou 2019)



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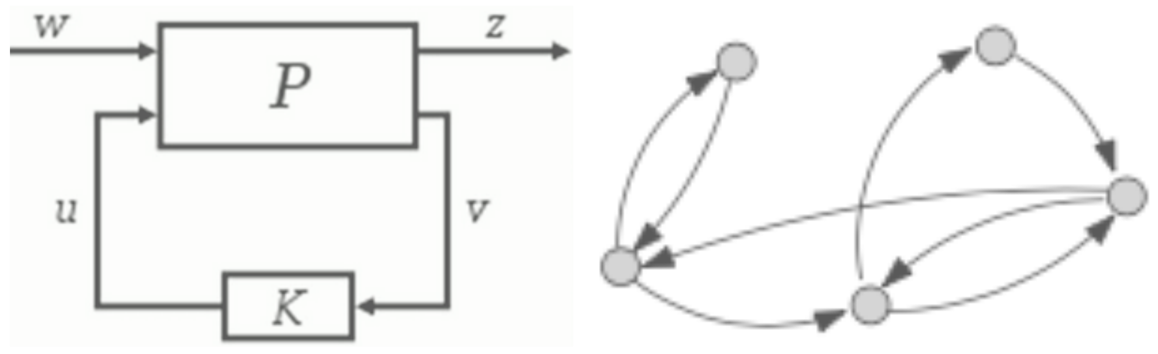

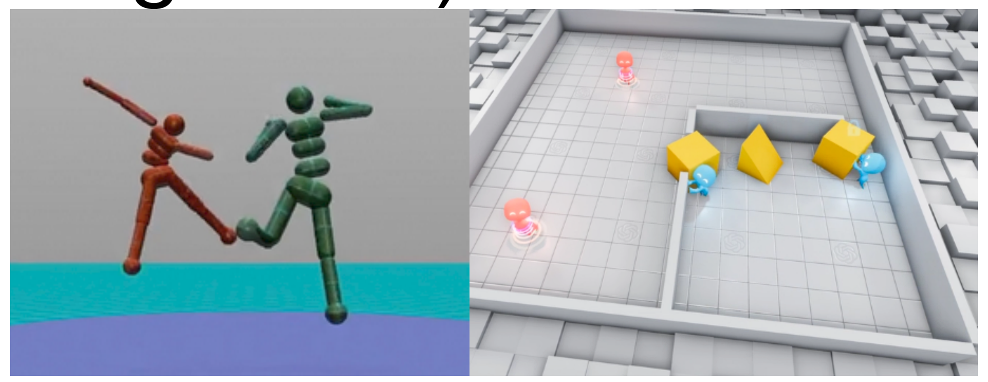

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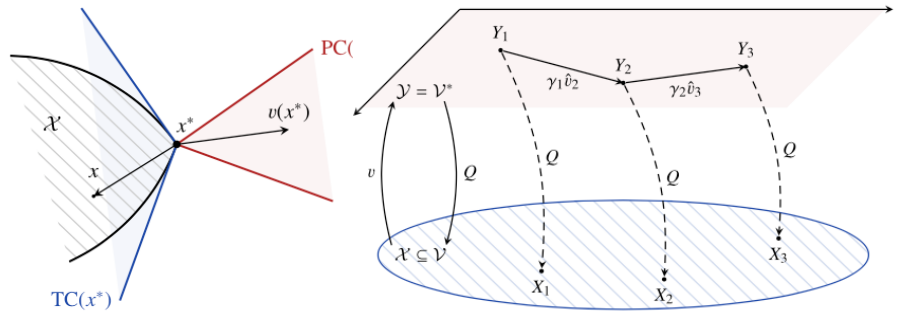

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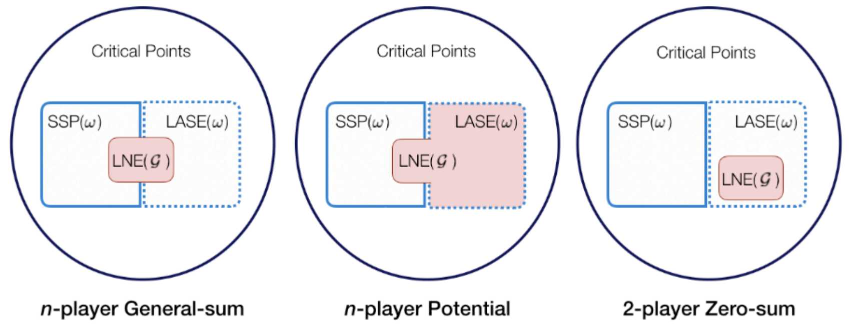

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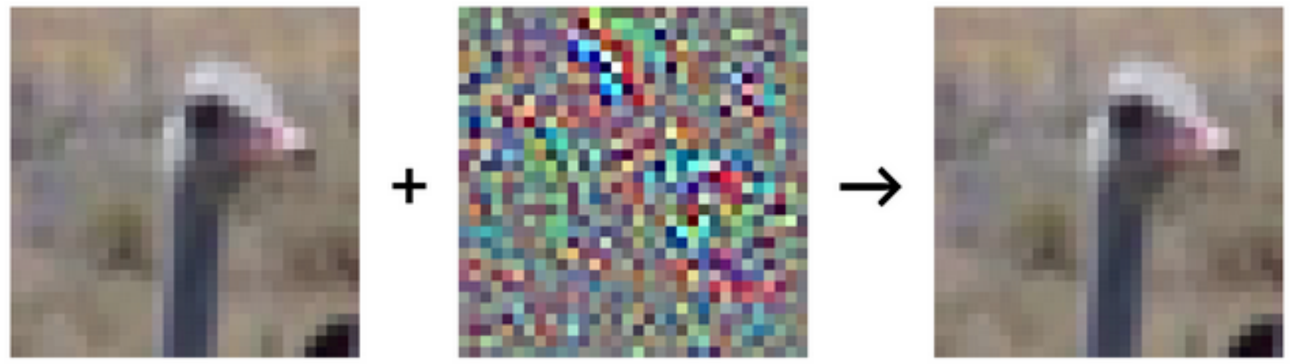

(https://gradientscience.org/)

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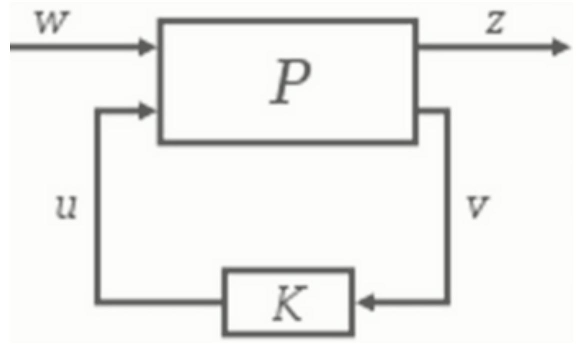

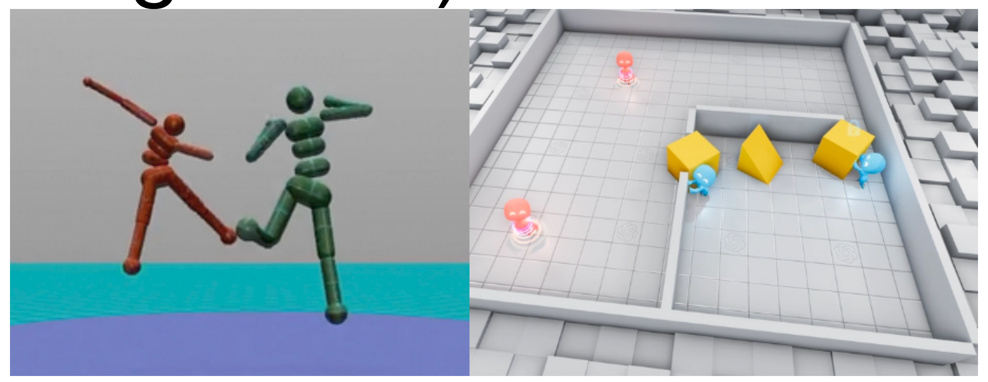

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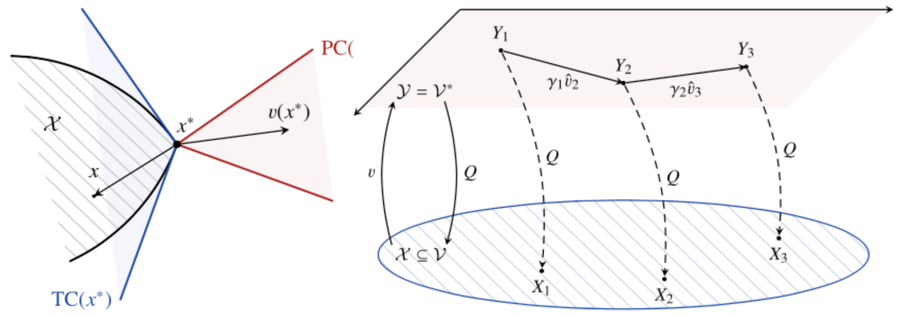

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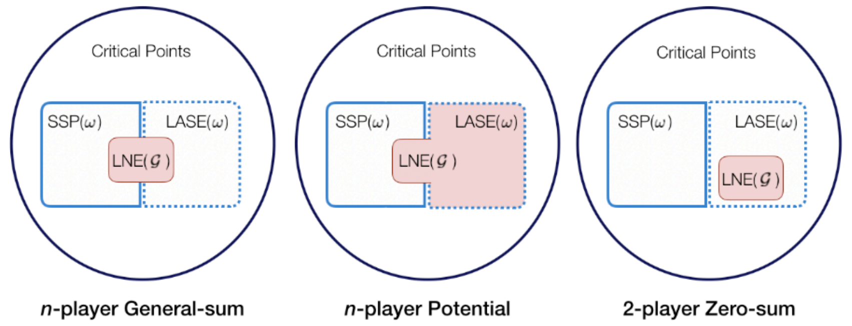

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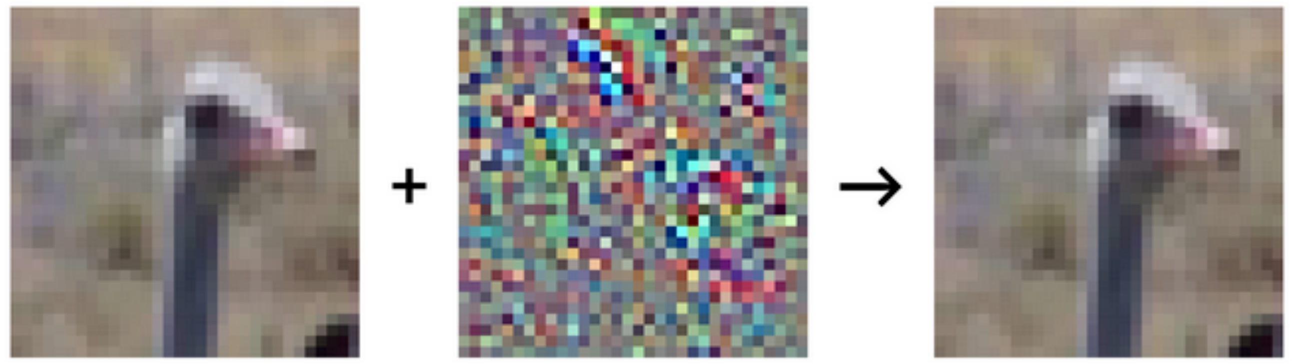

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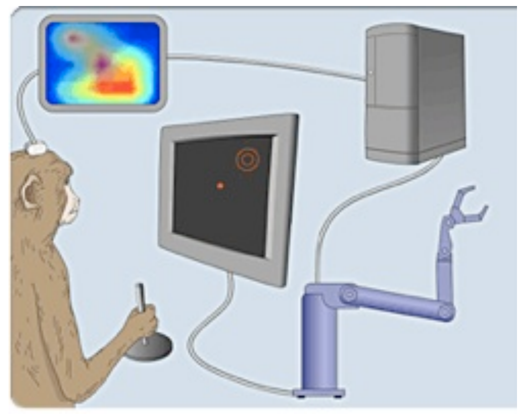
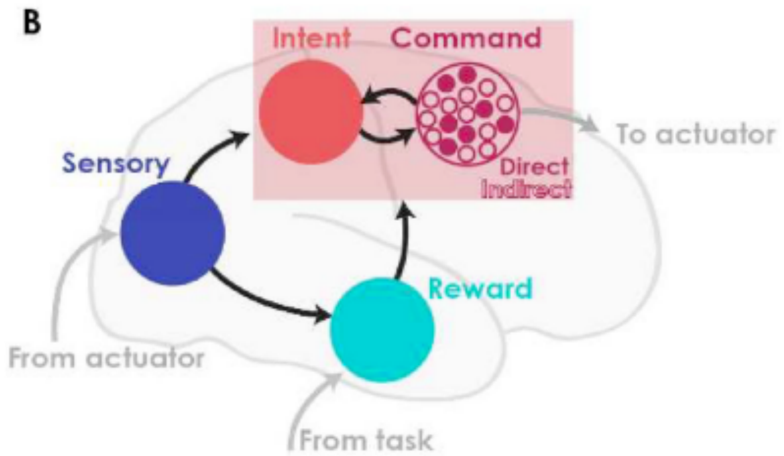
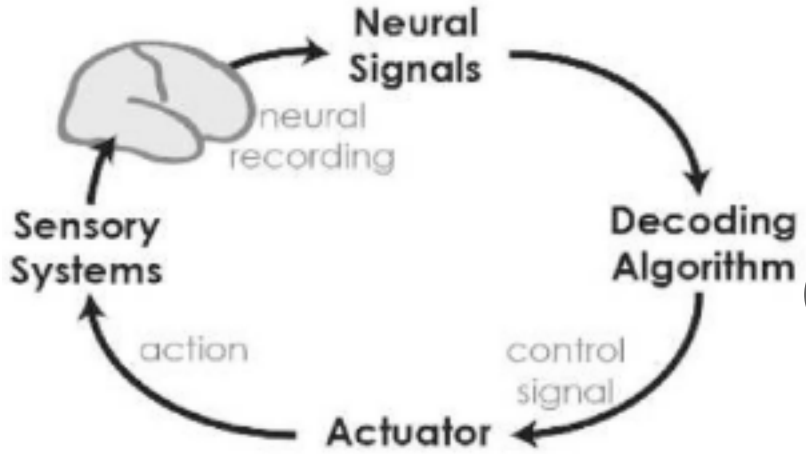
- **Neuroscience** (e.g. computational models for learning, brain-machine interfaces)



(Lebedev, Nicolelis, 2006)





(Orsborn, Pesaran, 2017)

Gradient dynamics in games

Background

Gradient dynamics in games

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    - Unconstrained actions
    - Feedback to agents: deterministic gradients

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- For costs f_1 and f_2, we consider the setting:

cooperative "game": $\min_{x,y} w_1 f_1(x,y) + w_2 f_2(x,y), \quad w_1, w_2 > 0.$

2-player zero-sum game: $\min_x \max_y f(x,y), \quad f = f_1 = -f_2.$

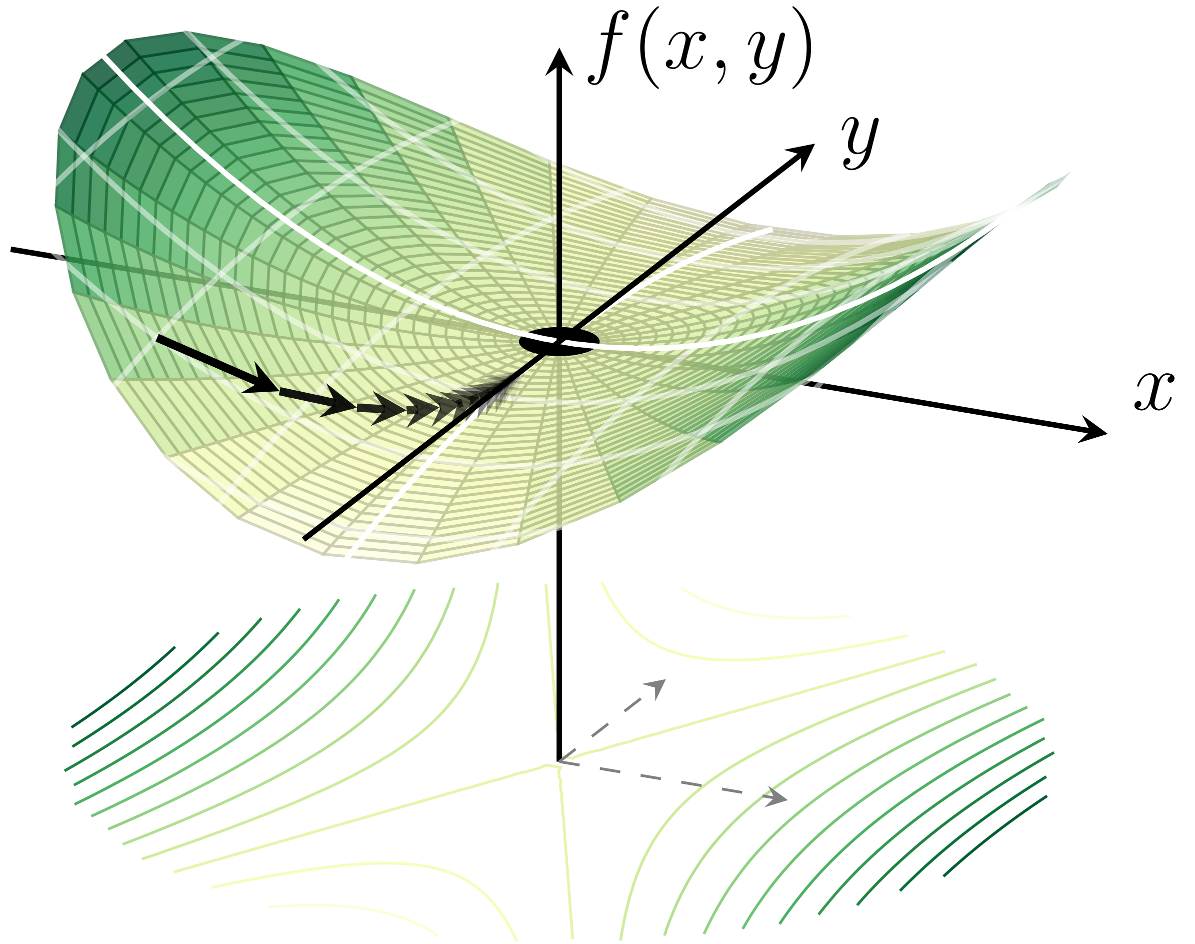
2-player potential game: $\min_{x,y} \ \phi(x,y), \quad D_1\phi = D_1 f_1,\ D_2\phi = D_2 f_2.$

This paper: $\{$ 2-player general-sum: $\min_x \ f_1(x,y), \ \min_y \ f_2(x,y)$

n-player game: $\min_{x_i} \ f_i(x_i, x_{-i}), \ \forall i = 1,\ldots,n$

Gradient descent on multiple losses

Problem formulation



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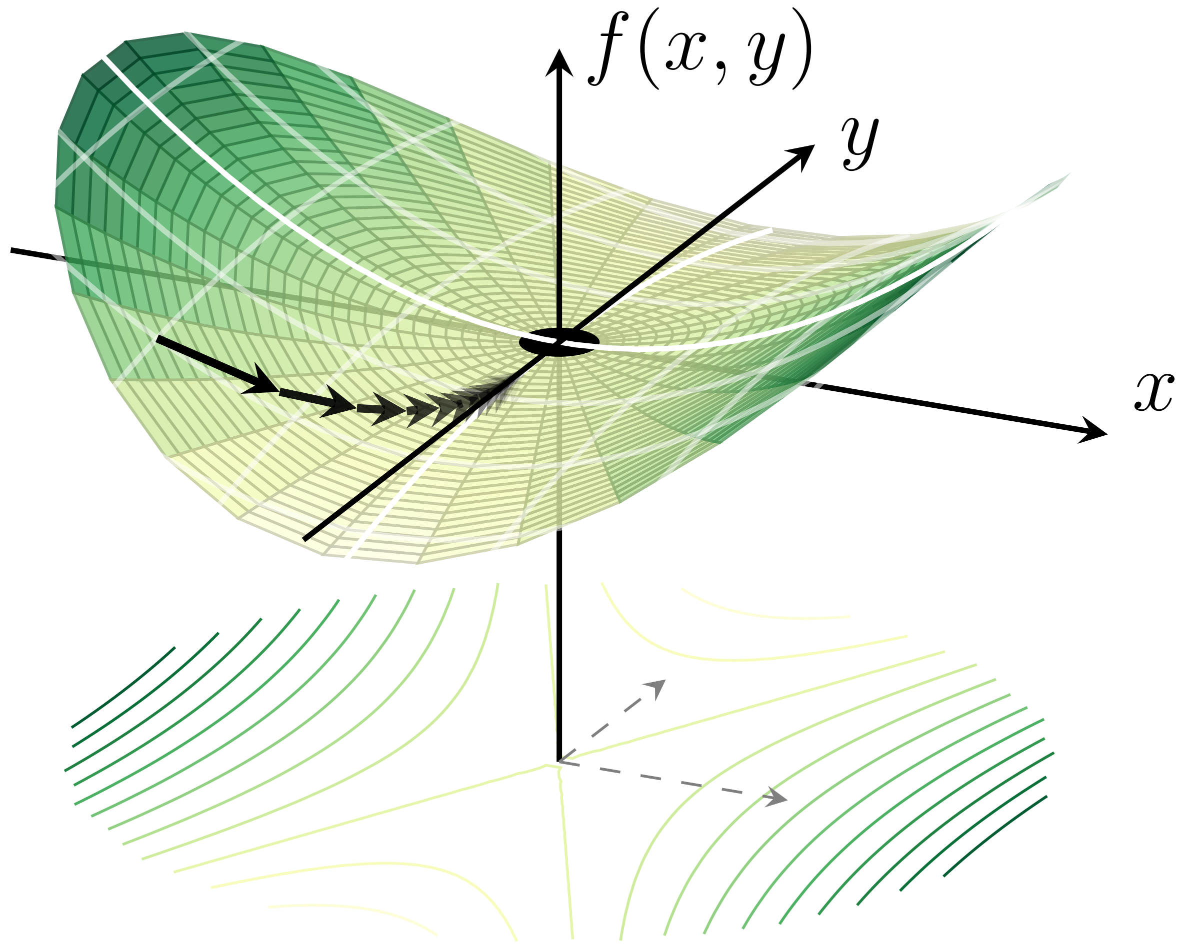

