Final Exam Review - Math 243

1. Consider the linear system below.

$$\frac{dx}{dt} = x + 2y$$
$$\frac{dy}{dt} = x$$

   (a) Re-write the system in matrix form.

   (b) Find the general solution for the system.

   (c) Find the solution to the initial value problem $x(0) = 1$ and $y(0) = 2$.

2. Laplace transform example!

   (a) Find $Y(s)$

   (b) Partial fraction decomposition?

   (c) Solve the equation by computing the inverse Laplace transform of each term.

3. Consider the population model

$$\frac{dP}{dt} = -\tfrac{1}{25}P^2 + 4P - C$$

for a species of fish in a lake (here dP/dt is measured in fish/year). Here C is a parameter that represents the number of fish caught per year.

(a) Draw a bifurcation diagram showing how the equilibrium fish populations depend on C. Include the phase lines at three different values of C.

(b) What is the maximum catch C that would still allow the fish to have a chance of survival in the lake?

(c) Given the possibility of unexpected perturbations of the population not included in the model, what do you think would happen to the actual fish population if the yearly catch determined in part (b) was actually taken every year?

4. Consider the following one-parameter family of linear systems

$$\frac{dY}{dt} = AY \quad \text{where} \quad A = \begin{pmatrix} 2 & -a \\ 1 & 0 \end{pmatrix}$$

(a) Sketch the the curve determined by the parameter a in the trace-determinant plane.

(b) Describe the different types of equilibrium behaviors exhibited by the system as a increases.

5. Consider the system below.

$$\frac{dx}{dt} = 1$$
$$\frac{dy}{dt} = 2x$$

(a) Verify that the system is a Hamiltonian system.

(b) Find the Hamiltonian function H.

(c) Use the Hamiltonian to sketch the solution curves in the phase plane for this system.

6. Consider the forced harmonic oscillator

$$y'' + 5y' + 6y = 6t - 1$$

(a) Find the general solution.

(b) If the forcing term were removed, would this oscillator be over-damped, under-damped, or critically damped?

7. A 1000 L tank is filled with salt water. The water currently contains 20 kg of dissolved salt. Suppose that fresh water is added to the tank at a rate of 10 L per minute and an equal amount of water drains out of the bottom so that the water level stays constant. Assume that the water draining from the bottom is well mixed.

   (a) Find a differential equation that models the amount of salt in the tank.

   (b) Solve the differential equation to find an equation for the amount of salt in the tank as a function of time.

8. Consider the differential equation $y' = 2y + \cos t$.

   (a) Find the general solution.

   (b) Solve the initial value problem $y(0) = 0$.

9. Consider the differential equation $\frac{dy}{dt} = 3y^{2/3}$.

   (a) Verify that $y(t) = 0$ is a solution to the differential equation.

   (b) Verify that $y(t) = t^3$ is also a solution to the differential equation.

   (c) One of the most important theorems about differential equations is the following:

   Uniqueness Theorem. *If* $\frac{dy}{dt} = f(t, y)$ *is a differential equation where both* f *and its partial derivative* f_y *are continuous for all points* (t, y) *near* (t_0, y_0)*, then there exists a unique solution* $y(t)$ *defined on an open interval around* t_0 *such that* $y(t_0) = y_0$*.*

   Given that solutions of a differential equation are supposed to be unique, how is it possible that the differential equation above have two different solutions with $y(0) = 0$? Explain how this does not contradict the Uniqueness Theorem.

10. Consider the following nonlinear system of equations.

$$\frac{dx}{dt} = x^2 - y$$
$$\frac{dy}{dt} = 2y^2 - 2$$

(a) Identify all of the equilibrium points of the system and classify the type of equilibrium for each (sink, source, spiral sink, spiral source, or saddle).

(b) Describe what will happen to the solution with initial condition $(x_0, y_0) = (-1, 0)$ as $t \to \infty$? This one still needs to be fixed... maybe a graph question instead?

11. Find the general solution to the linear system

$$\frac{dY}{dt} = \begin{pmatrix} -4 & 3 \\ -3 & -4 \end{pmatrix} Y$$

If it helps, every matrix of the form $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ has $\begin{pmatrix} 1 \\ i \end{pmatrix}$ as an eigenvector.