e in a box of cereal

Finding the number e in history and everyday life.

Brian Lins

Hampden-Sydney College

October 23, 2025

How to explain e?

Only one of the following is accessible without calculus.

$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$	$e = \sum_{n=0}^{\infty} \frac{1}{n!}$
$\frac{d}{dx} e^x = e^x$	$\int_1^e \frac{1}{x}\, dx = 1$

$$e \approx 2.718281828459$$

e in a box of cereal

In the 1980's, Golden Grahams cereal had a promotion where 1 out of every 15 boxes of cereal had a digital watch.

I really wanted a digital watch, so I tried to talk my mom into buying 15 boxes of cereal!

e in a box of cereal

Suppose we did buy 15 boxes of cereal.

- First box's probability of **not** winning $= \frac{14}{15}$.
- Second box's probability of **not** winning $= \frac{14}{15}$.
- $\ldots$
- Fifteenth box's probability of **not** winning $= \frac{14}{15}$.

e in a box of cereal

Suppose we did buy 15 boxes of cereal.

- First box's probability of **not** winning $= \frac{14}{15}$.
- Second box's probability of **not** winning $= \frac{14}{15}$.
- $\ldots$
- Fifteenth box's probability of **not** winning $= \frac{14}{15}$.

So the combined probability of **not winning any watches** is:

$$\left(\frac{14}{15}\right)^{15} = 35.526\%$$

e in a box of cereal

Suppose we did buy 15 boxes of cereal.

- First box's probability of **not** winning $= \frac{14}{15}$.
- Second box's probability of **not** winning $= \frac{14}{15}$.
- $\ldots$
- Fifteenth box's probability of **not** winning $= \frac{14}{15}$.

So the combined probability of **not winning any watches** is:

$$\left(\frac{14}{15}\right)^{15} = 35.526\%$$

This is approximately $\dfrac{1}{e}$ $(= 36.788\%)$!

e in a box of cereal

If only 1 out of n boxes has a prize, and you buy n boxes, then the probability you'll win nothing is:

$$\Pr(\text{zero prizes}) = \left(1 - \frac{1}{n}\right)^n .$$

This is one of the formulas for e:

$$e^r = \lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^n .$$

How e is usually introduced

The formula

$$e^r = \lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^n$$

is usually explained with compound interest.

Suppose you invest in a bank that pays $r = 6\%$ APR.

- Compounded once 1.06.
- Compounded quarterly $(1 + \frac{0.06}{4})^4 = 1.06136$.
- Compounded monthly $(1 + \frac{0.06}{12})^{12} = 1.061678$.
- Compounded daily $(1 + \frac{0.06}{365})^{365} = 1.061831$.
- Infinite compounding $e^{0.06} = 1.061837$.

The discovery of e

- Jacob Bernoulli (1683) discovered that the limit

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

  converges to a number between 2.5 and 3.

- Leonhard Euler started using the letter e for this constant in the late 1720s.

History question

Why have different APR and APY numbers? The right way to split a 6% annual rate into 12 equal payments would be to calculate:

$$(1.06)^{(1/12)} = 1.00487.$$

The correct monthly rate should be 0.487%, not 0.5%.

Why do we have the system we have?

- Is it to confuse borrowers?
- Is it because bankers were confused?
- Are there laws that make it hard to change?

History digression: why compound interest?

Simple Interest Only pay interest on the money you borrow, not the interest you still owe.

$$A = P(1 + nr)$$

Compound Interest Pay interest on both the money borrowed and any interest accrued.

$$A = P(1 + r)^n$$

Annuities

Mortgage Payments If you pay back a fixed amount every month for 30 years, how much total interest should you pay?

Fixed Income Retirement Accounts How much should you pay now to get a fixed income of $3000 per month for the next 20 years?

Why compound interest?

Suppose you deposit money (called the present value PV) in an annuity and make fixed withdrawals (payments P) each year.

Simple Interest Annuity is a harmonic sum

$$PV = \frac{P}{1+r} + \frac{P}{1+2r} + \ldots \frac{P}{1+nr}$$

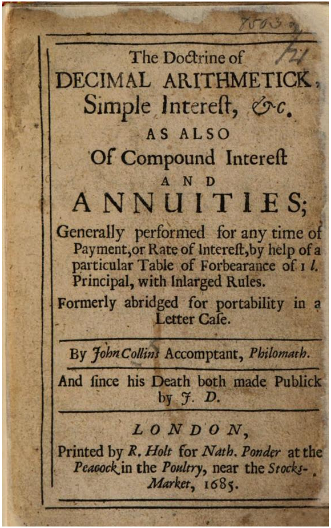
Compound Interest Annuity is a geometric sum

$$PV = \frac{P}{(1+r)} + \frac{P}{(1+r)^2} + \ldots + \frac{P}{(1+r)^n}$$

John Collins

John Collins (1625-1683) was an English mathematician who is known for the letters he wrote to other famous mathematicians including Newton & Leibniz.

