Thompson's metric nonexpansive maps with bounded fixed point sets and applications to nonlinear matrix equations

Brian Lins

Hampden-Sydney College

January 4, 2026

Motivating examples

Ferrante-Levy 1996:

$$X = A + N^* X^{-1} N.$$

Lawson-Lim 2012:

$$X = A + N^* (B + X^{-1})^{-1} N.$$

where $A, B \in \mathbb{C}^{n \times n}$ are positive semidefinite and $N \in \mathbb{C}^{n \times n}$.

Question. Under what circumstances will fixed point iteration converge to a unique positive definite solution?

Closed cones

A **closed cone** is a closed convex set C in a real Banach space such that

1. $\lambda C \subseteq C$ for all $\lambda \geq 0$.
2. $C \cap (-C) = \{0\}$.

A closed cone is **solid** if it has nonempty interior.

Example

The **standard cone** $\mathbb{R}^n_{\geq 0}$ of nonnegative vectors in $\mathbb{R}^n$.

Example

The **cone of positive semidefinite matrices** $\mathcal{P}_n \subset \mathcal{H}_n \subset \mathbb{C}^{n \times n}$.

Example

The **cone of nonnegative functions** in $C([0,1])$.

Partial order induced by a cone

A closed cone C in a real Banach space induces a partial order:

$$x \leq y \text{ when } y - x \in C.$$

$$x \ll y \text{ when } y - x \in C^\circ.$$

For $x \leq y$, we define the closed **order interval**

$$[x, y] := \{z \in X : x \leq z \leq y\}.$$

Thompson's metric

Thompson's metric on the interior of a solid closed cone is

$$d_T(x, y) := \inf\{R > 0 : e^{-R}x \leq y \leq e^R x\}.$$

The **closed ball** of radius R around x in Thompson's metric is

$$B_R(x) = [e^{-R}x, e^R x],$$

(C°, d_T) is a complete metric space with topology equivalent to the norm topology when C is finite dimensional (also true in infinite dimensions for normal cones).

Order-preserving and (sub)homogeneous maps

Let $f : C^\circ \to C^\circ$.

- f is **order-preserving** if $x \leq y \Rightarrow f(x) \leq f(y)$.
- f is **homogeneous** if $f(\lambda x) = \lambda f(x)$ for all $\lambda > 0$, $x \in C^\circ$.
- f is **subhomogeneous** if $f(\lambda x) \leq \lambda f(x)$ for all $\lambda > 1$, $x \in C^\circ$.

Nonexpansive maps

Let f be any map on a metric space (M, d) such that

$$d(f(x), f(y)) \leq \alpha d(x, y)$$

for all $x, y \in M$. If $\alpha = 1$, we say that f is **nonexpansive**. If $\alpha < 1$, then f is a **strict contraction**.

Proposition

If $f : C^\circ \to C^\circ$ is order-preserving, then f is nonexpansive with respect to Thompson's metric if and only if f is subhomogeneous.

Strict contractions

A strict contraction on (C°, d_T) always has a unique fixed point by the Banach contraction mapping principle.

Fact. If F is a d_T-nonexpansive map on positive definite matrices, and $0 < \alpha < 1$, then $X \mapsto F(X)^\alpha$ is a strict contraction.

Example

You can apply fixed point iteration to solve

$$X = (A + N^* X N)^\alpha.$$

Krasnoselskii iteration

Nonexpansive maps don't always have fixed points, or if they do, the fixed points might not be unique and/or attracting.

Theorem (L., 2023)

Let C be a solid closed cone in a finite dimensional[1] Banach space, let $f : C^\circ \to C^\circ$ be d_T-nonexpansive, and let $0 < \alpha < 1$. If $\mathrm{Fix}(f)$ *is nonempty, then the sequence*

$$x_{n+1} = \alpha f(x_n) + (1 - \alpha) x_n$$

converges to a fixed point of f for any $x_0 \in C^\circ$.

[1] Also true in infinite dimensions with compactness assumptions on f.

Some consequences

Corollary

If $\mathrm{Fix}(f)$ *is nonempty, then*

$$g := \lim_{n\to\infty} (\alpha f + (1-\alpha)\,\mathrm{id})^n$$

is a nonexpansive retraction from C° *onto* $\mathrm{Fix}(f)$.

Corollary

$\mathrm{Fix}(f)$ *is connected and contractible.*

Another fixed point condition

Proposition (Krasnoselskii)

Let C be a closed cone in a finite dimensional[2] Banach space. If $x \leq y$, and f is an order-preserving map such that

$$f([x, y]) \subseteq [x, y],$$

then f has a fixed point in $[x, y]$ and both $f^k(x)$ and $f^k(y)$ converge to fixed points in $[x, y]$ as $k \to \infty$.

