

# Statistical Modeling

CH.4 - Qualitative Variables

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Wir geben Impulse

- 1 Organizational Information
- 2 Qualitative Variables as Predictors

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- 1 Organizational Information
- 2 Qualitative Variables as Predictors

- Qualitative or categorical variables (such as gender, marital status, etc) are useful predictors and are usually called **indicator** or **dummy variables**.
- Those variables usually only take two values, 0 and 1, which signify that the observation belongs to one of two possible categories.
- The numerical values of indicator variables **do not reflect quantitative ordering**.
- **Example Variable:** Gender, coded as 1 for *female* and 0 for *male*.
- Indicator variables can also be used in a regression equation to distinguish between three or more groups.
- The response variable is still a quantitative continuous in all discussed cases.

# Example: Salary Survey Data

P130

```
##      S  X  E  M
## 1 13876 1 1 1
## 2 11608 1 3 0
## 3 18701 1 3 1
## 4 11283 1 2 0
## 5 11767 1 3 0
## 6 20872 2 2 1
## 7 11772 2 2 0
## 8 10535 2 1 0
## 9 12195 2 3 0
## 10 12313 3 2 0
## 11 14975 3 1 1
## 12 21371 3 2 1
## 13 19800 3 3 1
## 14 11417 4 1 0
## 15 20263 4 3 1
## 16 13231 4 3 0
## 17 12884 4 2 0
## 18 13245 5 2 0
## 19 13677 5 3 0
## 20 15965 5 1 1
## 21 12336 6 1 0
## 22 21352 6 3 1
## 23 13839 6 2 0
## 24 22884 6 2 1
## 25 16978 7 1 1
## 26 14803 8 2 0
## 27 17404 8 1 1
## 28 22184 8 3 1
## 29 13548 8 1 0
## 30 14467 10 1 0
## 31 15942 10 2 0
```

## Your turn

Salary survey of computer professionals with objective to identify and quantify variables that determine salary differentials.

S Salary (Response)

X Experience, measured in years

E Education, 1 (High School/HS), 2 (Bachelor/BS), 3 (Advanced Degree/AD)

M Management 1 (is Manager), 0 (no Management Responsibility)

## Example: Salary Survey Data

- **Experience:** We assume linearity, which means that each additional year is worth a fixed salary increment.
- **Education:** Can be used in a linear or categorical form.
  - ▶ Using the the variable in its raw form would assume that each step up in education is worth a fixed increment in salary. This may be too restrictive.
  - ▶ Using education as categorical variable can be done by defining **two indicator variables**. This allows to pick up the effect of education wether it is linear or not.
- **Management:** Is also an indicator variable, that allows to distinguish between management (1) an regular staff positions (0).

## Indicator Variables

When using indicator variables to represent a set of categories, the number of these variables required is **one less than the number of categories**. For *education* we can create two indicators variables:

$$E_{i1} = \begin{cases} 1, & \text{if the } i\text{-th person is in the HS category} \\ 0, & \text{otherwise.} \end{cases}$$

$$E_{i2} = \begin{cases} 1, & \text{if the } i\text{-th person is in the BS category} \\ 0, & \text{otherwise.} \end{cases}$$

These two variables allow representing the three groups (HS, BS, AD).

HS:  $E_1 = 1, E_2 = 0$ , BS:  $E_1 = 0, E_2 = 1$ , AD:  $E_1 = 0, E_2 = 0$



- The regression equation from the Salary Survey Data is:

$$S = \beta_0 + \beta_1 X + \gamma_1 E_1 + \gamma_2 E_2 + \delta_1 M + \epsilon$$

# Indicator Variables

- The regression equation from the Salary Survey Data is:

$$S = \beta_0 + \beta_1 X + \gamma_1 E_1 + \gamma_2 E_2 + \delta_1 M + \epsilon$$

- There is a different valid regression equation for each of the six (three education and two management) categories.

