Cubical models of $(\infty, 1)$-categories

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December 5, 2020

Conclusions

Theorem

The category cSet *of cubical sets with connections carries a model structure that presents the homotopy theory of* $(\infty, 1)$*-categories, which is equivalent to the Joyal model structure via triangulation.*

References

- D., Kapulkin, Lindsey, Sattler, *Cubical models of* $(\infty, 1)$*-categories*, 2020. arXiv:2005.04853

Cubical sets

The **box category** $\square$:

- objects are $[1]^n = \{0 \leq 1\}^n$;
- morphisms are some subset of order-preserving maps.

Cubical sets are presheaves on $\square$

$$\mathrm{cSet} := \mathrm{Fun}(\square^{\mathrm{op}}, \mathrm{Set}),$$

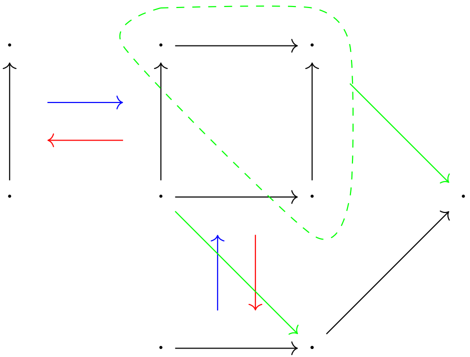
and are pieced together from **standard cubes** $\square^n$, $n \geq 0$:

$$
0 \quad , \quad 0 \longrightarrow 1 \quad , \quad
\begin{tikzcd}
10 \arrow[r] & 11 \\
00 \arrow[r] \arrow[u] & 01 \arrow[u]
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
& 110 \arrow[rr] & & 111 \\
100 \arrow[rr] \arrow[ur] & & 101 \arrow[ur] & \\
& 010 \arrow[rr] \arrow[uu] & & 011 \arrow[uu] \\
000 \arrow[rr] \arrow[uu] \arrow[ur] & & 001 \arrow[uu] \arrow[ur] &
\end{tikzcd}
\quad , \quad \cdots
$$

Cubical sets

In our work, maps in $\square$ are generated by:

- **face** and **degeneracy** maps
- **connections** (max and min)



The geometric product

The Cartesian product of cubical sets is not well-behaved, e.g. $\Box^m \times \Box^m \ncong \Box^{m+n}$.

Instead we work with the **geometric product**.

$$\begin{tikzcd}
\Box \times \Box \arrow[rr, "{([1]^m,[1]^n) \mapsto \Box^{m+n}}"] \arrow[d, hook] & & \mathsf{cSet} \\
\mathsf{cSet} \times \mathsf{cSet} \arrow[urr, dashed, "\otimes"'] & &
\end{tikzcd}$$

Why study cSet?

- **Simplification**
  - $\square^m \otimes \square^n \cong \square^{m+n}$ (easier to construct homotopies)
  - Straightening (Kapulkin-Voevodsky '18)
- **Applications to type theory** (CCHM '16)

Grothendieck model structure

cSet carries a model structure:

- cofibrations = monomorphisms;
- homotopy: geometric product with $\square^1$
  - I.e. a homotopy of maps $X \to Y$ is a map $\square^1 \otimes X \to Y$
  - From this we get homotopy equivalences, weak equivalences by standard techniques
- fibrations = RLP with respect to **open box fillings**

$$
\begin{tikzcd}
01 \arrow[r] & 11 & & 01 \arrow[r] & 11 \\
 & & \to & & \\
00 \arrow[uu] & 10 \arrow[uu] & & 00 \arrow[uu] \arrow[r] & 10 \arrow[uu]
\end{tikzcd}
$$

Comparing cSet and sSet: Triangulation

We define $T \colon \mathrm{cSet} \to \mathrm{sSet}$ by Kan extension:

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \square \arrow[r, "{[1]^n \mapsto (\Delta^1)^n}"] \arrow[d] & \mathrm{sSet} \\ \mathrm{cSet} \arrow[ur, "T"'] & \end{tikzcd}$$

T has a right adjoint U given by $(UX)_n = \mathrm{sSet}((\Delta^1)^n, X)$.

Theorem (Cisinski)

$T \dashv U$ is a Quillen equivalence between the Grothendieck model structure on cSet *and the Quillen model structure on* sSet.

Inner open boxes

Goal: construct a cubical analogue of $\mathrm{sSet}_{\mathrm{Joyal}}$, Quillen-equivalent via triangulation.

What's an **inner** open box?

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \cdot \arrow[d] & & \cdot \arrow[r] \arrow[d] & \cdot & & \cdot \arrow[d] & \cdot \arrow[d] & & \cdot \arrow[r] & \cdot \arrow[d] \\
\cdot & \cdot & & \cdot \arrow[r] & \cdot & & \cdot \arrow[r] & \cdot & & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

Solution: require critical edges to be degenerate!

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, equal, red] & \cdot \arrow[d] & & \cdot \arrow[r, equal, red] \arrow[d] & \cdot & & \cdot \arrow[d] & \cdot \arrow[d, equal, red] & & \cdot \arrow[r] & \cdot \arrow[d] \\
\cdot & \cdot & & \cdot \arrow[r] & \cdot & & \cdot \arrow[r] & \cdot & & \cdot \arrow[r, equal, red] & \cdot
\end{tikzcd}$$

Cubical quasicategories

A **cubical quasicategory** is $X \in \mathrm{cSet}$ having fillers for inner open boxes.

In particular, this lets us "compose" edges.

$$
\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, equal] & y \arrow[d, "g"] \\
x & z
\end{tikzcd}
$$

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$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, equal] & y \arrow[d, "g"] \\
x \arrow[r, dashed, "gf"] & z
\end{tikzcd}$$

The cubical Joyal model structure

Theorem (D.-Kapulkin-Lindsey-Sattler)

cSet *carries a model structure:*

- *Cofibrations are monomorphisms;*
- *Fibrant objects are cubical quasicategories;*
- *Homotopy: geometric product with*

$$K = \begin{tikzcd} \bullet \arrow[r] \arrow[d, equal] & \bullet \arrow[r, equal] \arrow[d] & \bullet \arrow[d, equal] \\ \bullet \arrow[r, equal] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

i. e. a homotopy of maps $X \to Y$ *is* $H \colon K \otimes X \to Y$.

Main result

Theorem (D.-Kapulkin-Lindsey-Sattler)

The adjunction $T : \mathsf{cSet} \rightleftarrows \mathsf{sSet} : U$ *is a Quillen equivalence between the Joyal and cubical Joyal model structures.*

category \ theory	∞-groupoids	$(\infty, 1)$-categories
sSet	Quillen	Joyal
cSet	Grothendieck	cubical Joyal

Other results

- Two more models of $(\infty, 1)$-categories: **marked cubical sets** and **structurally marked cubical sets**.
- Theory of **cones in cubical sets**.
- Definitions of **homotopy categories** and **mapping spaces** for cubical quasicategories, and characterization of equivalences of cubical quasicategories in terms of these concepts.
  - New proof of the analogous result for simplicial sets.