He wrote a book about calculating interest and annuities.



The Doctrine of
DECIMAL ARITHMETICK,
Simple Intereſt, &c.
AS ALSO
Of Compound Intereſt
AND
ANNUITIES;
Generally performed for any time of Payment, or Rate of Intereſt, by help of a particular Table of Forbearance of 1 *l.* Principal, with Inlarged Rules.
Formerly abridged for portability in a Letter Caſe.

By *John Collins* Accomptant, *Philomath.*

And ſince his Death both made Publick by *J. D.*

LONDON,
Printed by *R. Holt* for *Nath. Ponder* at the *Peacock* in the *Poultry*, near the *Stocks-Market*, 1685.

John Collins

In his book *The Doctrine of Decimal Arithmetick*, Collins writes:

If you are to Equate an Annuity at Simple Interest, I presume a Compendium may be found in [Pietro Mengoli's] *Arithmetical Quadratures (a Book I never saw)...*

Pietro Mengoli

Pietro Mengoli (1626 - 1686) did prove that the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

converges to $\ln 2$, but he didn't have a practical formula for general harmonic sums.

Sums of harmonic series

Unfortunately, there doesn't seem to be a nice formula for sums of harmonic series. In 1734, Euler observed that for large n:

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \approx \ln(n) + \gamma$$

where $\gamma \approx 0.577216$ is the Euler-Mascheroni constant.

γ is even more mysterious than e. We still don't even know if γ is rational, for example!

Sums of geometric series

Unlike simple interest annuities, there is a "nice" formula for the sum of a compound interest annuity.

$$\frac{P}{(1+r)} + \frac{P}{(1+r)^2} + \ldots + \frac{P}{(1+r)^n} = \frac{P[(1+r)^n - 1]}{r(1+r)^n}$$

Sums of geometric series

Unlike simple interest annuities, there is a “nice” formula for the sum of a compound interest annuity.

$$\frac{P}{(1+r)} + \frac{P}{(1+r)^2} + \ldots + \frac{P}{(1+r)^n} = \frac{P[(1+r)^n - 1]}{r(1+r)^n}$$

It just requires calculating powers, multiplication, and division.

A brief history of compound interest

- Early compound interest tables were published in Italy starting in the 14th Century.
- John Napier 1614 *Mirifici Logarithmorum Canonis Descriptio* - showed how to use logarithms to calculate products and roots.
- Henry Briggs 1624 *Arithmetica Logarithmica* - included examples of compound interest and annuity calculations using logarithms.

John Napier

John Napier (1550 – 1617) was a Scottish landowner and mathematician.

He wrote *A Plaine Discovery of the Whole Revelation of St. John* (1593) where he predicted that the world would end in either 1688 or 1700.

He also wrote *Mirifici Logarithmorum Canonis Descriptio* (1614).

The wonderful method of logarithms

The phrase *mirifici logarithmorum canonis* means “the wonderful method of logarithms”.

The wonderful method of logarithms

The phrase *mirifici logarithmorum canonis* means "the wonderful method of logarithms".

Here's how Napier starts the preface.

Since nothing is more tedious, fellow mathematicians, in the practice of the mathematical arts, than the great delays suffered in the tedium of lengthy multiplications and divisions, the finding of ratios, and in the extraction of square and cube roots – and in which not only is there the time delay to be considered, but also the annoyance of the many slippery errors that can arise: I had therefore been turning over in my mind, by what sure and expeditious art, I might be able to improve upon these said difficulties.

Why logarithms are great

- Logarithms turn multiplication into addition...

$$\log(ab) = \log(a) + \log(b)$$

- ...and division into subtraction:

$$\log\left(\frac{a}{b}\right) = \log(a) - \log(b).$$

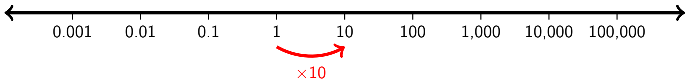
- Logarithms also make powers easier:

$$\log(a^n) = n\log(a).$$

What are logarithms?

