Cubical models of $(\infty, 1)$-categories

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Overview

Our goal: a cubical analogue of the Joyal model structure, filling in the bottom corner of the table:

category \ theory	∞-groupoids	$(\infty, 1)$-categories
sSet	Quillen	Joyal
cSet	Grothendieck	present work

Throughout this talk, we work with cubical sets having only:

- faces $\partial_{i,\varepsilon} \colon [1]^n \to [1]^{n+1}$;
- degeneracies $\sigma_i \colon [1]^n \to [1]^{n-1}$;
- max-connections $\gamma_i \colon [1]^n \to [1]^{n-1}$.

We write cubical structure maps on the right, e.g. $x\partial_{i,\epsilon}$.

Main result

Theorem

The category cSet *of cubical sets carries a model structure in which:*

- *the cofibrations are the monomorphisms;*
- *the fibrant objects are defined by having fillers for all inner open boxes.*

This model structure is Quillen equivalent to the Joyal model structure on sSet *via the triangulation functor* $T \colon \mathrm{cSet} \to \mathrm{sSet}$.

Review: The Grothendieck model structure

In the **Grothendieck model structure** on cSet:

- Cofibrations are monomorphisms;
- Fibrations are defined by the right lifting property with respect to open box inclusions $\sqcap^n_{i,\varepsilon} \hookrightarrow \square^n$;
- Weak equivalences $X \to Y$ induce bijections on homotopy classes $[Y, Z] \to [X, Z]$ where Z is fibrant.

Review: The Grothendieck model structure

Here open box fillings play the role of horn fillings in the Quillen model structure on sSet

$$
\begin{tikzcd}
(0,1) \arrow[r] & (1,1) & & (0,1) \arrow[r] & (1,1) \\
& & \arrow[r, hook] & & \\
(0,0) \arrow[uu] & (1,0) \arrow[uu] & & (0,0) \arrow[uu] \arrow[r] & (1,0) \arrow[uu]
\end{tikzcd}
$$

$\sqcap^2_{2,0}$ $\qquad\qquad$ $\square^2$

Review: The Grothendieck model structure

Theorem (Cisinski)

The adjunction $T : \mathrm{cSet} \rightleftarrows \mathrm{sSet} : U$ is a Quillen equivalence between the Grothendieck and Quillen model structures. $\square$

So the Grothendieck model structure presents the theory of ∞-groupoids – is there a model structure on cSet for $(\infty, 1)$-categories?

We begin with a model structure on **marked cubical sets**.

Structurally marked cubical sets

Define a new category $\Box_\sharp$ by adding an object $[1]_e$ to $\Box$.
New generating maps:

$$\varphi \colon [1] \to [1]_e$$

$$\zeta \colon [1]_e \to [0]$$

such that $\zeta\varphi = \sigma_1^1$.

$$\begin{tikzcd}
 & {[1]_e} \arrow[dl, "\zeta"'] & & \\
{[0]} \arrow[r, shift left=3] \arrow[r, shift right=3] & {[1]} \arrow[l] \arrow[u, "\varphi"] \arrow[r, shift left=7] \arrow[r, shift left=4] \arrow[r, shift right=4] \arrow[r, shift right=7] & {[1]^2} \arrow[l, shift left=1.5] \arrow[l, shift right=1.5] & \cdots
\end{tikzcd}$$

Structurally marked cubical sets

cSet$''$: category of presheaves on $\square_\sharp$. **Structurally marked cubical sets**.

"Cubical sets with (possibly multiple) markings on their edges".

$\mathrm{hom}(-, [1]_e) := (\square^1)^\sharp$
For $X \in$ cSet$''$, $X([1]_e) := X_e$. "Markings in X".

- $\alpha \in X_e \Rightarrow \alpha\varphi \in X_1$. Underlying edge of marking α.
- $x \in X_0 \Rightarrow x\zeta \in X_e$ with $x\zeta\varphi = x\sigma_1$. "Distinguished marking on $x\sigma_1$".

Marked cubical sets

Marked cubical sets: structurally marked cubical set with at most one marking on each edge.

cSet': category of marked cubical sets. Maps are simply cubical set maps preserving marked edges.

Think of marked edges as "equivalences".

