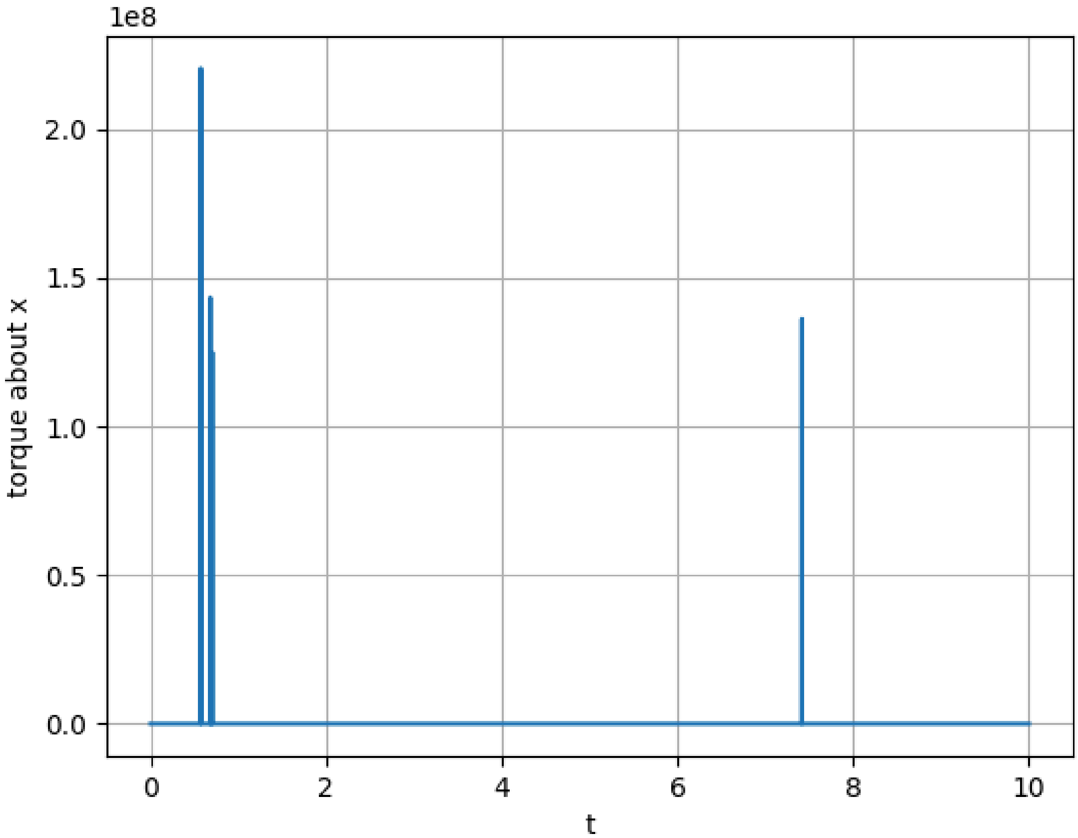
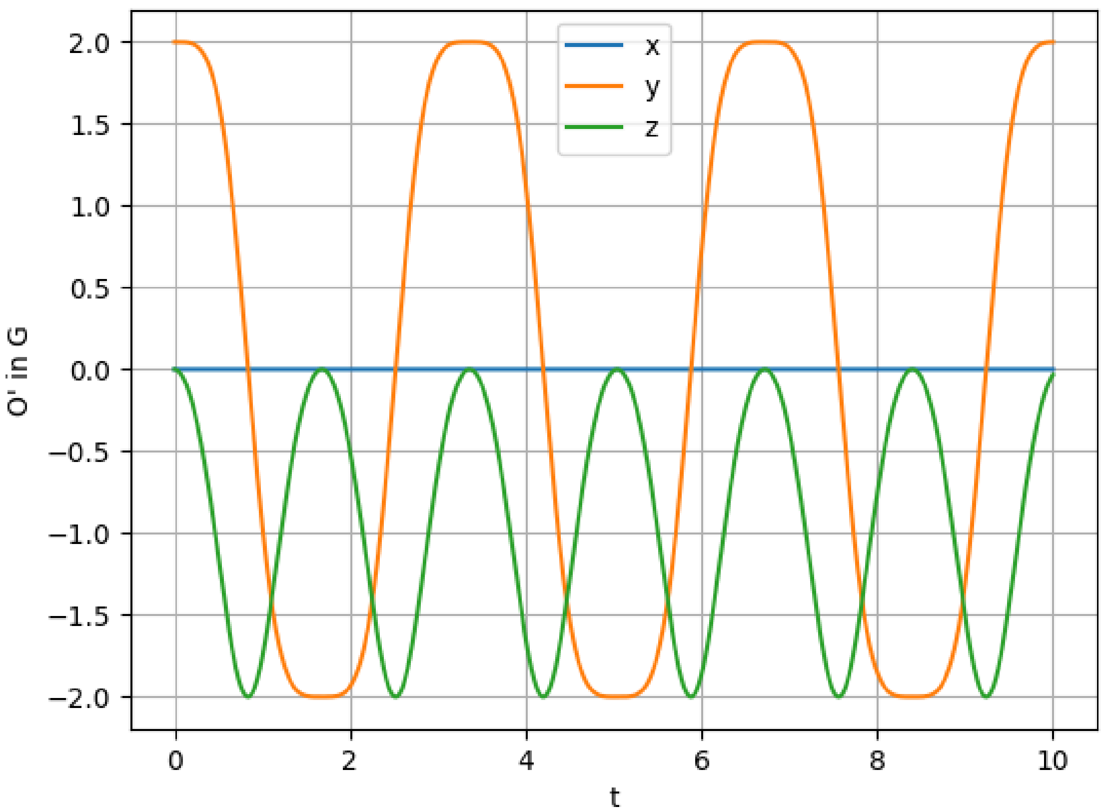