The base-b logarithm $\log_b(x)$ is both:

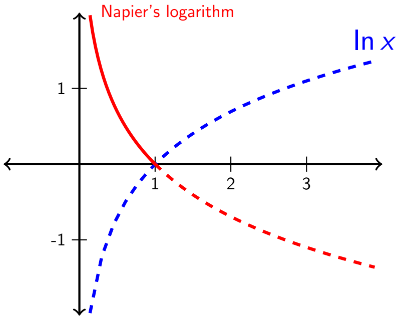
- The power you need to raise b to in order to get x.
- The number of steps x is away from 1 on a logarithmic scale.



Base-10 logarithmic scale

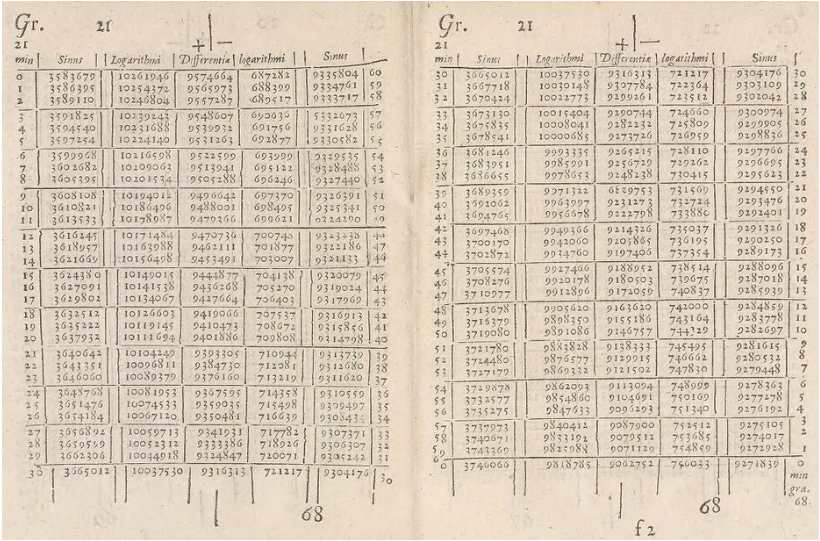
Napier's logarithm

Napier's logarithm is essentially the negative of the modern natural logarithm function.



It's also the base-$(1/e)$ logarithm.

Logarithm table



Gr. 21

21

min	Sinus	Logarithmi	Differentia +	logarithmi −	Sinus	
0	3583679	10261946	9574664	687282	9335804	60
1	3586395	10254372	9565973	688399	9334761	59
2	3589110	10246804	9557287	689517	9333717	58
3	3591825	10239243	9548607	690636	5332673	57
4	3594540	10231688	9539932	691756	9331628	56
5	3597254	10224140	9531263	692877	9330582	55
6	3599968	10216598	9522599	693999	9329535	54
7	3602682	10209063	9513941	695122	9328488	53
8	3605395	10201534	9505288	696246	9327440	52
9	3608108	10194012	9496642	697370	9326391	51
10	3610821	10186496	9488001	698495	9325341	50
11	3613533	10178987	9479366	699621	9324290	49
12	3616245	10171484	9470736	700748	9323238	48
13	3618957	10163988	9462111	701877	9322186	47
14	3621669	10156498	9453491	703007	9321133	46
15	3624380	10149015	9444877	704138	9320079	45
16	3627091	10141538	9436268	705270	9319024	44
17	3629802	10134067	9427664	706403	9317969	43
18	3632512	10126603	9419066	707537	9316913	42
19	3635222	10119145	9410473	708672	9315856	41
20	3637932	10111694	9401886	709808	9314798	40
21	3640642	10104249	9393305	710944	9313739	39
22	3643351	10096811	9384730	712081	9312680	38
23	3646060	10089379	9376160	713219	9311620	37
24	3648768	10081953	9367595	714358	9310559	36
25	3651476	10074533	9359035	715498	9309497	35
26	3654184	10067120	9350481	716639	9308434	34
27	3656892	10059713	9341931	717782	9307371	33
28	3659599	10052312	9333386	718926	9306307	32
29	3662306	10044918	9324847	720071	9305242	31
30	3665012	10037530	9316313	721217	9304176	30

