Recent developments in nonlinear Perron-Frobenius theory

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The classic Perron-Frobenius theorem

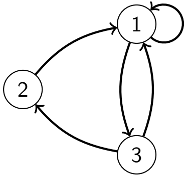
Theorem (Perron-Frobenius)

If a nonnegative matrix $A \in \mathbb{R}^{n \times n}$ is irreducible, then A has a unique (up to scaling) eigenvector with all positive entries. The corresponding eigenvalue has the maximum absolute value of all eigenvalues of A.

Irreducible means that the **adjacency graph** of the matrix is **strongly connected**. The adjacency graph $G(A)$ of a nonnegative matrix $A \in \mathbb{R}^{n \times n}$ has an edge from j to i when the (i, j)-entry of A is positive.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$G(A)$



Notation

- $[n] = \{1, \ldots, n\}$.
- $x \geq y$ in $\mathbb{R}^n$ when $x_i \geq y_i$ for all $i \in [n]$.
- The standard cone in $\mathbb{R}^n$ is the set of vectors with nonnegative entries:

$$\mathbb{R}^n_{\geq 0} = \{x \in \mathbb{R}^n : x \geq 0\}.$$

- The interior of $\mathbb{R}^n_{\geq 0}$ is the set of vectors with positive entries:

$$\mathbb{R}^n_{>0} = \{x \in \mathbb{R}^n : x_i > 0 \text{ for all } i \in [n]\}.$$

Proving Perron-Frobenius

I want to outline a proof of the Perron-Frobenius theorem. But I only want to use some of the properties of the map $x \mapsto Ax$.

A function f defined on a subset of $\mathbb{R}^n$ is

1. Order-preserving when $x \geq y$ implies that $f(x) \geq f(y)$ for all x, y in the domain.
2. Homogeneous if $f(tx) = tf(x)$ for all $t > 0$.

Parts of a cone

For vectors $x \in \mathbb{R}^n_{\geq 0}$, the support of x is

$$\operatorname{supp}(x) = \{i \in [n] : x_i > 0\}.$$

Two vectors $x, y \in \mathbb{R}^n_{\geq 0}$ are comparable, denoted $x \sim y$, if there are constants $\alpha, \beta > 0$ such that

$$\alpha x \leq y \leq \beta x$$

Comparability is an equivalence relation, and $x \sim y$ if and only if $\operatorname{supp}(x) = \operatorname{supp}(y)$. The equivalence classes are the parts of the standard cone.

Parts of a cone

If $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ is order-preserving and homogeneous, then f preserves comparability, i.e.,

$$x \sim y \text{ implies } f(x) \sim f(y).$$

After all, if

$$\alpha x \leq y \leq \beta x,$$

then

$$\alpha f(x) \leq f(y) \leq \beta f(x).$$

Invariant parts

Lemma

If $A \in \mathbb{R}^{n\times n}$ is a nonnegative irreducible matrix, then the only parts of $\mathbb{R}^n_{\geq 0}$ that are invariant under multiplication by A are $\{0\}$ and $\mathbb{R}^n_{>0}$.

Proof.

A non-trivial part has the form

$$K_I = \{x \in \mathbb{R}^n_{\geq 0} : \mathrm{supp}(x) = I\}$$

where $I \subsetneq [n]$ is not empty. Since $G(A)$ is strongly connected, there is an edge from some $i \in I$ to $j \notin I$. Then $(Ax)_j > 0$ for every $x \in K_I$, so K_I is not invariant under A. $\square$

Proof of Perron-Frobenius (existence)

Every column of A has a positive entry so $Ax \neq 0$ for every nonzero $x \in \mathbb{R}^n_{\geq 0}$.

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$$f : \Sigma \to \Sigma, \qquad f(x) = Ax/(\sum_i (Ax)_i).$$

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The fixed point x is an eigenvector of A and it must be in $\mathbb{R}^n_{>0}$ because no other non-trivial part is invariant.