Marked cubical sets

Two obvious ways of marking a cubical set X:

- **Maximal marking** $X^\sharp$: all edges marked
- **Minimal marking** $X^\flat$: only degenerate edges marked

These are functorial, and we have adjunctions:

$$\begin{tikzcd}
\mathrm{cSet}'(') \arrow[r] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathrm{cSet} \arrow[l, bend right, "(-)^\flat"'] \arrow[l, bend left, "(-)^\sharp"]
\end{tikzcd}$$

Geometric product of structurally marked cubical sets

Extend $\otimes\colon \Box \times \Box \to \mathsf{cSet}$ to $\otimes\colon \Box_\sharp \times \Box_\sharp \to \mathsf{cSet}''$ as follows:

- $[1]^n \otimes [1]_e$ has $\Box^{n+1}$ as underlying cubical set with edges $(\varepsilon_1, ..., \varepsilon_n, 0) \to (\varepsilon_1, ..., \varepsilon_n, 1)$ marked;
- $[1]_e \otimes [1]^n$ has $\Box^{n+1}$ as underlying cubical set with edges $(0, \varepsilon_1, ..., \varepsilon_n) \to (1, \varepsilon_1, ..., \varepsilon_n)$ marked;
- $[1]_e \otimes [1]_e = (\Box^2)^\sharp$.

Example: $[1] \otimes [1]_e =$

$$\begin{tikzcd} (1,0) \arrow[r] & (1,1) \\ (0,0) \arrow[u, "\sim"] \arrow[r] & (0,1) \arrow[u, "\sim"'] \end{tikzcd}$$

Geometric product of structurally marked cubical sets

Kan extend as with the geometric product of cubical sets:

$$\begin{tikzcd} \Box_\sharp \times \Box_\sharp \arrow[r] \arrow[d, hook] & \mathrm{cSet}'' \\ \mathrm{cSet}'' \times \mathrm{cSet}'' \arrow[ur, dashed, "\otimes"'] & \end{tikzcd}$$

This defines a monoidal product on cSet'', and restricts to a monoidal product on cSet'.

For each X, the functor $X \otimes - \colon \mathrm{cSet}'(') \to \mathrm{cSet}'(')$ has a right adjoint $\underline{\mathrm{hom}}_R(X, -)$.

- $\underline{\mathrm{hom}}_R(X, Y)_n = \mathrm{cSet}''(X \otimes \square^n, Y)$;
- $\underline{\mathrm{hom}}_R(X, Y)_e = \mathrm{cSet}''(X \otimes (\square^1)^\sharp, Y)$.

Similarly, $- \otimes X$ has a right adjoint $\underline{\mathrm{hom}}_L(X, Y)$.

First goal: a model structure on cSet', analogous to the marked model structure on sSet'.

What do we need?

- Generating anodyne maps
- A concept of homotopy

The critical edge

What kinds of open boxes represent composition?

Certain **critical edges** should be marked.

$$
\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "\sim"'] & \cdot \arrow[d] \\
\cdot & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "\sim"] \arrow[d] & \cdot \\
\cdot \arrow[r] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[d] & \cdot \arrow[d, "\sim"] \\
\cdot \arrow[r] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r] & \cdot \arrow[d] \\
\cdot \arrow[r, "\sim"] & \cdot
\end{tikzcd}
$$

For $n \geq 1, 1 \leq i \leq n, \varepsilon \in \{0, 1\}$, the **critical edge** of $\square^n$ with respect to face $\partial_{i,\varepsilon}$ is the unique edge which:

- is adjacent to $\partial_{i,\varepsilon}$;
- together with $\partial_{i,\varepsilon}$, contains vertices $(0, ..., 0)$ and $(1, ..., 1)$.

Special open boxes

For $n \geq 1, 1 \leq i \leq n, \varepsilon \in \{0,1\}$ we have the (i,ε) **special open box inclusion** $\iota^n_{i,\varepsilon}$:

- Underlying cubical set map is $\sqcap^n_{i,\varepsilon} \hookrightarrow \square^n$;
- Critical edge wrt face (i,ε) is marked in domain and codomain.

$$(0) \xhookrightarrow{\iota^1_{1,0}} (0) \xrightarrow{\sim} (1)$$

$$\begin{tikzcd}
(0,1) \arrow[r] & (1,1) & & (0,1) \arrow[r] & (1,1) \\
(0,0) \arrow[u] & (1,0) \arrow[u, "\sim"'] \arrow[rr, phantom, "\xhookrightarrow{\iota^2_{2,0}}"] & & (0,0) \arrow[u] \arrow[r] & (1,0) \arrow[u, "\sim"']
\end{tikzcd}$$

The saturation map

$$K := \begin{tikzcd} 1 \arrow[r] \arrow[d, equal] & 0 \arrow[r, equal] \arrow[d] & 0 \arrow[d, equal] \\ 1 \arrow[r, equal] & 1 \arrow[r] & 0 \end{tikzcd}$$

An edge $\square^1 \to X$ factoring through the middle edge of K is an **equivalence**.