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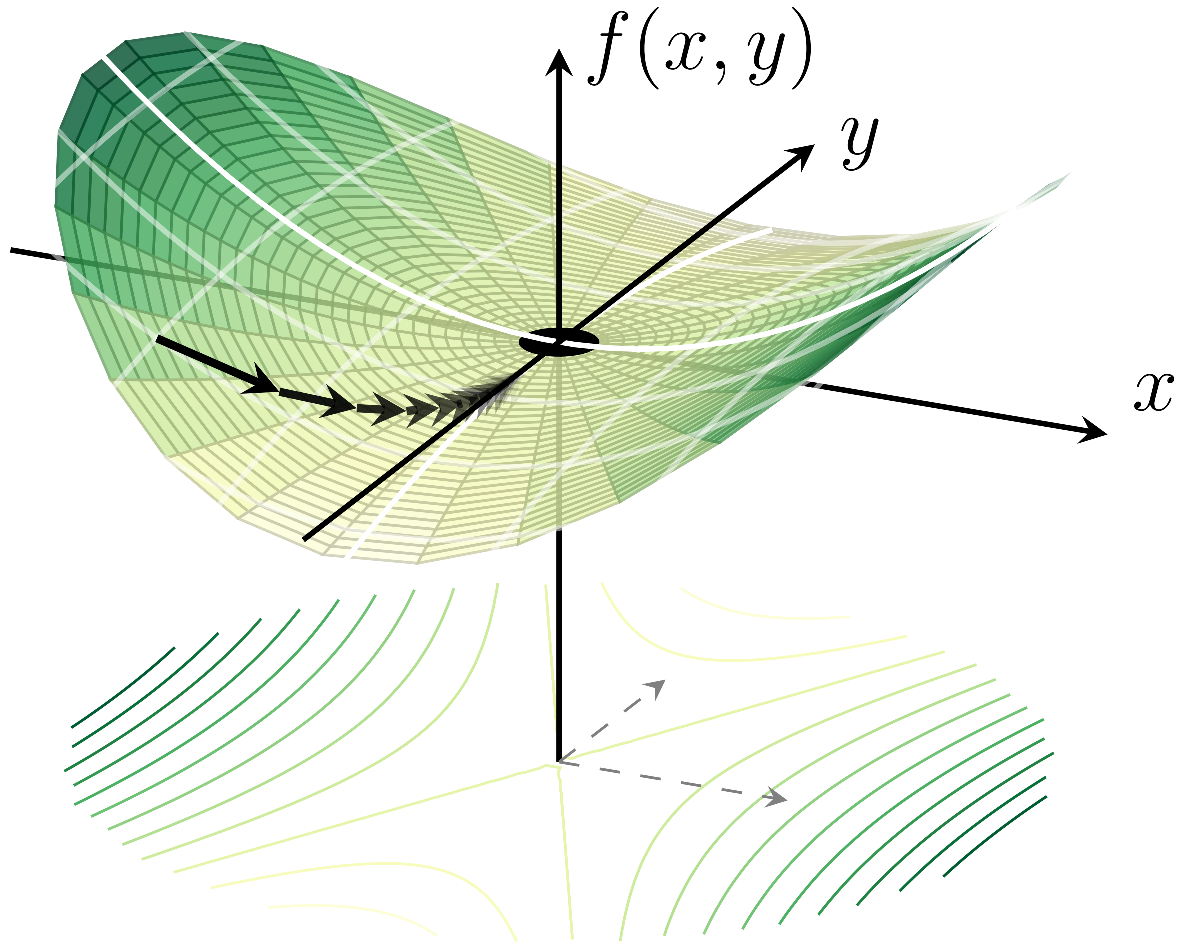

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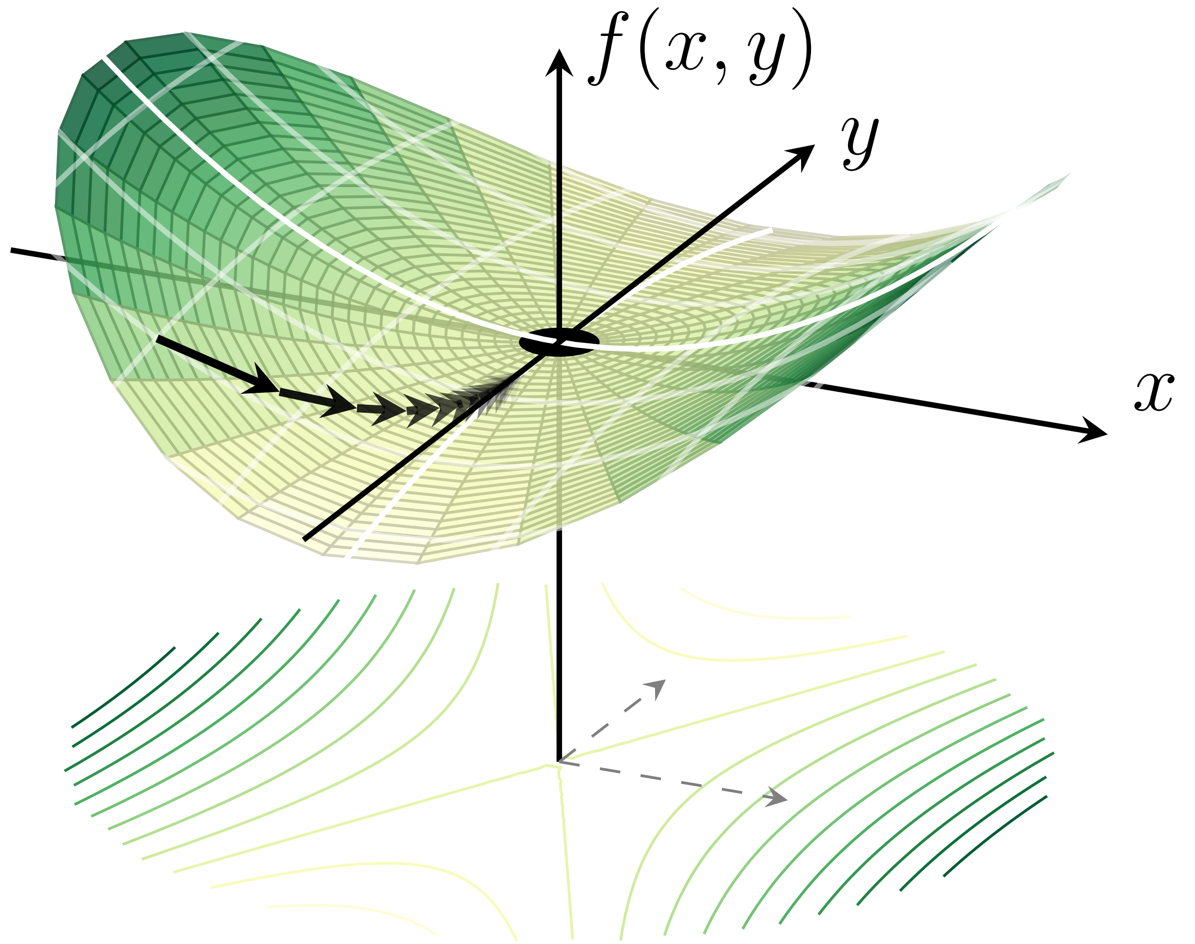

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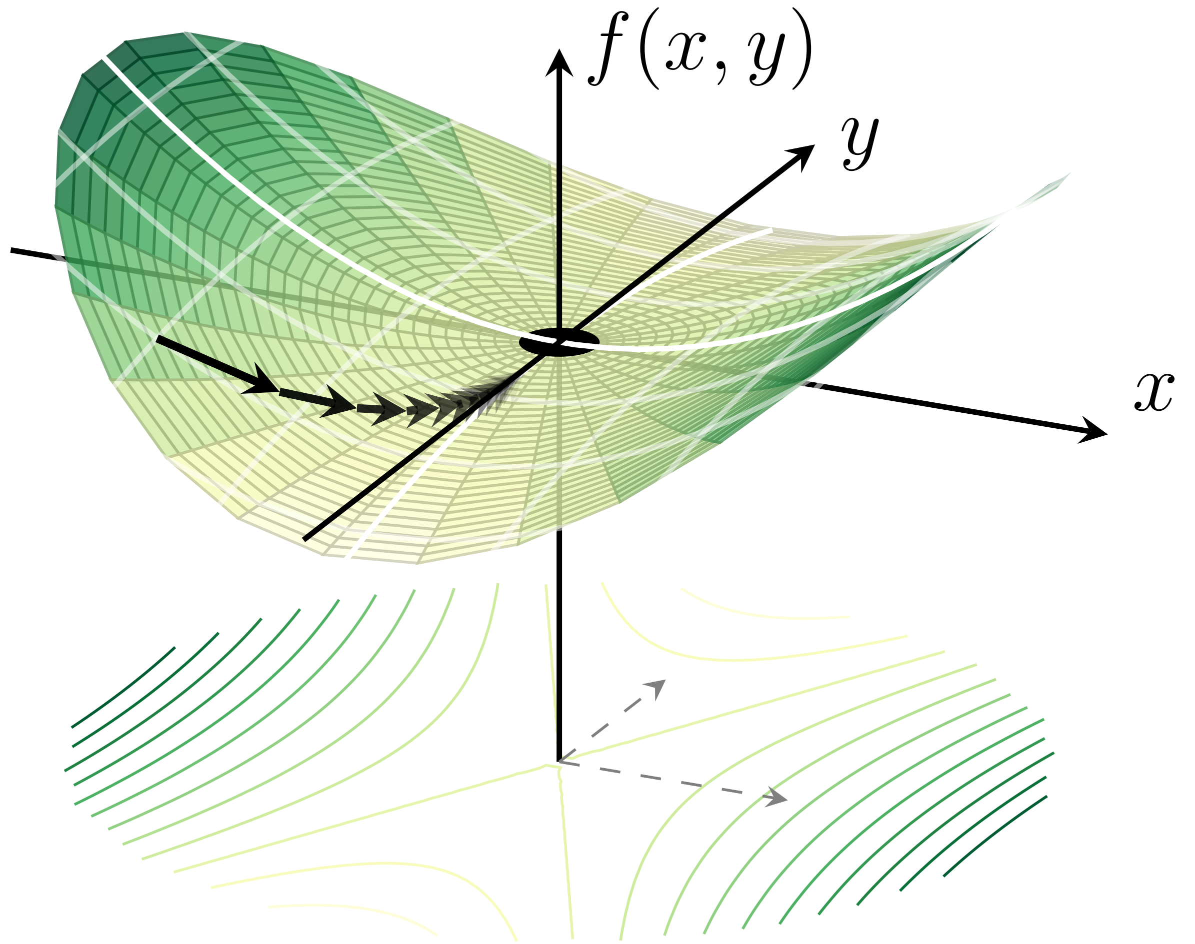

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- Agent i's goal: minimize$_{x_i}$ $f_i(x_i, x_{-i})$ via gradient descent.

Two local equilibrium concepts

Definitions and known results

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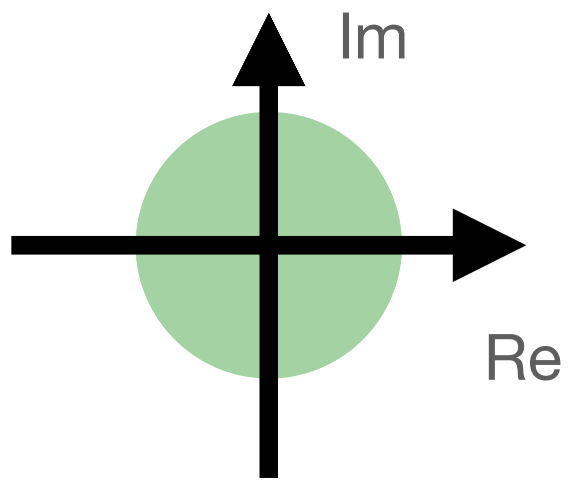
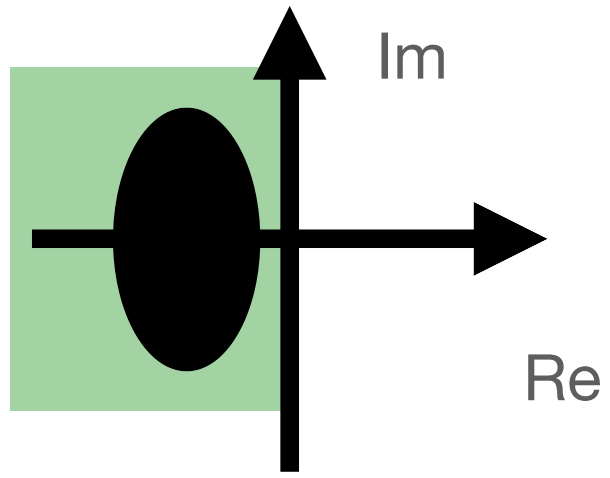
$$x(t+1) = x(t) - \gamma g(x(t))$$

$$\underbrace{\rho(I + \gamma J(x^*))}_{spectral\ radius} < 1$$



$$\dot{x} = g(x)$$

$$\boxed{\underbrace{\sigma(J(x^*))}_{spectrum} \subset \mathbf{C}_-^\circ}$$



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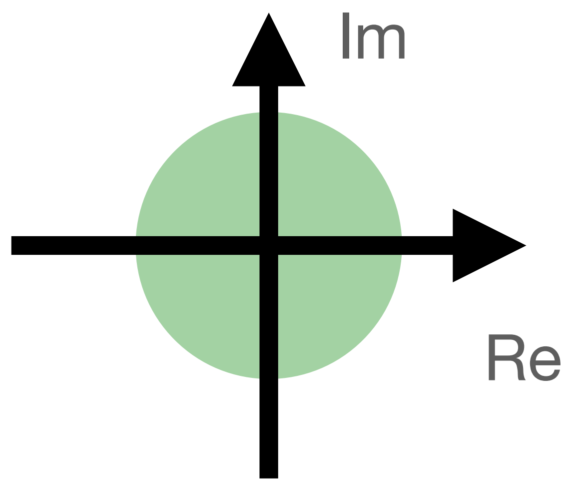

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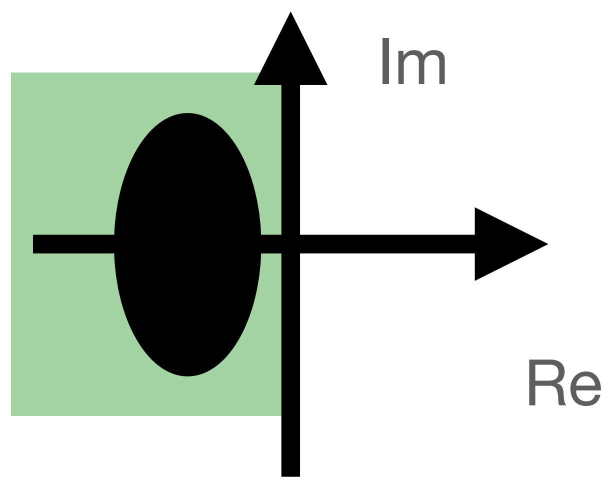
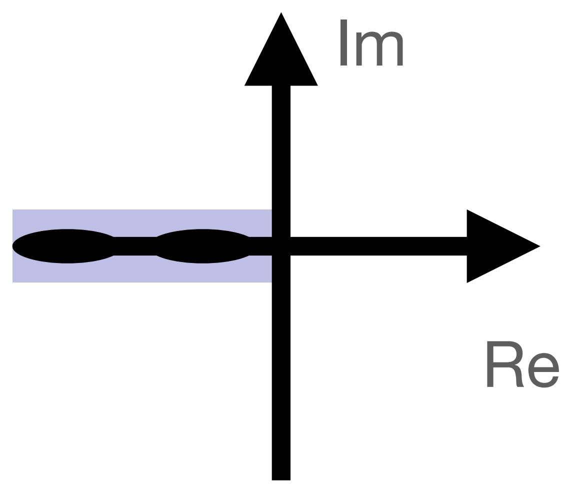
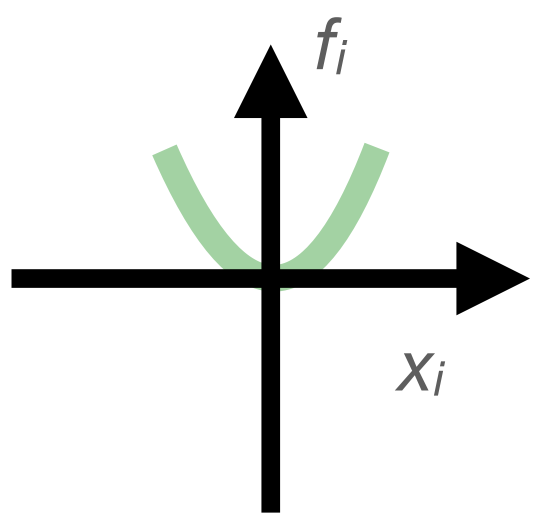


- Differential Nash equilibrium $x^* = (x_i^*, x_{-i}^*)$ (Ratliff, Burden, Sastry, 2014)

$$\boxed{\underbrace{D_i g_i(x_i^*, x_{-i}^*)}_{\textit{individual Hessian}} \succ 0,}$$



$$\underbrace{f_i(x_i^*, x_{-i}^*) < f_i(x_i, x_{-i}^*)}_{\textit{strict local Nash}}, \quad x_i \in U_i \backslash \{x_i^*\}, \quad \forall i$$



Stability and spectrum of a block matrix

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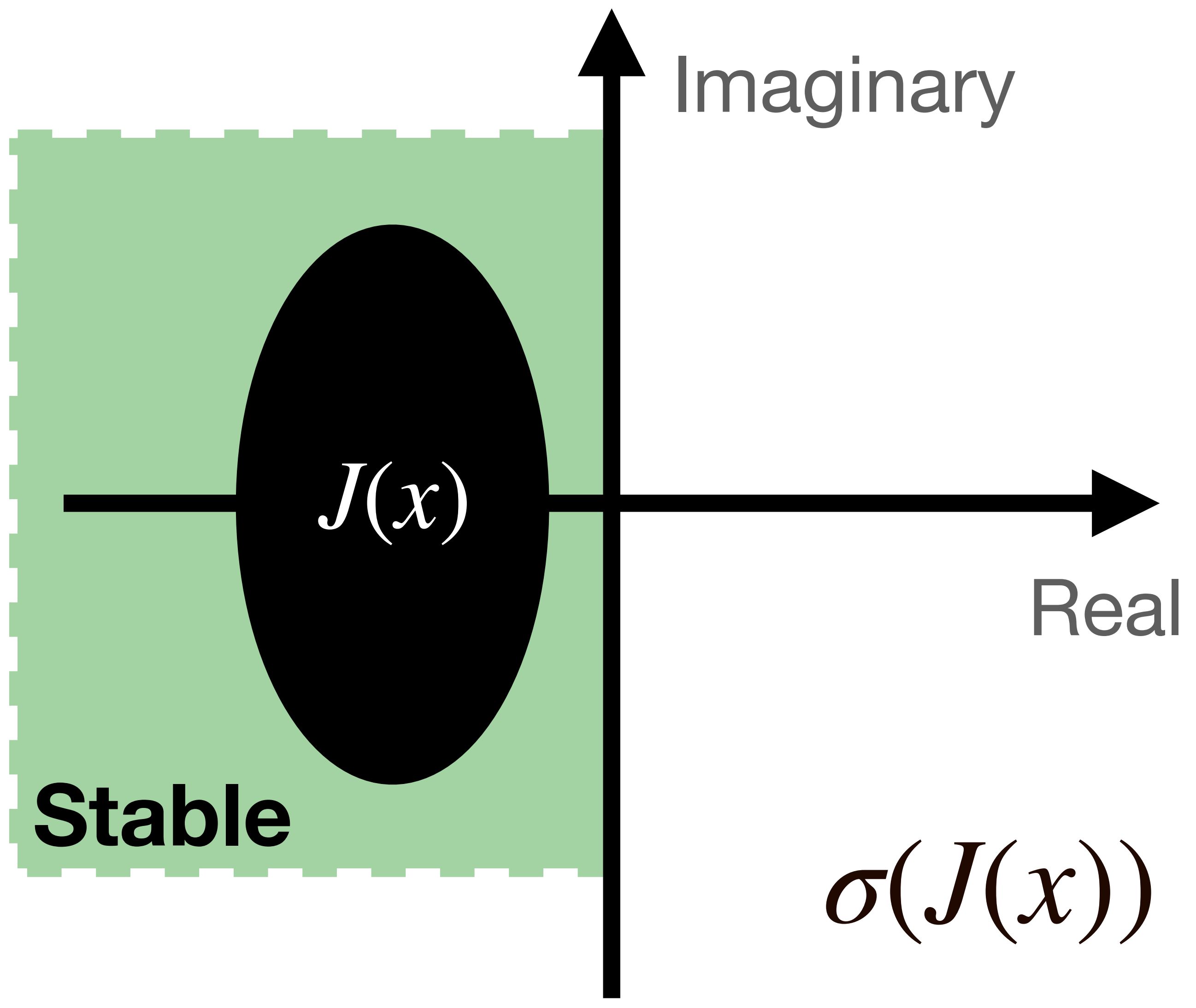