Perron-Frobenius (maximal eigenvalue)

Let λ be the eigenvalue corresponding to x. We can assume that $\lambda = 1$ by replacing A with $\lambda^{-1}A$. Consider any other $y \in \mathbb{R}^n_{>0}$.

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So $A^k y$ is bounded, which implies that 1 is the maximum eigenvalue (in absolute value).

Nonexpansiveness

The idea in the last proof is important. If $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ is order-preserving & homogeneous, and

$$\alpha x \leq y \leq \beta x,$$

then

$$\alpha f(x) \leq f(y) \leq \beta f(x),$$

In a sense, $f(x)$ and $f(y)$ are no farther apart than x and y.

Hilbert's projective metric

For $x, y \in \mathbb{R}^n_{>0}$, Hilbert's projective metric is

$$d_H(x, y) = \min \left\{ \log\left(\frac{\beta}{\alpha}\right) : 0 \leq \alpha x \leq y \leq \beta x \right\}.$$

It is a metric on the rays from the origin in $\mathbb{R}^n_{>0}$. Points in the boundary of $\mathbb{R}^n_{\geq 0}$ are infinitely far away.

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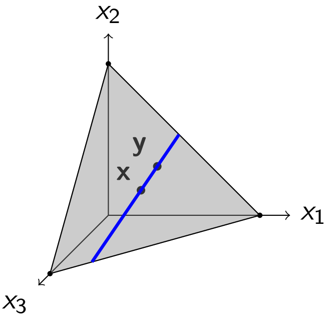
Any order-preserving, homogeneous map $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ is nonexpansive with respect to d_H:

$$d_H(f(x), f(y)) \le d_H(x, y)$$

for all $x, y \in \mathbb{R}^n_{>0}$.

Perron-Frobenius (uniqueness)

All eigenvectors of A in $\mathbb{R}^n_{>0}$ must have the same eigenvalue.



If x and y are linearly independent eigenvectors in $\mathbb{R}^n_{>0}$, then every vector in their span is an eigenvector, which is impossible if A is irreducible.

Applications of the Perron-Frobenius Theorem

- Markov chains
- Google PageRank Algorithm
- Monotone dynamical systems in physics and biology

Topical functions

We can extend the proof of the Perron-Frobenius theorem to nonlinear maps with almost no changes.

A function $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ that is order-preserving and homogeneous is multiplicatively topical.

A function $T : \mathbb{R}^n \to \mathbb{R}^n$ is additively topical if

$$T = \log \circ f \circ \exp$$

where $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ is multiplicatively topical and exp and log denote the entrywise natural exponential and logarithms functions.

Examples of topical functions

Additively topical examples

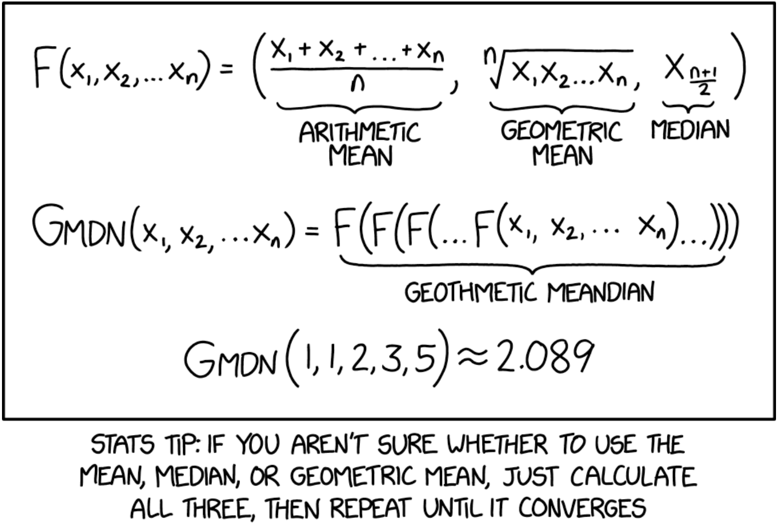
- Max-plus linear maps
- Min-max-plus operators (e.g., Shapley operators from stochastic game theory)