[2] This also holds in infinite dimensions if you make additional assumptions on either the cone (i.e., regularity) or on the map f (some kind of compactness).

Necessary and sufficient fixed point conditions

Krasnoselskii's result gives a sufficient condition for fixed points, but what about necessary conditions?

We'll describe a simple necessary and sufficient condition for $\mathrm{Fix}(f)$ to be nonempty and bounded (in Thompson's metric).

Collatz-Wielandt Theorem

Theorem (Collatz-Wielandt)

For any nonnegative matrix $A \in \mathbb{R}^{n\times n}$, its spectral radius is

$$r(A) = \inf_{x \in (\mathbb{R}^n_{\geq 0})^\circ} \max_{1\leq i\leq n} \frac{(Ax)_i}{x_i}.$$

Observe that $\beta = \max_{1\leq i\leq n} \dfrac{(Ax)_i}{x_i}$ is the smallest positive number such that $Ax \leq \beta x$.

Super and sub-eigenvectors

For any map $f : C^\circ \to C$,

$x \in C$ is a **super-eigenvector** with **super-eigenvalue** $\beta > 0$ if

$$f(x) \leq \beta x,$$

$x \in C^\circ$ is a **sub-eigenvector** with **sub-eigenvalue** $\alpha > 0$ if

$$\alpha x \leq f(x).$$

Collatz-Wielandt numbers

Let $f : C^\circ \to C$. The **upper Collatz-Wielandt number** is

$$r(f) := \inf_{x \in C^\circ} \inf\{\beta > 0 : f(x) \leq \beta x\}$$

i.e., the infimum of the super-eigenvalues of f.

If $f : C^\circ \to C^\circ$, then the **lower Collatz-Wielandt number** is

$$\lambda(f) := \sup_{x \in C^\circ} \sup\{\alpha > 0 : \alpha x \leq f(x)\}$$

i.e., the supremum of the sub-eigenvalues of f.

Collatz-Wielandt numbers

If f is homogeneous (not just subhomogeneous), then $\lambda(f) \leq r(f)$.

If a homogeneous f has an eigenvector in C°, then $\lambda(f) = r(f)$ (the Perron-Frobenius eigenvalue or **cone spectral radius**).

A homogeneous map can't have a d_T-bounded set of fixed points.

A necessary & sufficient condition

Theorem (L., 2023)

Let C be a closed cone in a finite dimensional[3] *Banach space. If $f : C^\circ \to C^\circ$ is order-preserving and subhomogeneous, then TFAE.*

(a) $\mathrm{Fix}(f)$ *is nonempty and bounded in* (C°, d_T).

(b) *There exist* $x, y \in C^\circ$ *such that* $f(x) \gg x$ *and* $f(y) \ll y$.

(c) $\lambda(f) > 1$ *and* $r(f) < 1$.

If f is also real analytic, then the following is also equivalent.

(d) *f has a unique globally attracting fixed point in C°.*

[3] Also holds in infinite dimensions with C regular or f sufficiently compact.

Notes on the proof

(b) $\Rightarrow$ (a). If there exist $x, y \in C^\circ$ with $f(x) \gg x$ and $f(y) \ll y$, then you can prove that $x \ll y$ so there is a fixed point in $[x, y]$. In fact, $\mathrm{Fix}(f)$ must be contained in $[x, y]^\circ$ since $\mathrm{Fix}(f)$ is connected.

(b) $\Leftrightarrow$ (c). Follows from the definitions of $\lambda(f)$ and $r(f)$.

(a) $\Rightarrow$ (b). The original intuition comes from topological degree theory, but the published proof is more elementary.

Real analytic functions

Let X, Y be real Banach spaces and U an open subset of X. Then $f : U \to Y$ is **real analytic** if for every $x \in U$, there is an $r > 0$ and continuous symmetric n-linear forms $A_n : X^n \to Y$ such that $\sum_{n=1}^{\infty} \|A_n\| r^n < \infty$ and

$$f(x+h) = f(x) + \sum_{n=1}^{\infty} A_n(h^n)$$

for all h in a neighborhood of 0 in X.

Real analyticity

If $\mathrm{Fix}(f)$ is nonempty and bounded, then it is compact and contractible.

If f is real analytic, then $\mathrm{Fix}(f)$ is a real analytic variety. Using a result of Sullivan, you can show that any compact, contractible real analytic variety must be a single point.

An alternative (longer, but more elementary) proof leans on f being order-preserving, but holds in infinite dimensions.

Detecting fixed points

Two ways to confirm existence (and uniqueness!) of fixed points.

1. **Monte Carlo algorithm** Randomly generate points $x \in C^{\circ}$. Stop when you find $f(x) \gg x$ and $f(x) \ll x$.

2. **Direct computation.** Find the Cauchy-Wielandt numbers.

