The fourth order Runge-Kutta method (RK4) is commonly used to numerically approximate the solution of an initial value problem

$$\frac{dy}{dt} = f(t, y) \quad \text{with } t \in [a, b] \text{ and initial condition } y(a) = y_0.$$

The formula for updating the y-values in RK4 is

$$y_{i+1} = y_i + \tfrac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$\begin{aligned} k_1 &= f(t_i, y_i), \\ k_2 &= f(t_i + h/2, y_i + hk_1/2), \\ k_3 &= f(t_i + h/2, y_i + hk_2/2), \\ k_4 &= f(t_i + h, y_i + hk_3). \end{aligned}$$

1. Write a function in Python to implement RK4.

2. Use your RK4 function to approximate the solution to the IVP

$$\frac{dy}{dt} = \frac{y}{t} - \left(\frac{y}{t}\right)^2$$

   on the interval $[1, 2]$ with initial condition $y(1) = 1$.

3. The exact solution to this IVP is $y(t) = \dfrac{t}{1 + \ln t}$. Find the absolute error in the RK4 approximation of $y(2)$ and compare it with the absolute error in the Euler's method approximation (both with $h = 0.1$).

4. Use RK4 with $h = 0.01$ to approximate the solution of the differential equation

$$\frac{dy}{dt} = \frac{\cos t}{y^2 + 1}, \qquad y(0) = 1$$

on the interval $[0, 4\pi]$. Graph your solution.

5. Solve this IVP by hand.

6. How could you (numerically) compute the value of $y(\pi/3)$ without using a numerical method for differential equations (like Euler's method or a Runge-Kutta)?