Multiplicatively topical examples

- The homogeneous eigenvalue problem for nonnegative tensors
- Examples from economics and population biology
- The arithmetic-geometric mean function

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{2}(x_1 + x_2) \\ \sqrt{x_1 x_2} \end{pmatrix}.$$

XKCD #2435 by Randall Munroe



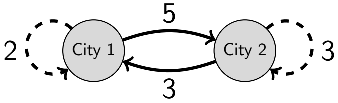
The geothmetic meandian

When $n = 3$, the geothmetic meandian is a multiplicatively topical map

$$F(x_1, x_2, x_3) = \begin{pmatrix} \frac{1}{3}(x_1 + x_2 + x_3) \\ \sqrt[3]{x_1 x_2 x_3} \\ \text{median}(x_1, x_2, x_3) \end{pmatrix}$$

Another nonlinear example

Two cities are connected by rail. The cities also have local trains to serve their suburbs.



The edges are tracks (labeled with the transit time). A train can only depart a station after both the local and inter-city trains have arrived, so that passengers can switch trains.

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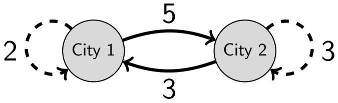
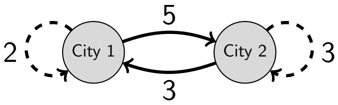


The edges are tracks (labeled with the transit time). A train can only depart a station after both the local and inter-city trains have arrived, so that passengers can switch trains.
If the first departure time from city 1 is $x_1(0)$ and the first departure time from city 2 is $x_2(0)$, then we get the following model for subsequent departures:

$$x_1(k+1) = \max(x_1(k) + 2, x_2(k) + 3)$$
$$x_2(k+1) = \max(x_1(k) + 5, x_2(k) + 3).$$

Model trains



Same model in vector notation:

$$x(k+1) = \begin{pmatrix} \max(x_1(k)+2, x_2(k)+3) \\ \max(x_1(k)+5, x_2(k)+3) \end{pmatrix}.$$

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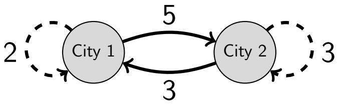

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We can simplify this formula using **max-plus algebra** as

$$x(k+1) = \begin{pmatrix} 2 & 3 \\ 5 & 3 \end{pmatrix} \otimes \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix}.$$

Max-plus algebra

The **max-plus algebra** consists of $\mathbb{R} \cup \{-\infty\}$ with two operations

- **max-plus addition**

$$a \oplus b = \max(a, b)$$

- **max-plus multiplication**

$$a \otimes b = a + b.$$

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Max-plus algebra is associative and matrix multiplication works, so:

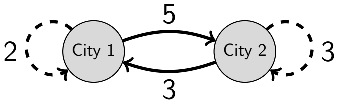
$$x(k) = \begin{pmatrix} 2 & 3 \\ 5 & 3 \end{pmatrix}^{\otimes k} \otimes x(0).$$

Max-plus eigenvectors

$A = \begin{pmatrix} 2 & 3 \\ 5 & 3 \end{pmatrix}$ has max-plus eigenvector $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ because

$$\begin{pmatrix} 2 & 3 \\ 5 & 3 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 4 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}.$$

In our train model, this means if the first departure in city 1 is noon and in city 2 is 1:00pm, then we can have departures every 4 hours from that point on.



Max-plus algebra in Julia

```
julia> using MaxPlus

julia> A = MP([2 3; 5 3])
2×2 (max,+) dense matrix:
  2   3
  5   3


julia> x = MP([0,0])
2-element (max,+) vector:
  0
  0


julia> A*x
2-element (max,+) vector:
  3
  5


julia> A^2*x
2-element (max,+) vector:
  8
  8


julia> A^3*x
2-element (max,+) vector:
  11
  13
```

Eigenvectors of topical functions

For $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$, the eigenspace of f is

$$E(f) := \{x \in \mathbb{R}^n_{>0} : x \text{ is an eigenvector of } f\}.$$

Note that $E(f)$ only includes eigenvectors with all positive entries.