Category	E	M	Regression Equation
1	1	0	$S = (\beta_0 + \gamma_1) + \beta_1 X + \epsilon$
2	1	1	$S = (\beta_0 + \gamma_1 + \delta_1) + \beta_1 X + \epsilon$
3	2	0	$S = (\beta_0 + \gamma_2) + \beta_1 X + \epsilon$
4	2	1	$S = (\beta_0 + \gamma_2 + \delta_1) + \beta_1 X + \epsilon$
5	3	0	$S = \beta_0 + \beta_1 X + \epsilon$
6	3	1	$S = (\beta_0 + \delta_1) + \beta_1 X + \epsilon$

# Indicator Variables

```
d <- P130
d$E1 <- as.numeric(d$E == 1)
d$E2 <- as.numeric(d$E == 2)
mod <- lm(S ~ 1 + X + E1 + E2 + M, data=d)
summary(mod)
```

```
##
## Call:
## lm(formula = S ~ 1 + X + E1 + E2 + M, data = d)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -1884.6  -653.6    22.2    844.9   1716.5
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  11031.8      383.2    28.79 < 2e-16 ***
## X              546.2       30.5    17.90 < 2e-16 ***
## E1           -2996.2      411.8    -7.28 6.7e-09 ***
## E2              147.8      387.7     0.38  0.7
## M             6883.5      313.9    21.93 < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1030 on 41 degrees of freedom
## Multiple R-squared:  0.957, Adjusted R-squared:  0.953
## F-statistic: 227 on 4 and 41 DF, p-value: <2e-16
```

## Your turn

Interpret the regression coefficients. Assume that the residual patterns are satisfactory.

**Table 3**

	<i>Dependent variable:</i>	
	S	
	(1)	(2)
X	546.200*** (30.520)	570.100*** (38.560)
E1	−2,996.000*** (411.800)	
E2	147.800 (387.700)	
E		1,579.000*** (262.300)
M	6,884.000*** (313.900)	6,688.000*** (398.300)
Constant	11,032.000*** (383.200)	6,963.000*** (665.700)
Observations	46	46
R <sup>2</sup>	0.957	0.928
Adjusted R <sup>2</sup>	0.953	0.923
Residual Std. Error	1,027.000 (df = 41)	1,313.000 (df = 42)
F Statistic	226.800*** (df = 4; 41)	179.600*** (df = 3; 42)

Note:

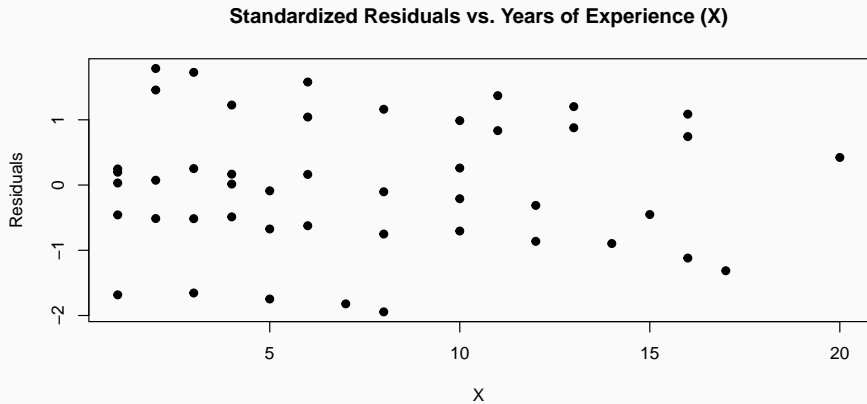
\* p<0.1; \*\* p<0.05; \*\*\* p<0.01

**Before we continue we check the residuals**

- 1) Residuals vs. Years of Experience
- 2) Residuals vs. Categories from Dummies

# Regression Diagnostics

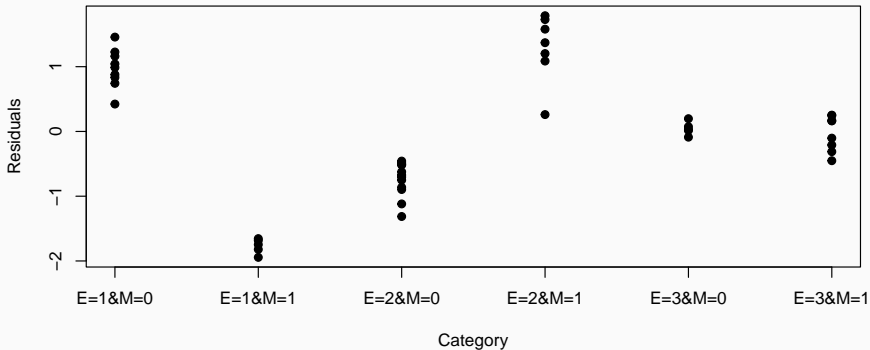
```
plot(x = d$X, y = rstandard(mod), pch=19,  
     ylab="Residuals", xlab = "X",  
     main = "Standardized Residuals vs. Years of Experience (X)")
```