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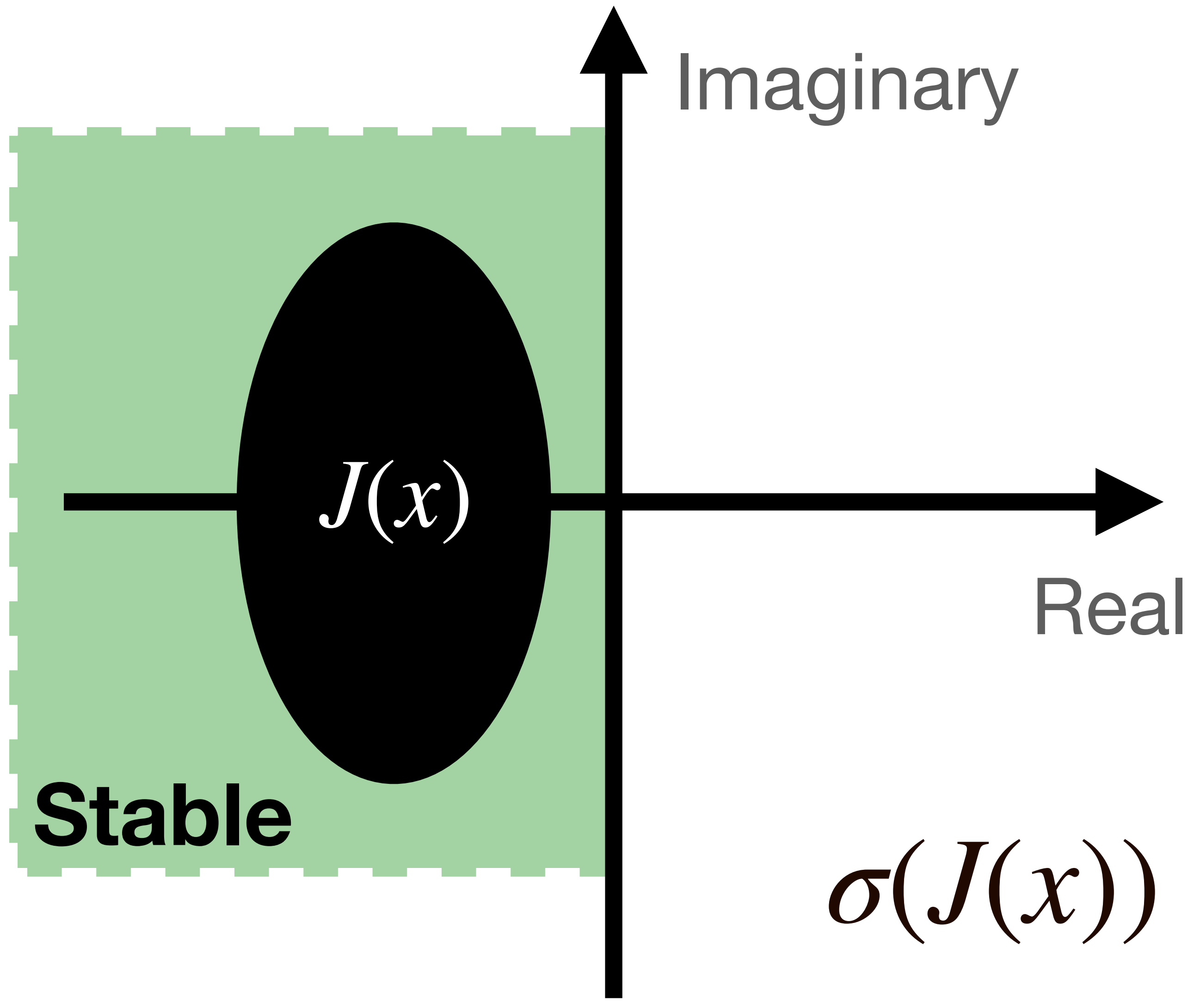

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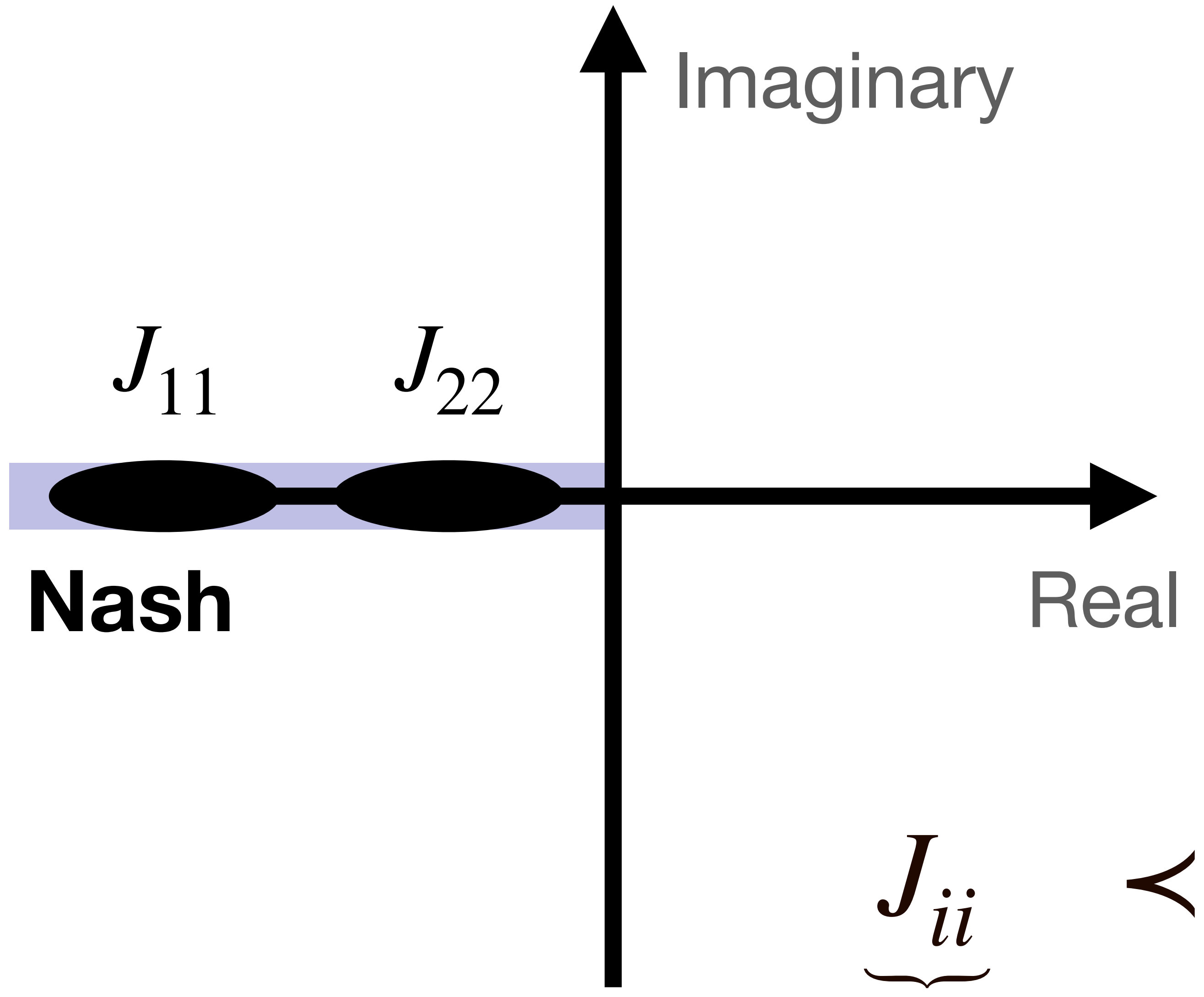

$$\underbrace{J_{ii}}_{Hessian} \prec 0,\ \forall i$$

Stability of game dynamics

Locally stable equilibrium

Examples: $\min_x \max_y f(x, y),$ for different costs f

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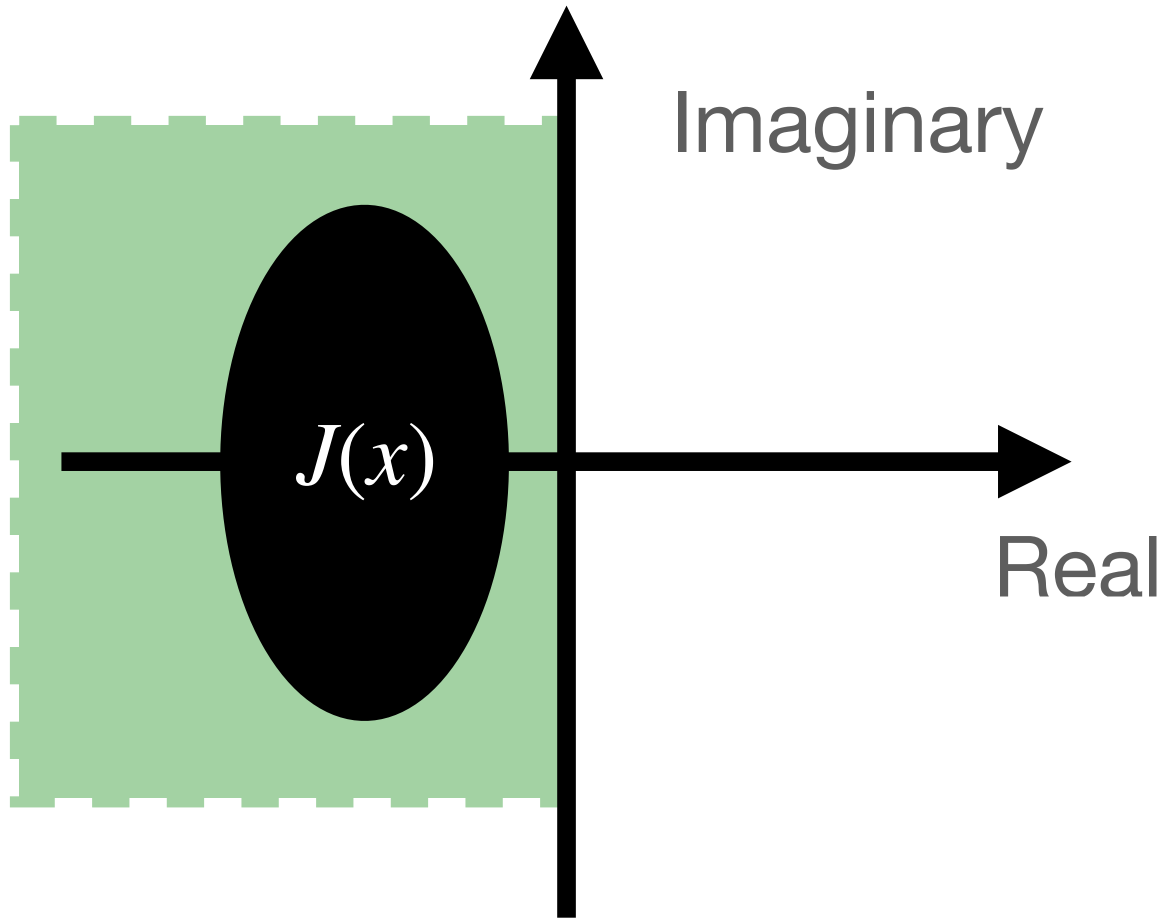

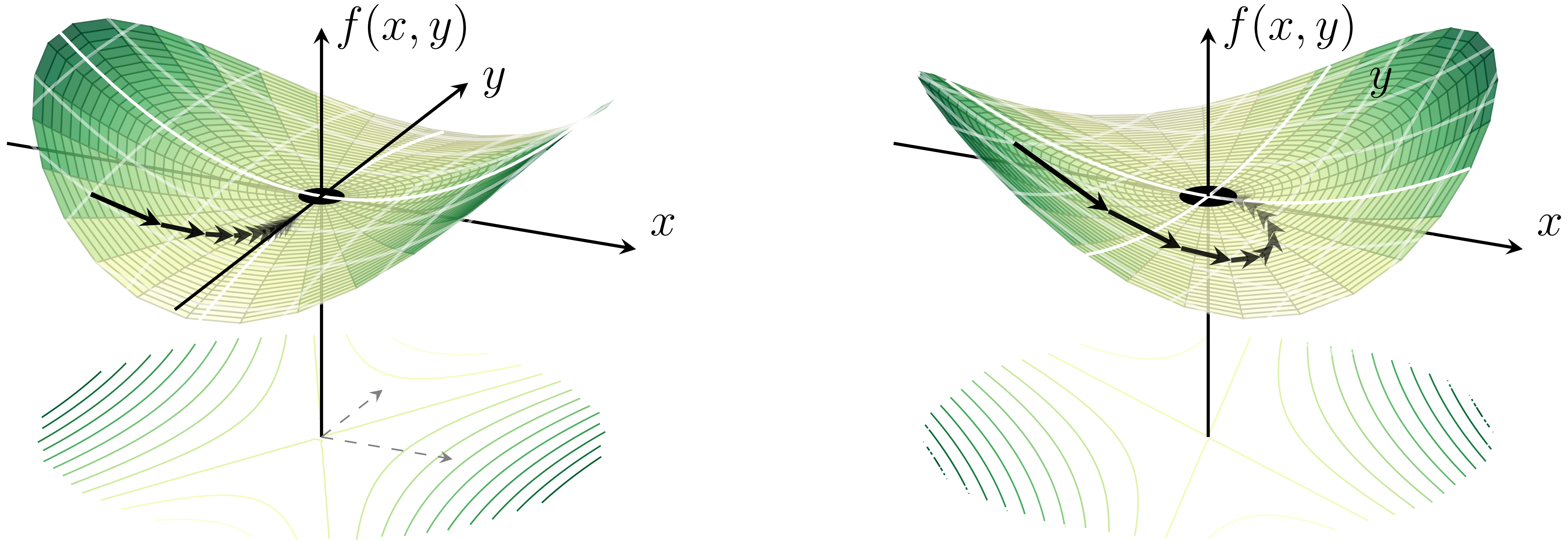

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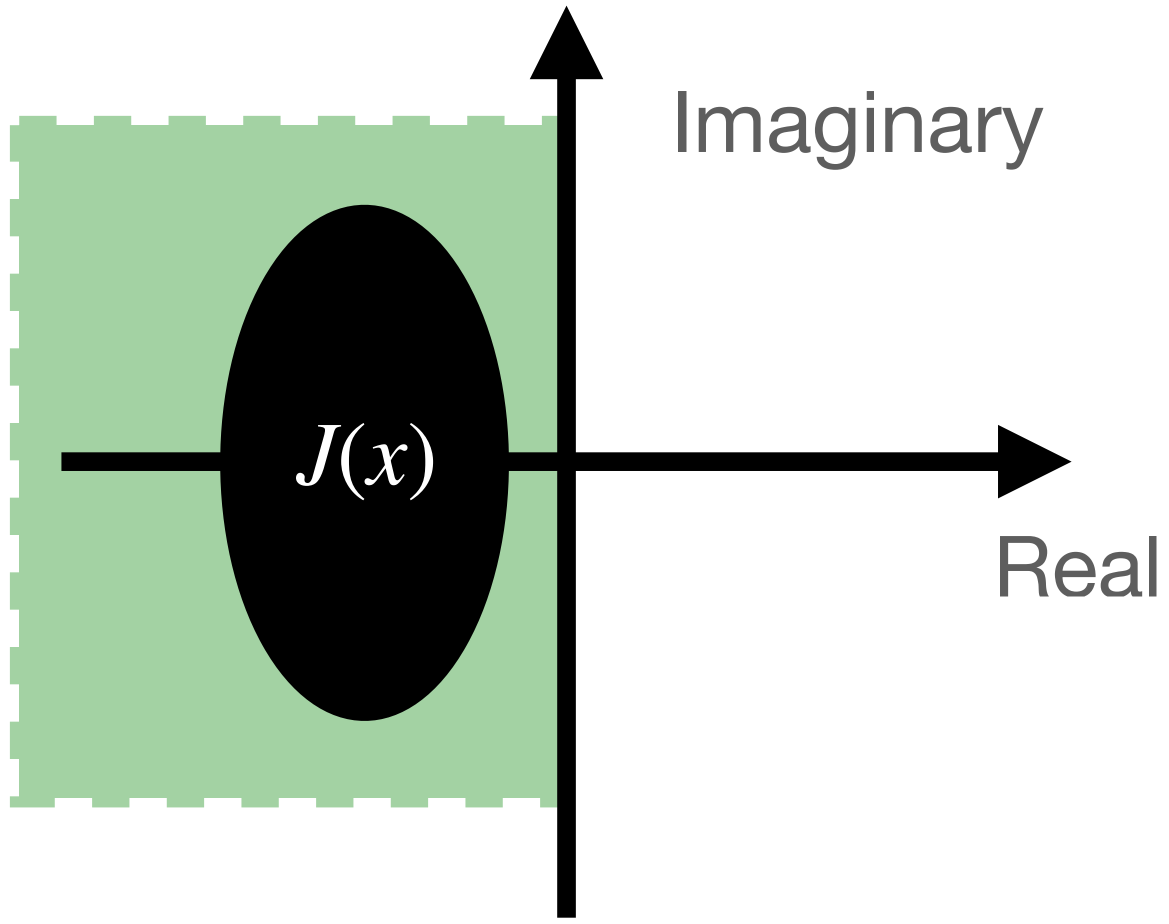
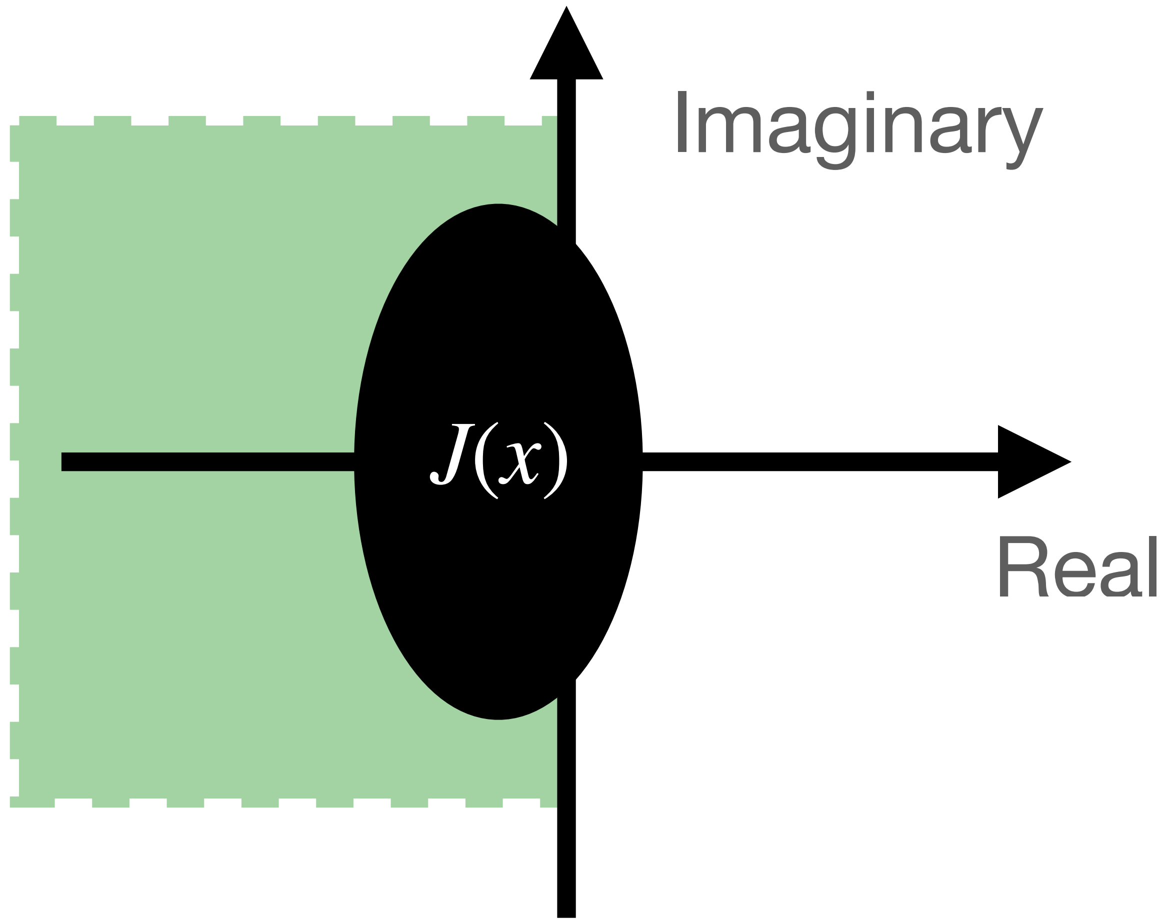
- 

- 

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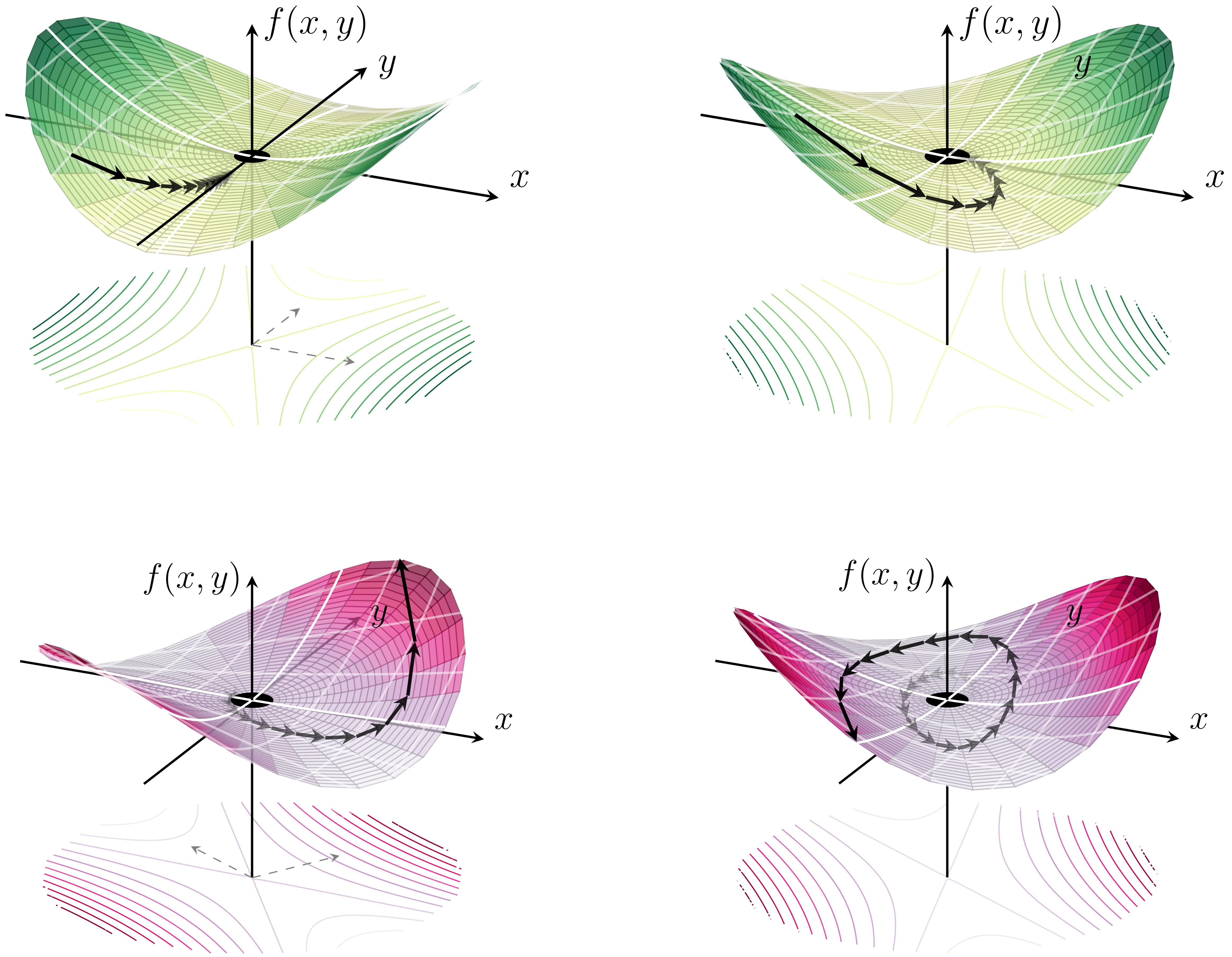
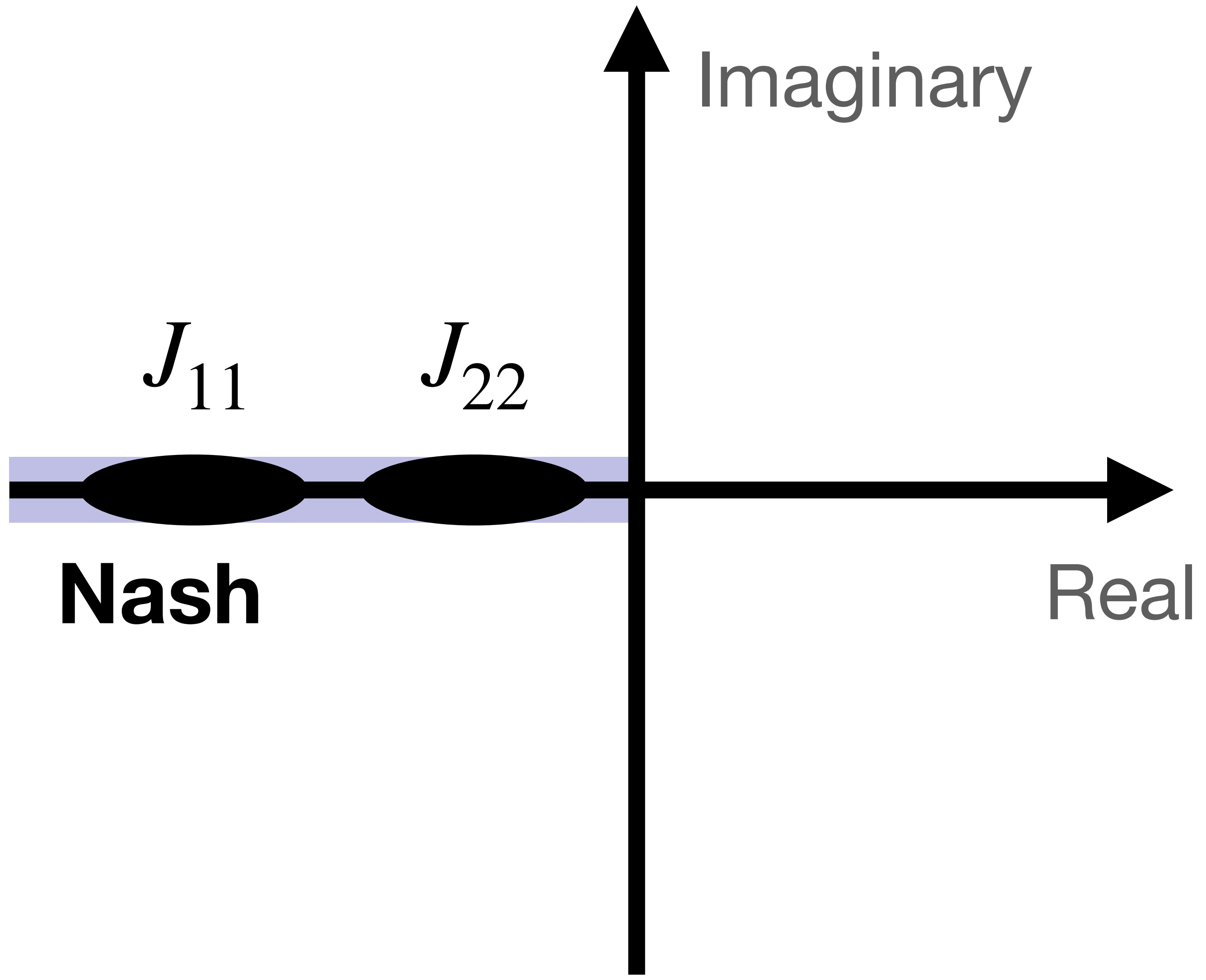


