Midterm 1 Review - Math 243

1. Consider the initial value problem

$$\frac{dy}{dt} - 3(y-1)^{2/3} = 0, \quad y(0) = 1$$

   (a) Verify that $y(t) = 1$ is a solution of the initial value problem above.

   (b) Verify that $y(t) = t^3 + 1$ is also a solution to the initial value problem.

   (c) Why doesn't this contradict the Uniqueness Theorem? Explain.

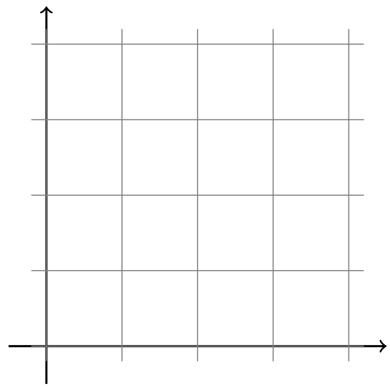
2. Solve the initial value problem

$$y' = \frac{2t+1}{y^2}; \qquad y(1) = 2.$$

3. Consider the equation

$$\frac{dy}{dt} = (y-1)^2 + t.$$

(a) Sketch the slope field for this differential equation. Just use the points (t, y) with integer coordinates and $0 \leq t \leq 4$ and $0 \leq y \leq 4$.



(b) Use Euler's method with step size $\Delta t = 1$ to estimate the values of $y(t)$ for $t = 1, 2, 3$ given the initial condition $y(0) = 1$. Add a sketch the Euler's method solution to the slope field above.

4. The following coupled system is a predator-prey population model.

$$\frac{dA}{dt} = 5A - \frac{A^2}{1000} - 3AB$$

$$\frac{dB}{dt} = B\sqrt{A}$$

(a) Which of the variables A or B represents the predator and which represents the prey? Explain your answer.

(b) In the model, what would happen to the predator population if the prey is extinct?

(c) What would happen to the prey population if there were no predators?

5. Let $\dfrac{dy}{dt} = f_\alpha(y)$ be a family of autonomous differential equations parametrized by α where $f_\alpha(y) = y^2 - 2y + \alpha$.

   (a) Draw phase lines for $\alpha = 0$, $\alpha = 1$, $\alpha = 2$. In each case identify the equilibria and say whether they are stable, unstable, or nodes.

   (b) Use the phase lines in part (a) to sketch solutions to the initial value problem $y(0) = 0$ for the three cases, $\alpha = 0$, $\alpha = 1$, and $\alpha = 2$. (Use a different graph for each α).

   (c) What is (are) the bifurcation value(s) for this family of equations?

   (d) Draw the bifurcation diagram.

6. Find the general solutions for the following differential equations.

   (a) $\dfrac{dy}{dt} + \dfrac{2}{t}y = 4t^2$.

   (b) $\dfrac{dy}{dt} + 2y = 2t + 1$

7. Solve the following initial value problems.

(a) $y' - 5y = e^{5t}$, $y(0) = 4$.

(b) $\dfrac{dy}{dx} = x(y-1)$, $y(0) = 3$

(c) $\dfrac{dy}{dt} + \dfrac{y}{t+1} = 6t$, $y(1) = 4$.

8. Find the general solution to the partially coupled system

$$\frac{dx}{dt} = 2x + 3y$$
$$\frac{dy}{dt} = -4y.$$

9. Suppose the population of fish in a pond obeys a logistic growth model $\dfrac{dP}{dt} = 0.3\left(1 - \dfrac{P}{2000}\right)$ where t is measured in years.

(a) How would you change the model if 100 fish were harvested from the pond each year?

(b) How would you change the model if a quarter of the fish were harvested from the pond each year?

10. Consider the 2nd order homogeneous linear differential equation

$$y'' + 2y' + 5y = 0.$$

(a) Find the general solution.

(b) Show that $y(3) = (1 + 3i)e^{(-1+2i)t} + (1 - 3i)e^{(-1-2i)t}$ is a solution that satisfies the initial conditions $y(0) = 2$ and $y'(0) = 10$.

(c) Show that $y(t) = 2e^{-t}\cos(2t) + 6e^{-t}\sin(2t)$ is also a solution that satisfies the initial conditions $y(0) = 2$ and $y'(0) = 10$.