## Regression Diagnostics

```
d$cat <- factor((paste0("E=", d$E, "&M=", d$M)))
plot(x = as.numeric(d$cat), y = rstandard(mod), pch=19, xaxt="n",
     ylab="Residuals", xlab = "Category",
     main = "Standardized Residuals vs. Education-Management Category")
axis(1, at=1:6, labels=levels(d$cat))
```

### Standardized Residuals vs. Education–Management Category



### What is wrong with the residuals:

- Depending on the category the residuals are almost entirely positive or negative.
- The **pattern of the residuals is highly moderated by the associated group** (education-management category). This makes it clear that the combinations of education and management have not been treated sufficiently in the model.
- The residual plots provide evidence that the effects of education and management status on salary determination are **not additive**.

The multiplicative pattern needs to be embedded in the model!



- Interaction effects are *multiplicative* effects that allow capturing nondadditive effects in variables.
- Interaction variables are products of existing indicator variables.
- Using the Salary Survey Data this can be achieved by creating the two interaction effects ( $E_1 \cdot M$ ) and ( $E_2 \cdot M$ ) and **adding** them to the model.
- The interaction effects **do not replace** the indicator variables.

# Interaction Effects

```
mod <- lm(S ~ 1 + X + E1 + E2 + M + E1*M + E2*M, data=d)
summary(mod)
```

**Your turn**

Is that model sufficient?

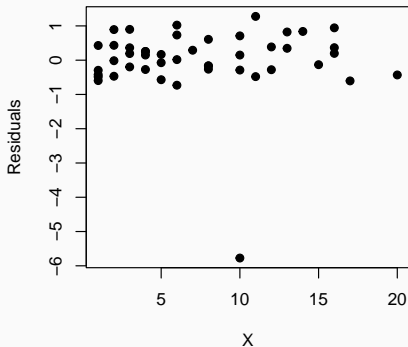
```
##
## Call:
## lm(formula = S ~ 1 + X + E1 + E2 + M + E1 * M + E2 * M, data = d)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -928.1   -46.2    24.3    65.9   204.9
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) 11203.43      79.07   141.70 < 2e-16 ***
## X              496.99       5.57    89.28 < 2e-16 ***
## E1            -1730.75     105.33   -16.43 < 2e-16 ***
## E2             -349.08      97.57    -3.58 0.00095 ***
## M              7047.41     102.59    68.70 < 2e-16 ***
## E1:M          -3066.04     149.33   -20.53 < 2e-16 ***
## E2:M           1836.49     131.17    14.00 < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 174 on 39 degrees of freedom
## Multiple R-squared:  0.999, Adjusted R-squared:  0.999
## F-statistic: 5.52e+03 on 6 and 39 DF, p-value: <2e-16
```

# Regression Diagnostics

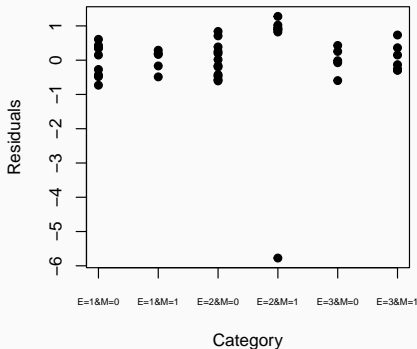
```
summary(rstandard(mod))
```

```
##      Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
## -5.773  -0.286   0.150   0.001   0.418   1.277
```

