Fachhochschule Südwestfalen
University of Applied Sciences

Statistical Modeling

CH.4 - Qualitative Variables

SS 2021 || Prof. Dr. Buchwitz

Wir geben Impulse

Outline

1. Organizational Information
2. Qualitative Variables as Predictors

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Outline

Introduction

- Qualitative or categorical variables (such as gender, marital status, etc) are useful predictors and are usually called **indicator** or **dummy variables**.
- Those variables usually only take two values, 0 and 1, which signify that the observation belongs to one of two possible categories.
- The numerical values of indicator variables**do not reflect quantitative ordering**.
- **Example Variable:** Gender, coded as 1 for *female* and 0 for *male*.
- Indicator variables can also be used in a regression equation to distinguish between three or more groups.
- The response variable is stil a quantiative continuous in all discussed cases.

Example: Salary Survey Data

```
P130
```

```
##        S  X E M
## 1  13876  1 1 1
## 2  11608  1 3 0
## 3  18701  1 3 1
## 4  11283  1 2 0
## 5  11767  1 3 0
## 6  20872  2 2 1
## 7  11772  2 2 0
## 8  10535  2 1 0
## 9  12195  2 3 0
## 10 12313  3 2 0
## 11 14975  3 1 1
## 12 21371  3 2 1
## 13 19800  3 3 1
## 14 11417  4 1 0
## 15 20263  4 3 1
## 16 13231  4 3 0
## 17 12884  4 2 0
## 18 13245  5 2 0
## 19 13677  5 3 0
## 20 15965  5 1 1
## 21 12336  6 1 0
## 22 21352  6 3 1
## 23 13839  6 2 0
## 24 22884  6 2 1
## 25 16978  7 1 1
## 26 14803  8 2 0
```

Your turn

Salary survey of computer professionals with objective to identify and quantify variables that determine salary differentials.

S Salary (Response)

X Experience, measured in years

E Education, 1 (High School/HS), 2 (Bachelor/BS), 3 (Advanced Degree/AD)

M Management 1 (is Manager), 0 (no Management Responsibility)

Example: Salary Survey Data

- **Experience:** We assume linearity, which means that each additional year is worth a fixed salary increment.
- **Education:** Can be used in a linear or categorial form.
  - Using the the variable in its raw form would assume that each step up in education is worth a fixed increment in salary. This may be too restrictive.
  - Using education as categorical variable can be done by defining **two indicator variables**. This allows to pick up the effect of education wether it is linear or not.
- **Management:** Is also an indicator variable, that allows to distinguish between management (1) an regular staff positions (0).

Indicator Variables

When using indicator variables to represent a set of categroies, the number of these variables required is **one less than the number of categories**. For *education* we can create two indicators variables:

$$E_{i1} = \begin{cases} 1, \text{if the i-th person is in the HS category} \\ 0, \text{otherwise.} \end{cases}$$

$$E_{i2} = \begin{cases} 1, & \text{if the i-th person is in the BS category} \\ 0, & \text{otherwise.} \end{cases}$$

These two variables allow representing the three groups (HS, BS, AD).

$$\text{HS: } E_1 = 1, E_2 = 0, \text{ BS: } E_1 = 0, E_2 = 1, \text{ AD: } E_1 = 0, E_2 = 0$$

Indicator Variables

- The regression equation from the Salary Survey Data is:

$$S = \beta_0 + \beta_1 X + \gamma_1 E_1 + \gamma_2 E_2 + \delta_1 M + \epsilon$$

Indicator Variables

- The regression equation from the Salary Survey Data is:

$$S = \beta_0 + \beta_1 X + \gamma_1 E_1 + \gamma_2 E_2 + \delta_1 M + \epsilon$$

- There is a different valid regression equation for each of the six (three education and two managmeent) categories.

Category	E	M	Regression Equation
1	1	0	$S = (\beta_0 + \gamma_1) + \beta_1 X + \epsilon$
2	1	1	$S = (\beta_0 + \gamma_1 + \delta_1) + \beta_1 X + \epsilon$
3	2	0	$S = (\beta_0 + \gamma_2) + \beta_1 X + \epsilon$
4	2	1	$S = (\beta_0 + \gamma_2 + \delta_1) + \beta_1 X + \epsilon$
5	3	0	$S = \beta_0 + \beta_1 X + \epsilon$
6	3	1	$S = (\beta_0 + \delta_1) + \beta_1 X + \epsilon$