Optimality of individual agents

Differential Nash equilibrium



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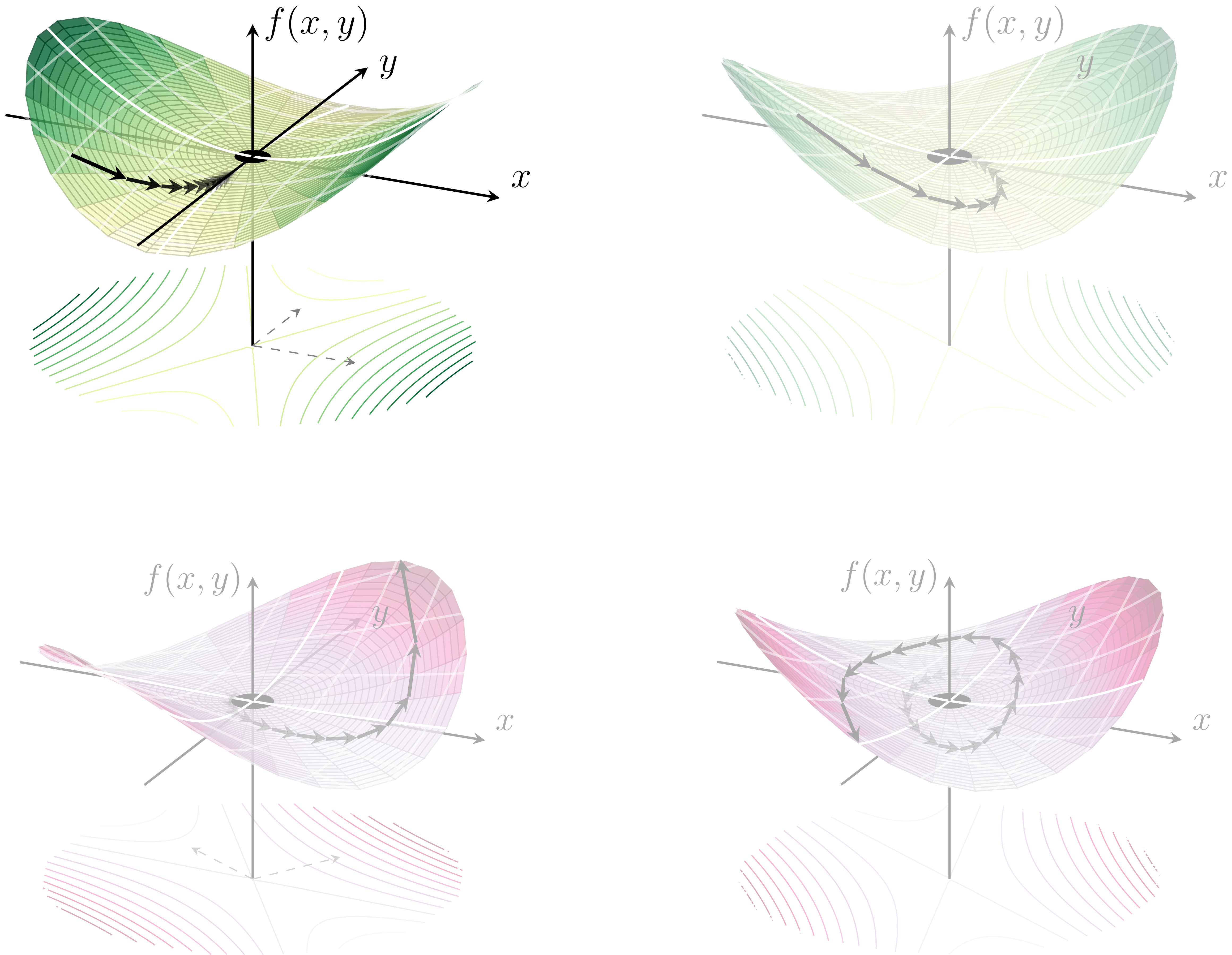

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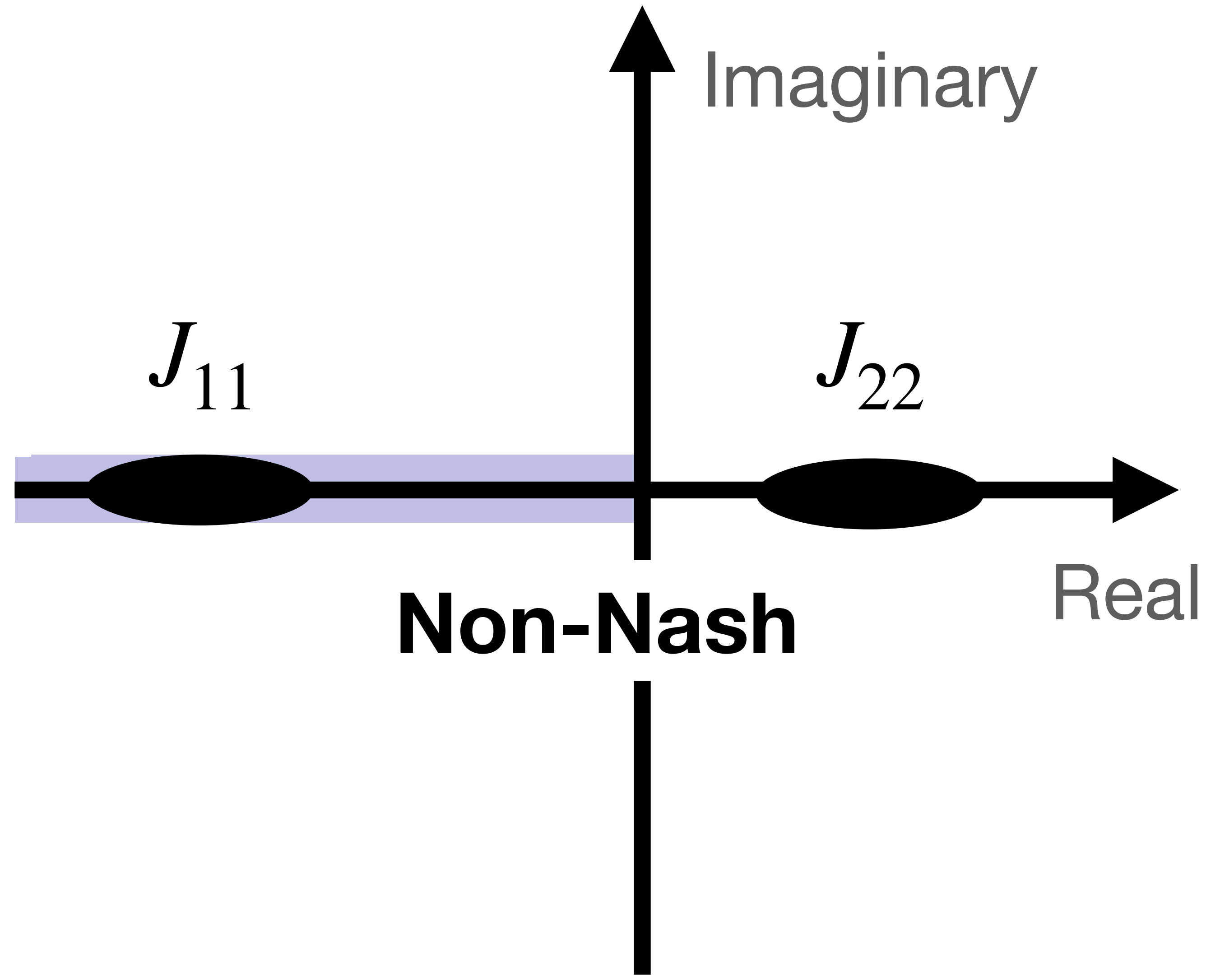

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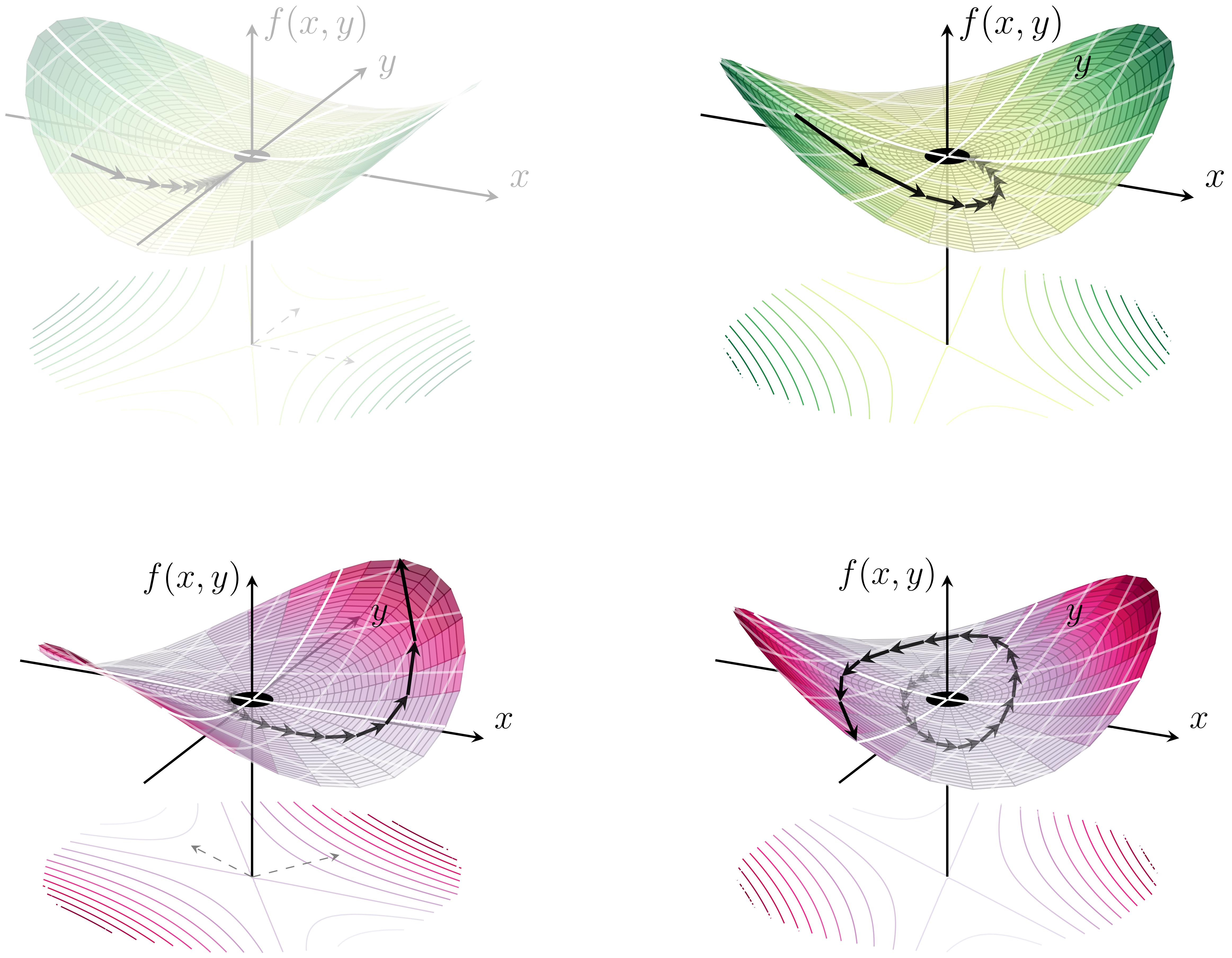

Decomposition of the game Jacobian

Game types

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-

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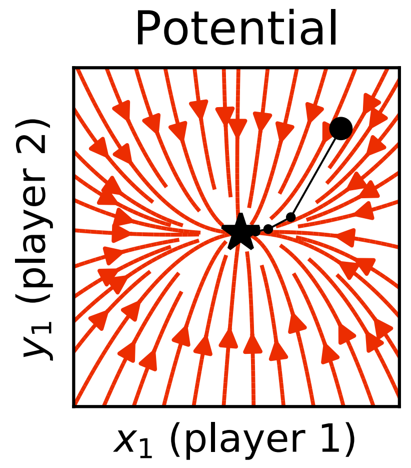
Game types

- $$J(x) = \overbrace{\underbrace{\begin{bmatrix} J_{11} & 0 \\ 0 & J_{22} \end{bmatrix}}_{individual} + \begin{bmatrix} 0 & P \\ P^\top & 0 \end{bmatrix}}^{symmetric} + \overbrace{\begin{bmatrix} 0 & -Z \\ Z^\top & 0 \end{bmatrix}}^{anti-symmetric}$$

  with $\begin{bmatrix} 0 & P \\ P^\top & 0 \end{bmatrix} + \begin{bmatrix} 0 & -Z \\ Z^\top & 0 \end{bmatrix}$ labelled *interaction*

- Symmetric: $$\frac{1}{2}\left(J + J^\top\right) = \begin{bmatrix} J_{11} & \frac{1}{2}\left(J_{12} + J_{21}^\top\right) \\ \frac{1}{2}\left(J_{21} + J_{12}^\top\right) & J_{22} \end{bmatrix} = \begin{bmatrix} J_{11} & P \\ P^\top & J_{22} \end{bmatrix}$$



Decomposition of the game Jacobian

Game types

- $$J(x) = \overbrace{\underbrace{\begin{bmatrix} J_{11} & 0 \\ 0 & J_{22} \end{bmatrix}}_{individual} + \begin{bmatrix} 0 & P \\ P^\top & 0 \end{bmatrix}}^{symmetric} + \underbrace{\overbrace{\begin{bmatrix} 0 & -Z \\ Z^\top & 0 \end{bmatrix}}^{anti-symmetric}}_{interaction}$$

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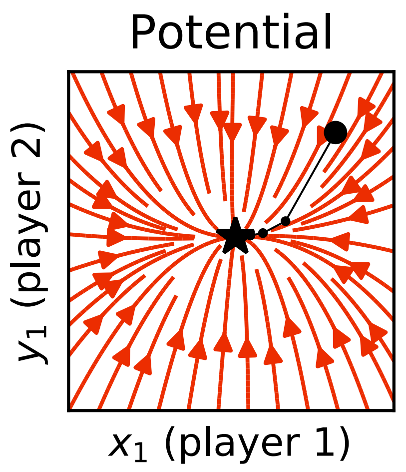
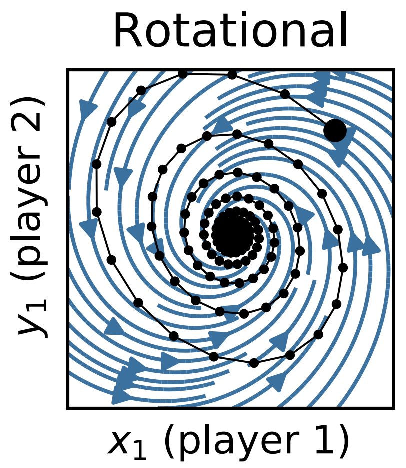


- Anti-symmetric: $$\frac{1}{2}\left(J - J^\top\right) = \begin{bmatrix} 0 & \frac{1}{2}\left(J_{12} - J_{21}^\top\right) \\ \frac{1}{2}\left(J_{21} - J_{12}^\top\right) & 0 \end{bmatrix} = \begin{bmatrix} 0 & -Z \\ Z^\top & 0 \end{bmatrix}$$



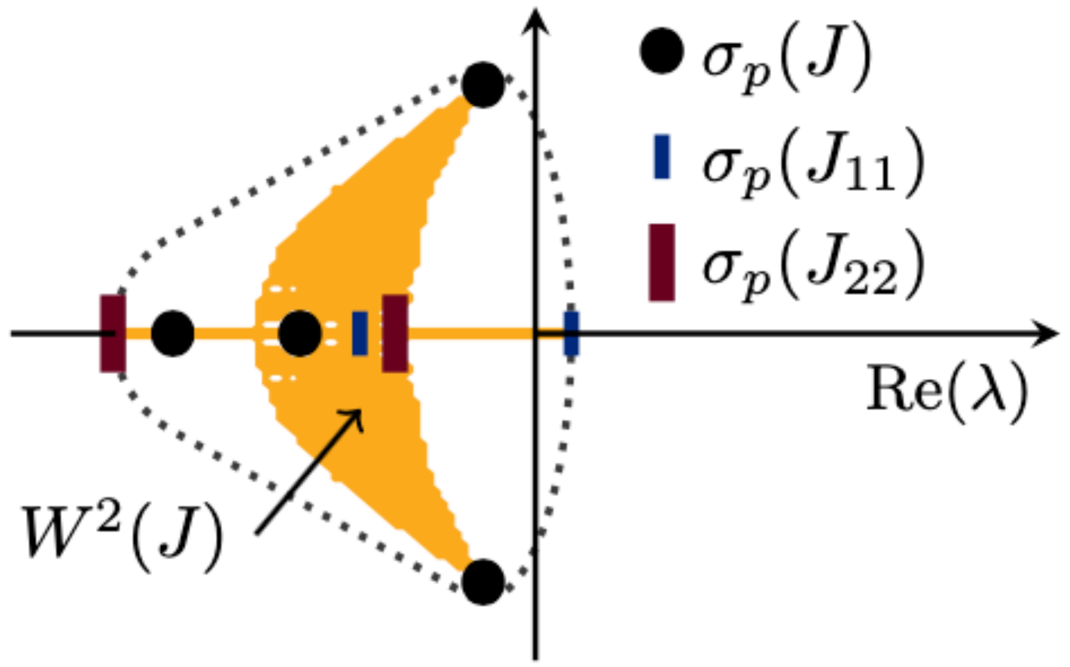
Spectrum of zero-sum and potential games

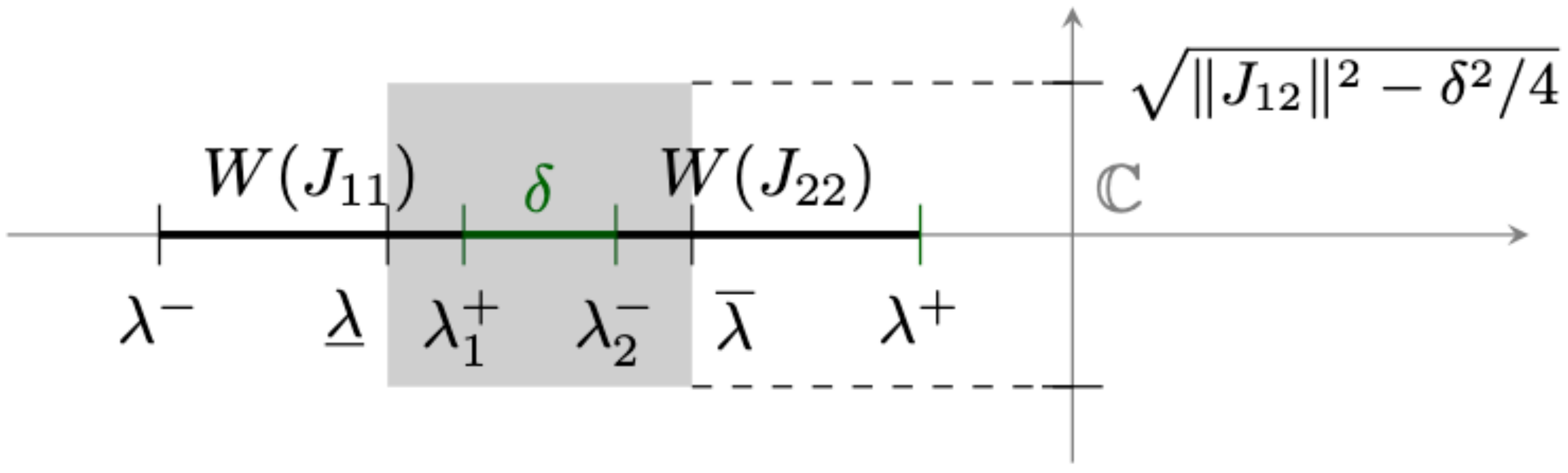
Results: vector actions

Spectrum of zero-sum and potential games

Results: vector actions

- Zero-sum game $J(x) = \begin{bmatrix} J_{11} & -Z \\ Z^\top & J_{22} \end{bmatrix}$