**Standardized Residuals vs.  
Years of Experience (X)**



**Standardized Residuals vs.  
Education-Management Category**



# Regression Diagnostics

```
d$res <- residuals(mod)
d$res_std <- rstandard(mod)
tail(d, n=15)
```

```
##      S  X  E  M  E1  E2      cat      res res_std
## 32 23174 10 3 1  0  0 E=3&M=1 -46.72 -0.2885
## 33 23780 10 2 1  0  1 E=2&M=1 -928.13 -5.7735
## 34 25410 11 2 1  0  1 E=2&M=1  204.89  1.2773
## 35 14861 11 1 0  1  0 E=1&M=0  -78.54 -0.4796
## 36 16882 12 2 0  0  1 E=2&M=0   63.80  0.3866
## 37 24170 12 3 1  0  0 E=3&M=1  -44.69 -0.2784
## 38 15990 13 1 0  1  0 E=1&M=0   56.48  0.3465
## 39 26330 13 2 1  0  1 E=2&M=1  130.91  0.8226
## 40 17949 14 2 0  0  1 E=2&M=0  136.83  0.8383
## 41 25685 15 3 1  0  0 E=3&M=1  -20.65 -0.1316
## 42 27837 16 2 1  0  1 E=2&M=1  146.95  0.9437
## 43 18838 16 2 0  0  1 E=2&M=0   31.85  0.1983
## 44 17483 16 1 0  1  0 E=1&M=0   58.52  0.3648
## 45 19207 17 2 0  0  1 E=2&M=0  -96.14 -0.6047
## 46 19346 20 1 0  1  0 E=1&M=0  -66.43 -0.4310
```

```
d <- d[-33, ] # Remove problematic observation
```

# Interaction Effects

**Note:** The level accuracy with which the model explains the data is very rare! Usually Goodness of fit indicators are worse.

Note: The level accuracy with which is very rare! Usually Goodness of fit

Model Summary							
-----							
R	1.000	RMSE	67.119				
R-Squared	1.000	Coef. Var	0.392				
Adj. R-Squared	1.000	MSE	4504.951				
Pred R-Squared	1.000	MAE	51.794				
-----							
ANOVA							
-----							
	Sum of						
	Squares	DF	Mean Square	F	Sig.		
-----							
Regression	957607113.080	6	159601185.513	35427.955	0.0000		
Residual	171188.120	38	4504.951				
Total	957778301.200	44					
-----							
Parameter Estimates							
-----							
model	Beta	Std. Error	Std. Beta	t	Sig	lower	upper
-----							
(Intercept)	11199.714	30.533		366.802	0.000	11137.902	11261.525
X	498.418	2.152	0.557	231.640	0.000	494.062	502.774
E1	-1741.336	40.683	-0.304	-42.803	0.000	-1823.693	-1658.979
E2	-357.042	37.681	0.052	-9.475	0.000	-433.324	-280.761
M	7040.580	39.619	0.738	177.707	0.000	6960.376	7120.785
E1:M	-3051.763	57.674	-0.149	-52.914	0.000	-3168.519	-2935.008
E2:M	1997.531	51.785	0.103	38.574	0.000	1892.697	2102.364
-----							

# Interaction Effects

**Note:** The notation is slightly different here as the equations are automatically generated. However, it does not really matter whether you use a  $\beta$ ,  $\delta$  or any other greek letter for the (interaction) effects.

```
mod <- lm(S ~ 1 + X + E1 + E2 + M + E1*M + E2*M, data=d)
equatiomatic::extract_eq(mod, use_coefs=F, intercept="beta", wrap=T)
```

$$S = \beta_0 + \beta_1(X) + \beta_2(E1) + \beta_3(E2) + \beta_4(M) + \beta_5(E1 \times M) + \beta_6(E2 \times M) + \epsilon \quad (1)$$

```
equatiomatic::extract_eq(mod, use_coefs=T, coef_digits=4, wrap=T)
```

$$\hat{S} = 11199.7138 + 498.4178(X) - 1741.3359(E1) - 357.0423(E2) + 7040.5801(M) - 3051.7633(E1 \times M) + 1997.5306(E2 \times M) \quad (2)$$

## Your Turn

Compare the models `mod1`, `mod2` and `mod3`. Use them to calculate the base salaries (no experience) for each of the six possible education-management categories.

```
# Data Preparation
d <- P130[-33, ]
d$cat <- factor((paste0("E=", d$E, "&M=", d$M)))
d$E.fac <- factor(d$E)

# Model estimation
mod1 <- lm(S ~ 1 + X + E.fac + M + E.fac*M, data=d)
mod2 <- lm(S ~ 1 + X + cat, data=d)
mod3 <- lm(S ~ 1 + X + E.fac*M, data=d)
```