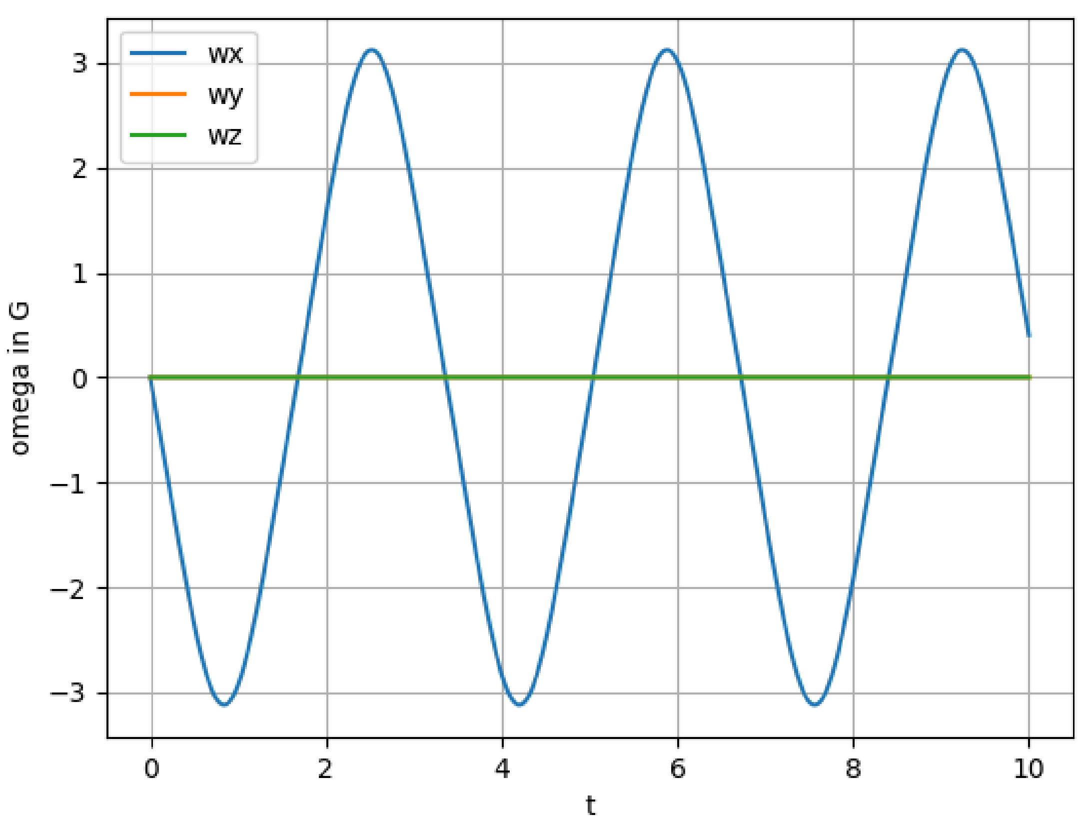
Problem 7.1

