Linear Programming Practice Problems

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Problems

1. Canada supplies 13 hackers per year, Russia supplies 103. The United States requires 85 hackers a year, and Germany requires 52. However, sending hackers across seas costs money. The transport costs are:

From / to	U.S.	Ger.
Canada	5	8
Russia	15	2

   Write down a linear program to meet hacker demand but minimize transport costs.

2. For more examples of concretely formulating LPs from some given information, there are lots of resources online. For example, you can check out: `https://web.sonoma.edu/users/w/wilsonst/courses/math_131/lp/default.html`, or `http://www.durban.gov.za/Documents/City_Government/Maths_Science_Technology_Programme/mathematics-newsletter.pdf`.

3. (Courtesy of Will Evans). Describe linear constraints on the three variables x, y and z so that if x, y are boolean variables (i.e., $x, y \in \{0, 1\}$), then $z = x \vee y$ (logical or).

4. We denote a weighted graph by a triple, $G = (V, E, w)$ where $V = \{v_1, \dots, v_n\}$ is a finite set of vertices, $E \subseteq V \times V$ are the edges and $w : E \to \mathbb{R}^+$ assigns to each edge a positive weight.
   (a) Let $s, t \in V$. Write down the linear program to find the shortest path from s to t in V on a weighted graph.
   (b) Write down the linear program for the minimum spanning tree on a weighted graph.

5. Given a graph $G = (V, E)$, a matching is a subset of the edges $M \subseteq E$ such that no two edges of M are incident on the same vertex. A maximum matching is a matching with at least as many edges as any other matching (i.e., a largest matching). G is bipartite if the vertices V can be partitioned into two disjoint subsets, U, W ($U \cup W = V$) such that $E \subseteq U \times W$ (i.e., all edges are from U to W). Write the maximum matching problem in a bipartite graph as a linear program.

Dual Problems: Some Intuition

There are lots of resources which show you how to dualize a linear program algorithmically. However, let's think for a few seconds about the intuition. Suppose we have the following linear program:

$$\max_{x=(x_1,x_2)} 3x_1 + 2x_2 \quad \text{s.t.} \quad x_1 + x_2/2 \le 1,\ 4x_1 + 5x_2 \le 5,\ x_1, x_2 \ge 0.$$

What's an upper bound on the solution? Well, certainly 5 is. Why? If $4x_1 + 5x_2 \le 5$ then because $3x_1 \le 4x_1$, $2x_2 \le 5x_2$, we have $3x_1 + 2x_2 \le 5$ as well. We could also divide the second constraint through by $3/4$ and get that $3x_1 + 15x_2/4 \le 15/4$, hence $15/4$ is an upper bound. If we divide the second constraint by anything smaller than $3/4$ however, the coefficient of x_1 will become smaller than 3 (the coefficient of x_1 in the objective function), hence it will not be an upper bound. However, we can get an even tighter bound via linear combinations of the first and second constraints. Dividing the second constraint by 2, and then adding the first constraint yields $3x_1 + 3x_2 \le 7/2$, so we get an upper bound of $7/2$.

Okay, let's try and formalize what's going on here. In all cases, we multiply the two constraints by y_1 and y_2:

$$\begin{aligned} y_1(x_1 + x_2/2) &\le y_1, \\ y_2(4x_1 + 5x_2) &\le 5y_2, \end{aligned}$$

and sum the equations. (In the first two attempts discussed, $y_1 = 0$). So that the inequalities don't flip, we need $y_1, y_2 \ge 0$. We also need to ensure that y_1 and y_2 are large enough so that the coefficients of the result are larger than the coefficients of the objective function. That is, we need that $y_1 + 4y_2 \ge 3$ and $y_1/2 + 5y_2 \ge 2$. The bound is then $y_1 + 5y_2$. We want the smallest such upper bound, so we want to minimize that sum. That is, we want to solve

$$\min_{y=(y_1,y_2)} y_1 + 5y_2 \quad \text{s.t.} \quad y_1 + 4y_2 \ge 3,\ y_1/2 + 5y_2 \ge 2,\ y_1, y_2 \ge 0.$$

That's the dual linear program! If we started with the minimization version, we could repeat much the same process and obtain the maximization version. This can be generalized to arbitrary LPs. The LP

$$\max_{\mathbf{x}\in\mathbb{R}^n} \sum_{k=1}^{n} c_k x_k \quad \text{s.t.} \quad \forall j \in [m] : \sum_{k=1}^{n} a_{jk} x_k \le b_j,\ \mathbf{x} \ge 0,$$

has dual

$$\min_{\mathbf{y}\in\mathbb{R}^m} \sum_{k=1}^{m} b_k y_k \quad \text{s.t.} \quad \forall j \in [n] : \sum_{k=1}^{m} a_{kj} y_k \ge c_j,\ \mathbf{y} \ge 0,$$

and vice versa. More concisely:

$$\max_{\mathbf{x}} \mathbf{c}^T\mathbf{x} \quad \text{s.t.} A\mathbf{x} \le \mathbf{b},\ \mathbf{x} \ge 0,$$

has dual

$$\min_{\mathbf{y}} \mathbf{b}^T\mathbf{y} \quad \text{s.t.} \quad A^T\mathbf{y} \ge \mathbf{c},\ \mathbf{y} \ge 0.$$

Additional Problems

These problems have less bearing on the course material, and are somewhat more advanced.

1. What's the dual of the shortest path LP from problem 2?
2. What's the dual of the MST LP from problem 2?
3. What down the linear program to maximize flow in a flow network (this should be similar to the shortest path LP).
4. (Courtesy of Will Evans via Algorithms by Dasgupta, Papadimitriou and Vazirani). Prove that finding the shortest path from s to t in a directed, weighted graph $G = (V, E, w)$ is equivalent to finding a flow f that minimizes $\sum_{e\in E} w(e)f(e)$ such that $size(f) = 1$.
5. Write down the dual from problem 5? What do the dual variables and constraints represent? What's this new problem asking?