NP-Completeness Practice Problems

Please report any typos to Ben Chugg at `ben.chugg@alumni.ubc.ca`

1. Let $G = (V, E)$ be an undirected graph. A subset of the vertices $C \subseteq V$ is called a clique if all $u, v \in C$ are connected by an edge $(u, v) \in E$. The CLIQUE problem is the following: Given G and an integer k, does G contain a clique of size at least k? A subset of the vertices $I \subseteq V$ is called an independent set if no two vertices in I are connected by an edge in E. The INDEPENDENT-SET problem is as follows: Given G and an integer k, does G contain an independent set of size at least k?

   (a) Prove that both CLIQUE and INDEPENDENT-SET are NP-Complete.

   (b) If CLIQUE is NP-Complete show that INDEPENDENT SET is NP-Complete.

   (c) If INDEPENDENT-SET is NP-Complete show that CLIQUE is NP-Complete.

2. (Courtesy of Will Evans via Algorithms by Dasgupta, Papadimitriou, Vazirani). Consider the CLIQUE problem restricted to graphs in which every vertex has degree at most 3. Call this problem CLIQUE-3.

   (a) Prove that CLIQUE-3 is in NP.

   (b) What is wrong with the following proof of NP-completeness for CLIQUE-3? We know that the CLIQUE problem in general graphs is NP-complete, so it is enough to present a reduction from CLIQUE-3 to CLIQUE. Given a graph G with vertices of degree ≤ 3, and a parameter k, the reduction leaves the graph and the parameter unchanged. Clearly this is a possible input for the CLIQUE problem. Furthermore, the answer to both problems is identical. This proves the correctness of the reduction and, therefore, the NP-completeness of CLIQUE-3.

   (c) It is true that the VERTEX COVER problem remains NP-complete even when restricted to graphs in which every vertex has degree at most 3. Call this problem VC-3. What is wrong with the following proof of NP-completeness for CLIQUE-3?

   We present a reduction from VC-3 to CLIQUE-3. Given a graph $G = (V, E)$ with node degrees bounded by 3, and a parameter b, we create an instance of CLIQUE-3 by leaving the graph unchanged and switching the parameter to $|V| - b$. Now, a subset $C \subseteq V$ is a vertex cover in G if and only if the complementary set $V - C$ is a clique in G. Therefore G has a vertex cover of size $\leq b$ if and only if it has a clique of size $\geq |V| - b$. This proves the correctness of the reduction and, consequently, the NP-completeness of CLIQUE-3.

   (d) Describe an $O(|V|^4)$-time algorithm for CLIQUE-3.

3. A vertex cover of a graph $G = (V, E)$ is a subset of the vertices $U \subseteq V$ such that each edge is incident to at least one vertex U. Clearly, therefore, taking $U = V$ yields a vertex cover. The VERTEX-COVER problem asks, given a graph G and an integer k, if G has a vertex cover of size at most k.

   (a) Prove that VERTEX-COVER is NP-Complete. (Hint: Think about CLIQUE).

   (b) A set cover of a set X is a collection of subsets of X, $\{S_1, S_2, \ldots, S_m\}$ such that $\bigcup_{n=1}^{m} S_m = X$ (i.e., their union is X). The SET-COVER decision problem is as follows: Given a set X, a collection of subsets of X, $\mathcal{S}$, and an integer k, does there exist a subset $\mathcal{C} \subseteq \mathcal{S}$ which is a set cover of X with size at most k?

4. (Courtesy of Will Evans via Jeff Erickson's Approximation Algorithms notes) The chromatic number $\chi(G)$ of a graph G is the minimum number of colors required to color the vertices of the graph, so that every edge has endpoints with different colors. Computing the chromatic number exactly is NP-hard. Prove that the following problem is also NP-hard: Given an n-vertex graph G, return any integer between $\chi(G)$ and $\chi(G) + 573$.

5. (Courtesy of Nick Harvey). Let A, B be problems (formally, languages) such that $A \leq_P B$. Which of the following are true?

   (a) If Y is in NP then X is in NP.
   (b) If X is in NP then Y is in NP.
   (c) If Y is in P then X is in P.
   (d) If X is in P then Y is in P.
   (e) If Y is NP-Complete then so is X.
   (f) If X is NP-Complete then so is Y.
   (g) If Y is NP-Complete and X is in NP then X is NP-Complete.
   (h) If X is NP-Complete and Y is in NP then Y is NP-Complete.

6. (Courtesy of Will Evans via Jeff Erickson's online course notes). A boolean formula is in disjunctive normal form (or DNF) if it consists of a disjunction (Or) of several terms, each of which is the conjunction (And) of one or more literals. For example, the formula

$$(x \wedge y \wedge z) \vee (y \wedge z) \vee (x \wedge y \wedge z),$$

   is in disjunctive normal form. DNF-SAT asks, given a boolean formula in disjunctive normal form, whether that formula is satisfiable.

   (a) Describe a polynomial-time algorithm to solve DNF-SAT.
   (b) What is the error in the following argument that P=NP?

   Suppose we are given a boolean formula in conjunctive normal form with at most three literals per clause, and we want to know if it is satisfiable. We can use the distributive law to construct an equivalent formula in disjunctive normal form. For example,

$$(x \vee y \vee z) \wedge (x \vee y) \Leftrightarrow (x \wedge y) \vee (y \wedge x) \vee (z \wedge x) \vee (z \wedge y).$$

   Now, we can use the algorithm from part (a) to determine, in polynomial time, whether the resulting DNF formula is satisfiable. We have just solved 3SAT in polynomial time. Since 3SAT is NP-hard, we must conclude that P=NP!

7. Is P=NP? If yes, provide a proof. If not, provide a counterexample.