See attached `program7-1.py`

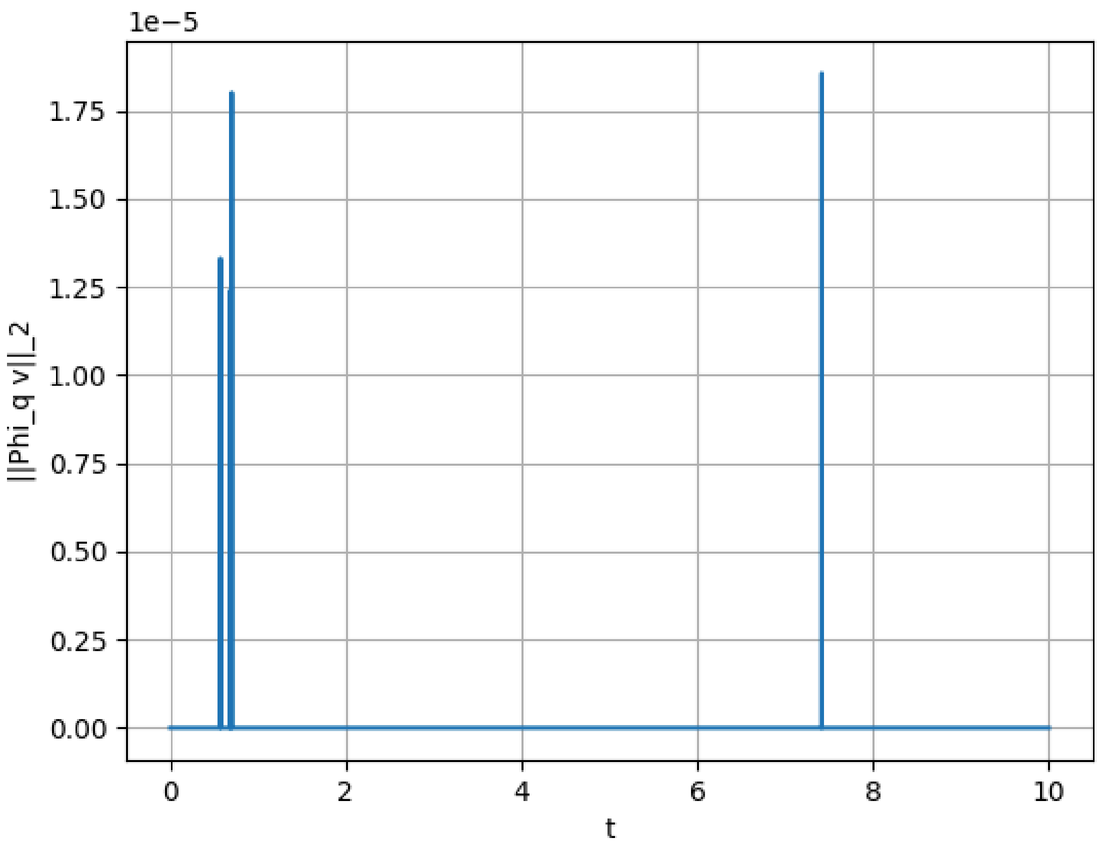




x
y
z
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
O' in G
0
2
4
6
8
10
t



wx
wy
wz
3
2
1
0
−1
−2
−3
omega in G
0
2
4
6
8
10
t



Step size = 0.001

Time = 12.94987

Problem 7.2

Solution(s)

(a)

B3-24 is a 2-node, 3D ANCF beam element. Each node carries:

- $\mathbf{r}$ — position (3 components),
- $\mathbf{r}_{,u}$, $\mathbf{r}_{,v}$, $\mathbf{r}_{,w}$ — position gradients w.r.t. the element's local coordinates u, v, w (each a 3-vector).