$K' := K$ with the middle edge marked.
The **saturation map** is the inclusion $K \hookrightarrow K'$.

The 3-out-of-4-maps

$$
\begin{tikzcd}
(0,1) \arrow[r, "\sim"] & (1,1) \\
(0,0) \arrow[u, "\sim"] \arrow[r] & (1,0) \arrow[u, "\sim"]
\end{tikzcd}
\quad \hookrightarrow \quad
\begin{tikzcd}
(0,1) \arrow[r, "\sim"] & (1,1) \\
(0,0) \arrow[u, "\sim"] \arrow[r, "\sim"] & (1,0) \arrow[u, "\sim"]
\end{tikzcd}
$$

and 3 others for other sides.

Anodyne maps: Saturation of special open box inclusions, saturation map, 3-out-of-4.

Naive fibrations: RLP(Anodyne maps).

Marked cubical quasicategory: $X \in \mathrm{cSet}'$ such that $X \to \square^0$ is a naive fibration. (Suffices to check special open boxes and saturation map.)

Proposition

In a marked cubical quasicategory X, the marked edges are exactly the equivalences.

Proof.

$X \to \square^0$ lifts against $K \to K'$ by assumption.
The inclusion $(\square^1)^\sharp \to K'$ is a pushout of special open box fillings, so $X \to \square^0$ lifts against this map as well. $\square$

$$
\begin{tikzcd}
1 \arrow[r] \arrow[d, equal] & 0 \arrow[r, equal] \arrow[d, "\sim"] & 0 \arrow[d, equal] \\
1 \arrow[r, equal] & 1 \arrow[r] & 0
\end{tikzcd}
$$

An **elementary right homotopy** of maps $f, g \colon X \to Y$ in $\mathrm{cSet}'(')$ is a map $H \colon X \otimes (\Box^1)^\sharp \to Y$ with $H|_{\{0\}} = f, H|_{\{1\}} = g$.

A **right homotopy** is a zigzag of elementary right homotopies.

By adjointness, right homotopies correspond to zigzags of marked edges in $\underline{\mathrm{hom}}_R(X, Y)$.

The cubical marked model structure

Theorem

cSet' *carries a model structure in which:*

- *Cofibrations are monomorphisms;*
- *Fibrant objects are marked cubical quasicategories;*
- *Fibrations between fibrant objects are naive fibrations;*
- *Weak equivalences* $X \to Y$ *induce bijections on homotopy classes* $[Y, Z] \to [X, Z]$ *for* Z *fibrant.*

$\square$

This resembles a Cisinski model structure, except that cSet' is not a presheaf category. We construct it using Jeff Smith's theorem.

By (HKRS,2017) we can transfer this model structure along $\mathrm{cSet} \rightleftarrows \mathrm{cSet}'$, where the left adjoint is the minimal marking and the right is the forgetful functor.

We obtain the **cubical Joyal model structure** on cSet. Cofibrations and weak equivalences created by minimal marking.

Theorem

The adjunction $\mathrm{cSet} \rightleftarrows \mathrm{cSet}'$ *is a Quillen equivalence.*

Proof.

The left adjoint $(-)^\flat$ preserves and reflects cofibrations and weak equivalences by definition.
For a marked cubical quasicategory X, the counit is a composite of pushouts of the saturation map. $\square$

Analysis of the cubical Joyal model structure

What can we say about this model structure on cSet?

- Cofibrations are monomorphisms.
- Goal: characterize weak equivalences, fibrant objects, fibrations between fibrant objects.
- Goal: show it is Quillen-equivalent to the Joyal model structure.

Inner open boxes

What are the cubical analogues of inner horns?

The **inner open box** $\widehat{\sqcap}^n_{i,\varepsilon}$ is $\sqcap^n_{i,\varepsilon}$ with the critical edge quotiented to a point.

Inner cube $\widehat{\square}^n_{i,\varepsilon}$: the corresponding quotient of $\square^n$.

Have an inclusion $\widehat{\sqcap}^n_{i,\varepsilon} \hookrightarrow \widehat{\square}^n_{i,\varepsilon}$.

Cubical quasicategories

A **cubical quasicategory** is $X \in \mathrm{cSet}$ having the RLP against inner open box fillings.