68

Gr. 21

21

min	Sinus	Logarithmi	Differentia +	logarithmi −	Sinus	
30	3665012	10037530	9316313	721217	9304176	30
31	3667718	10030148	9307784	722364	9303109	29
32	3670424	10022773	9299261	723512	9302042	28
33	3673130	10015404	9290744	724660	9300974	27
34	3675835	10008041	9282232	725809	9299905	26
35	3678541	10000685	9273726	726959	9298836	25
36	3681246	9993335	9265215	728110	9297766	24
37	3683951	9985991	9256729	729262	9296695	23
38	3686655	9978653	9248238	730415	9295623	22
39	3689359	9971322	6229753	731569	9294550	21
40	3692062	9963997	9231273	732724	9293476	20
41	3694765	9956678	9222798	733880	9292401	19
42	3697468	9949366	9214326	735037	9291326	18
43	3700170	9942060	9205865	736195	9290250	17
44	3702872	9934760	9197406	737354	9289173	16
45	3705574	9927466	9188952	738514	9288096	15
46	3708276	9920178	9180503	739675	9287018	14
47	3710977	9912896	9172059	740837	9285939	13
48	3713678	9905620	9163620	742000	9284859	12
49	3716379	9898350	9155186	743164	9283778	11
50	3719080	9891086	9146757	744329	9282697	10
51	3721780	9883828	9138333	745495	9281615	9
52	3724480	9876577	9129915	746662	9280532	8
53	3727179	9869332	9121502	747830	9279448	7
54	3729878	9862093	9113094	748999	9278363	6
55	3732577	9854860	9104691	750169	9277278	5
56	3735275	9847633	9096293	751340	9276192	4
57	3737973	9840412	9087900	752512	9275105	3
58	3740671	9833192	9079512	753685	9274017	2
59	3743369	9825988	9071129	754859	9272928	1
60	3746066	9818785	9062752	756033	9271839	0

min gra. 68.

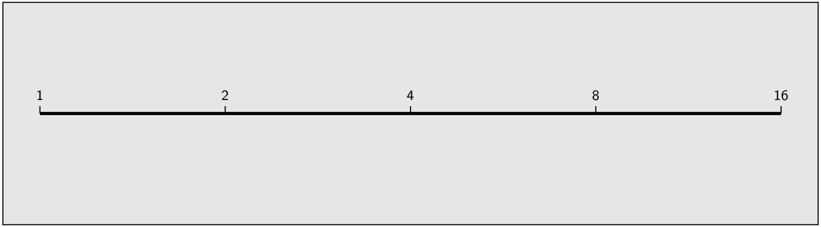
68

f2

Logarithmic scales

Each step on a log-scale represents multiplication by a fixed factor.

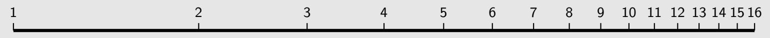
Here is a log-scale where each step is a factor of 2.



Logarithmic scales

Each step on a log-scale represents multiplication by a fixed factor.

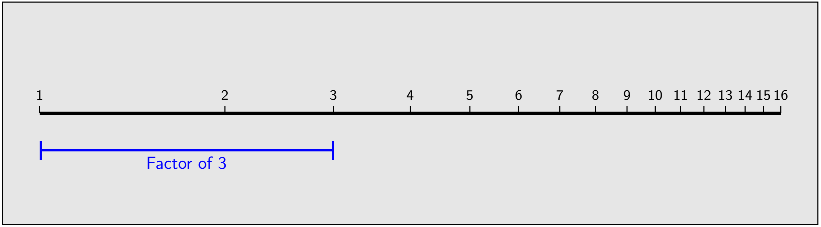
Here is the same log-scale with the other integers included.



Logarithmic scales

Multiply by moving to the right.

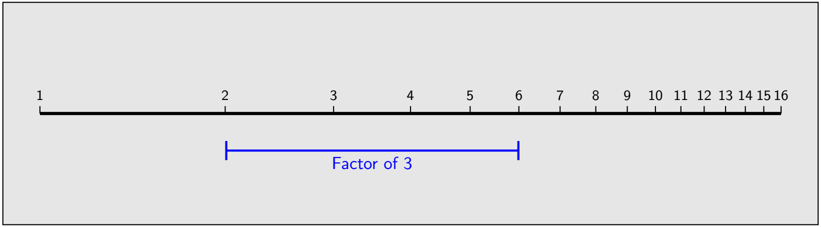
Divide by moving to the left.



Logarithmic scales

Multiply by moving to the right.