There might also be eigenvectors on the boundary of the cone $\mathbb{R}^n_{\geq 0}$, but that is not our focus.

The hypergraphs $\mathcal{H}_0^-(f)$ and $\mathcal{H}_\infty^+(f)$

For a multiplicatively topical function f, $\mathcal{H}_0^-(f)$ and $\mathcal{H}_\infty^+(f)$ are directed hypergraphs with nodes $[n]$ that were introduced by Akian, Gaubert, and Hochart.

The hyperarcs of $\mathcal{H}_0^-(f)$ are the pairs $(I, \{j\})$ such that $I \subset [n]$, $j \in [n] \setminus I$, and

$$\lim_{t\to\infty} f(\exp(-tx_I))_j = 0$$

where exp is the entrywise natural exponential function and $x_I \in \mathbb{R}^n_{\geq 0}$ is any vector with $\mathrm{supp}(x_I) = I$.

The hyperarcs of $\mathcal{H}_\infty^+(f)$ are $(I, \{j\})$ such that $I \subset [n]$, $j \in [n] \setminus I$ and

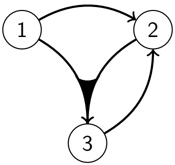
$$\lim_{t\to\infty} f(\exp(tx_I))_j = \infty.$$

Example

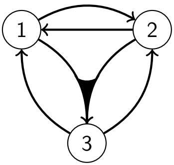
The geothmetic meandian function $F(x) = \begin{pmatrix} \frac{1}{3}(x_1 + x_2 + x_3) \\ \sqrt[3]{x_1 x_2 x_3} \\ \text{median}(x_1, x_2, x_3) \end{pmatrix}$

has

$\mathcal{H}_0^-(F)$



$\mathcal{H}_\infty^+(F)$



These show the minimal hyperarcs of $\mathcal{H}_0^-(F)$ and $\mathcal{H}_\infty^+(F)$.

Invariant nodes and reach

A subset $I \subseteq [n]$ is invariant in $\mathcal{H}_0^-(f)$ or $\mathcal{H}_\infty^+(f)$ if there are no hyperarcs $(I, \{j\})$ that originate from I in the hypergraph.

The reach of $J \subset [n]$ in a hypergraph $\mathcal{H}$, denoted $\text{reach}(J, \mathcal{H})$, is the smallest invariant subset of the nodes of $\mathcal{H}$ containing J.

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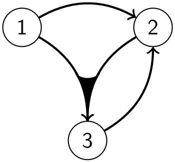
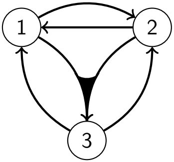


$\mathcal{H}_\infty^+(F)$



$I = \{2, 3\}$ is invariant in $\mathcal{H}_0^-(F)$, but $\mathcal{H}_\infty^+(F)$ has no invariant subsets.

Super & sub-eigenspaces

For any $\alpha, \beta > 0$, the sub-eigenspace corresponding to α is the set

$$S_\alpha(f) := \{x \in \mathbb{R}^n_{>0} : \alpha x \leq f(x)\}$$

and the super-eigenspace corresponding to β is

$$S^\beta(f) := \{x \in \mathbb{R}^n_{>0} : f(x) \leq \beta x\}.$$

The intersection $S^\beta_\alpha(f) := S_\alpha(f) \cap S^\beta(f)$ is called a slice space.

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Idea: These sets are all invariant under f. If any of these sets is nonempty and bounded in Hilbert's projective metric, then f has a positive eigenvector.

An irreducibility condition

Theorem (Gaubert-Gunawardena, 2004)

Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ be order-preserving and homogeneous. Then all super-eigenspaces $S^\beta(f)$ are bounded in $(\mathbb{R}^n_{>0}, d_H)$ if and only if $\operatorname{reach}(J, \mathcal{H}^+_\infty(f)) = [n]$ *for every nonempty $J \subsetneq [n]$.*

A corresponding condition involving the hypergraph $\mathcal{H}^-_0(f)$ is equivalent to all sub-eigenspaces of f being d_H-bounded.