Category	E	M	Estimated Base Salary	95% CI Low	95% CI High
1	1	0	9458	9396	9521
2	2	1	19881	19814	19947
3	3	0	11200	11138	11262
4	1	1	13447	13383	13511
5	2	0	10843	10790	10896
6	3	1	18240	18183	18298

- All models lead to the **same estimates for the base salaries**. This shows that from a technical point using the `cat` variable (instead of the interaction effects) allows to capture the variation in the data.
- It is still **beneficial to use interaction effects** as we did, because this allows to separate the effects of the three sets of predictor variables education, management and education-management interaction.



A dataset may consists of **two or more distinct subsets**, which may require individual regression equations to avoid bias. Subsets may occur cross-sectional or over time and need to be treated differently:

## ■ Cross-Sectional Data

- 1 Each group has a separate regression model.
- 2 The models have the same intercept but different slopes.
- 3 the models have the same slope but different intercepts.

## ■ Time Series Data

- 1 Calendar Effects, e.g. Seasonality
- 2 Stability of regression parameters over time

# Example: Preemployment Test

P140

##	TEST	RACE	JPERF
## 1	0.28	1	1.83
## 2	0.97	1	4.59
## 3	1.25	1	2.97
## 4	2.46	1	8.14
## 5	2.51	1	8.00
## 6	1.17	1	3.30
## 7	1.78	1	7.53
## 8	1.21	1	2.03
## 9	1.63	1	5.00
## 10	1.98	1	8.04
## 11	2.36	0	3.25
## 12	2.11	0	5.30
## 13	0.45	0	1.39
## 14	1.76	0	4.69
## 15	2.09	0	6.56
## 16	1.50	0	3.00
## 17	1.25	0	5.85
## 18	0.72	0	1.90
## 19	0.42	0	3.85
## 20	1.53	0	2.95

## Your turn

**TEST** Score on the preemployment test.

**RACE** Dummy to indicate if individual is part of a minority (1) or not (0).

**JPERF** Job Performance Ranking after 6 weeks on the job.

## Example: Preemployment Test

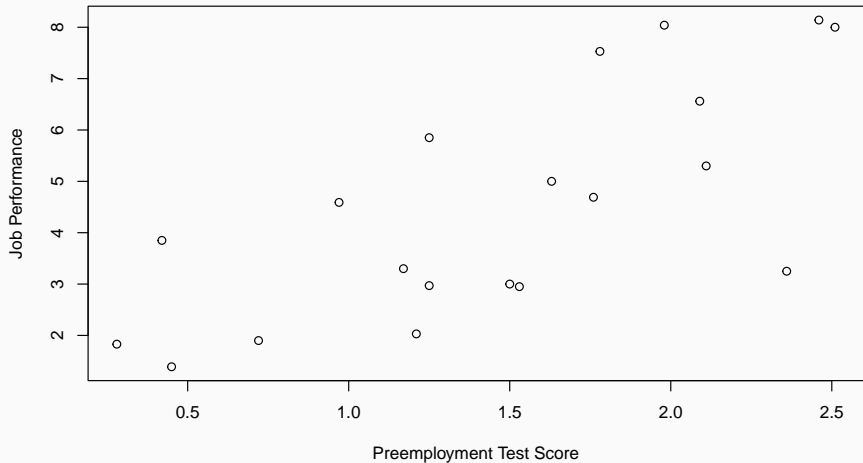
For simplicity and generality we refer to the job performance as  $Y$  and the score on the preemployment test as  $X$ . We want to compare the following two models:

