Homework 2 - Computer Science 461 **Name:** ______________________

Due Monday, January 27.

1. Find the cardinalities of the following sets. If the set is infinite, say whether the cardinality is equal to $|\mathbb{N}| = \aleph_0$ or not.
   (a) $[10] \times [10] \times \{a, b, c\}$

   (b) $\{f : f : \{0,1\}^3 \to \{\text{“yes”}, \text{“no”}, \text{“maybe”}\}\}$

   (c) $[3]^*$.

2. The function IF-THEN-ELSE: $\{0,1\}^3 \to \{0,1\}$ is defined:

$$\text{IF-THEN-ELSE}(x, y, z) = \begin{cases} y & \text{if } x = 1, \\ z & \text{otherwise.} \end{cases}$$

   Prove that if you combine this function with the constant functions 0 and 1, then you get a universal set, i.e., you can construct any function $f : \{0,1\}^n \to \{0,1\}$ using just these three basic functions. Hint: prove that you can use $\{\text{IF-THEN-ELSE}, 0, 1\}$ to construct all of the functions in another universal set such as $\{\text{AND}, \text{OR}, \text{NOT}\}$ or $\{\text{NAND}\}$.

3. Any function $f : \{0,1\}^* \to \{0,1\}^*$ can be encoded by a Boolean function $g : \{0,1\}^* \to \{0,1\}$. One way to do this is to let g input two binary strings $s, t \in \{0,1\}^*$ and return 1 if $t = f(s)$ and 0 otherwise. Suppose someone else wrote a computer program that could compute the value of $g(s,t)$ for all possible binary input strings. Explain in words how you could use their code to write a new program that would evaluate the function $f(s)$ for any binary input string s.

Let A, B be sets and let $|A|$ and $|B|$ denote their cardinalities. We say that $|B| \geq |A|$ if there is an onto function $f : A \to B$. We say that $|B| > |A|$ if $|B| \geq |A|$ and there is no bijection from B to A.

4. Let 2^A denote the power set of A, i.e., the set of all subsets of A. Show that $|2^A| \geq |A|$ by describing an onto function $g : 2^A \to A$.

5. Suppose that there is a bijection $f : A \to 2^A$. Let $B = \{a \in A : a \notin f(a)\}$ and let b be the unique element of A such that $f(b) = B$. Then either $b \in B$ or $b \notin B$. Explain why both possibilities lead to a contradiction.

6. What does this mean about the cardinality $|2^A|$?

7. The majority function $\text{MAJ} : \{0,1\}^3 \to \{0,1\}$ returns 1 if at least two of the inputs are 1, and returns 0 otherwise. Write a formula or psuedocode program that just uses the NAND function to compute $\text{MAJ}(x, y, z)$. Your program can use as many variables as you need.