Practice Final Exam - Math 140

1. Find the intervals of increase/decrease for $f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - 6x^2$.

Solution: $f'(x) = x^3 - x^2 - 12x$ which factors as $x(x-4)(x+3)$. Therefore decreasing on $(-\infty, -3)$ and $(0, 4)$, increasing on $(-3, 0)$ and $(4, \infty)$.

2. Use the logarithm rules to simplify, then differentiate $y = \ln\left(\frac{x}{(x+4)^2}\right)$.

Solution: $y = \ln x - 2\ln(x+4)$, therefore $y' = \frac{1}{x} - \frac{2}{x+4}$.

3. Find a formula for the linear function $f(x)$ with $f(0) = -1$ and $f(3) = 5$.

Solution: The slope is 2 and the y-intercept is -1, so the line is $f(x) = 2x - 1$.

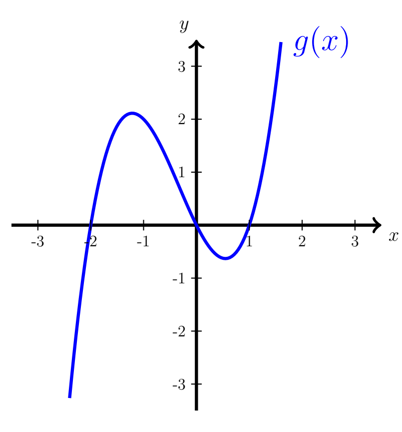
4. The graph of a function $y = g(x)$ is shown. Use the graph to find the following.

(a) What are the roots of $g(x)$?

Solution: The graph hits the x-axis at -2, 0, and 1.

(b) What is $g(-1)$?

Solution: It looks like $g(-1) = 2$.



(c) Is $g'(-1)$ positive, negative, or zero?

Solution: It is negative.

5. Solve $\dfrac{1}{x} - \dfrac{2}{x+4} = 0$.

Solution: With common denominators you get: $\dfrac{x+4-2x}{x(x+4)} = \dfrac{4-x}{x(x+4)} = 0$, so the solution is $x = 4$.

6. Find the following derivatives.

(a) $\dfrac{d}{dx} x^{1/5}$

Solution: Power rule:

$$\tfrac{1}{5}x^{-4/5}$$

(b) $\dfrac{d}{dx} e^{(x^2+3x)}$

Solution: Chain rule:

$$(2x+3)e^{(x^2+3x)}$$

(c) $\dfrac{d}{dx} \dfrac{x+5}{x^2-4}$

Solution: Quotient rule:

$$\frac{(x^2-4)-(x+5)(2x)}{(x^2-4)^2}$$

(d) $\dfrac{d}{dx} x^2 e^x$

Solution: Product rule:

$$2xe^x + x^2e^x$$

7. Find the (x, y) coordinates of the local max of $f(x, y) = 9 - x^2 - y^2 + 6y$.

Solution: Set the partial derivatives equal to zero and solve:

$$f_x = -2x = 0$$

$$f_y = -2y + 6 = 0$$

So $x = 0$ and $y = 3$. To check that this is a max, find the 2nd partial derivatives and calculate the determinant $f_{xx}f_{yy} - (f_{xy})^2$.

$$f_{xx} = -2, \quad f_{yy} = -2, \quad f_{xy} = 0,$$

So $D = 4$ and therefore $(0, 3)$ is a local max by the second derivative test.

8. What is the slope of the tangent line to $h(x) = \frac{1}{3}x^3 - 3x^2 + 5x$ at the point $(3, -3)$?

Solution: The derivative is $h'(x) = x^2 - 6x + 5$. Plug in $x = 3$ to get $9 - 18 + 5 = -4$ as the slope of the tangent line.

9. Find the differential of $y = e^x$ at $x = 0$ when $dx = 0.1$ and use it to estimate the value of $e^{0.1}$.

Solution:

$$dy = e^x dx$$

Substituting $x = 0$ and $dx = 0.1$, we get $dy = 0.1$. We also know that when $x = 0$, $y = e^0 = 1$, so we adjust that by adding the dy to get

$$e^{0.1} \approx 1.1.$$

10. Simplify $(\sqrt{2} + \sqrt{50})^2$. *Remember: Powers don't distribute to terms!*

Solution: FOIL: $(\sqrt{2} + \sqrt{50})(\sqrt{2} + \sqrt{50}) = 2 + 10 + 10 + 50 = 72$.

11. Find the absolute maximum and minimum y-values of the function $y = x + \dfrac{100}{x}$ on the interval $[5, 25]$.

Solution: $y' = 1 - 100x^{-2}$. Therefore $x = 10$ is a critical point. So we calculate the y-values at the critical point and the endpoints:

x	y
5	25
10	20
25	29

Therefore the absolute max is at $(25, 29)$ and the absolute min is at $(10, 20)$.

12. Suppose that the population of a town is growing by 5% per year and the current population is 10,000, so the formula for the population after t years is

$$P(t) = 10{,}000(1.05)^t.$$

How long will it be until the population of the town reaches 30,000 people? It is okay to write your answer as a formula using logarithms.

Solution: Solving $10{,}000(1.05)^t = 30{,}000$ is the same as solving $(1.05)^t = 3$. Take the natural-log of both sides, then solve for t to get

$$t = \frac{\ln 3}{\ln 1.05}.$$

13. Solve $2e^{5x} = 64$.

Solution: $5x = \ln 32$, so $x = \frac{1}{5}\ln 32 = \ln 2$.

14. A fruit importer will sell $q(p) = 800 - 20p$ boxes of fruit when the price of a box is p dollars. What price would maximize the fruit importer's revenue?

Solution: $R = 800p - 20p^2$ so $R' = 800 - 40p = 0$ when $p = 800/40 = 20$. Since the second derivative is -40, this is a max.

15. Calculate the following logarithms.

(a) $\log_3(9)$

Solution: 2

(b) $\log_2\left(\frac{8}{\sqrt{2}}\right)$

Solution: $3 - 0.5 = 2.5$

16. Find the partial derivatives (both f_x and f_y) for $f(x, y) = x^2 + y^4 + 2xy^3$.

Solution: $f_x = 2x + 2y^3$ and $f_y = 4y^3 + 6xy^2$.

17. The production level at a factory is $Q(x, y) = 40x^{0.25}y^{0.75}$ where x is hours of labor and y is capital expenditure. Find the partial derivatives of Q when $x = 100$ and $y = 100$.

Solution: $Q_x = 10x^{-0.75}y^{0.75}$ and $Q_y = 30x^{0.25}y^{-0.25}$. Plugging in 100 for x and y, we get $Q_x = 10$ units of output per hour of labor and $Q_y = 30$ units of output per dollar of capital expenditure.

18. A box with a square bottom and an open top needs to have exactly 4 cubic feet of volume. Find the dimensions of the box with the smallest possible surface area. Recall that the volume of a box is $V = x^2y$ where x is the length and width and y is the height. The surface area is $S = 4xy + x^2$.

Solution: Substituting $y = 4/x^2$ into S, we get $S = 16/x + x^2$. So $S' = -16/x^2 + 2x = 0$ when $x = 2$. Then $y = 1$. And that is a minimum since $S'' = 32/x^3 + 2$ which is positive.

Solution using Lagrange multipliers
Constraint: $V = x^2y = 4$ Objective: $S = 4xy + x^2$.