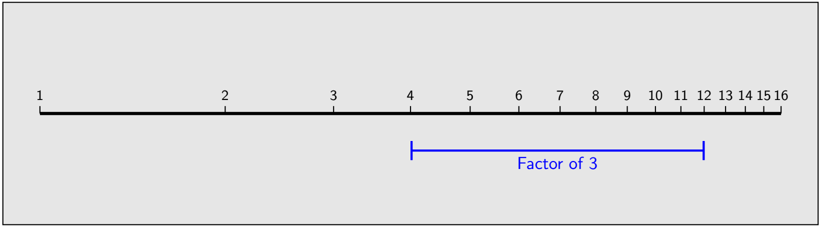
Divide by moving to the left.



Logarithmic scales

Multiply by moving to the right.

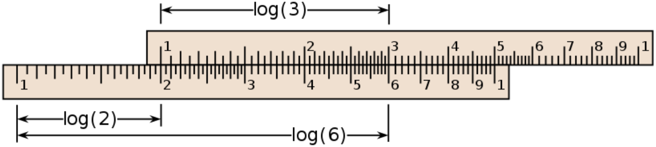
Divide by moving to the left.



History of logarithmic scales

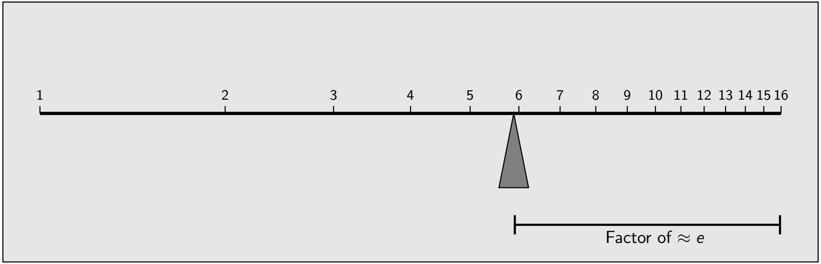
- Edmund Gunter created a wooden ruler marked with a log-scale in 1620.

- By 1622, William Oughtred invented the first slide rule by putting two wooden log-scales side-by-side.



The fulcrum of a logarithmic scale

If you position equal weights at the integers $1, \ldots, n$ on a logarithmic scale, then it will balance on a point $\approx \frac{n}{e}$.



Why?

The **factorial** of n is $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots n$.

Why?

The **factorial** of n is $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots n$.

Stirling's formula says

$$n! \approx \left(\frac{n}{e}\right)^n.$$

Why?

The **factorial** of n is $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots n$.

Stirling's formula says

$$n! \approx \left(\frac{n}{e}\right)^n .$$

Take n-th root and then the logarithm of both sides:

$$\ln(\sqrt[n]{n!}) \approx \ln\left(\frac{n}{e}\right) .$$

Why?

The **factorial** of n is $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots n$.

Stirling's formula says

$$n! \approx \left(\frac{n}{e}\right)^n .$$

Take n-th root and then the logarithm of both sides:

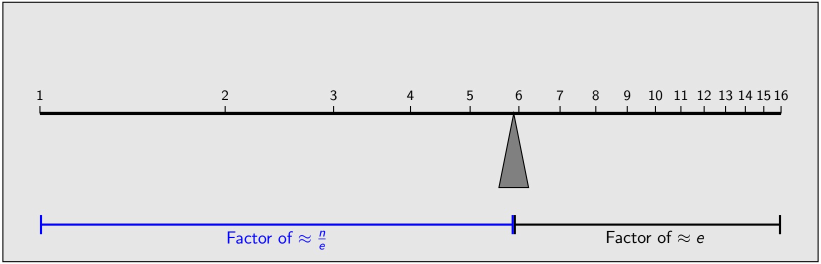
$$\ln(\sqrt[n]{n!}) \approx \ln\left(\frac{n}{e}\right) .$$

By the properties of logarithms:

$$\frac{\ln(1) + \ln(2) + \ldots + \ln(n)}{n} \approx \ln\left(\frac{n}{e}\right) .$$

Why?

The center of mass is the average position of the weights.