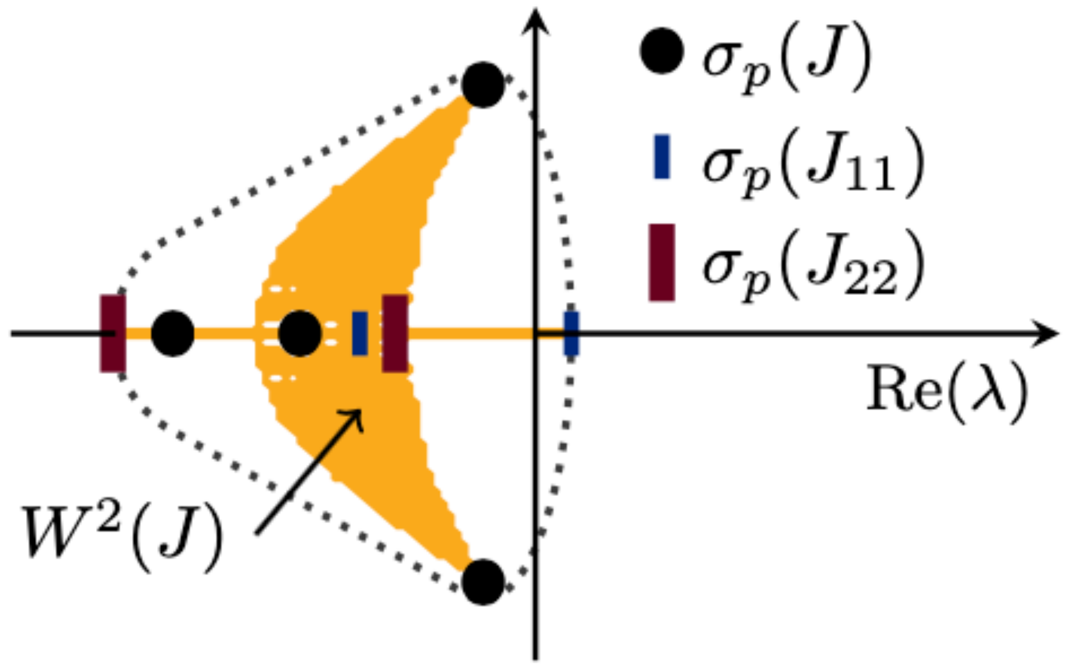
(a) Zero-sum game where $\delta = \lambda_2^- - \lambda_1^+ > 0$ and $\|J_{12}\| > \delta/2$

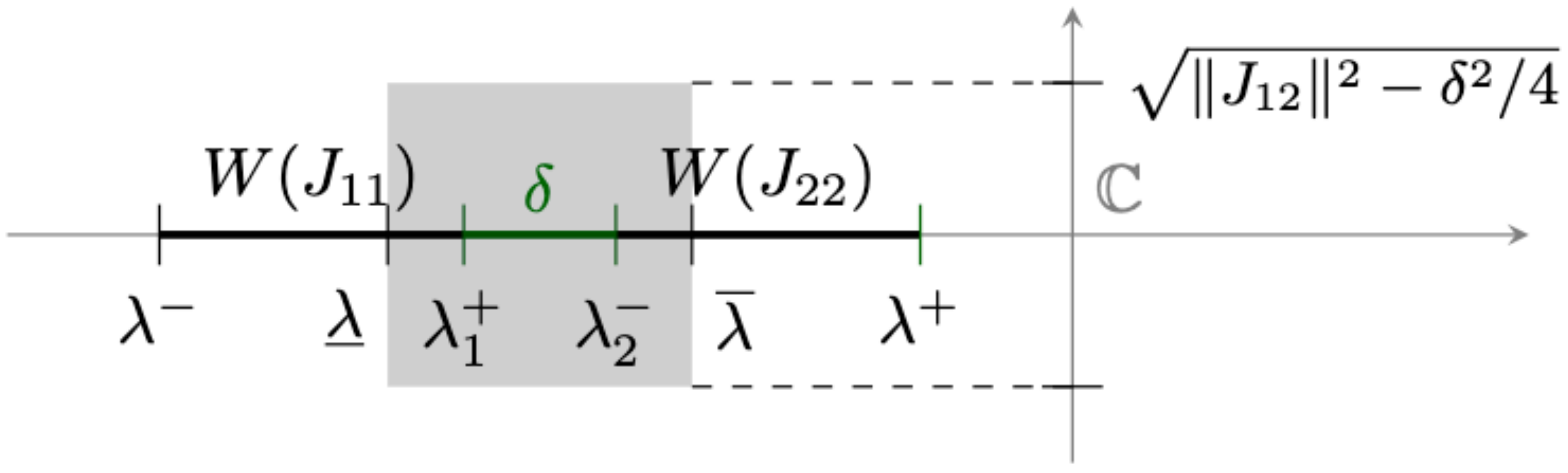
Spectrum of zero-sum and potential games

Results: vector actions

- Zero-sum game

$$J(x) = \begin{bmatrix} J_{11} & -Z \\ Z^\top & J_{22} \end{bmatrix}$$

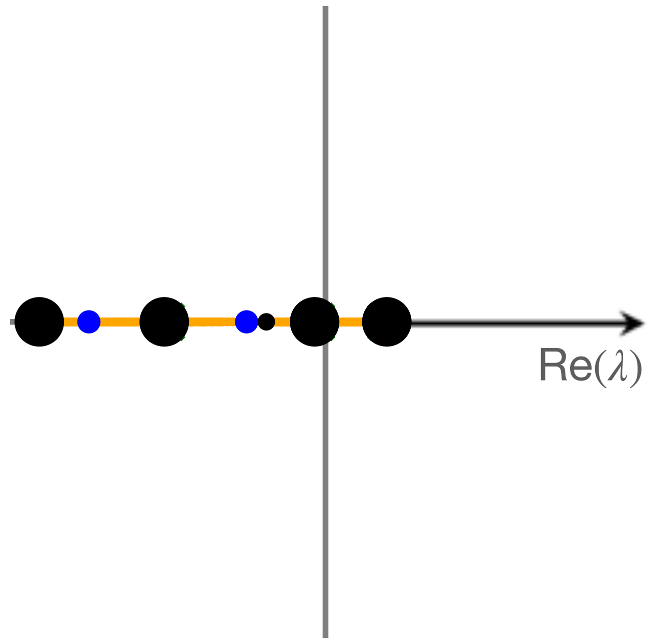


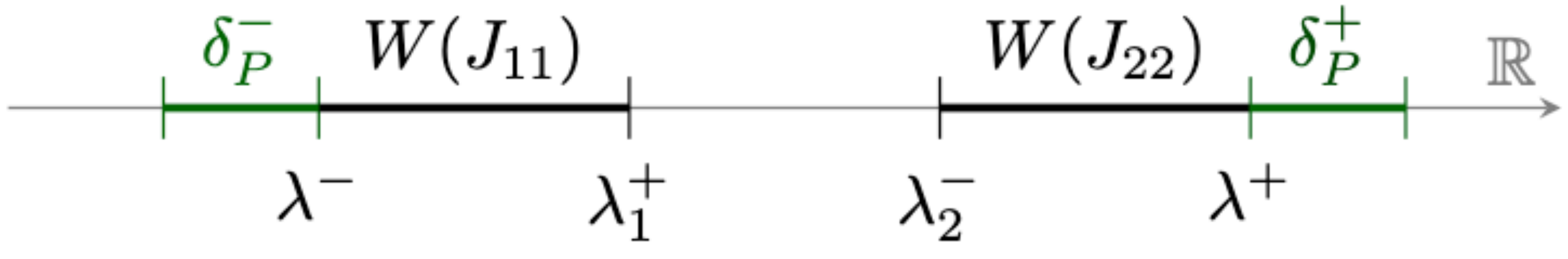


(a) Zero-sum game where $\delta = \lambda_2^- - \lambda_1^+ > 0$ and $\|J_{12}\| > \delta/2$

- Potential game

$$J(x) = \begin{bmatrix} J_{11} & P \\ P^\top & J_{22} \end{bmatrix}$$





(b) Potential game where $\lambda_2^- - \lambda_1^+ > 0$.

Stability of general-sum games

Results: scalar actions

$$J(x) = \underbrace{\begin{bmatrix} m & -z \\ z & m \end{bmatrix}}_{\substack{\textit{complex} \\ \textit{numbers}}} + \underbrace{\begin{bmatrix} h & p \\ p & -h \end{bmatrix}}_{\textit{circle}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$m = \tfrac{1}{2}(a+d) \qquad h = \tfrac{1}{2}(a-d)$$

$$p = \tfrac{1}{2}(b+c) \qquad z = \tfrac{1}{2}(c-b)$$

Stability of general-sum games

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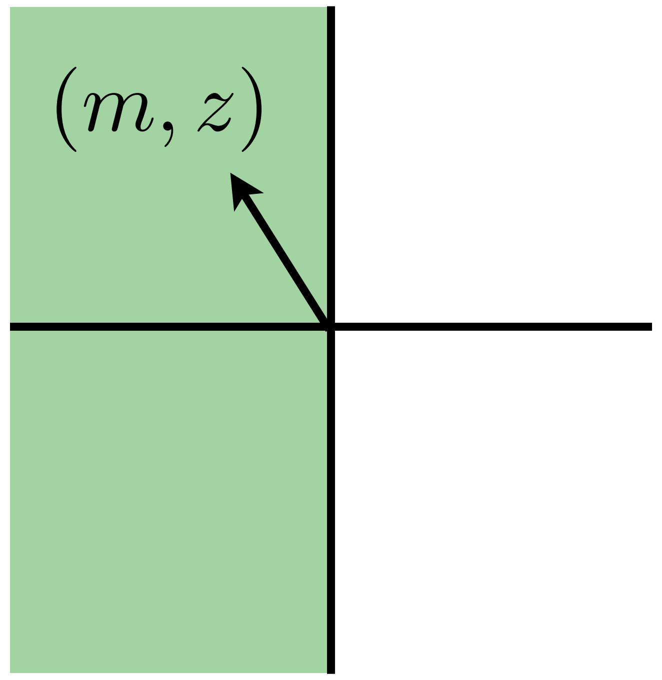

$y = m + zi$

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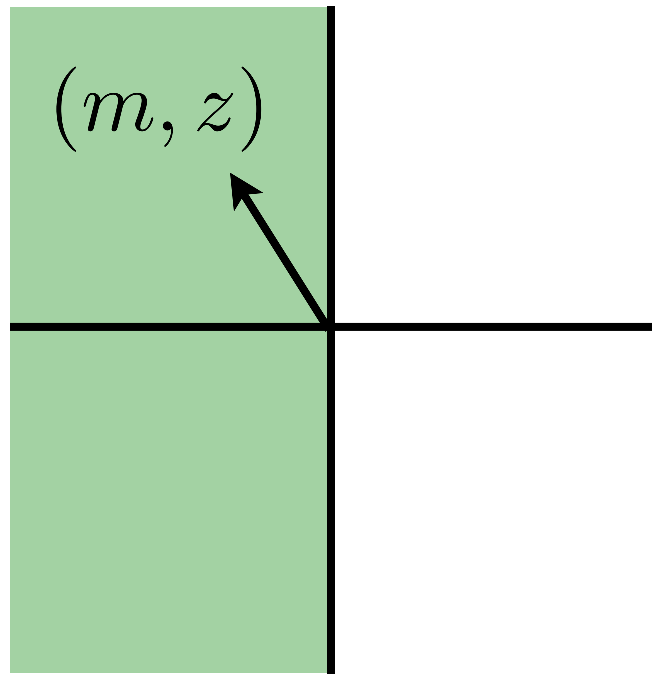
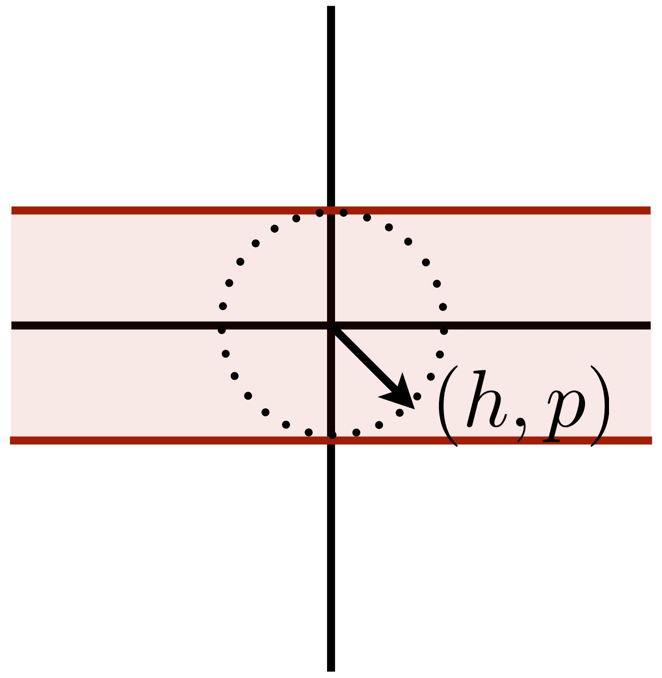


$+$



$=$

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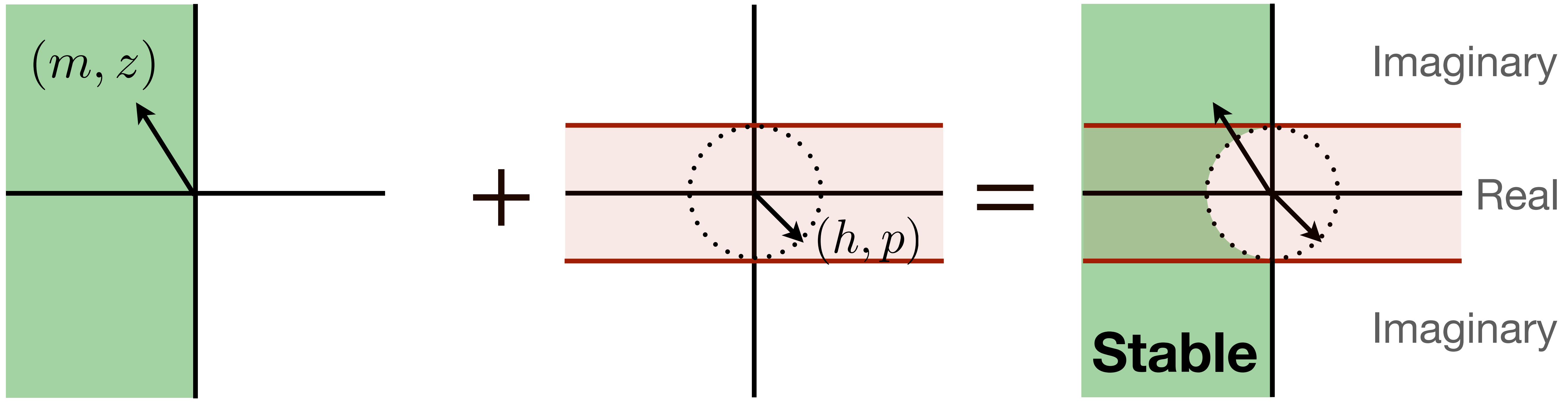

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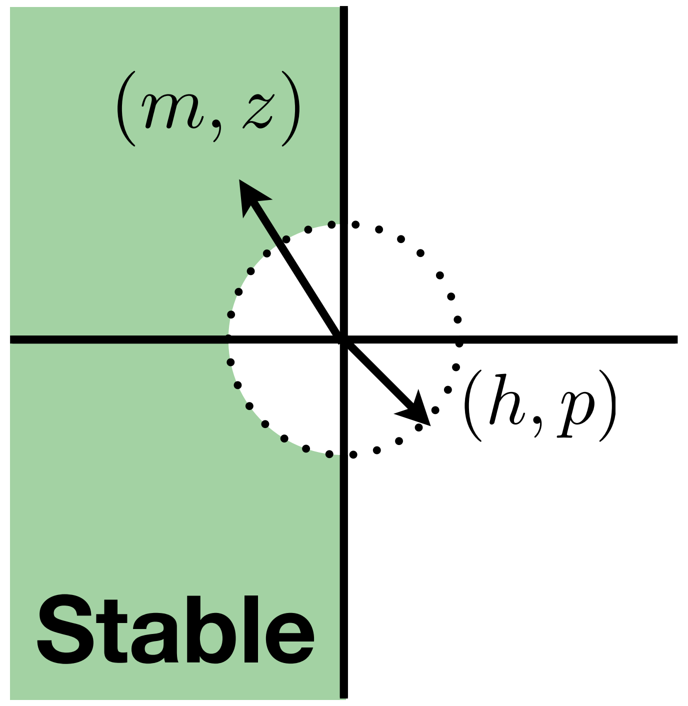
Necessary and sufficient conditions

Results: scalar actions

Necessary and sufficient conditions

Results: scalar actions

- Two-by-two matrix decomposition: $$J(x) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} m+h & p-z \\ p+z & m-h \end{bmatrix}$$
- Stability: tr$(J) < 0$, det$(J) > 0$



Necessary and sufficient conditions

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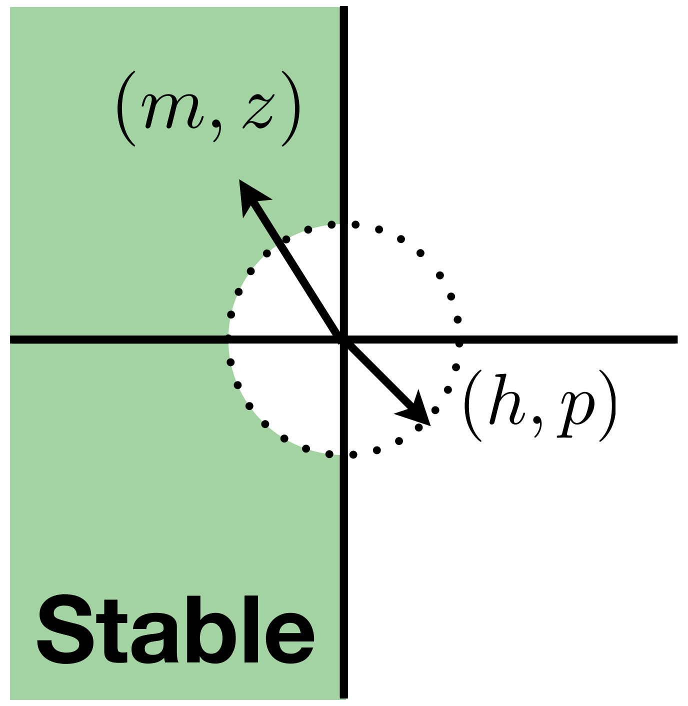
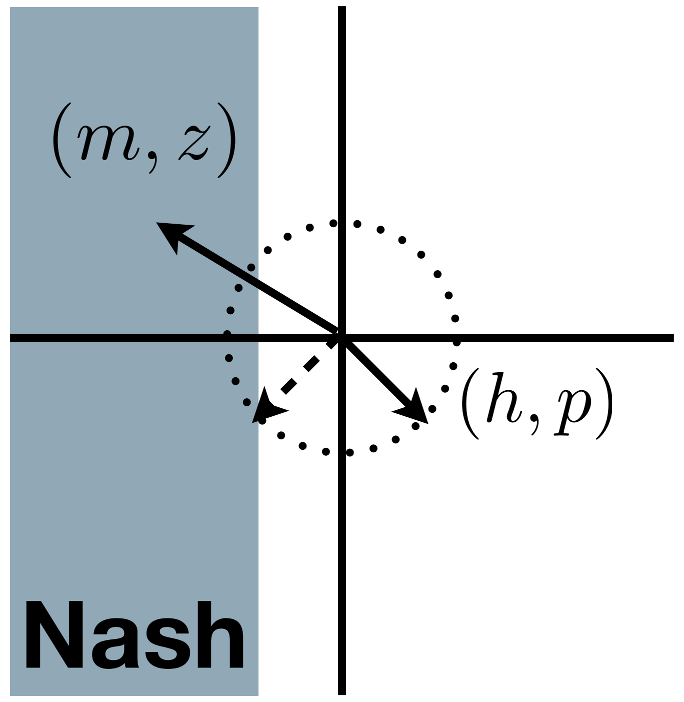


- Nash: $\underbrace{a, d}_{Hessians} < 0 \iff m < -|h|$



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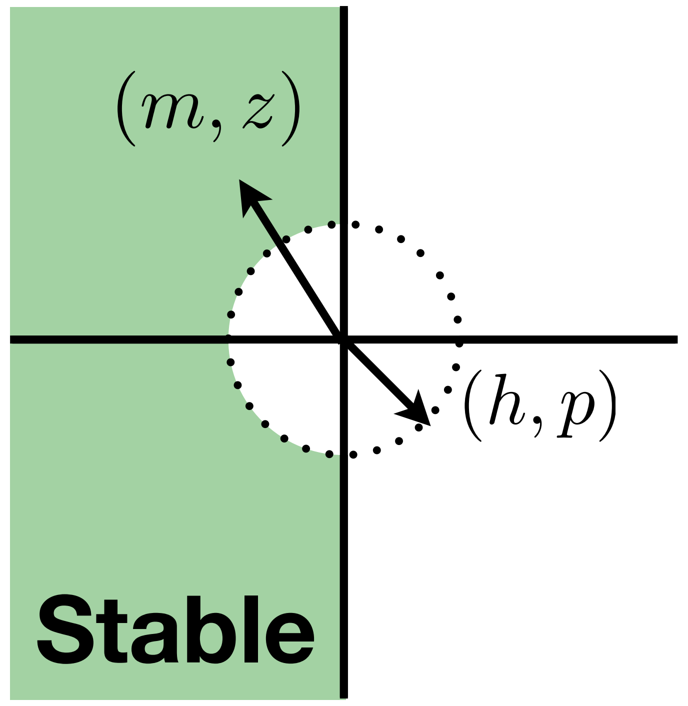