Total DOFs: $2 \text{ nodes} \times (1+3) \text{ vectors} \times 3 = 24.$

(b)

Let

$$N(t) = \left[\, \mathbf{r}_1 \;\; \mathbf{r}_{1,u} \;\; \mathbf{r}_{1,v} \;\; \mathbf{r}_{1,w} \;\; \mathbf{r}_2 \;\; \mathbf{r}_{2,u} \;\; \mathbf{r}_{2,v} \;\; \mathbf{r}_{2,w} \,\right] = \left[\, \mathbf{e}_1 \; \mathbf{e}_2 \; \mathbf{e}_3 \; \mathbf{e}_4 \; \mathbf{e}_5 \; \mathbf{e}_6 \; \mathbf{e}_7 \; \mathbf{e}_8 \right].$$

With scalar shape functions $s_i(u, v, w)$,

$$\mathbf{r}(u,v,w,t) = N(t)\, s(u,v,w) = \sum_{i=1}^{8} \mathbf{e}_i(t)\, s_i(u,v,w).$$

(c)
Let $\xi := u/L = u/10 \in [0,1]$. Use cubic Hermite in the axial coordinate and linear dependence in v, w:

$$\begin{aligned} h_1(\xi) &= 1 - 3\xi^2 + 2\xi^3, \qquad & h_2(\xi) &= \xi - 2\xi^2 + \xi^3, \\ h_3(\xi) &= 3\xi^2 - 2\xi^3, & h_4(\xi) &= -\xi^2 + \xi^3. \end{aligned}$$

The eight scalar shape functions are

$$\begin{aligned} s_1 &= h_1, \qquad & s_2 &= L\, h_2, \qquad & s_3 &= v\, h_1, \qquad & s_4 &= w\, h_1, \\ s_5 &= h_3, & s_6 &= L\, h_4, & s_7 &= v\, h_3, & s_8 &= w\, h_3. \end{aligned}$$

These enforce at $u = 0$: $\mathbf{r} = \mathbf{r}_1,\ \mathbf{r}_{,u} = \mathbf{r}_{1,u},\ \mathbf{r}_{,v} = \mathbf{r}_{1,v},\ \mathbf{r}_{,w} = \mathbf{r}_{1,w}$, and similarly at $u = L$ for node 2. For the cross-section, $v \in [-W/2, W/2] = [-0.5, 0.5]$, $w \in [-H/2, H/2] = [-0.5, 0.5]$.

(d)
Define $H \in \mathbb{R}^{8\times 3}$ by rows $i = 1, \ldots, 8$: $[\, s_{i,u} \;\; s_{i,v} \;\; s_{i,w} \,]$.
Using $\partial/\partial u = (1/L)\, d/d\xi$ with $L = 10$ and

$$h_1' = -6\xi + 6\xi^2, \quad h_2' = 1 - 4\xi + 3\xi^2, \quad h_3' = 6\xi - 6\xi^2, \quad h_4' = -2\xi + 3\xi^2,$$

the entries are

$$\begin{aligned} s_{1,u} &= 0.1\, h_1', & s_{1,v} &= 0, & s_{1,w} &= 0, \\ s_{2,u} &= h_2', & s_{2,v} &= 0, & s_{2,w} &= 0, \\ s_{3,u} &= 0.1\, v\, h_1', & s_{3,v} &= h_1, & s_{3,w} &= 0, \\ s_{4,u} &= 0.1\, w\, h_1', & s_{4,v} &= 0, & s_{4,w} &= h_1, \\ s_{5,u} &= 0.1\, h_3', & s_{5,v} &= 0, & s_{5,w} &= 0, \\ s_{6,u} &= h_4', & s_{6,v} &= 0, & s_{6,w} &= 0, \\ s_{7,u} &= 0.1\, v\, h_3', & s_{7,v} &= h_3, & s_{7,w} &= 0, \\ s_{8,u} &= 0.1\, w\, h_3', & s_{8,v} &= 0, & s_{8,w} &= h_3. \end{aligned}$$