Indicator Variables

```r
d <- P130
d$E1 <- as.numeric(d$E == 1)
d$E2 <- as.numeric(d$E == 2)
mod <- lm(S ~ 1 + X + E1 + E2 + M, data=d)
summary(mod)
```

```
##
## Call:
## lm(formula = S ~ 1 + X + E1 + E2 + M, data = d)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -1884.60  -653.60    22.23   844.85  1716.47
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) 11031.81     383.22  28.787  < 2e-16 ***
## X             546.18      30.52  17.896  < 2e-16 ***
## E1          -2996.21     411.75  -7.277 6.72e-09 ***
## E2            147.82     387.66   0.381    0.705
## M            6883.53     313.92  21.928  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1027 on 41 degrees of freedom
## Multiple R-squared:  0.9568, Adjusted R-squared:  0.9525
## F-statistic: 226.8 on 4 and 41 DF,  p-value: < 2.2e-16
```

Your turn

Interpret the regression coefficients. Assume that the residual patterns are satisfactory.

Table 3

	Dependent variable:	
	S	
	(1)	(2)
X	546.184*** (30.519)	570.087*** (38.559)
E1	−2,996.210*** (411.753)	
E2	147.825 (387.659)	
E		1,578.750*** (262.322)
M	6,883.531*** (313.919)	6,688.130*** (398.276)
Constant	11,031.810*** (383.217)	6,963.478*** (665.695)
Observations	46	46
R^2	0.957	0.928
Adjusted R^2	0.953	0.923
Residual Std. Error	1,027.437 (df = 41)	1,312.789 (df = 42)
F Statistic	226.836*** (df = 4; 41)	179.627*** (df = 3; 42)
Note:		*p<0.1; **p<0.05; ***p<0.01

Before we continue we check the residuals

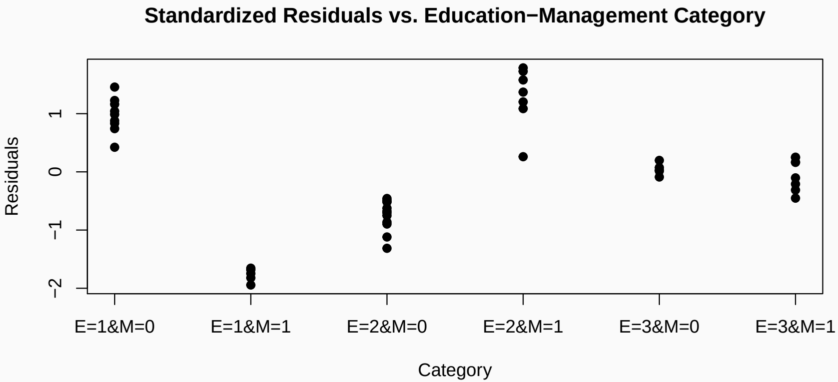
1) Residuals vs. Years of Experience
2) Residuals vs. Categories from Dummys

Regression Diagnostics

```
plot(x = d$X, y = rstandard(mod), pch=19,
     ylab="Residuals", xlab = "X",
     main = "Standardized Residuals vs. Years of Experience (X)")
```

Regression Diagnostics

```r
d$cat <- factor((paste0("E=",d$E,"&M=",d$M)))
plot(x = as.numeric(d$cat), y = rstandard(mod), pch=19, xaxt="n",
     ylab="Residuals", xlab = "Category",
     main = "Standardized Residuals vs. Education-Management Category")
axis(1,at=1:6,labels=levels(d$cat))
```



What is wrong with the residuals:

- Depending on the category the residuals are almost entirely positive or negative.
- The **pattern of the residuals is highly moderated by the associated group** (education-management category). This makes it clear that thath the combinations of education and management have not been tretaed sufficiently in the model.
- The residual plots provide evidence that the effects of education and management status on salary determination are **not additive**.

The multiplicative pattern needs to be embedded in the model!

Interaction Effects

- Interaction effects are *multiplicative* effects that allow capturing nondditive effects in variables.
- Interaction variables are products of existing indicator variables.
- Using the Salary Survey Data this can be achieved by creating the two interaction effects $(E_1 \cdot M)$ and $(E_2 \cdot M)$ and **adding** them to the model.
- The interaction effects **do not replace** the indicator variables.