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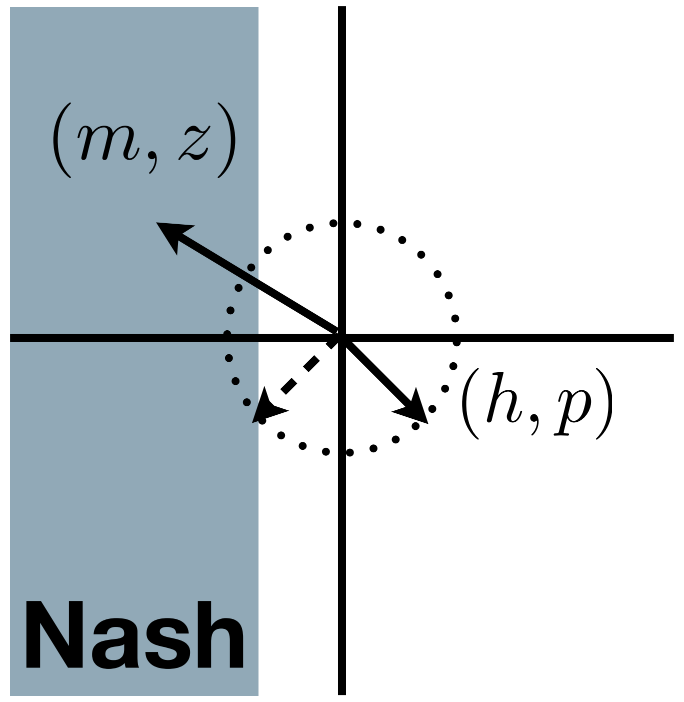
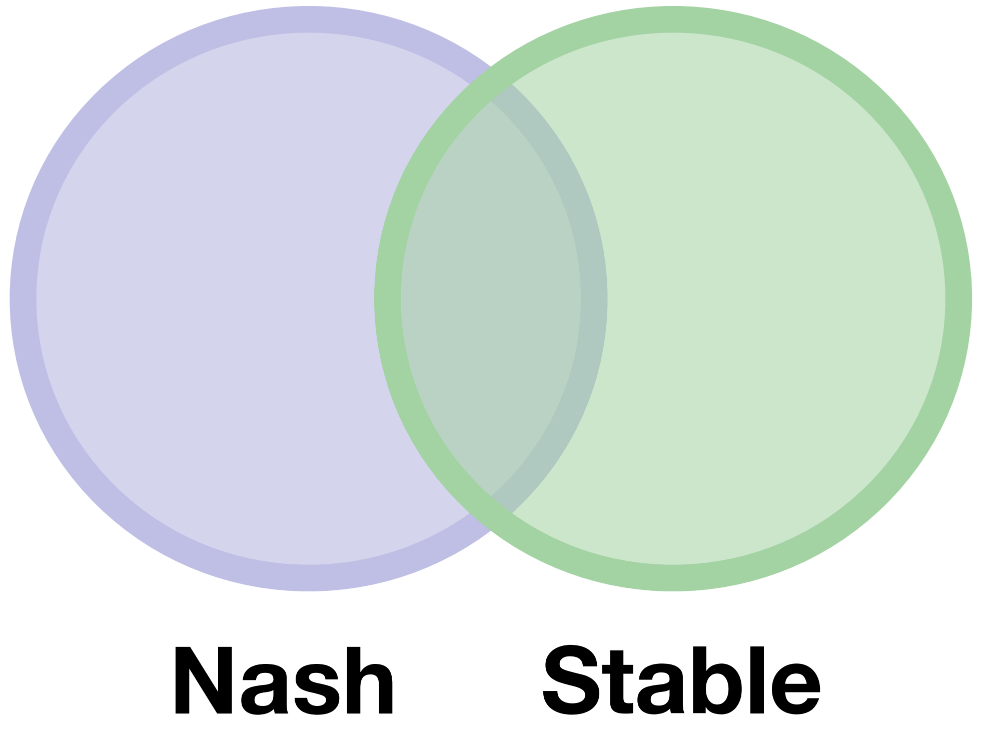
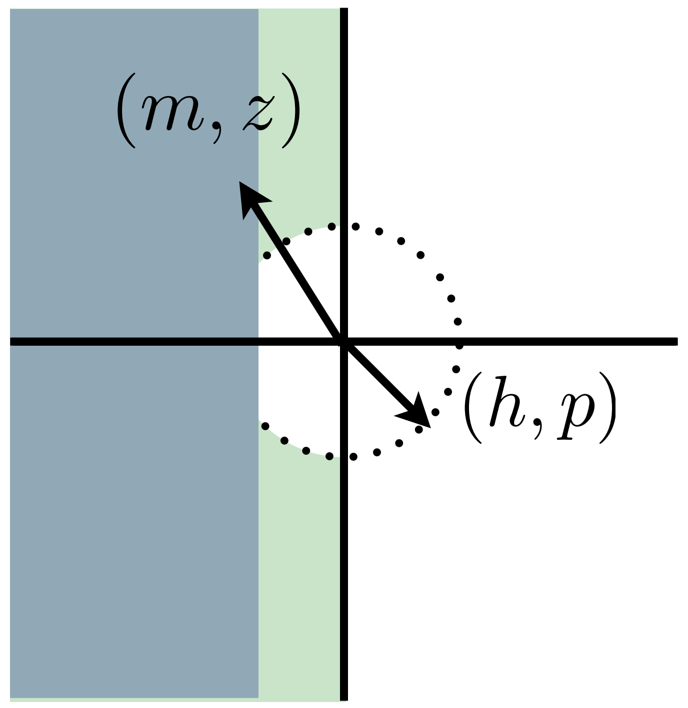






Types of games

Results: scalar actions

- Potential games (p)

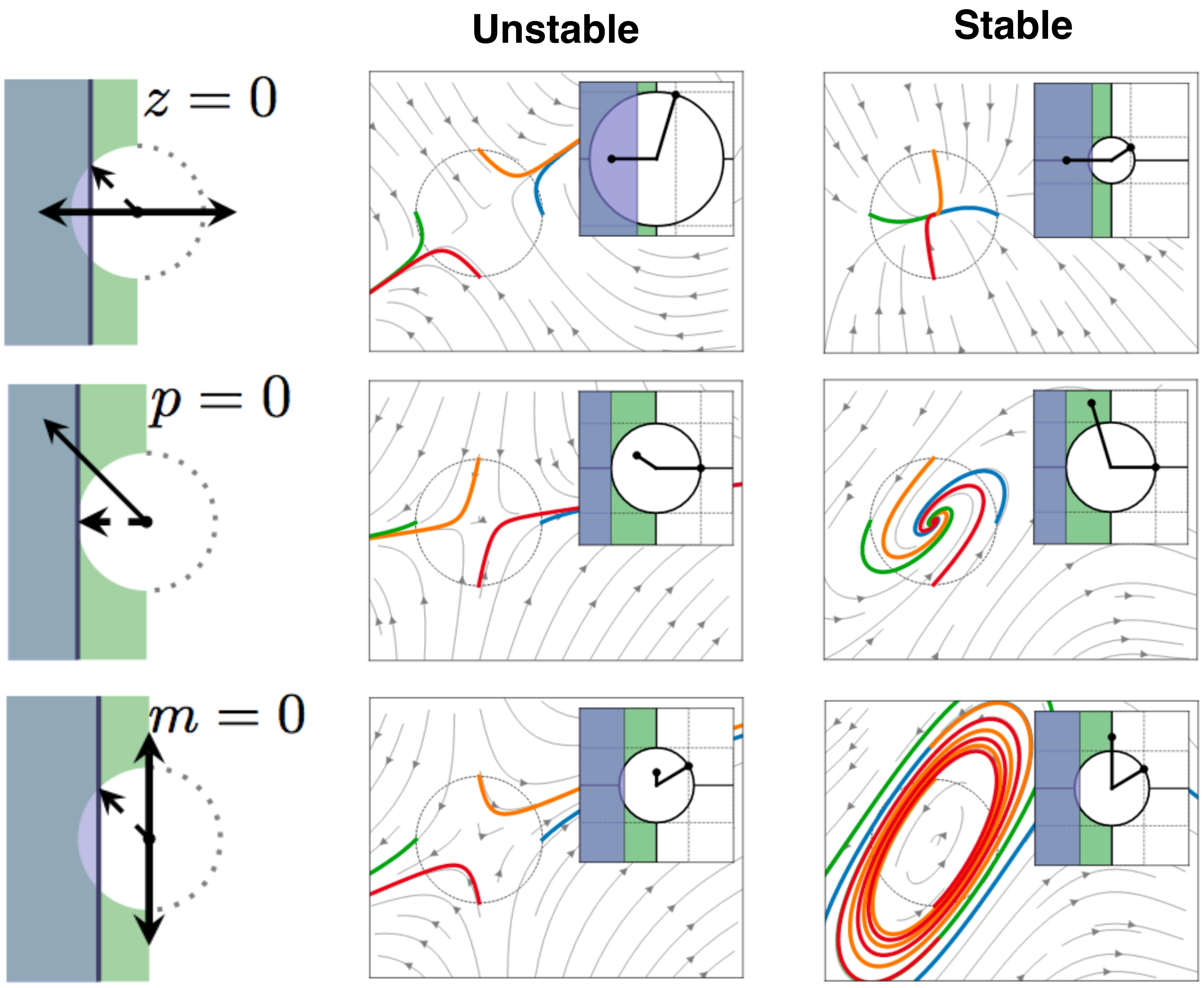
$$J = \begin{bmatrix} m+h & \mathbf{p} \\ \mathbf{p} & m-h \end{bmatrix}$$

- Zero-sum games (z)

$$J = \begin{bmatrix} m+h & \mathbf{z} \\ \mathbf{z} & m-h \end{bmatrix}$$

- Hamiltonian games (h)

$$J = \begin{bmatrix} h & p-\mathbf{z} \\ p+\mathbf{z} & -h \end{bmatrix}$$

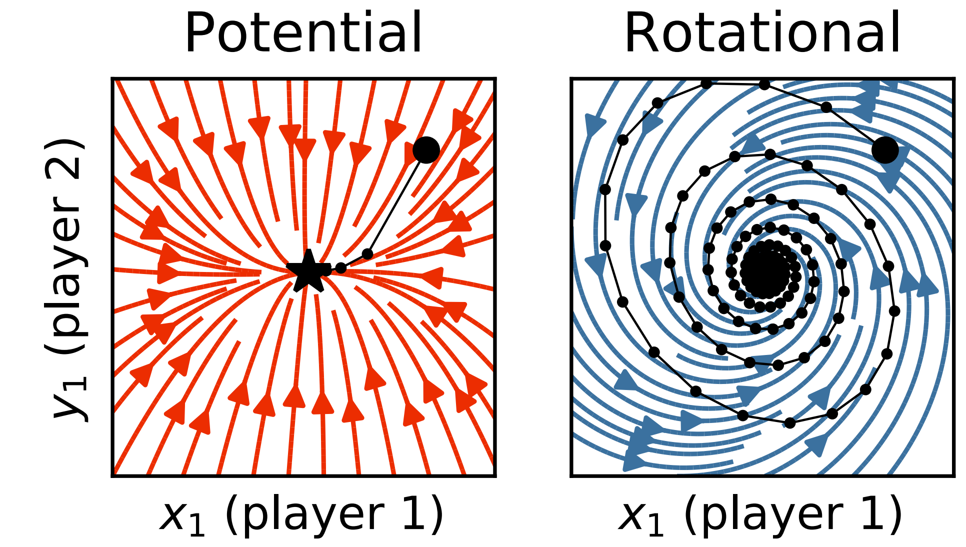


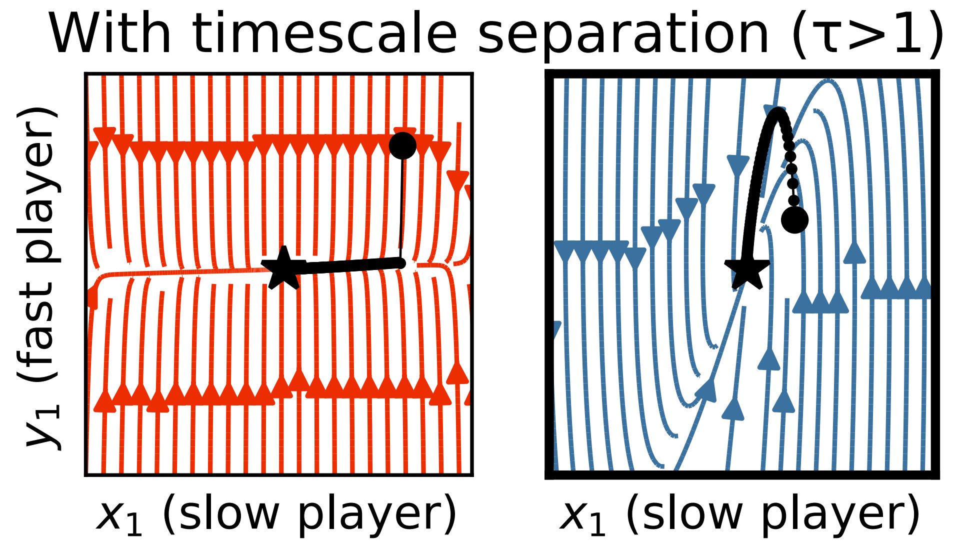
(*marginally stable*)

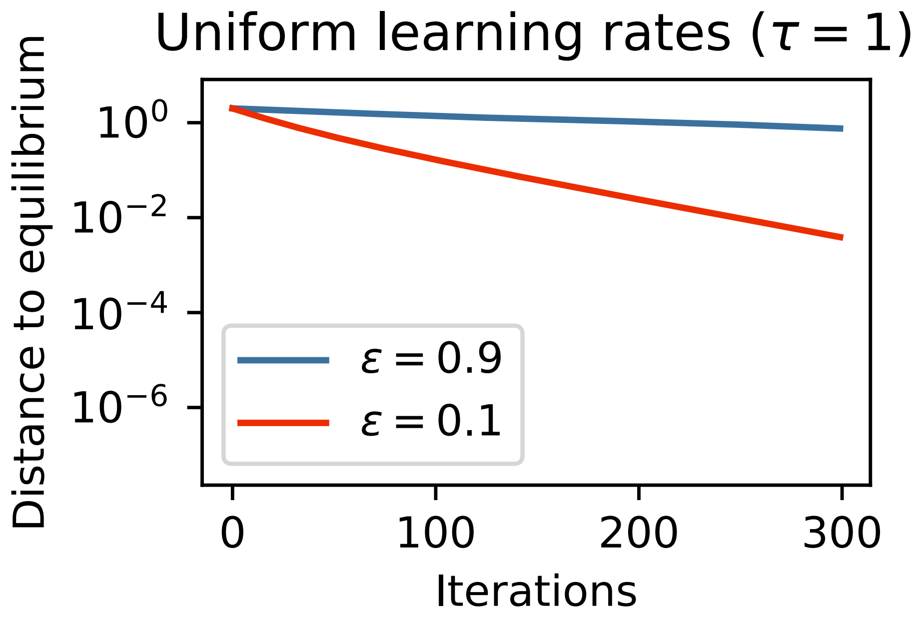
Potential and rotational vector fields

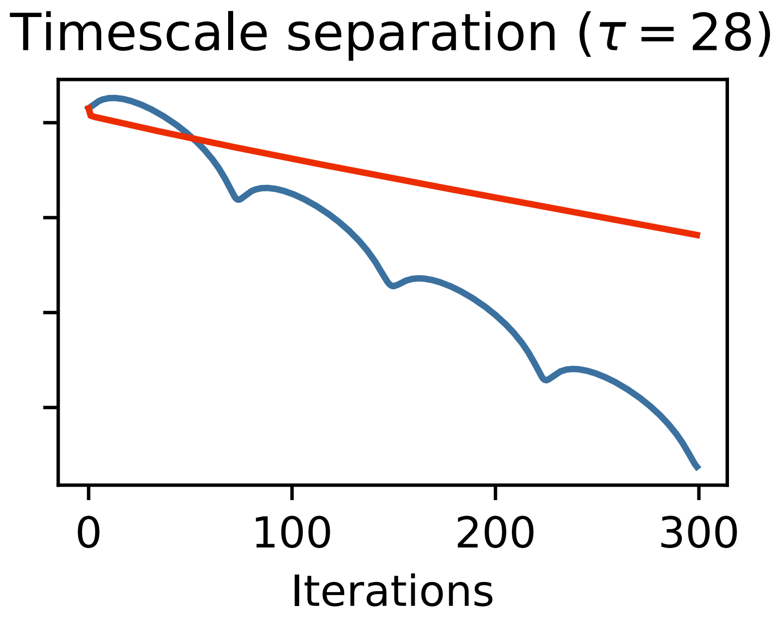
Numerical Experiments

- Timescale separation $(\tau = \frac{\gamma_2}{\gamma_1} > 0)$: $\quad J(x) = \begin{bmatrix} J_{11} & J_{12} \\ \tau J_{21} & \tau J_{22} \end{bmatrix}$
- Symmetric/anti-symmetric Jacobian (zero-sum): $\quad J = (1-\varepsilon)S + \varepsilon A, \quad S = S^\top, A = -A^\top$









Potential and rotational vector fields

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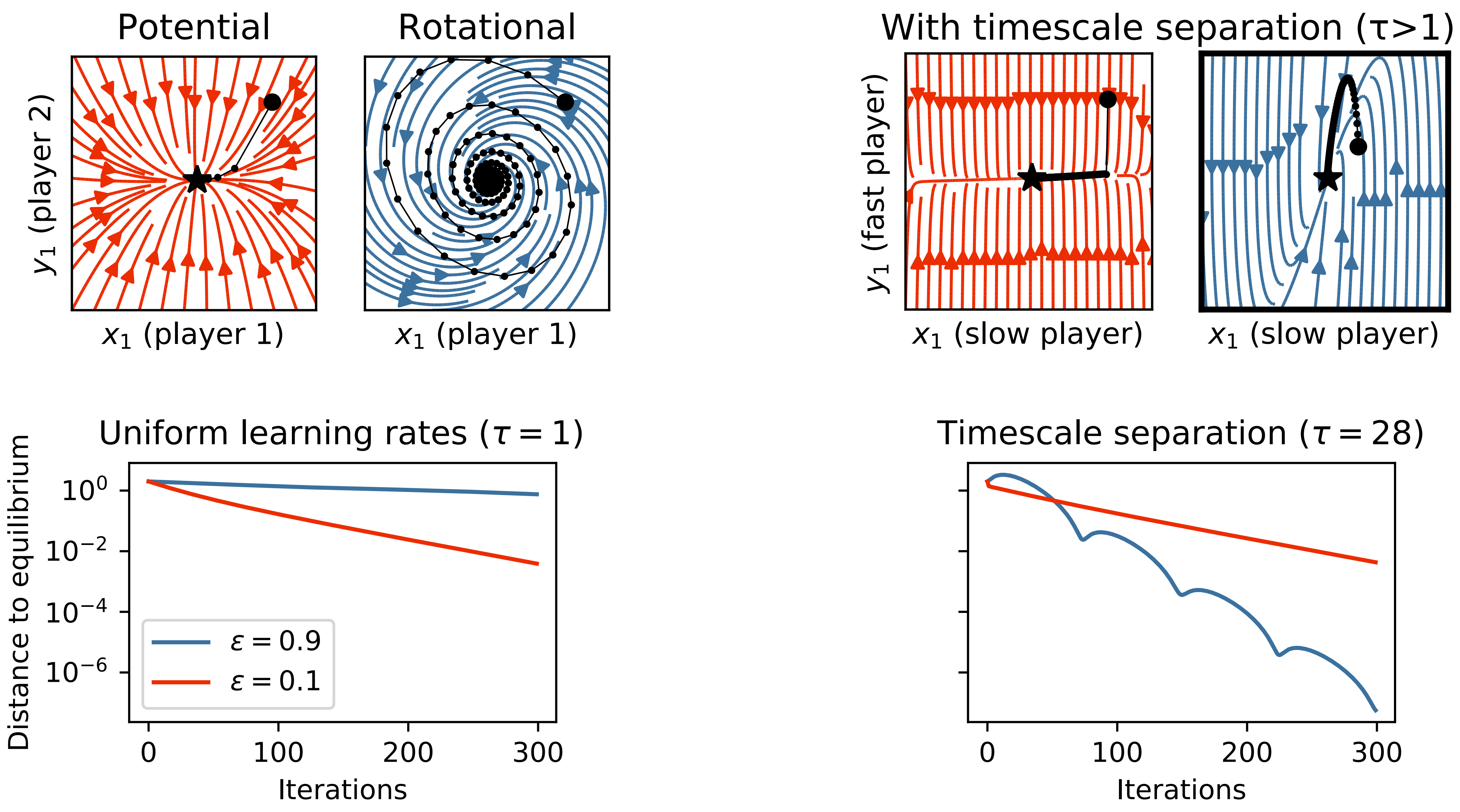
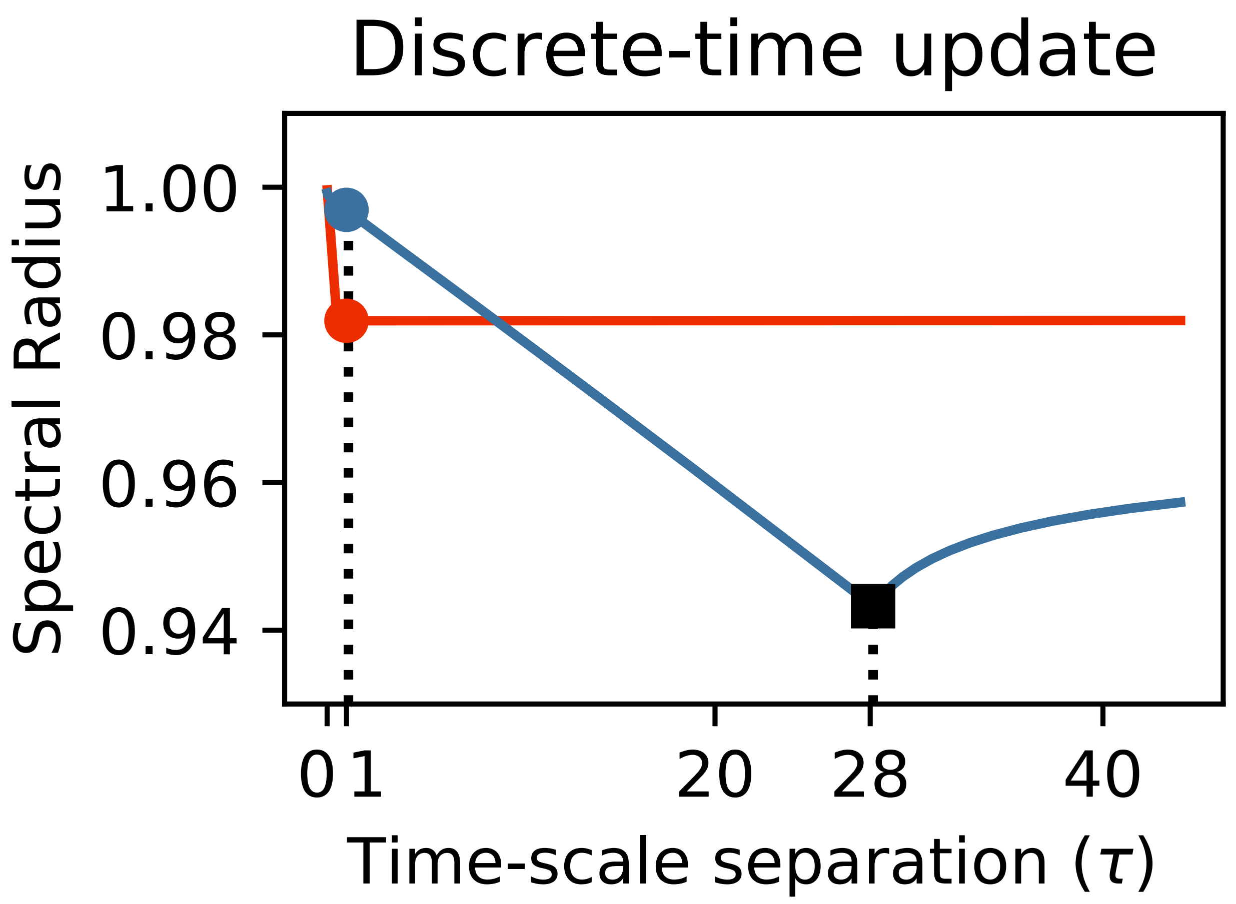


