Submodular Stochastic Probing with Prices

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Query and Commit

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- Stochastic Matching and Packing
- The Secretary Problem
- Buying socks, airplane tickets, booking a hotel room.

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  - Solution, S, must lie in $\mathcal{F}_{in}$.
- Maximize objective function f subject to these constraints.

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Objective:

$$\begin{aligned} \max_{P} \quad & \mathbb{E}_{\mathcal{A}}[f(P \cap \mathcal{A})] \\ \text{s.t.} \quad & P \in \mathcal{F}_{out} \\ & S = P \cap \mathcal{A} \in \mathcal{F}_{in}. \end{aligned}$$

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$\mathcal{F}_{in}$ and $\mathcal{F}_{out}$ are *downward-closed*.

Stochastic Probing

Objective set function f is either linear or *submodular*:

$$f(U \cup \{e\}) - f(U) \geq f(V \cup \{e\}) - f(V) \quad \text{whenever} \quad U \subset V.$$

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Monotone if $f(V) \geq f(U)$ whenever $U \subset V$.

Stochastic Probing (SP)

Why should you care?

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Useful for modelling:

- Kidney Exchange;
- Online Dating;
- Posted Price Mechanisms;
- etc.

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Associate with each $e \in N$ a *price* $\Delta_e \in \mathbb{R}^+$.

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New objective:

$$\begin{aligned} \max_{P} \quad & \mathbb{E}_{\mathcal{A}}[f(P \cap \mathcal{A})] - \sum_{e \in P} \Delta_e \\ \text{s.t.} \quad & P \in \mathcal{F}_{out} \\ & S = P \cap \mathcal{A} \in \mathcal{F}_{in}. \end{aligned}$$

PAUSE

Contention Resolution Schemes (CRSs)

First introduced by Chekuri, Vondrák, and Zenklusen in 2011 for submodular maximization.

$\mathcal{P}$: Polytope relaxation of the constraints of an optimization problem (i.e., fractional solutions).

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Online CRSs were introduced by Feldman, Svensson, and Zenklusen in 2016.

Back to main show.

Results

Maximum facility Location is inapproximable to within a constant factor. (Feige et al. 2009)

Therefore unlikely that SPP has a constant factor approximation.

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Therefore unlikely that SPP has a constant factor approximation.

Instead, we look for bi-criteria approximations:

(α, β) approximation

If Opt is optimal set of probed elements, and Alg satisfies $\mathbb{E}[\mathsf{Alg}] \geq \alpha \mathbb{E}[f(\mathsf{Opt} \cap \mathcal{A})] - \beta \sum_{e \in \mathsf{Opt}} \Delta_e$, then we say Alg is an (α, β) approximation for SPP.

Results

Nomenclature:

- Offline SPP — we choose how we query the elements.
- Online SPP — order is given to us, either in (uniformly) *random order* or in *adversarial order*.

  Not to be confused with Offline and Online CRSs! (I know ...)

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- Linear SPP: Objective function f is linear.
- Submodular SPP: Objective function f is submodular. (Note: Cost function is always linear.)

Results

Theorem 1

If there exists an α-approximation to linear SP then there exists an α-approximation to linear SPP.

Theorem 2

Suppose $\mathcal{P}(\mathcal{F}_{in})$ and $\mathcal{P}(\mathcal{F}_{out})$ admit (b, c_{in}) and (b, c_{out}) CRSs and let $\gamma = \max\{c_{out} + c_{in} - 1, c_{out} \cdot c_{in}\}$. There exists a $(\gamma\alpha(b), b)$-approximation to adversarial, submodular SPP where $\alpha(b) = 1 - e^{-b}$ if the objective function is monotone, and be^{-b} otherwise.

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Theorem 2 applies to both offline and online CRSs.

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- Synthesis of different techniques into two algorithms: one for offline CRSs, one for online CRSs.

SP for Matroids

Suppose $\mathcal{F}_{in}$ and $\mathcal{F}_{out}$ are intersections of k and ℓ matroids.

f is submodular.

$\checkmark =$ best or equal approximation ratio.

		This work	Previous
Monotone	Offline		✓(ASW '16)
	Random Order	✓	
	Adversarial	✓	
Non-Monotone	Offline	✓	
	Random Order	✓	
	Adversarial	✓	

Invaluable Prior Work

We owe much to

- A recent technique of Sviridenko, Vondrák, and Ward (2014) to maximize the sum of a concave and linear function, which can take negative values (!).
- The Measured Continuous Greedy algorithm of Feldman, Noar, and Schwartz (2011) which applies to non-monotone submodular functions.