Bounded slice spaces

Theorem (Akian-Gaubert-Hochart, 2020)

Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ be order-preserving and homogeneous. All slice spaces $S_\alpha^\beta(f)$ are bounded in $(\mathbb{R}^n_{>0}, d_H)$ if and only if

$$\operatorname{reach}(J, \mathcal{H}^+_\infty(f)) = [n] \text{ or } \operatorname{reach}(J^c, \mathcal{H}^-_0(f)) = [n]$$

for every nonempty $J \subsetneq [n]$.

A graph condition

For any multiplicatively topical map f, define a digraph $G(f)$ with an edge from j to i if

$$\lim_{t \to \infty} f(x + te_j)_i = \infty \text{ for any } x \in \mathbb{R}^n_{>0}.$$

This is just the adjacency graph when f is a nonnegative matrix.

Theorem (Gaubert-Gunawardena, 2004)

Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ be order-preserving and homogeneous. If the graph $G(f)$ is strongly connected, then all super-eigenspaces $S^\beta(f)$ are bounded in $(\mathbb{R}^n_{>0}, d_H)$.

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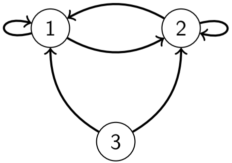
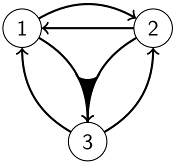
$G(f)$



$\mathcal{H}^+_\infty(F)$



$G(f)$ is not strongly connected, but the super-eigenspaces are all d_H-bounded anyway since reach $\mathcal{H}^+_\infty(F) = [n]$.

Types of irreducibility for topical maps

There are several generalizations of irreducibility for topical maps. Here are four:

1. f has no non-trivial invariant parts.
2. The graph $G(f)$ is strongly connected.
3. $\text{reach}(J, \mathcal{H}^+_\infty(f)) = [n]$ (super-eigenspaces are d_H-bounded)
4. All slice spaces are d_H-bounded

These are all equivalent for nonnegative matrices, but not for topical maps. In general $(2) \Rightarrow (3) \Rightarrow (4)$ and $(1) \Rightarrow (4)$.

All four guarantee that $E(f)$ is d_H-bounded and nonempty.

Positive eigenvectors for non-irreducible matrices

A nonnegative matrix can have a unique positive eigenvector even if it is not irreducible.

Example

$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ has unique positive eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

A corresponding notion for topical maps is when $E(f)$ is nonempty and d_H-bounded even though none of the irreducibility conditions hold.

Nonempty & bounded eigenspace

Theorem (Lemmens-L-Nussbaum, 2018)

Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ be order-preserving and homogeneous. The eigenspace $E(f)$ is nonempty and bounded in $(\mathbb{R}^n_{>0}, d_H)$ if and only if for every nonempty $J \subsetneq [n]$, there exists $x \in \mathbb{R}^n$ such that

$$\max_{j \in J} \frac{f(x)_j}{x_j} < \min_{i \in J^c} \frac{f(x)_i}{x_i}.$$

You can check this condition by testing random points in $\mathbb{R}^n_{>0}$, but that is a very slow algorithm.

Upper & lower Collatz-Wielandt numbers

The upper Collatz-Wielandt number for f is

$$r(f) := \inf\{\beta > 0 : S^{\beta}(f) \text{ is nonempty}\},$$

and the lower Collatz-Wielandt number for f is

$$\lambda(f) := \sup\{\alpha > 0 : S_{\alpha}(f) \text{ is nonempty}\}.$$

Alternatively, $r(f)$ is the infimum of the super-eigenvalues and $\lambda(f)$ is the supremum of the sub-eigenvalues.

If $E(f)$ is nonempty, then $\lambda(f) = r(f)$, but the converse is not always true.

The upper Collatz-Wielandt number $r(f)$ is equal to the *cone spectral radius*, i.e., the largest eigenvalue of f as a map on $\mathbb{R}^n_{\geq 0}$.