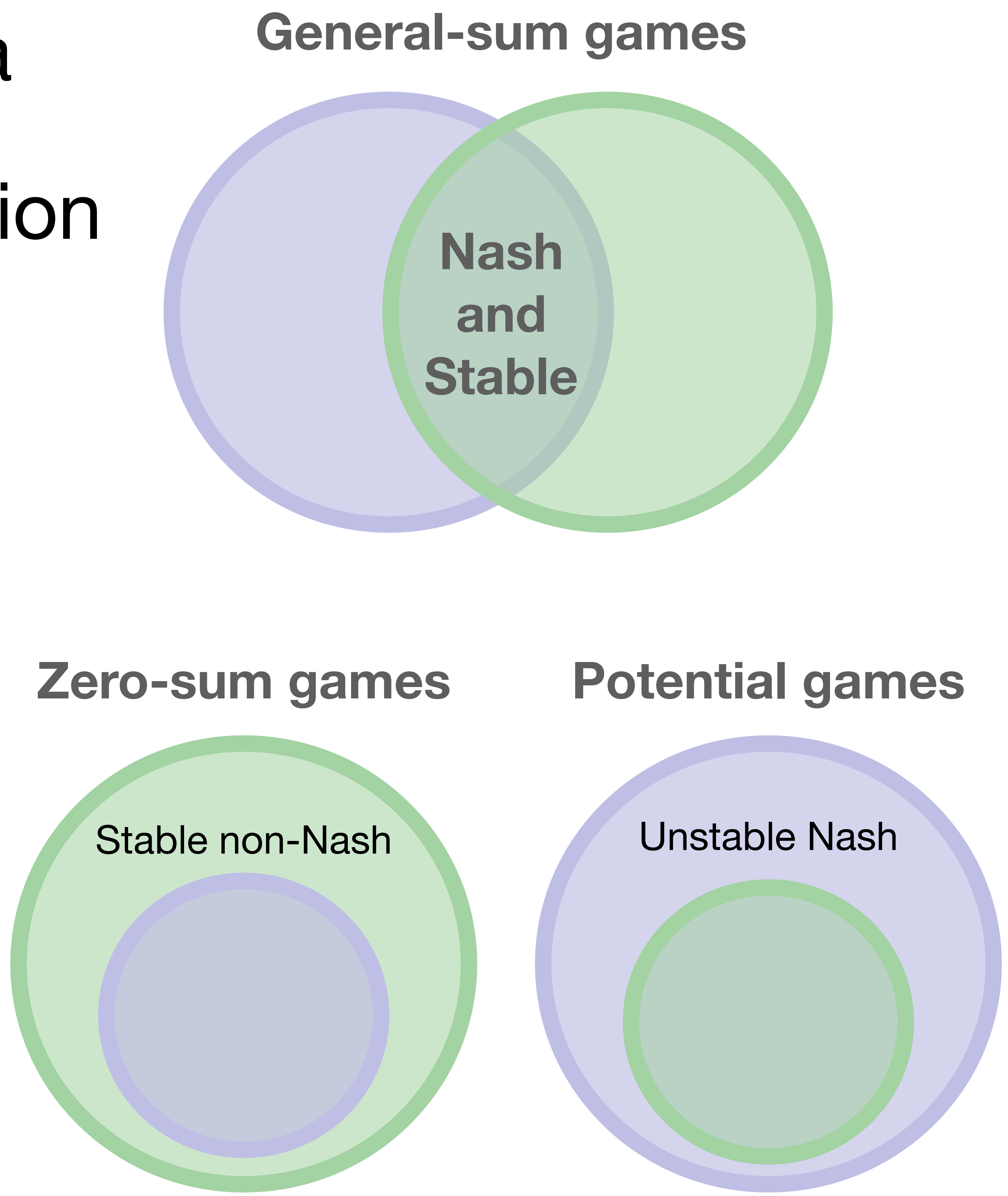
$$\rho(I + \gamma J(x^*))$$



Stability of differential Nash equilibria

Conclusion and future work

- Computing game-theoretically meaningful equilibria
- Optimize convergence rates with timescale separation
- Limited and noisy feedback
- Other solution concepts/boundedly-rational agents



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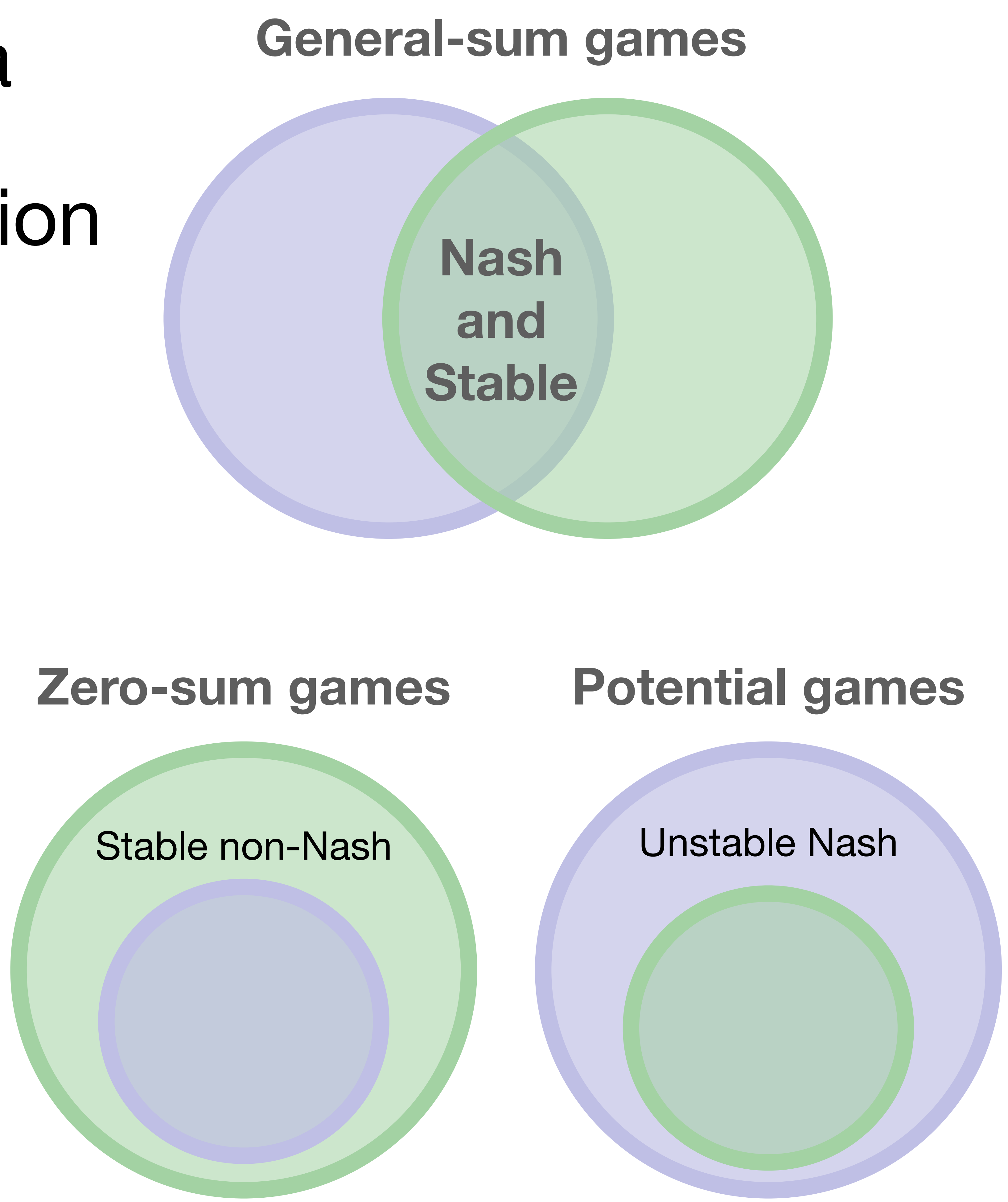


Thanks! Questions?

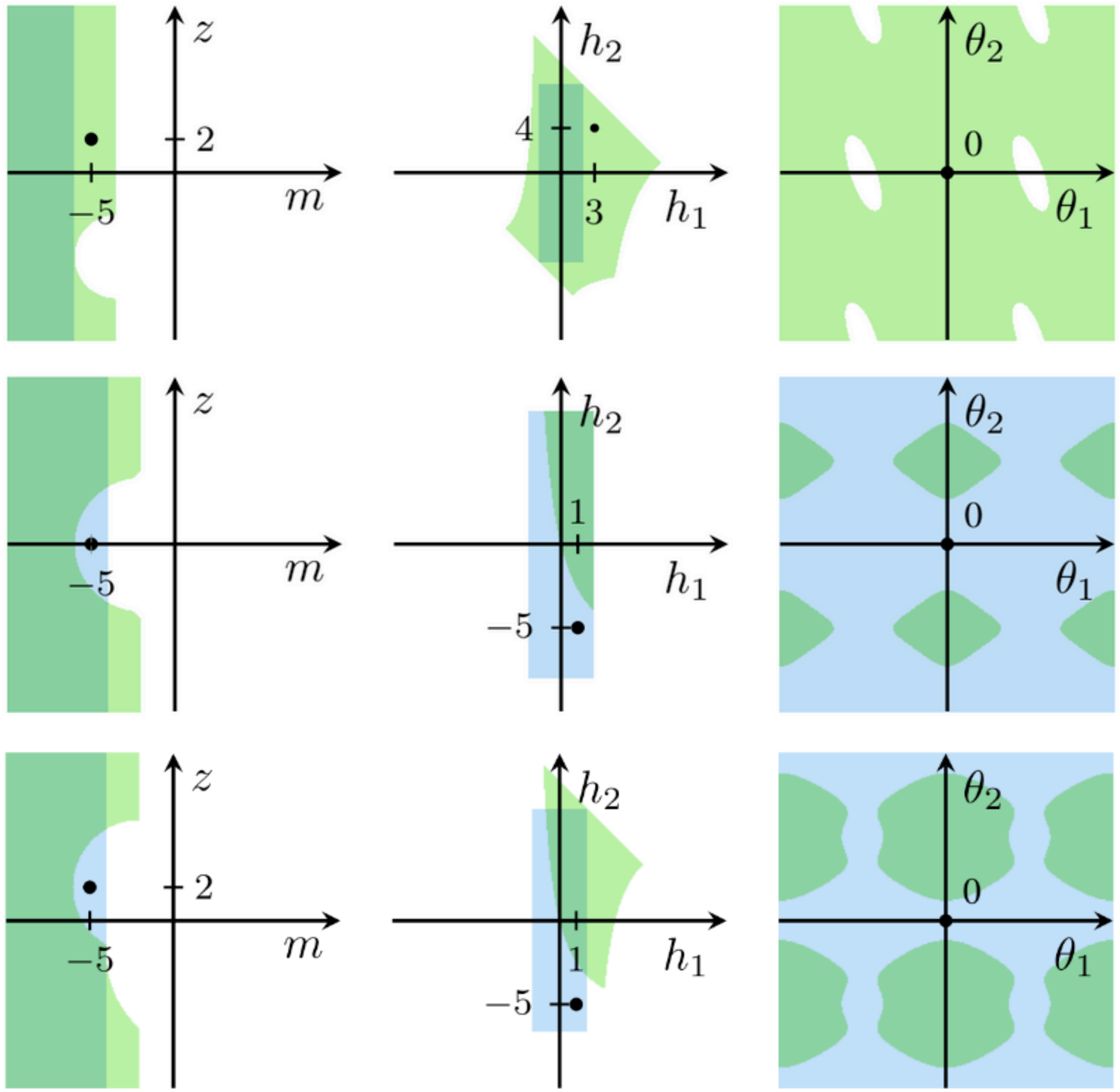
Thanks

Backup slides

Overlapping regions of Nash and stability

Numerical Experiments

- Zero-sum game
- Potential game
- General game



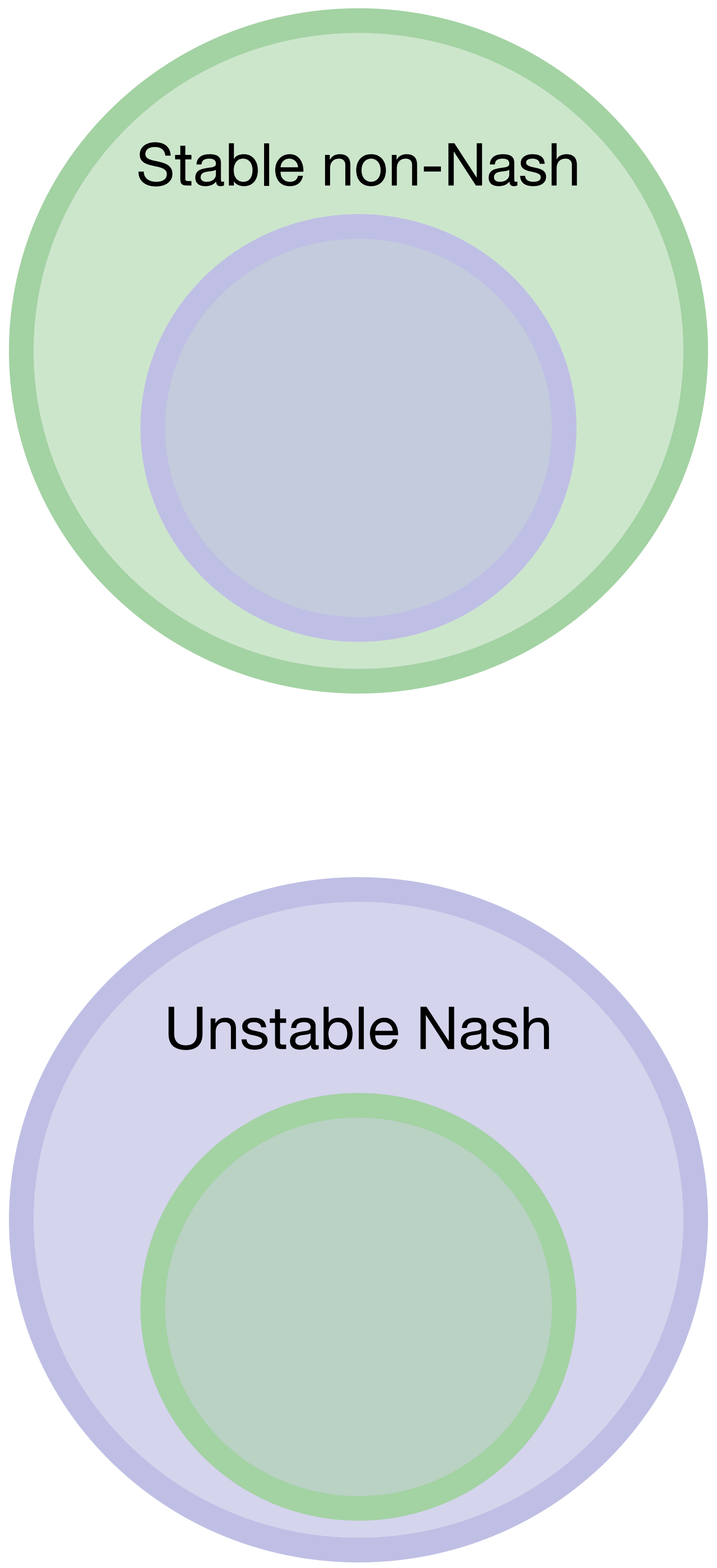
Robustness to variations in learning rates

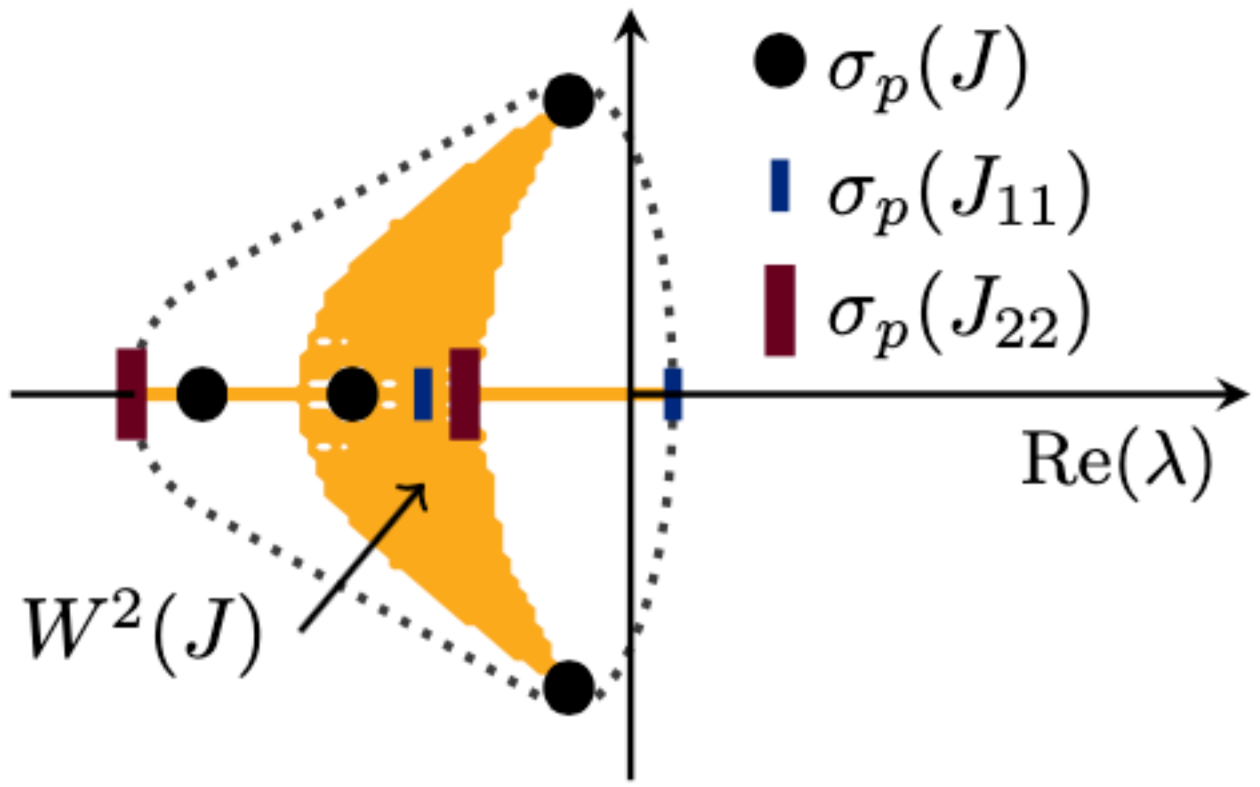
Results: vector actions

- Timescale separation $(\tau > 0)$: $J(x) = \begin{bmatrix} J_{11} & J_{12} \\ \tau J_{21} & \tau J_{22} \end{bmatrix}$