The deformation gradient used in the element kinematics is $F(u,v,w,t) = N(t)\, H(u,v,w) = \sum_{i=1}^{8} \mathbf{e}_i(t)\, \nabla s_i(u,v,w)$.
"""

Problem 7.3

Solution(s)

(a)

S3-44 is a 4-node, 3D shell (ANCF-style) element. Each corner node carries **four vector nodal unknowns**:

- $\mathbf{r}$ — position (3 components),
- $\mathbf{r}_{,u}$ — position gradient w.r.t. the element's local u coordinate (3 components),
- $\mathbf{r}_{,v}$ — position gradient w.r.t. the element's local v coordinate (3 components),
- $\mathbf{r}_{,uv}$ — mixed position gradient (cross-derivative) w.r.t. u and v (3 components).

Total DOFs: $4 \text{ nodes} \times 4 \text{ vectors/node} \times 3 = 48.$

The thickness direction $w \in [-H/2, H/2]$ is included for integration; the interpolation below depends on u, v (surface) and is independent of w.

(b)

Number the four corner nodes in the (u, v) parent domain as
$(0,0) \to \text{node } 1,\ (L,0) \to \text{node } 2,\ (L,W) \to \text{node } 3,\ (0,W) \to \text{node } 4.$
Let the vector of time-dependent unknowns be

$$N(t) = \left[\mathbf{r}_1\ \mathbf{r}_{1,u}\ \mathbf{r}_{1,v}\ \mathbf{r}_{1,uv}\ \mathbf{r}_2\ \mathbf{r}_{2,u}\ \mathbf{r}_{2,v}\ \mathbf{r}_{2,uv}\ \mathbf{r}_3\ \mathbf{r}_{3,u}\ \mathbf{r}_{3,v}\ \mathbf{r}_{3,uv}\ \mathbf{r}_4\ \mathbf{r}_{4,u}\ \mathbf{r}_{4,v}\ \mathbf{r}_{4,uv}\right] = \left[\mathbf{e}_1\ \mathbf{e}_2\ \dots\ \mathbf{e}_{16}\right].$$

With scalar shape functions $s_i(u, v, w)$, the interpolation is

$$\mathbf{r}(u, v, w, t) = N(t)\, s(u, v, w) = \sum_{i=1}^{16} \mathbf{e}_i(t)\, s_i(u, v, w).$$

(c)

Introduce normalized coordinates $\xi := u/L = u/10 \in [0,1]$ and $\eta := v/W = v/10 \in [0,1]$.
Use **tensor-product Hermite** polynomials in ξ and η:

$$\begin{aligned}
h_1(\xi) &= 1 - 3\xi^2 + 2\xi^3, & h_2(\xi) &= \xi - 2\xi^2 + \xi^3,\\
h_3(\xi) &= 3\xi^2 - 2\xi^3, & h_4(\xi) &= -\xi^2 + \xi^3,\\
g_1(\eta) &= 1 - 3\eta^2 + 2\eta^3, & g_2(\eta) &= \eta - 2\eta^2 + \eta^3,\\
g_3(\eta) &= 3\eta^2 - 2\eta^3, & g_4(\eta) &= -\eta^2 + \eta^3.
\end{aligned}$$

At each node we associate four scalar shape functions multiplying the nodal sets $\{\mathbf{r}, \mathbf{r}_{,u}, \mathbf{r}_{,v}, \mathbf{r}_{,uv}\}$.
Using the standard Hermite scaling, the sixteen s_i are