Interaction Effects

```r
mod <- lm(S ~ 1 + X + E1 + E2 + M + E1*M + E2*M, data=d)
summary(mod)
```

Your turn

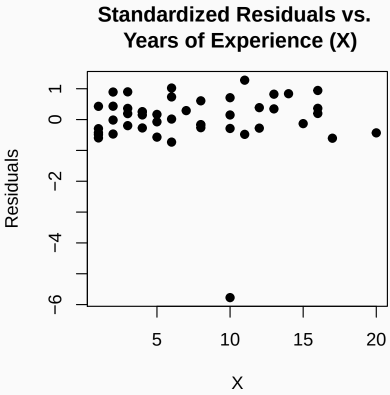
Is that model sufficient?

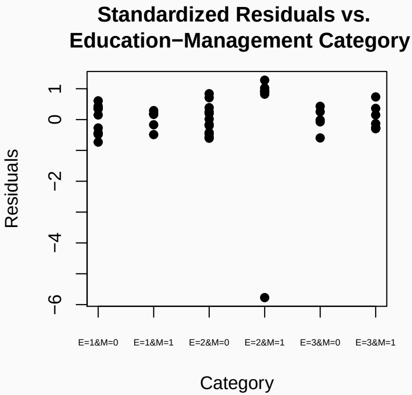
```
##
## Call:
## lm(formula = S ~ 1 + X + E1 + E2 + M + E1 * M + E2 * M, data = d)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -928.13  -46.21   24.33   65.88  204.89
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) 11203.434     79.065 141.698  < 2e-16 ***
## X             496.987      5.566  89.283  < 2e-16 ***
## E1          -1730.748    105.334 -16.431  < 2e-16 ***
## E2           -349.078     97.568  -3.578 0.000945 ***
## M            7047.412    102.589  68.695  < 2e-16 ***
## E1:M        -3066.035    149.330 -20.532  < 2e-16 ***
## E2:M         1836.488    131.167  14.001  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 173.8 on 39 degrees of freedom
## Multiple R-squared:  0.9988, Adjusted R-squared:  0.9986
## F-statistic:  5517 on 6 and 39 DF,  p-value: < 2.2e-16
```

Regression Diagnostics

```r
summary(rstandard(mod))
```

```
##      Min.   1st Qu.    Median      Mean   3rd Qu.      Max.
## -5.773499 -0.285974  0.149530  0.000711  0.418380  1.277256
```





Regression Diagnostics

```r
d$res     <- residuals(mod)
d$res_std <- rstandard(mod)
tail(d, n=15)
```

```
##         S  X E M E1 E2     cat        res    res_std
## 32 23174 10 3 1  0  0 E=3&M=1  -46.71590 -0.2885031
## 33 23780 10 2 1  0  1 E=2&M=1 -928.12616 -5.7734989
## 34 25410 11 2 1  0  1 E=2&M=1  204.88683  1.2772560
## 35 14861 11 1 0  1  0 E=1&M=0  -78.54257 -0.4795797
## 36 16882 12 2 0  0  1 E=2&M=0   63.79979  0.3865825
## 37 24170 12 3 1  0  0 E=3&M=1  -44.68992 -0.2783867
## 38 15990 13 1 0  1  0 E=1&M=0   56.48341  0.3464916
## 39 26330 13 2 1  0  1 E=2&M=1  130.91281  0.8226405
## 40 17949 14 2 0  0  1 E=2&M=0  136.82577  0.8383399
## 41 25685 15 3 1  0  0 E=3&M=1  -20.65096 -0.1315808
## 42 27837 16 2 1  0  1 E=2&M=1  146.95178  0.9436856
## 43 18838 16 2 0  0  1 E=2&M=0   31.85175  0.1983361
## 44 17483 16 1 0  1  0 E=1&M=0   58.52238  0.3647875
## 45 19207 17 2 0  0  1 E=2&M=0  -96.13526 -0.6047045
## 46 19346 20 1 0  1  0 E=1&M=0  -66.42566 -0.4309873
```

```r
d <- d[-33, ] # Remove problematic observation
```