Boundary projections

For $\alpha \in [0, \infty]$ and $J \subseteq [n]$, let P_α^J be the projection

$$P_\alpha^J(x)_j := \begin{cases} x_j & \text{if } j \in J \\ \alpha & \text{otherwise.} \end{cases}$$

For any order-preserving homogeneous function $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$, we define

$$f_0^J := P_0^J f P_0^J \qquad \text{and} \qquad f_\infty^J := P_\infty^J f P_\infty^J.$$

Both $f_0^J : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ and $f_\infty^J : (0, \infty]^n \to (0, \infty]^n$ are order-preserving and homogeneous functions.

Bounded nonempty eigenspaces - revisited

Theorem (L, 2023)

Let $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ be order-preserving and homogeneous. The eigenspace $E(f)$ is nonempty and bounded in $(\mathbb{R}^n_{>0}, d_H)$ if and only if

$$r(f_0^J) < \lambda(f_\infty^{[n]\setminus J})$$

for every nonempty $J \subsetneq [n]$.

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Lemma

For $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ be order-preserving and homogeneous,

- $\mathrm{reach}(J^c, \mathcal{H}_0^-(f)) = [n] \iff r(f_0^J) = 0.$
- $\mathrm{reach}(J, \mathcal{H}_\infty^+(f)) = [n] \iff \lambda(f_\infty^{[n]\setminus J}) = \infty.$

So you can check the hypergraphs first, and only check the Collatz-Wielandt numbers for J where the reach condition fails.

Example

For geothmetic meandian function $F(x) = \begin{pmatrix} \frac{1}{3}(x_1 + x_2 + x_3) \\ \sqrt[3]{x_1 x_2 x_3} \\ \text{median}(x_1, x_2, x_3) \end{pmatrix}$:

- $F_{\infty}^{\{2,3\}}(x) = P_{\infty}^{\{2,3\}} F P_{\infty}^{\{2,3\}}(x) = \begin{pmatrix} \infty \\ \infty \\ \max(x_2, x_3) \end{pmatrix}$,

$$\lambda(F_{\infty}^{\{2,3\}}) = \infty.$$

- $F_{0}^{\{1,3\}}(x) = P_{0}^{\{1,3\}} F P_{0}^{\{1,3\}}(x) = \begin{pmatrix} \frac{1}{3}(x_1 + x_3) \\ 0 \\ \min(x_1, x_3) \end{pmatrix}$,

$$r(F_{0}^{\{1,3\}}) = \frac{1}{6}(1 + \sqrt{13})$$

Convex maps

1. Checking that $E(f)$ is nonempty and bounded requires checking an exponential number of subsets $J \subsetneq [n]$. This can be reduced dramatically if the additively topical map $\log \circ f \circ \exp$ is convex.

2. In addition, if $\log \circ f \circ \exp$ is convex and real analytic, or convex and piecewise affine, then we can give complete necessary and sufficient conditions for $E(f)$ to be nonempty.

Unique fixed points of real analytic nonexpansive maps

Theorem (L, 2023)

Let X be a real Banach space with the fixed point property. Let $f : X \to X$ be nonexpansive and real analytic. If f has more than one fixed point, then the set of fixed points of f is unbounded.

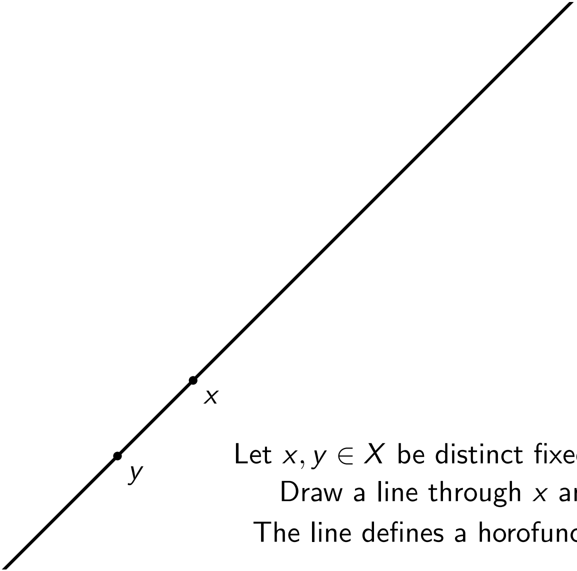
Corollary

If $f : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ is order-preserving, homogeneous, and real analytic, then f has a unique eigenvector (up to scaling) if and only if