Recession maps

Proposition

Let C be a closed cone with nonempty interior in a finite dimensional Banach space and let $f : C^\circ \to C^\circ$ be order-preserving and subhomogeneous. Then the **recession map**

$$f_\infty(x) := \lim_{t\to\infty} t^{-1} f(tx)$$

is a well-defined, order-preserving, homogeneous map $f_\infty : C^\circ \to C$.

Furthermore, $r(f) = r(f_\infty)$.

Computing upper Collatz-Wielandt numbers

Since f_∞ is homogeneous, there are iterative formulas to compute its upper Collatz-Wielandt number, for example

$$r(f_\infty) = \lim_{k\to\infty} \|(f_\infty + \mathrm{id})^k(u)\|^{1/k} - 1$$

for any $u \in C^\circ$.

A necessary and sufficient condition for $r(f) < 1$ is

$$\|(f_\infty + \mathrm{id})^k(u)\| \leq 2^k \text{ for some } k \in \mathbb{N}.$$

Computing lower Collatz-Wielandt numbers

In general, $\lambda(f)$ is harder to compute.

Lemma

If there is an order-reversing, bijective, isometry L on (C°, d_T) and $f : C^\circ \to C^\circ$ is order-preserving and subhomogeneous, then

$$\lambda(f) = r((L\, f\, L)_\infty)^{-1}.$$

A detailed example

Consider the map $f : \mathcal{P}_n{}^\circ \to \mathcal{P}_n{}^\circ$

$$f(X) = A + N^*(B + X^{-1})^{-1}N.$$

where A, B are positive semidefinite and $X \in \mathbb{C}^{n\times n}$.

This is order-preserving and subhomogeneous, so it is d_T-nonexpansive. But it is not necessarily a strict contraction.

A detailed example: recession maps

The recession maps for $f(X) = A + N^*(B + X^{-1})^{-1}N$ are

- $f_\infty(X) = N^*(Q_B X^{-1} Q_B)^\dagger N$,
- $(L\,f\,L)_\infty(X) = (Q_A N^* X^{-1} N Q_A)^\dagger$.

where † denotes the Moore-Penrose psuedoinverse and Q_A, Q_B are orthogonal projections onto the nullspaces of A and B respectively.

A detailed example: fixed points

The map $f(X) = A + N^*(B + X^{-1})^{-1}N$ has a unique, globally attracting positive definite fixed point if and only if

$$r(f_\infty) < 1 \text{ and } r((L\,f\,L)_\infty) < 1.$$

Substituting I into f_∞ and $(L\,f\,L)_\infty$, a sufficient condition is:

$$\|N^* Q_B N\| < 1 \text{ and } \|(Q_A N^* N Q_A)^\dagger\| < 1.$$

Thank you!

Some extra details: finding regression maps

Lemma

Let $A \in \mathcal{P}_n$ *and* $B(t) \in \mathcal{P}_n$ *for all* $t > 0$*, and suppose that* $\lim_{t\to\infty} B(t) = B(\infty) \in \mathcal{P}_n$ *such that* $A + B(\infty) \in \mathcal{P}_n^\circ$*. Then*

$$\lim_{t\to\infty} (tA + B(t))^{-1} = (Q_A B(\infty) Q_A)^\dagger$$

where Q_A *is the orthogonal projection onto the nullspace of* A*.*

You can prove this with the Schur complement formula.

- A. Ferrante and B. C. Levy. Hermitian solutions of the equation $X = Q + NX^{-1}N^*$. *Linear Algebra Appl.*, 247(1):359–373, 1996.
- M. A. Krasnosel'skiĭ. *Positive solutions of operator equations*. P. Noordhoff Ltd., Groningen, 1964. Translated from the Russian by Richard E. Flaherty; edited by Leo F. Boron.
- J. Lawson and Y. Lim. A Lipschitz constant formula for vector addition in cones with applications to Stein-like equations. *Positivity*, 16(1):81–95, 2012.
- Y. Lim. Solving the nonlinear matrix equation $X = Q + \sum_{i=1}^{m} M_i X^{\delta_i} M_i^*$ via a contraction principle. *Linear Algebra Appl.*, 430(4):1380–1383, 2009.
- B. Lins. Bounded fixed point sets and Krasnoselskii iterates of Thompson metric nonexpansive maps. *J. Korean Math. Soc.*, 62(2):285–304, 2025.
- D. Sullivan. Combinatorial invariants of analytic spaces. In *Proceedings of Liverpool Singularities—Symposium, I (1969/70)*, Lecture Notes in Mathematics, Vol. 192, pages 165–168. Springer, Berlin, 1971.
- A. C. Thompson. On certain contraction mappings in a partially ordered vector space. *Proc. Amer. Math. Soc.*, 14:438–443, 1963.