Interaction Effects

Note: The level accuracy with which the model explains the data is very rare! Usually Goodness of fit indicators are worse.

```
##                          Model Summary
## ---------------------------------------------------------------
## R                          1.000       RMSE                 67
## R-Squared                  1.000       Coef. Var             0.392
## Adj. R-Squared             1.000       MSE                4504.951
## Pred R-Squared             1.000       MAE                  51.794
## ---------------------------------------------------------------
##                                    ANOVA
## -----------------------------------------------------------------------------
##                      Sum of
##                     Squares        DF       Mean Square        F          Sig.
## -----------------------------------------------------------------------------
## Regression    957607113.080         6     159601185.513    35427.955    0.0000
## Residual         171188.120        38          4504.951
## Total         957778301.200        44
## -----------------------------------------------------------------------------
##                                       Parameter Estimates
## -------------------------------------------------------------------------------------------------
##       model          Beta    Std. Error    Std. Beta         t         Sig         lower        upper
## -------------------------------------------------------------------------------------------------
## (Intercept)     11199.714        30.533                 366.802    0.000    11137.902    11261.525
##           X       498.418         2.152        0.557    231.640    0.000      494.062      502.774
##          E1     -1741.336        40.683       -0.304    -42.803    0.000    -1823.693    -1658.979
##          E2      -357.042        37.681        0.052     -9.475    0.000     -433.324     -280.761
##           M      7040.580        39.619        0.738    177.707    0.000     6960.376     7120.785
##        E1:M     -3051.763        57.674       -0.149    -52.914    0.000    -3168.519    -2935.008
##        E2:M      1997.531        51.785        0.103     38.574    0.000     1892.697     2102.364
## -------------------------------------------------------------------------------------------------
```

Interaction Effects

Note: The notation is slightly different here as the equations are automatically generated. However, it does not really matter wether you use a β, δ or any other greek letter for the (interaction) effects.

```
mod <- lm(S ~ 1 + X + E1 + E2 + M + E1*M + E2*M, data=d)
equatiomatic::extract_eq(mod, use_coefs=F, intercept="beta", wrap=T)
```

$$
\begin{aligned}
\operatorname{S} &= \beta_{0} + \beta_{1}(\operatorname{X}) + \beta_{2}(\operatorname{E1}) + \beta_{3}(\operatorname{E2})\ + \\
&\quad \beta_{4}(\operatorname{M}) + \beta_{5}(\operatorname{E1} \times \operatorname{M}) + \beta_{6}(\operatorname{E2} \times \operatorname{M}) + \epsilon
\end{aligned}
$$

```
equatiomatic::extract_eq(mod, use_coefs=T, coef_digits=4, wrap=T)
```

$$
\begin{aligned}
\operatorname{\widehat{S}} &= 11199.7138 + 498.4178(\operatorname{X}) - 1741.3359(\operatorname{E1}) - 357.0423(\operatorname{E2})\ + \\
&\quad 7040.5801(\operatorname{M}) - 3051.7633(\operatorname{E1} \times \operatorname{M}) + 1997.5306(\operatorname{E2} \times \operatorname{M})
\end{aligned}
$$

Interaction Effects

Your Turn

Compare the models `mod1`, `mod2` and `mod3`. Use them to calculate the base salaries (no experience) for each of the six possible education-management categories.

```
# Data Preparation
d <- P130[-33, ]
d$cat <- factor((paste0("E=",d$E,"&M=",d$M)))
d$E.fac <- factor(d$E)

# Model estimation
mod1 <- lm(S ~ 1 + X + E.fac + M + E.fac*M, data=d)
mod2 <- lm(S ~ 1 + X + cat, data=d)
mod3 <- lm(S ~ 1 + X + E.fac*M, data=d)
```

Interaction Effects

Category	E	M	Estimated Base Salary	95% CI Low	95% CI High
1	1	0	9458.378	9395.539	9521.216
2	2	1	19880.782	19814.090	19947.474
3	3	0	11199.714	11137.902	11261.525
4	1	1	13447.195	13382.933	13511.456
5	2	0	10842.672	10789.719	10895.624
6	3	1	18240.294	18182.503	18298.084

- All models lead to the **same estimates for the base salaries**. This shows that from a technical point using the `cat` variable (instead of the intercation effects) allows to capture the variation in the data.
- It is still **beneficial to use interaction effects** as we did, because this allows to seperate the effects of the three sets of predictor variables education, managemengt and education-management interaction.

Systems of Regression Equations

A dataset may consists of **two or more distinct subsets**, which may require individual regression quations to avoid bias. Subsets may occure cross-sectional or over time and need to be treated differently:

- Cross-Sectional Data
  1. Each group has a separate regression model.
  2. The models have the same intercept but different slopes.
  3. the models have the same slope but different intercepts.
- Time Series Data
  1. Seasonality
  2. Stability of regression parameters over time

Example: Preemployment Test

```
P140
```

```
##    TEST RACE JPERF
## 1  0.28    1  1.83
## 2  0.97    1  4.59
## 3  1.25    1  2.97
## 4  2.46    1  8.14
## 5  2.51    1  8.00
## 6  1.17    1  3.30
## 7  1.78    1  7.53
## 8  1.21    1  2.03
## 9  1.63    1  5.00
## 10 1.98    1  8.04
## 11 2.36    0  3.25
## 12 2.11    0  5.30
## 13 0.45    0  1.39
## 14 1.76    0  4.69
## 15 2.09    0  6.56
## 16 1.50    0  3.00
## 17 1.25    0  5.85
## 18 0.72    0  1.90
## 19 0.42    0  3.85
## 20 1.53    0  2.95
```

Your turn

`TEST` Score on the preemployment test.