$$r(f_0^J) < \lambda(f_\infty^{[n]\setminus J})$$

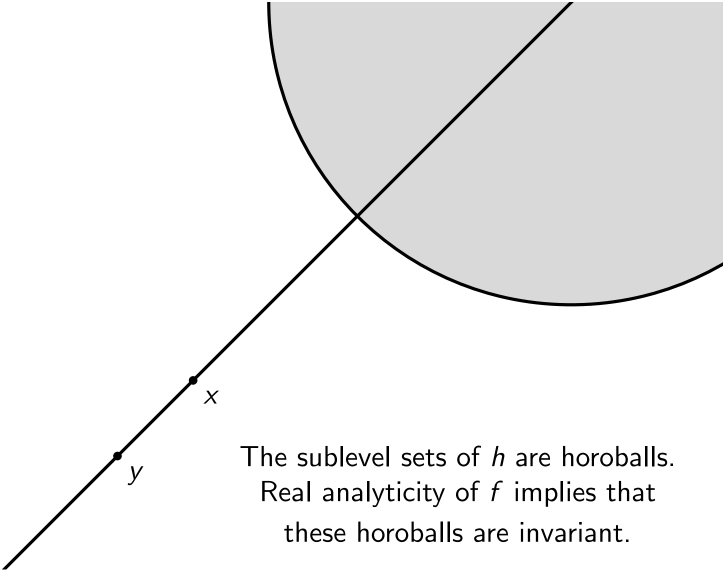
for every nonempty $J \subsetneq [n]$.

Intuition for uniqueness



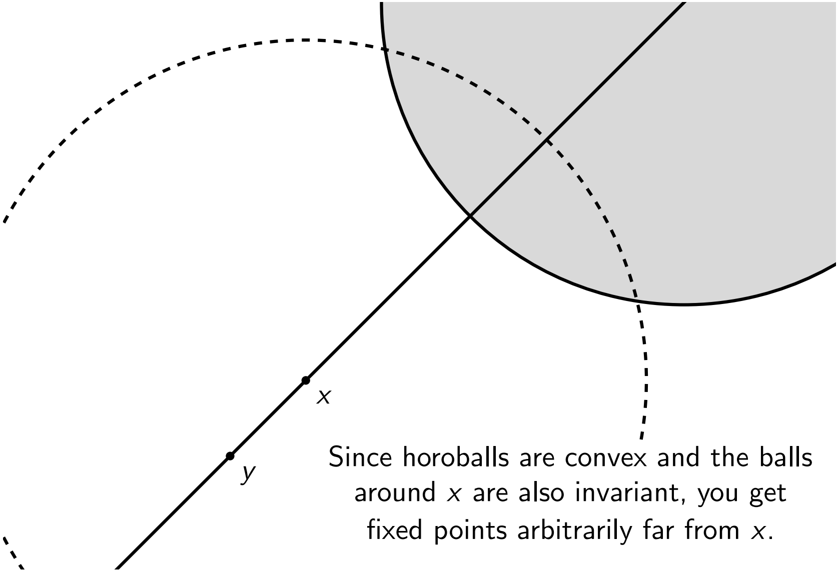
Let $x, y \in X$ be distinct fixed points.
Draw a line through x and y.
The line defines a horofunction h.

Intuition for uniqueness



The sublevel sets of h are horoballs. Real analyticity of f implies that these horoballs are invariant.

Intuition for uniqueness



Since horoballs are convex and the balls around x are also invariant, you get fixed points arbitrarily far from x.

Thanks & references

Thanks for your attention!

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