Math 444 - Homework 10 **Name:** ______________________

1. Suppose that f is entire and there exists $M > 0$ such that $|f(z)| \geq M$ for all $z \in \mathbb{C}$. Use Liouville's theorem to prove that f is constant.

2. Suppose that f is entire and $\operatorname{Re} f$ is bounded. Prove that f must be constant. Hint: Consider the function $\exp(f(z))$.

3. One of the roots of the polynomial $p(z) = z^3 - 6z + 20i$ is $z = 2i$. Factor $p(z)$ into a product of linear factors. Hint: Use polynomial long division to remove the factor $(z - 2i)$ first.

4. Suppose that f is entire and $|f(z)| \leq a + b|z|^n$ for all $z \in \mathbb{C}$ where n is a fixed positive integer and $a, b > 0$ are constants. Prove that f is a polynomial of degree at most n by showing that the coefficients a_k of its Maclaurin series are zero when $k > n$. Use Cauchy's integral formula for the Maclaurin series coefficients:

$$a_k = \frac{1}{2\pi i} \oint_{|z|=R} \frac{f(z)}{z^{k+1}}\, dz$$

and notice that the radius R can be arbitrarily large since f is entire.

5. Use mathematical induction to prove that $(z-1)(z^{n-1}+z^{n-2}+\ldots+z+1) = z^n - 1$ for every positive integer n.

6. Use the previous result to determine all of the roots of the polynomial $z^{n-1} + z^{n-2} + \ldots + z + 1$.