In particular, this lets us "compose" edges.

$$
\begin{tikzcd}
y \arrow[r, "g"] & z \arrow[d, equal] \\
x \arrow[u, "f"] & z
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\widehat{\sqcap}^2_{2,0} \arrow[r] \arrow[d, hook] & X \arrow[d] \\
\widehat{\square}^2 \arrow[r] & \square^0
\end{tikzcd}
$$

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$$
\begin{tikzcd}
y \arrow[r, "g"] & z \arrow[d, equal] \\
x \arrow[u, "f"] \arrow[r, dashed, "gf"] & z
\end{tikzcd}
\qquad
\begin{tikzcd}
\widehat{\sqcap}^2_{2,0} \arrow[r] \arrow[d, hook] & X \arrow[d] \\
\widehat{\square}^2 \arrow[r] \arrow[ur, dashed] & \square^0
\end{tikzcd}
$$

Special open boxes in cSet

For $X \in \mathrm{cSet}$, a **special open box** in X is $\sqcap^n_{i,\varepsilon} \to X$ sending the critical edge to an equivalence.

Proposition

Cubical quasicategories admit fillers for special open boxes. $\square$

Fibrant objects in cSet

Theorem

The fibrant objects in cSet *are precisely the cubical quasicategories.*

Proof.

Every fibrant object is a cubical quasicategory since inner open box inclusions are trivial cofibrations.
Every cubical quasicategory is the underlying cubical set of a marked cubical quasicategory. $\square$

A similar proof shows:

Theorem

Fibrations between fibrant objects are characterized by the RLP against inner open box inclusions and endpoint inclusions $\{\epsilon\} \hookrightarrow K$. $\square$

Weak equivalences in cSet

We define homotopy in this model structure using K as a cylinder object, i.e. **(right) homotopy** of maps $X \to Y$ is given by maps $X \otimes K \to Y$.

Theorem

A map $X \to Y$ is a weak equivalence in cSet *if and only if $[Y, Z] \to [X, Z]$ is a bijection for any cubical quasicategory Z.* $\square$

Mapping spaces

Let x_0 and x_1 be 0-cubes in a cubical quasicategory X.
$\mathrm{Map}_X(x_0, x_1)$ is the cubical set given by

$$\mathrm{Map}_X(x_0, x_1)_n = \left\{ \square^{n+1} \xrightarrow{s} X \;\middle|\; s\partial_{n+1,\varepsilon} = x_\varepsilon \right\},$$

with cubical operations given by those of X.

Example

- a 0-cube in $\mathrm{Map}_X(x_0, x_1)$ is a 1-cube from x_0 to x_1 in X;
- a 1-cube in $\mathrm{Map}_X(x_0, x_1)$ is a 2-cube in X of the form

$$\begin{tikzcd} x_0 \arrow[r, "f"] \arrow[d, equal] & x_1 \arrow[d, equal] \\ x_0 \arrow[r, "g"] & x_1 \end{tikzcd}$$

Mapping spaces

Proposition

Given a cubical quasicategory X and 0-cubes $x_0, x_1 : \Box^0 \to X$, the mapping space $\mathrm{Map}_X(x_0, x_1)$ is a cubical Kan complex. $\square$

Triangulation

Theorem

Triangulation and its right adjoint define a Quillen adjunction $T : \mathrm{cSet} \rightleftarrows \mathrm{sSet}_{\mathrm{Joyal}} : U$.

Proof.

T preserves cofibrations.
T sends $\{\varepsilon\} \hookrightarrow K$ to a trivial cofibration by direct computation.
$T\widehat{\sqcap}^n_{i,\varepsilon} \hookrightarrow T\widehat{\Box}^n$: use decomposition of $\sqcap^n_{i,\varepsilon} \hookrightarrow \Box^n$ as a pushout product, reduce to open prism filling in sSet. $\square$

Triangulation is difficult to work with. It would be hard to show directly that $T \dashv U$ is a Quillen equivalence.

We will develop another adjunction $Q : \mathrm{sSet} \rightleftarrows \mathrm{cSet} : \int$ and show that it is a Quillen equivalence, and that the derived functors of T and Q are inverses.

Cones

To define Q, we develop a theory of **cones** in cubical sets.
For $m, n \geq 0$, define the **standard** (m, n)-**cone** $C^{m,n}$ inductively as follows:

- $C^{m,0} = \Box^m$;
- For $n \geq 1$, $C^{m,n}$ is given by the following pushout:

$$\begin{tikzcd}
C^{m,n-1} \arrow[r] \arrow[d, "\partial_{1,1} \otimes C^{m,n-1}"'] & \Box^0 \arrow[d] \\
\Box^1 \otimes C^{m,n-1} \arrow[r] & C^{m,n} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

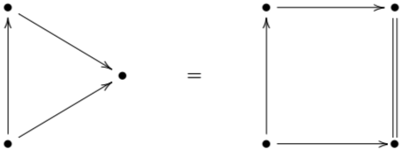
Each $C^{m,n}$ is a quotient of $\Box^{m+n}$.