Model 1 (Pooled):  $y_{ij} = \beta_0 + \beta_1 x_{ij} + \epsilon_{ij}$

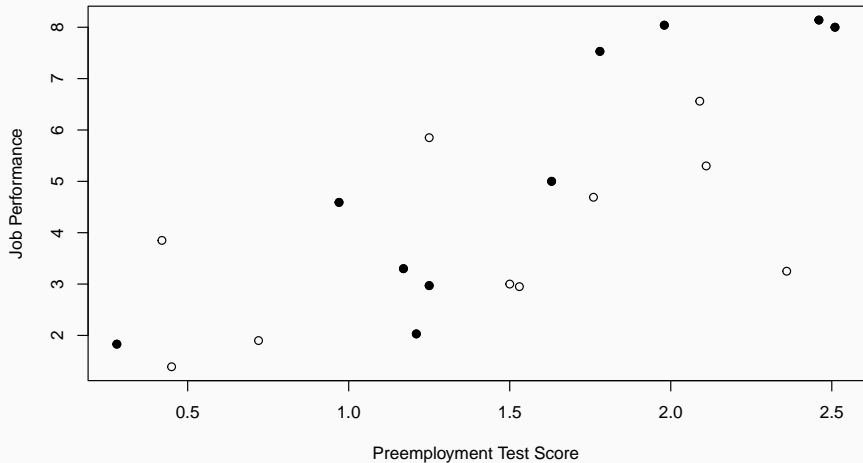
Model 2 (Minority):  $y_{i1} = \beta_{01} + \beta_{11} x_{i1} + \epsilon_{i1}$

Model 2 (non Minority):  $y_{i2} = \beta_{02} + \beta_{12} x_{i2} + \epsilon_{i2}$

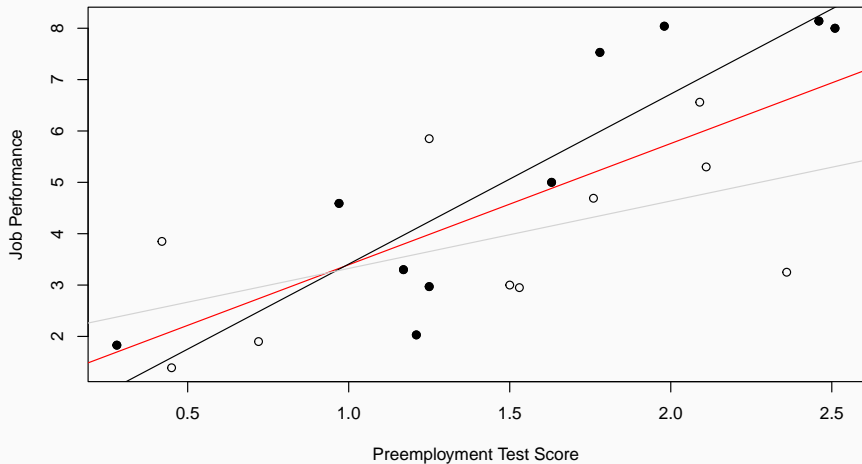
## Example: Preemployment Test



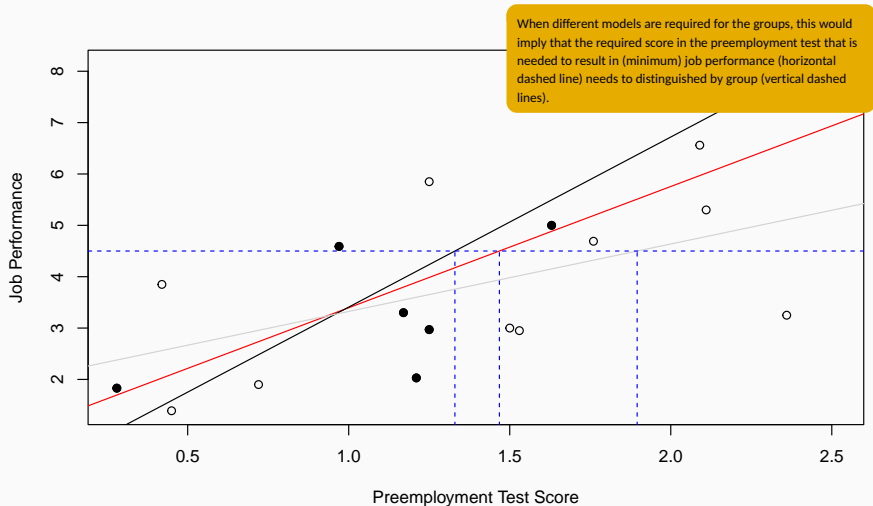
## Example: Preemployment Test



## Example: Preemployment Test



## Example: Preemployment Test



## Models with different Slopes and different Intercepts

- What we want to test for the Preemployment Test data is for differences in intercept and slope using the following Null.

