Midterm 2 Review - Math 243

1. Determine the type of the equilibrium (sink, source, spiral sink, spiral source, saddle, or center) at the origin for each of the following linear systems.
   (a) $\dfrac{dX}{dt} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} X$
   (b) $\dfrac{dY}{dt} = \begin{bmatrix} -1 & -5 \\ 4 & -2 \end{bmatrix} Y$
   (c) $\dfrac{dZ}{dt} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} Z$

2. Show that $\begin{bmatrix} 2 \\ i \end{bmatrix}$ is an eigenvector for the matrix $A = \begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix}$. What is the corresponding eigenvalue?

3. Use the eigenvalue and eigenvector from the previous problem to find the general solution of the system
$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix} \mathbf{x}.$$

4. Find the eigenvalues for the matrix $\begin{bmatrix} 2 & -6 \\ 2 & -3 \end{bmatrix}$. For each eigenvalue, find a corresponding eigenvector.

5. Use the eigenvectors and eigenvalues from the last problem to find the general solution of the linear system
$$\frac{dx}{dt} = 2x - 6y,$$
$$\frac{dy}{dt} = 2x - 3y.$$

6. The system
$$\frac{dx}{dt} = x - 2y,$$
$$\frac{dy}{dt} = 3y$$
is partially decoupled. Suppose that $x(0) = 1$ and $y(0) = 2$. Solve this initial value problem by solving for $y(t)$ first, and then use $y(t)$ to find $x(t)$.

7. The previous example can be expressed using matrix notation as $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$. It turns out that
$$e^{At} = \begin{bmatrix} e^t & -e^{3t} + e^t \\ 0 & e^{3t} \end{bmatrix}.$$
Use this to solve the initial value problem above.

8. Consider the one-parameter system $\dfrac{d\mathbf{y}}{dt} = \begin{bmatrix} 1 & a \\ 1 & 3 \end{bmatrix} \mathbf{y}$. Use the trace and determinant to find the values of a where the type of equilibrium changes. Describe in words how the type of equilibrium depends on a.