Numerical Analysis - Math 342 **Midterm 1 Review**

The following problems are similar to ones you might see on the midterm exam.

1. Use Newton's method to write down an iterative formula for finding the root of $f(x) = x^3 - a$ for any constant a. If you start with the initial guess $x_0 = \frac{1}{3}a$, then what is x_1?

2. The root of $x^3 - 2$ is $\sqrt[3]{2}$, which is located in the interval $[1, 2]$. If we use the bisection method to find this root, starting with the endpoints $a = 1$ and $b = 2$, then what is the worst case error in our estimate for the root after 10 steps?

3. Find values for the constants M and L such that $|f''(x)| \leq M$ and $|f'(x)| \geq L$ when $f(x) = x^3 - 2$ on the interval $[1, 2]$.

4. Based on your constants from the previous problem, and the Newton's method error formula

$$|x_{n+1} - r| \leq \left(\frac{M}{2L}\right) |x_n - r|^2,$$

how close to the root r would a guess x_n in $[1, 2]$ need to be in order to guarantee that the next iterate x_{n+1} is definitely closer to r?

5. Use the triangle inequality to find an upper bound M for $|f'(x)|$ when $f(x) = \sin 2x + \cos 3x$.

6. Find the fixed points of the function $f(x) = \dfrac{8}{3x - 2}$.

7. What is the derivative of the function $f(x) = \dfrac{8}{3x - 2}$ at each fixed point? Based on the derivative, determine whether each fixed point is attracting or repelling (or not enough information).

8. Let $A = \begin{pmatrix} 1 & 2 & 4 \\ 5 & 7 & 21 \\ 1 & 11 & 1 \end{pmatrix}$.

   (a) Find the LU-decomposition of A.

   (b) What is the rank of A? Is A invertible?

   (c) Compute $\|A\|_\infty$.

   (d) Use the LU-decomposition to solve $Ax = \begin{pmatrix} 2 \\ 11 \\ -1 \end{pmatrix}$.

9. Suppose that $x = 1.234 \times 10^{-3}$ and $y = 1.225 \times 10^{-3}$ each have four significant digits. How many significant digits are there in each of the following numbers?

   (a) $x + y$.

   (b) $x - y$.

   (c) xy.

   (d) x/y.

10. Let $f(x) = \dfrac{e^x - 1}{x}$.

    (a) Find a Maclaurin polynomial for f by replacing e^x by its 3rd degree Maclaurin polynomial.

    (b) What is the worst case error if you use the 3rd degree polynomial to approximate e^x on the interval $[-1, 1]$? Use Taylor's remainder formula to find an upper bound for the error on $[-1, 1]$.

11. If you use the secant method to find the root of $y = 2^x - 5$ starting with $x_0 = 1$ and $x_1 = 2$, what is x_2?

12. Express the following system of equations as a vector equation $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ and find the Jacobian $\mathbf{J}(\mathbf{x})$.

$$x^2 - 4y^2 = 1$$
$$x^2 + xy = 1$$

13. Draw a rough sketch of a cobweb diagram for the function $f(x) = -\frac{1}{2}x + 5$ starting with $x_0 = 1$.