So by Stirling's formula the center of mass is

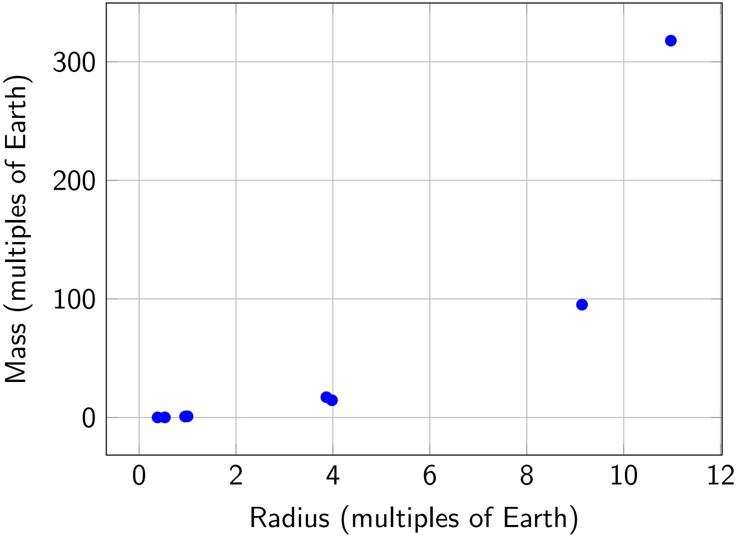
$$\frac{\ln(1) + \ln(2) + \ldots + \ln(n)}{n} \approx \ln\left(\frac{n}{e}\right).$$

More applications of logarithmic scales

- Displaying data
- Benford's law
- Apportioning seats in Congress

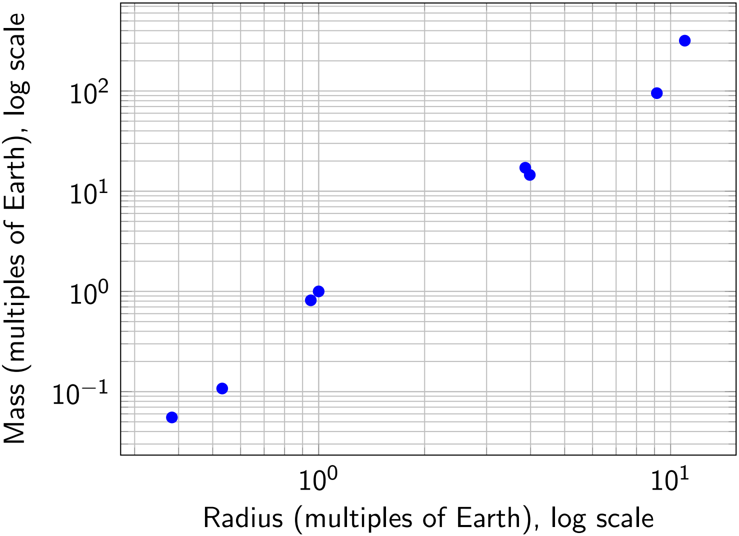
Displaying data

Here are the mass and radii of the planets in the solar sytem.



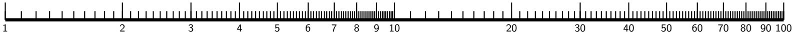
Displaying data

Here are the mass and radii of the planets in the solar sytem.



Benford's law

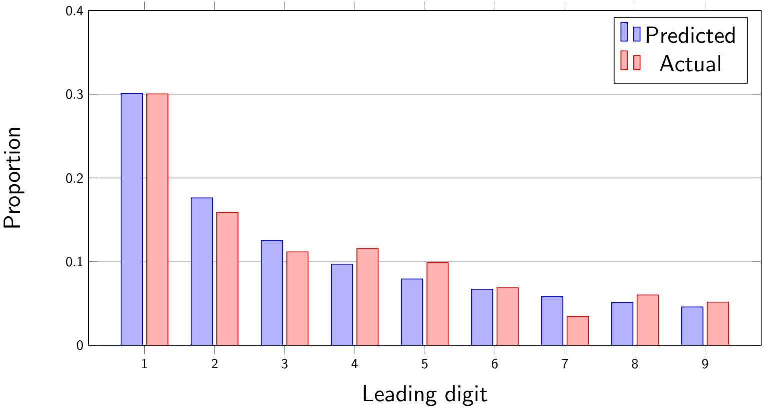
Benford's law asserts that in real world data, numbers beginning with a 1 (like 125, or 17, or 1903.72) are more common than numbers beginning with a 2 (like 28, or 207.4), which in turn are more common than numbers beginning with a 3, and so on.



Numbers starting with 1 make up about 30% of a logarithmic scale. Numbers starting with 9 are less than 4.6% of the scale.

Benford's law

Leading digits of the populations of countries (2017 UN data).



The apportionment problem

According to the Constitution: Representatives and direct Taxes shall be apportioned among the several States which may be included within this Union, according to their respective Numbers...