`RACE` Dummy to indicate if individual is part of a minority (1) or not (0).

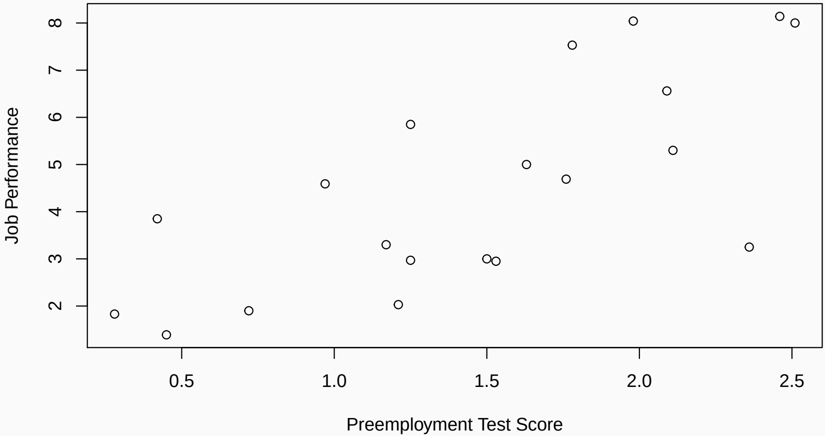
`JPERF` Job Performance Ranking after 6 weeks on the job.

Example: Preemployment Test

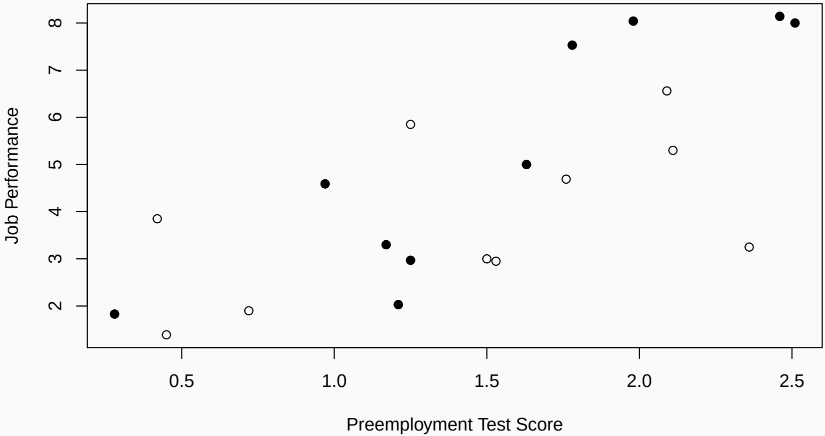
For simplicity and generality we refer to the job performance as Y and the score on the preemployment test as X. We want to compare the following two models:

Model 1 (Pooled):	$y_{ij} = \beta_0 + \beta_1 x_{ij} + \epsilon_{ij}$
Model 2 (Minority):	$y_{i1} = \beta_{01} + \beta_{11} x_{i1} + \epsilon_{i1}$
Model 2 (non Minority):	$y_{i2} = \beta_{02} + \beta_{12} x_{i2} + \epsilon_{i2}$