Node 1 at $(\xi, \eta) = (0, 0)$:

$$\boxed{\begin{aligned} s_1 &= h_1 g_1, \\ s_2 &= L\, h_2 g_1, \\ s_3 &= W\, h_1 g_2, \\ s_4 &= LW\, h_2 g_2. \end{aligned}}$$

Node 2 at $(1, 0)$:

$$\boxed{\begin{aligned} s_5 &= h_3 g_1, \\ s_6 &= L\, h_4 g_1, \\ s_7 &= W\, h_3 g_2, \\ s_8 &= LW\, h_4 g_2. \end{aligned}}$$

Node 3 at $(1, 1)$:

$$\boxed{\begin{aligned} s_9 &= h_3 g_3, \\ s_{10} &= L\, h_4 g_3, \\ s_{11} &= W\, h_3 g_4, \\ s_{12} &= LW\, h_4 g_4. \end{aligned}}$$

Node 4 at $(0, 1)$:

$$\boxed{\begin{aligned} s_{13} &= h_1 g_3, \\ s_{14} &= L\, h_2 g_3, \\ s_{15} &= W\, h_1 g_4, \\ s_{16} &= LW\, h_2 g_4. \end{aligned}}$$

These satisfy the Kronecker properties at the corners and enforce the prescribed first and mixed derivatives in u and v. The interpolation is independent of w; thickness $H = 0.1$ affects integration limits, not the s_i themselves.

(d)

Let $H \in \mathbb{R}^{16\times 3}$ stack the gradients $\left[\, s_{i,u} \quad s_{i,v} \quad s_{i,w} \,\right]$ row-wise. Using

$$\frac{\partial}{\partial u} = \frac{1}{L}\frac{\partial}{\partial \xi} = \tfrac{1}{10}\frac{\partial}{\partial \xi}, \qquad \frac{\partial}{\partial v} = \frac{1}{W}\frac{\partial}{\partial \eta} = \tfrac{1}{10}\frac{\partial}{\partial \eta}, \qquad \frac{\partial}{\partial w} = 0,$$

and the 1D derivatives

$$h_1' = -6\xi + 6\xi^2, \quad h_2' = 1 - 4\xi + 3\xi^2, \quad h_3' = 6\xi - 6\xi^2, \quad h_4' = -2\xi + 3\xi^2,$$
$$g_1' = -6\eta + 6\eta^2, \quad g_2' = 1 - 4\eta + 3\eta^2, \quad g_3' = 6\eta - 6\eta^2, \quad g_4' = -2\eta + 3\eta^2,$$

the nonzero entries are obtained by product rule. For example,

$$\begin{aligned}
&s_{1,u} = \tfrac{1}{10}\, h_1' g_1, && s_{1,v} = \tfrac{1}{10}\, h_1 g_1', && s_{1,w} = 0,\\
&s_{2,u} = h_2' g_1, && s_{2,v} = \tfrac{1}{10}\, L\, h_2 g_1' = \tfrac{1}{10}\, 10\, h_2 g_1' = h_2 g_1', && s_{2,w} = 0,\\
&s_{3,u} = \tfrac{1}{10}\, W\, h_1' g_2 = h_1' g_2, && s_{3,v} = \tfrac{1}{10}\, W\, h_1 g_2' = h_1 g_2', && s_{3,w} = 0,\\
&s_{4,u} = W\, h_2' g_2, && s_{4,v} = L\, h_2 g_2', && s_{4,w} = 0,
\end{aligned}$$

and similarly for $s_5, \ldots, s_{16}$ by replacing (h_1, h_2) with (h_3, h_4) when ξ is anchored at 1, and (g_1, g_2) with (g_3, g_4) when η is anchored at 1. Thus

$$H(u, v, w) = \begin{bmatrix} s_{1,u} & s_{1,v} & 0 \\ \vdots & \vdots & \vdots \\ s_{16,u} & s_{16,v} & 0 \end{bmatrix}.$$

Problem 7.3