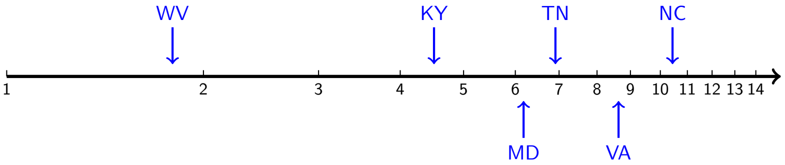
Since 1941, the United States has used the Huntington-Hill method to apportion the 435 seats of the House of Representatives.

Apportionment methods

- **Jefferson's method**: scale populations proportionally until you give out the correct number of seats when you round each number down.
- **Adams's method**: scale populations proportionally until you give out the correct number of seats when you round each number up.
- **Huntington-Hill method**: scale populations proportionally until you give out the correct number of seats by rounding to the nearest whole number (on a logarithmic scale).

Apportionment on a log-scale

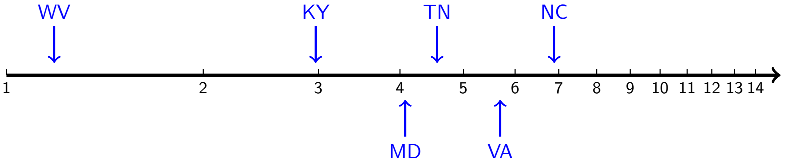
Population (in millions)



Need to find a divisor D that assigns exactly 435 seats to congress.

Apportionment on a log-scale

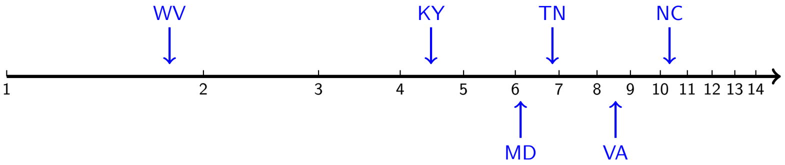
Quota (seats in congress)



If $D = 1.5$ million people per seat, then we only get 223 seats.

Apportionment on a log-scale

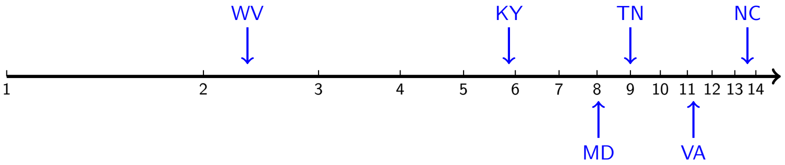
Quota (seats in congress)



If $D = 1$ million people per seat, then we only get 334 seats.

Apportionment on a log-scale

Quota (seats in congress)



If $D = 760{,}000$ people per seat, then we get exactly 435 seats.

First presidential veto

On April 5, 1792, George Washington issued the first presidential veto.

He objected to the apportionment method used to divide the seats after the 1790 census.

Gentlemen of the House of Representatives:

I have maturely considered the act passed by the two Houses entitled "An act for an apportionment of Representatives among the several States according to the first enumeration," and I return it to your House, wherein it originated, with the following objections:

First. The Constitution has prescribed that Representatives shall be apportioned among the several States according to their respective numbers, and there is no one proportion or divisor which, applied to the respective numbers of the States, will yield the number and allotment of Representatives proposed by the bill.

Second. The Constitution has also provided that the number of Representatives shall not exceed I for every 30,000, which restriction is by the context and by fair and obvious construction to be applied to the separate and respective numbers of the States; and the bill has allotted to eight of the States more than I for every 30,000.

-George Washington

e and the prime numbers

If you multiply all the prime numbers less than or equal to n together, you get the **primorial** of n, denoted $n\#$:

$$n\# = 2 \cdot 3 \cdot 5 \cdot 7 \cdots p.$$

Primorials are like a prime version of factorials.

e and the prime numbers

The primorial function $n\#$ grows **faster** than a^n for all positive $a < e$, and it grows **slower** than a^n for all $a > e$.

e and the prime numbers

The primorial function $n\#$ grows **faster** than a^n for all positive $a < e$, and it grows **slower** than a^n for all $a > e$.

Another way to express this fact is:

$$\lim_{n\to\infty} \sqrt[n]{n\#} = e.$$

e and the prime numbers

Knowing that $\sqrt[n]{n\#} \approx e$ tells us that prime numbers must be fairly common.

e and the prime numbers

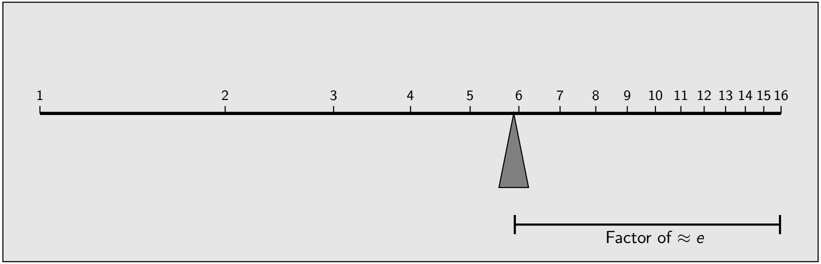
Knowing that $\sqrt[n]{n\#} \approx e$ tells us that prime numbers must be fairly common.