$$H_0 : \beta_{11} = \beta_{12}, \beta_{01} = \beta_{02}$$

- This test can be performed using an **interaction term** by using a variable  $z_{ij}$  that takes the value 1 if an individual is part of a minority group and 0 otherwise. This leads to two relevant models:

Model 1 (Pooled):  $y_{ij} = \beta_0 + \beta_1 x_{ij} + \epsilon_{ij}$

Model 3 (Interaction):  $y_{ij} = \beta_0 + \beta_1 x_{ij} + \gamma z_{ij} + \delta(z_{ij} \cdot x_{ij}) + \epsilon_{ij}$

- This model is **equivalent** to the previously discussed Model 2.



# Models with different Slopes and different Intercepts

	Model 1	Model 2	Model 2	Model 3
	Pooled	Minority	White	Interaction
(Intercept)	1.03 (0.87)	0.10 (1.04)	2.01 (1.13)	2.01 (1.05)
TEST	2.36*** (0.54)	3.31*** (0.62)	1.31 (0.72)	1.31 (0.67)
RACE				-1.91 (1.54)
TEST:RACE				2.00 (0.95)
R <sup>2</sup>	0.52	0.78	0.29	0.66
Adj. R <sup>2</sup>	0.49	0.75	0.20	0.60
Num. obs.	20	10	10	20

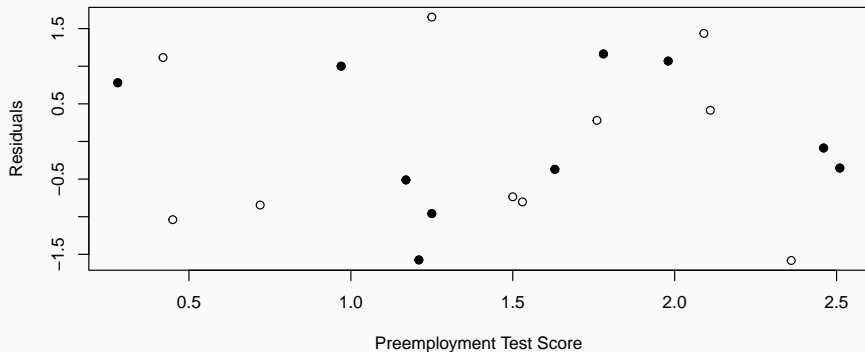
\*\*\*  $p < 0.001$ ; \*\*  $p < 0.01$ ; \*  $p < 0.05$

**Table 5**

- Model 1 can be seen as a restricted version (RM) of model 3, the full model (FM), with  $\gamma = \delta = 0$ .

# Models with different Slopes and different Intercepts

```
df <- cbind(P140,res = rstandard(mod3))  
plot(x=df$TEST, y=df$res,  
     ylab="Residuals", xlab="Preemployment Test Score")  
points(df$TEST[df$RACE == T], df$res[df$RACE == T],pch=19)
```



# Models with different Slopes and different Intercepts

- The framework using the models as FM and RM for comparison.

$$F = \frac{[SSE(RM) - SSE(FM)]/1}{SSE(FM)/16}$$

```
(SSE_RM <- sum(residuals(mod1)^2))
```

```
## [1] 45.57
```

```
(SSE_FM <- sum(residuals(mod3)^2))
```

```
## [1] 31.66
```

```
(F_stat <- ((SSE_RM - SSE_FM)/2)/(SSE_FM/16))
```

```
## [1] 3.516
```

```
pf(F_stat, df1=2, df2=16, lower.tail=FALSE)
```

```
## [1] 0.05424
```

## Your turn

Interpret the F-Test.

Can you conclude that the relationship is different for the two groups, so that two different equations (intercept + slope) are required?

## Models with same Slope and different Intercepts

- Assuming we have a reason to believe that only the intercepts for the two groups are different can be achieved using the indicator variable (and omitting the interaction term).