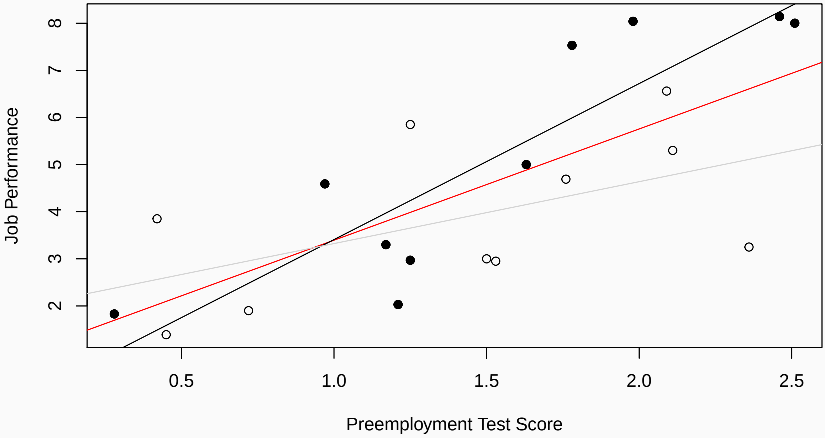
Example: Preemployment Test



Example: Preemployment Test

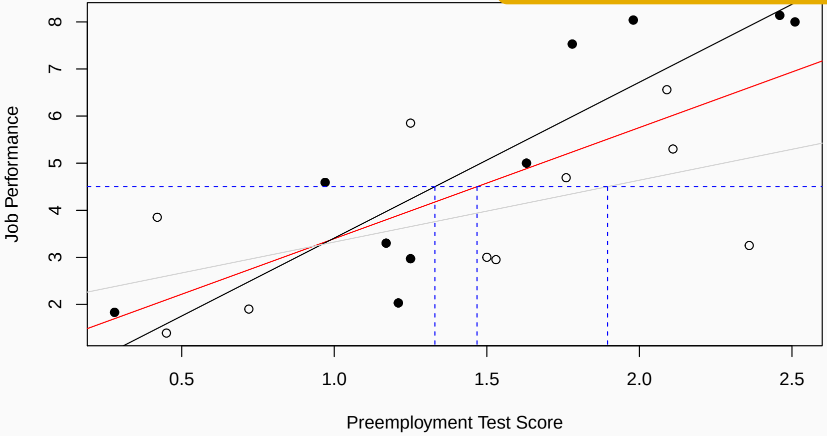


Example: Preemployment Test



Example: Preemployment Test

When different models are required for the groups, this would imply that the required score in the preemployment test that is needed to result in (minimum) job performance (horizontal dashed line) needs to distinguished by group (vertical dashed lines).



Models with different Slopes and different Intercepts

- What we want to test for the Preemployment Test data is for differences in intercept and slope using the following Null.

$$H_0 : \beta_{11} = \beta_{12}, \beta_{01} = \beta_{02}$$

- This test can be performed using an **interaction term** by using a variable z_{ij} that takes the value 1 if an individual is part of a minority group and 0 otherwise. This leads to two relevant models:

$$\begin{aligned} &\text{Model 1 (Pooled):} && y_{ij} = \beta_0 + \beta_1 x_{ij} + \epsilon_{ij} \\ &\text{Model 3 (Interaction):} && y_{ij} = \beta_0 + \beta_1 x_{ij} + \gamma z_{ij} + \delta(z_{ij} \cdot x_{ij}) + \epsilon_{ij} \end{aligned}$$

- This model is **equivalent** to the previously discussed Model 2.

Models with different Slopes and different Intercepts

	Model 1	Model 2	Model 2	Model 3
	Pooled	Minority	White	Interaction
(Intercept)	1.03	0.10	2.01	2.01
	(0.87)	(1.04)	(1.13)	(1.05)
TEST	2.36^{***}	3.31^{***}	1.31	1.31
	(0.54)	(0.62)	(0.72)	(0.67)
RACE				-1.91
				(1.54)
TEST:RACE				2.00
				(0.95)
R^2	0.52	0.78	0.29	0.66
Adj. R^2	0.49	0.75	0.20	0.60
Num. obs.	20	10	10	20

$^{***}p < 0.001$; $^{**}p < 0.01$; $^{*}p < 0.05$

Table 5

- Model 1 can be seen as a restriced verison (RM) of model 3, the full model (FM), with $\gamma = \delta = 0$.

Models with different Slopes and different Intercepts

- The framework using the models as FM and RM allows us to use the *F*-Test for comparison.

Your turn

Interpret the F-Test.

$$F = \frac{[SSE(RM) - SSE(FM)]/2}{SSE(FM)/16}$$

```r
(SSE_RM <- sum(residuals(mod1)^2))
```

```
## [1] 45.5683
```

```r
(SSE_FM <- sum(residuals(mod3)^2))
```

```
## [1] 31.65547
```

```r
(F_stat <- ((SSE_RM - SSE_FM)/2)/(SSE_FM/16))
```

```
## [1] 3.516061
```

```r
pf(F_stat, df1=2, df2=16, lower.tail=FALSE)
```

```
## [1] 0.05423606
```