Take the natural log of both sides:

$$\frac{\ln 2 + \ln 3 + \ln 5 + \ldots + \ln p}{n} \approx \ln e = 1.$$

Prime numbers and logarithmic scales

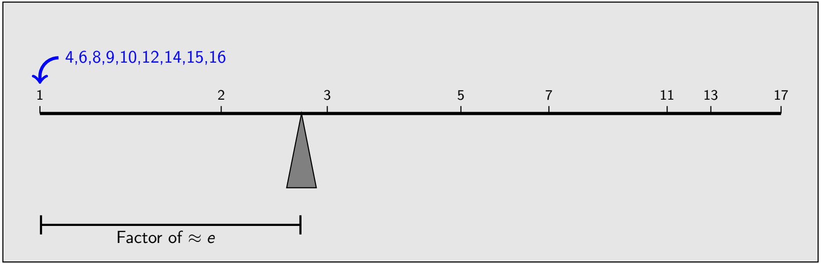
We've already seen that the integers on a logarithmic scale balance at a point $\approx \frac{n}{e}$.



Stirling's Formula

Prime numbers and logarithmic scales

If you move all of the weights for the composite numbers to one, then the fulcrum will be positioned at approximately e.



$$\sqrt[n]{n\#} \approx e$$

The Prime Number Theorem

This is closely related to the **Prime Number Theorem** which says that the fraction of integers between 1 and n that are prime is approximately $\frac{1}{\ln n}$.

- Versions of the Prime Number Theorem were first conjectured around 1800.
- The first proofs of the Prime Number Theorem used complex analysis and were published independently by Jacques Hadamard and Charles Jean de la Vallée Poussin in 1896.

Time for one more?

My daughter matches socks

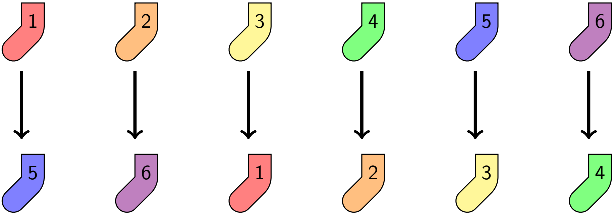
When my daughter was little, she loved to match socks. Unfortunately, she didn't care if the socks actually matched. She just randomly put two socks together. So given a stack of n freshly washed pairs of socks (all with different colors) what is the probability that she managed to **not match** any socks correctly?

This is called a **derangement** of the socks.

Permutations

A derangement is a permutation with no fixed points,

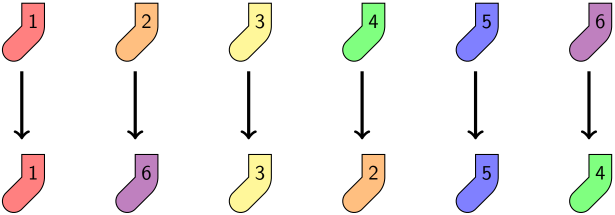
This is a derangement:



Permutations

A derangement is a permutation with no fixed points.

Not a derangement:



Probability of a derangement

The probability that a permutation of n pairs of socks is a derangment is

$$\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + \frac{(-1)^n}{n!}.$$

Probability of a derangement

The probability that a permutation of n pairs of socks is a derangment is

$$\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots + \frac{(-1)^n}{n!}.$$

This might look familiar if you've seen Taylor series.

Taylor series for e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots.$$

Taylor series for e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.$$

If you substitute $x = -1$, then

$$\frac{1}{e} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots.$$

The probability of getting a derangment approaches this limit quickly as n increases.

How good are these approximations?

- Estimate from 100 boxes of cereal:

$$e \approx 2.7320$$

- Estimate from the fulcrum of the log-scale up to 100:

$$e \approx 2.6321$$

- Estimate from n^{th} root of 100#:

$$e \approx 2.3101$$

- Estimate from matching 100 pairs of socks:

$$e \approx 2.7182818284590455$$

The last one is accurate to 15 decimal places.

Thank you!

Thanks for your attention!