Model 1 (Pooled):  $y_{ij} = \beta_0 + \beta_1 x_{ij} + \epsilon_{ij}$

Model 4 (Indicator only):  $y_{ij} = \beta_0 + \beta_1 x_{ij} + \gamma z_{ij} + \cancel{\delta(z_{ij} x_{ij})} + \epsilon_{ij}$

- In the case where  $z_{ij} = 1$  (which indicates the non-minority group) the coefficient  $\gamma$  can be added to the intercept  $\beta_0$  to obtain the effective intercept for that respective group.
- The resulting models represent **two parallel lines** (same slopes) with intercepts  $\beta_0$  and  $\beta_0 + \gamma$ .

# Models with same Slope and different Intercepts

```
mod4 <- lm(JPERF ~ 1 + TEST + RACE, data=P140)
```

- Significance can be tested using the *F*-Test. As the FM and RM differ by one parameter, results are equivalent to the *t*-Test.

	Model 1	Model 2	Model 2	Model 3	Model 4
	Pooled	Minority	White	Interaction	Indicator
(Intercept)	1.03 (0.87)	0.10 (1.04)	2.01 (1.13)	2.01 (1.05)	0.61 (0.89)
TEST	2.36*** (0.54)	3.31*** (0.62)	1.31 (0.72)	1.31 (0.67)	2.30*** (0.52)
RACE				-1.91 (1.54)	1.03 (0.69)
TEST:RACE				2.00 (0.95)	
R <sup>2</sup>	0.52	0.78	0.29	0.66	0.57
Adj. R <sup>2</sup>	0.49	0.75	0.20	0.60	0.52
Num. obs.	20	10	10	20	20

\*\*\*  $p < 0.001$ ; \*\*  $p < 0.01$ ; \*  $p < 0.05$

Table 6

## Models with different Slopes and same Intercept

- Finally we can hypothesize that the two groups have the same intercept  $\beta_0$  but different slopes, which can be done by including only the interaction.

Model 1 (Pooled):  $y_{ij} = \beta_0 + \beta_1 x_{ij} + \epsilon_{ij}$

Model 5 (Interaction only):  $y_{ij} = \beta_0 + \beta_1 x_{ij} + \gamma z_{ij} + \delta(z_{ij} \cdot x_{ij}) + \epsilon_{ij}$

```
mod5 <- lm(JPERF ~ 1 + TEST + RACE:TEST, data=P140)
```

- Inference for the  $\delta$  can be carried out using the *F*-Test or the *t*-Test. The FM and RM again only differ by one parameter.

# Systems of Regression Equations

- The final results for all discussed cases for the preemployment test data look like follows.

	Model 1	Model 2	Model 2	Model 3	Model 4	Model 5
	Pooled	Minority	White	Full Interaction	Indicator	Interaction
(Intercept)	1.03 (0.87)	0.10 (1.04)	2.01 (1.13)	2.01 (1.05)	0.61 (0.89)	1.12 (0.78)
TEST	2.36*** (0.54)	3.31*** (0.62)	1.31 (0.72)	1.31 (0.67)	2.30*** (0.52)	1.83** (0.54)
RACE				-1.91 (1.54)	1.03 (0.69)	
TEST:RACE				2.00 (0.95)		0.92* (0.40)
R <sup>2</sup>	0.52	0.78	0.29	0.66	0.57	0.63
Adj. R <sup>2</sup>	0.49	0.75	0.20	0.60	0.52	0.59
Num. obs.	20	10	10	20	20	20

\*\*\*  $p < 0.001$ ; \*\*  $p < 0.01$ ; \*  $p < 0.05$

**Table 7**

- Another interesting field of study is temporal structure in the data, which could fill a whole course by itself. Therefore we only briefly look at two ideas.

## 1) Calendar Effects, e.g. Seasonality

- Can be modeled by including time as regressor, e.g. in the form of (multiple) indicators for e.g. Week/Month/Quarter/Year
- The number of indicator variables is  $m - 1$  where  $m$  is the frequency of the time effects (e.g.  $m = 4$  for Quarters).

## 2) Stability of Parameters over Time

- By combining indicator and interaction terms one can model intertemporal and interspatial relationships. Insignificance of the interactions with all indicators then provides evidence stability over time.