Cones

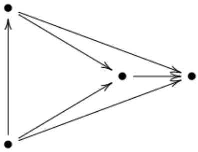
$C^{0,0}$

$C^{0,1} \cong C^{1,0}$

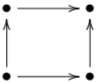
$C^{0,2} \cong C^{1,1}$



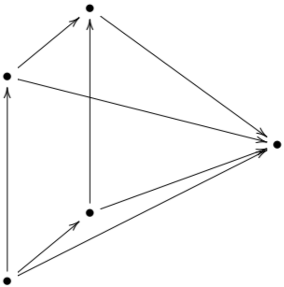
$C^{0,3} \cong C^{1,2}$



$C^{2,0}$



$C^{2,1}$



$Q \dashv \int$

Denote $C^{0,n}$ by Q^n. This defines a cosimplicial object $Q^\bullet \colon \Delta \to \mathrm{cSet}$.

a map $Q^{n-1} \to Q^n$	0^{th} face	1^{st} face	2^{nd} face	$\cdots$	j^{th} face	$\cdots$	n^{th} face
is induced by a map $\Box^{n-1} \to \Box^n$	$\partial_{n,1}$	$\partial_{n,0}$	$\partial_{n-1,0}$	$\cdots$	$\partial_{n-j+1,0}$	$\cdots$	$\partial_{1,0}$

a map $Q^n \to Q^{n-1}$	0^{th} deg.	1^{st} deg.	2^{nd} deg.	$\cdots$	j^{th} deg.	$\cdots$	$(n-1)^{\text{st}}$ deg.
is induced by a map $\Box^n \to \Box^{n-1}$	σ_n	γ_{n-1}	γ_{n-2}	$\cdots$	γ_{n-j}	$\cdots$	γ_1

This extends to a functor $Q \colon \mathrm{sSet} \to \mathrm{cSet}$ by left Kan extension.

Q has a right adjoint $\int$ given by $(\int X)_n = \mathrm{cSet}(Q^n, X)$.

Viewing sSet as $\mathrm{sSet} \downarrow \Delta^0$ and cSet as $\mathrm{cSet}^{[0]}$, $Q \dashv \int$ coincides with straightening $\dashv$ unstraightening.

$Q \dashv \int$

We'll show that $Q \dashv \int$ is a Quillen equivalence and use this to prove that $T \dashv U$ is a Quillen equivalence.

Theorem

The adjunction $Q : \mathsf{sSet} \rightleftarrows \mathsf{cSet} : \int$ is Quillen. □

Theorem

Q preserves and reflects weak equivalences.

Proof.

Both Q and T preserve weak equivalences.
We can define a natural weak equivalence $TQ \Rightarrow \mathrm{id}_{\mathsf{sSet}}$.

$$\begin{tikzcd} TQX \arrow[r] \arrow[d, "\sim"] & TQY \arrow[d, "\sim"] \\ X \arrow[r] & Y \end{tikzcd}$$

This shows Q reflects weak equivalences. □

The counit of $Q \dashv \int$

Our goal: show that the counit is a trivial cofibration for X a cubical quasicategory.

By (Kapulkin-Lindsey-Wong,2019) $Q \int X$ is the subcomplex of X whose cubes are those which factor through Q – the "maximal simplicial subcomplex" of X.

We factor the counit as a series of subcomplex inclusions:

$$Q \int X = X^1 \hookrightarrow X^2 \hookrightarrow ... \hookrightarrow X^m \hookrightarrow ... \hookrightarrow X$$

Non-degenerate cubes of each X^m are cubes factoring through some $C^{m',n'}$ with $m' \leq m$.

Each $X^m \hookrightarrow X^{m+1}$ is a transfinite composite of inner open box fillings.

$T \dashv U$ as a Quillen equivalence

Theorem

The adjunction $T : \mathrm{cSet} \rightleftarrows \mathrm{sSet} : U$ *is a Quillen equivalence.*

Proof.

The natural weak equivalence $TQ \Rightarrow \mathrm{id}_{\mathsf{sSet}}$ becomes a natural isomorphism in the homotopy category. The derived functor of Q is an equivalence of categories, hence so is that of T. $\square$