Stability of block matrices

Results: vector actions

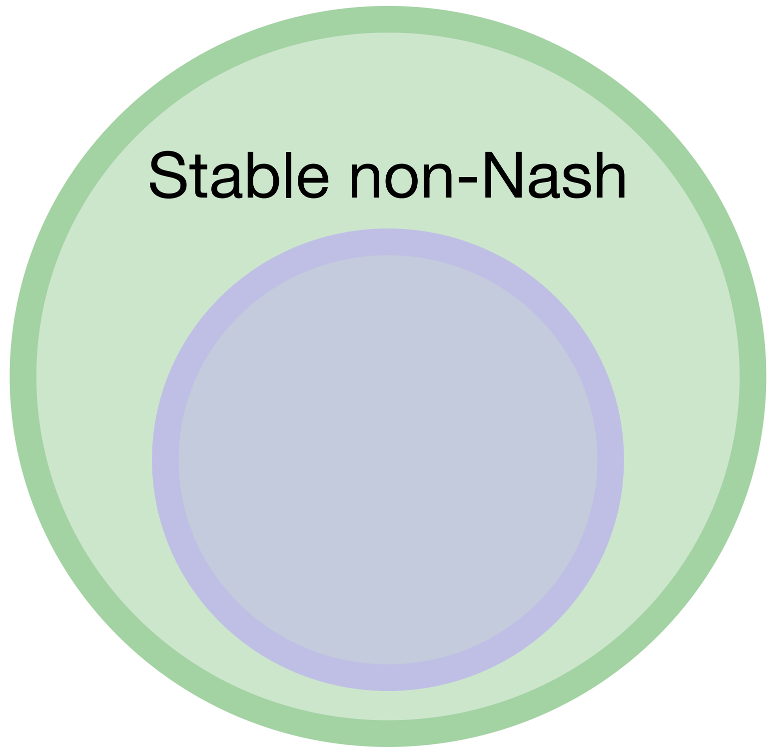


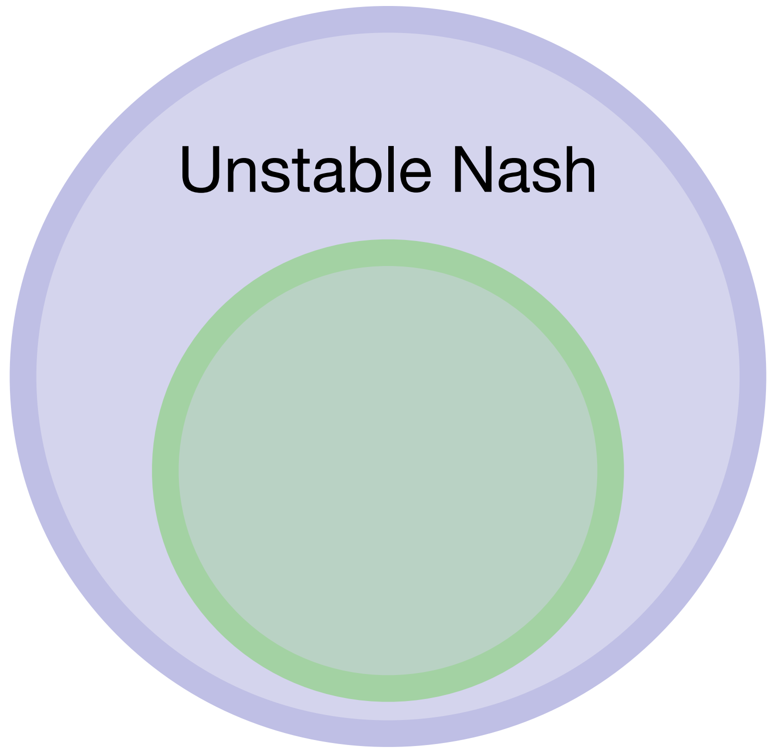


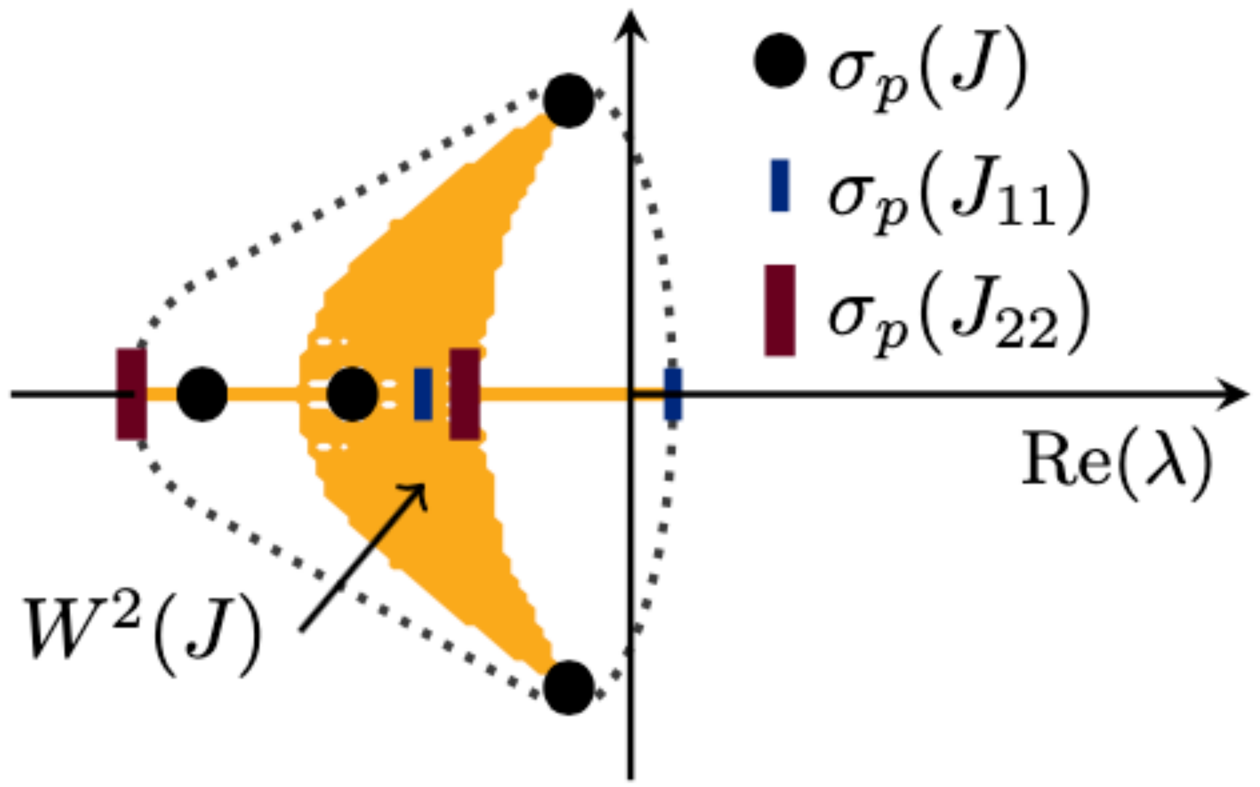
Stability of block matrices

Results: vector actions

- Zero-sum games: All Nash equilibria are stable.







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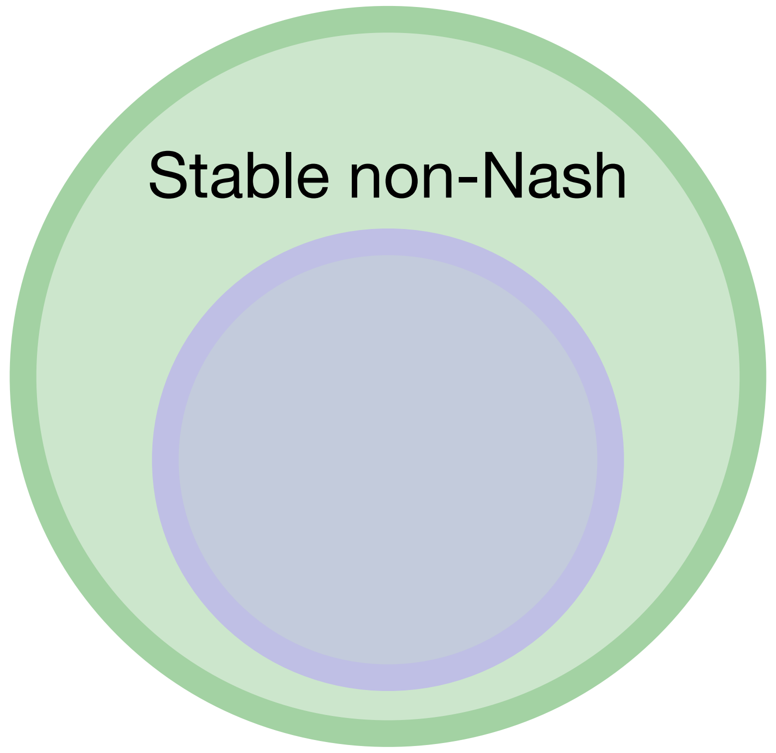
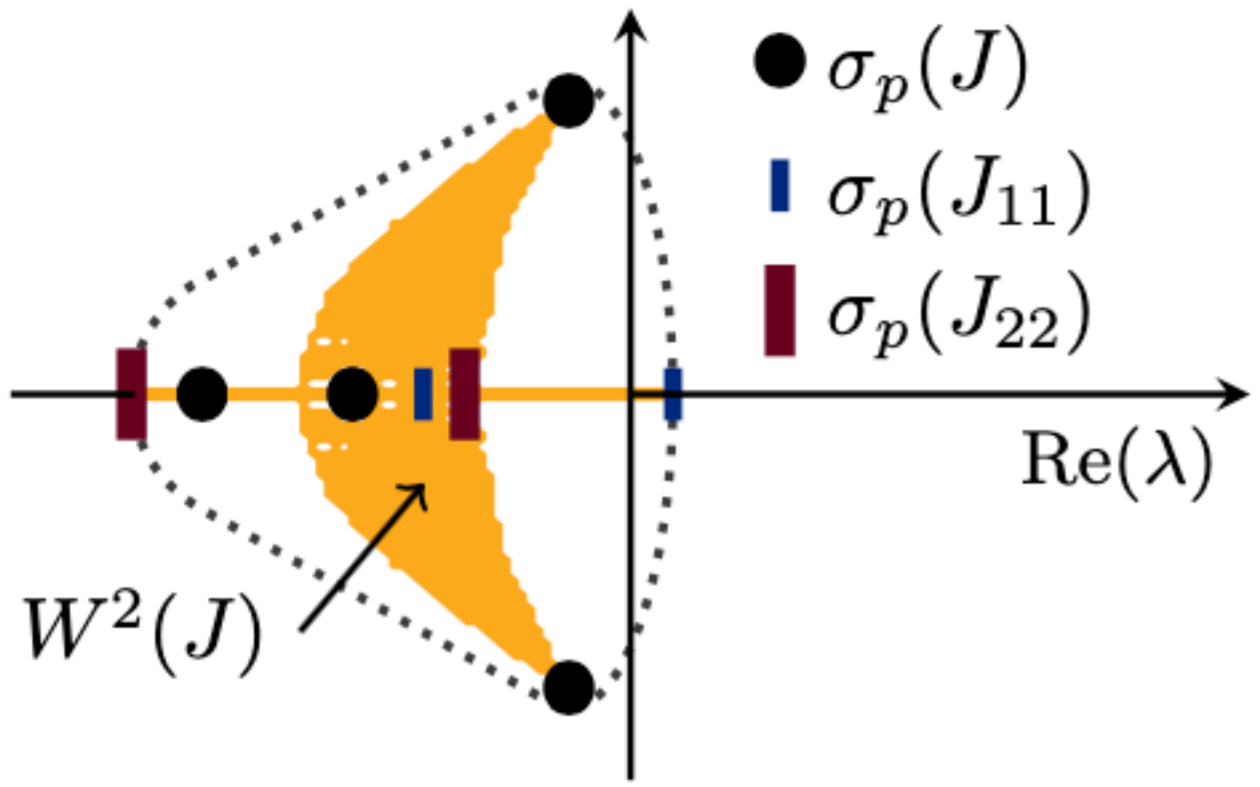
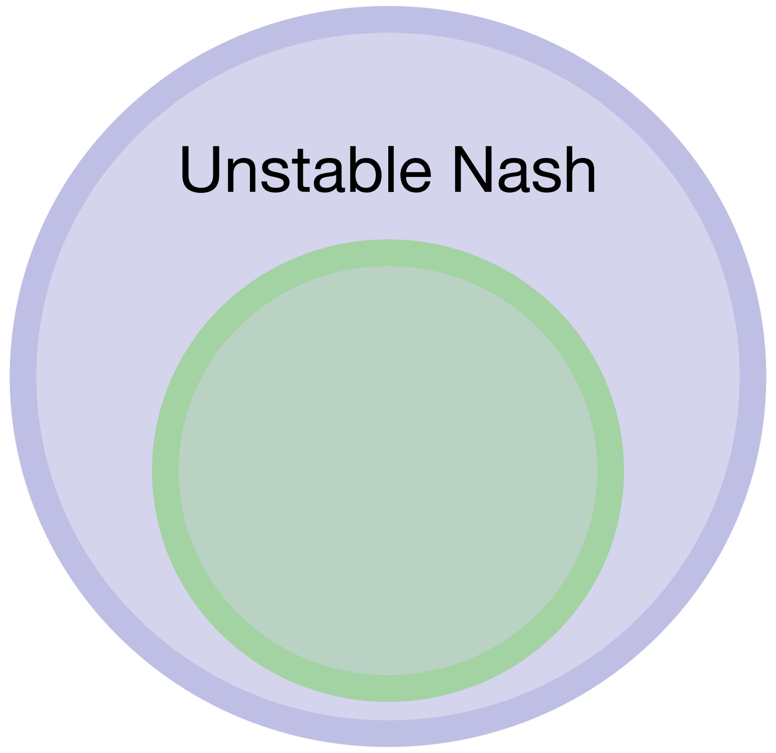
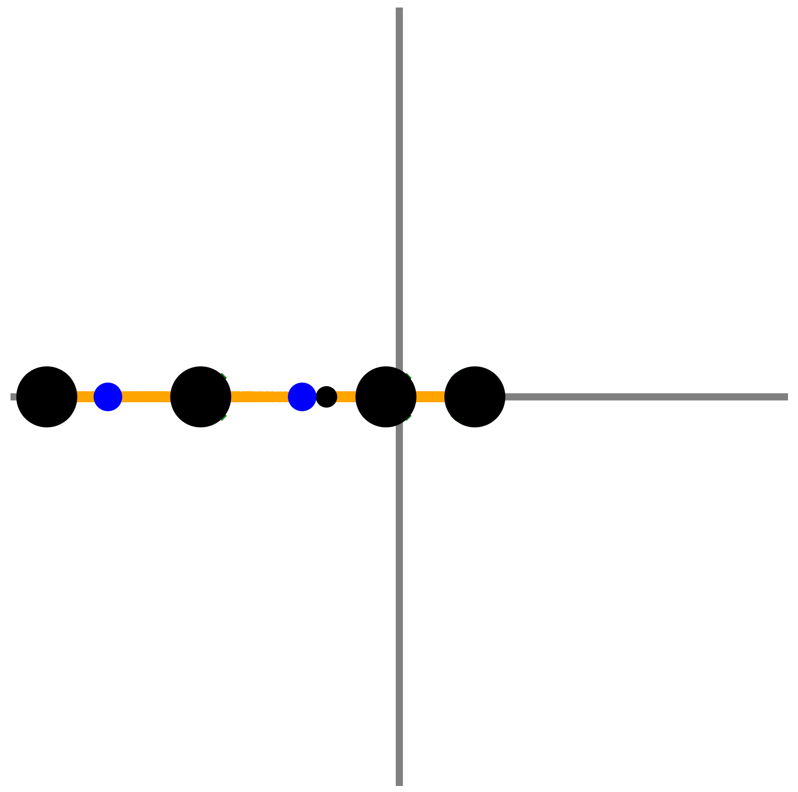




- Potential games: All stable equilibria are Nash.

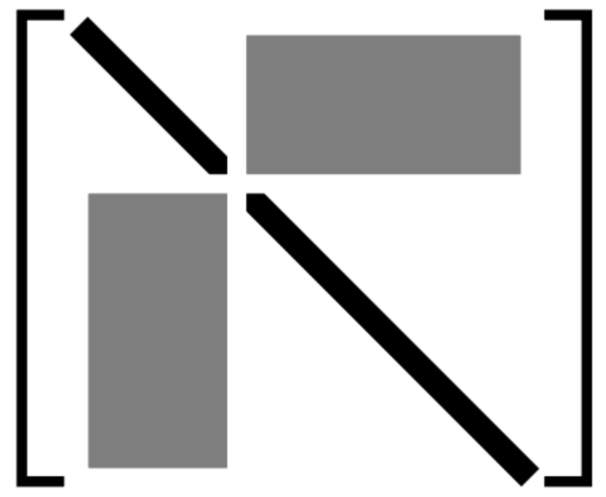




Decomposition of the game Jacobian

Game types

$$J(x) = \rlap{$\overbrace{\phantom{\begin{bmatrix} J_{11} & 0 \\ 0 & J_{22} \end{bmatrix} + \begin{bmatrix} 0 & P \\ P^\top & 0 \end{bmatrix}}}^{\textit{potential}}$}\underbrace{\begin{bmatrix} J_{11} & 0 \\ 0 & J_{22} \end{bmatrix}}_{\textit{individual}} + \underbrace{\begin{bmatrix} 0 & P \\ P^\top & 0 \end{bmatrix} + \overbrace{\begin{bmatrix} 0 & -Z \\ Z^\top & 0 \end{bmatrix}}^{\textit{rotational}}}_{\textit{interaction}} \sim$$



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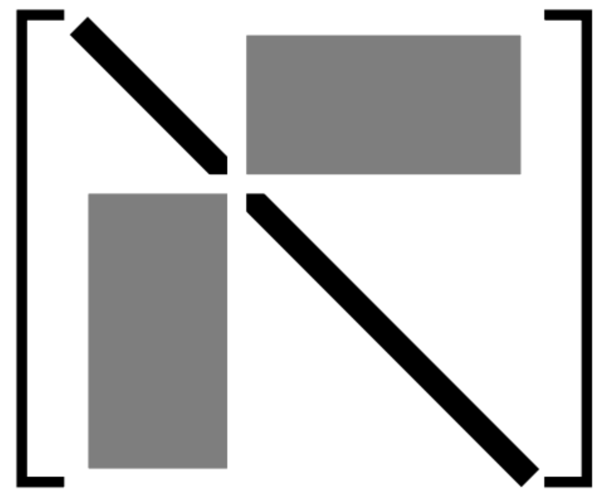

- Potential game $(\exists \phi \mid D_i \phi \equiv D_i f_i, \forall i)$: $\quad J(x) = \begin{bmatrix} J_{11} & P \\ P^\top & J_{22} \end{bmatrix}$

Decomposition of the game Jacobian

Game types

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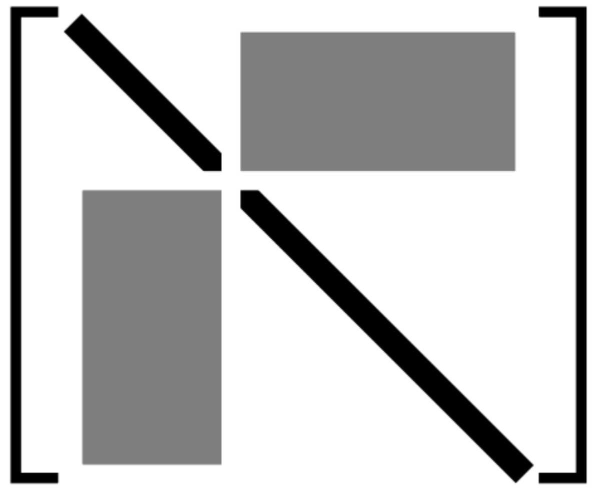

- Potential game $(\exists \phi \mid D_i \phi \equiv D_i f_i,\ \forall i)$: $\quad J(x) = \begin{bmatrix} J_{11} & P \\ P^\top & J_{22} \end{bmatrix}$

- Zero-sum game $(f_1 \equiv -f_2)$: $\quad J(x) = \begin{bmatrix} J_{11} & -Z \\ Z^\top & J_{22} \end{bmatrix}$

Decomposition of the game Jacobian

Game types

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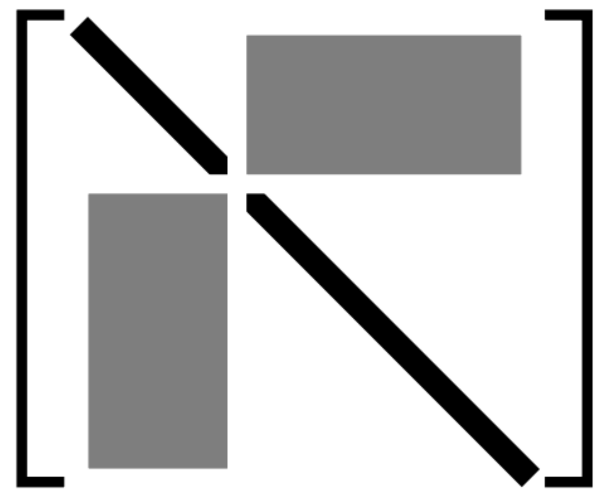

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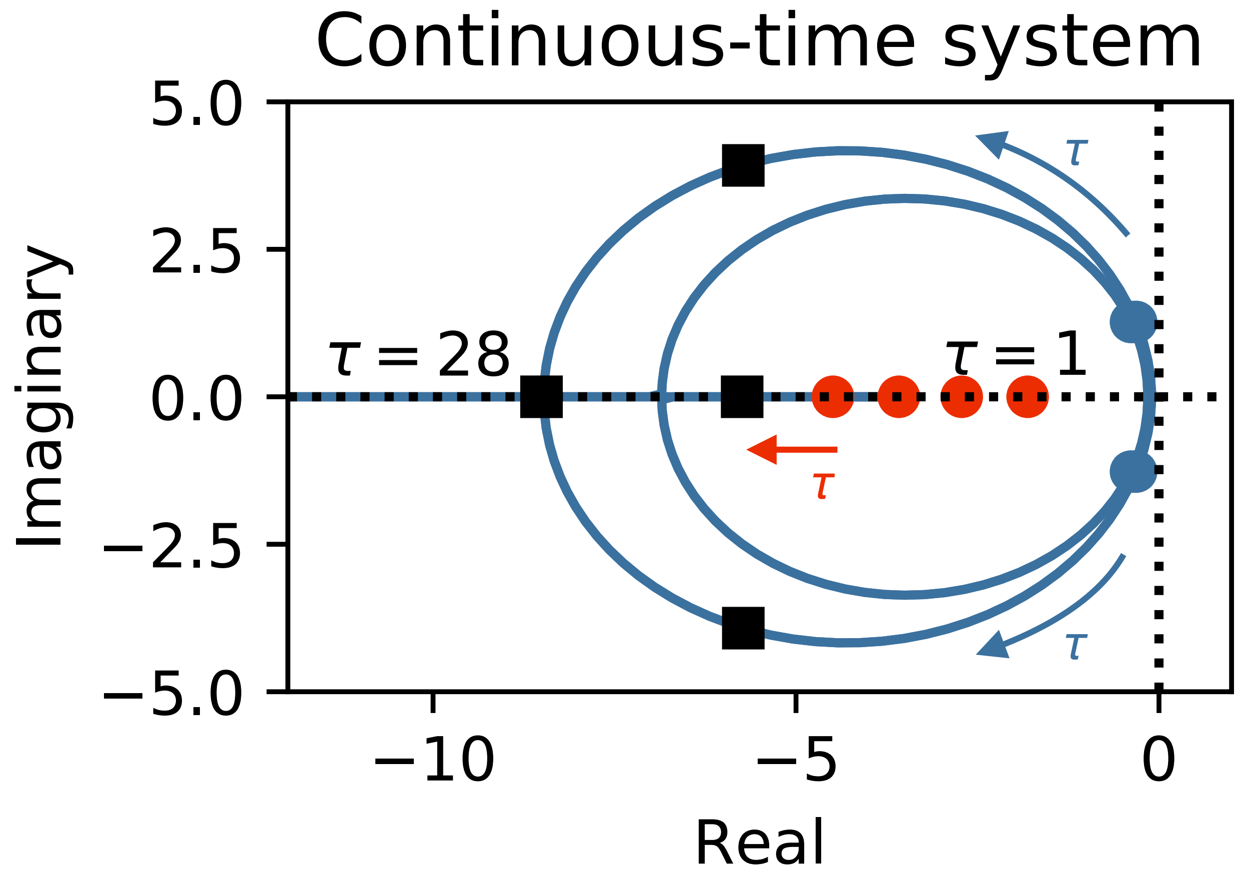

$\sigma(J(x^*))$

$$\underbrace{\sigma\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)}_{\textit{spectrum}} \subset \mathbf{C}_-^\circ \quad \iff \quad \begin{array}{l} \mathsf{Tr}(J) = 2m < 0 \\ \det(J) = m^2 + z^2 - p^2 - h^2 > 0 \end{array}$$