

Calculus I - Math 141

Final Exam

Name: _____

You must show all work to earn full credit. No calculators allowed. If you do not have room in the given space to answer a question, use the back of another page and indicate clearly which work goes with which problem.

Problem	Maximum Points	Your Score
1	8	
2	9	
3	8	
4	9	
5	8	
6	6	
7	8	
8	8	
9	6	
10	6	
11	6	
12	4	
13	8	
14	6	
Total:	100	

1. (8 points) Let $f(x) = x^3 - 3x^2 + 4$.

(a) Find the intervals of increase/decrease for $f(x)$.

Solution:

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

So the critical points are $x = 0$ and $x = 2$. Testing the value of $f'(x)$ at points to the left, right, and middle of these critical points, we find the intervals of increase $(-\infty, 0)$ and $(2, \infty)$ and the interval of decrease $(0, 2)$.

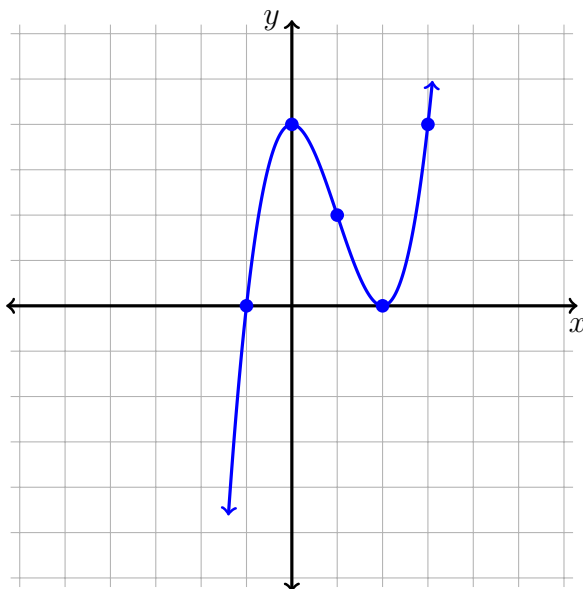
(b) Find the intervals of concavity.

Solution:

$$f''(x) = 6x - 6$$

which has critical point $x = 1$. Then $f(x)$ is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$.

(c) Sketch a graph of the function using the axis provided below.



Solution: Plot the y-values at the critical points $x = 0, 2$, and the inflection point $x = 1$, and also at some other easy points like $x = 3$ and $x = -1$. Then connect the dots, making sure it is increasing and decreasing when it should be, and concave the right way.

2. (9 points) Find the following derivatives.

(a) $\frac{d}{dx} \frac{x^2 + \sqrt{x}}{x}$

Solution: Break up the fraction first: $\frac{d}{dx} \left(\frac{x^2 + \sqrt{x}}{x} \right) = \frac{d}{dx} (x + x^{-1/2}) = 1 - \frac{1}{2}x^{-3/2}$

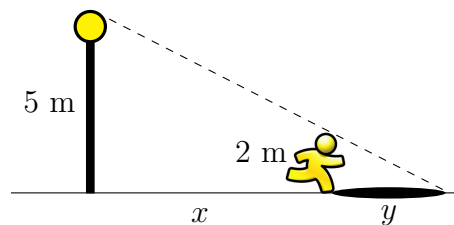
(b) $\frac{d}{d\theta} \cos(\theta^2 + 2)$

Solution: Chain rule: $-2\theta \sin(\theta^2 + 2)$.

(c) $\frac{d}{dx} x^2 \sin 2x$

Solution: Product rule (with a little chain rule too): $2x \sin 2x + 2x^2 \cos 2x$.

3. (8 points) A jogger, who is 2 meters tall, passes under a 5 meter street lamp running at 3 m/s.



- (a) Use similar triangles to find a formula relating x and y .

Solution:

$$\frac{x + y}{5} = \frac{y}{2}$$

or after cross-multiplying: $2x + 2y = 5y$ which is the same as: $2x = 3y$.

- (b) How fast is the jogger's shadow growing when the jogger has gone 6 meters past the lamppost?

Solution: The goal is to find $\frac{dy}{dt}$, since y is the shadow length. Start with the simplest version of the answer from part (a): $2x = 3y$ and take the derivative of both sides with respect to t :

$$2 \left(\frac{dx}{dt} \right) = 3 \left(\frac{dy}{dt} \right).$$

The runner is moving away from the lamppost at 3 m/s so that is $\frac{dx}{dt}$. Substituting, we can solve for $\frac{dy}{dt}$. Therefore the shadow length y is increasing at a rate of $\frac{dy}{dt} = 2$ meters per second.

4. (9 points) Find the following antiderivatives.

(a) $\int 6x^2 + 2x + 1 \, dx.$

Solution: $2x^3 + x^2 + x + C$

(b) $\int \sqrt{x} \, dx.$

Solution: $\frac{2}{3}x^{3/2} + C$

(c) $\int 2x \cos(x^2) \, dx.$

Solution: (u-Substitution) Let $u = x^2$, then $du = 2x \, dx$ and $dx = \frac{du}{2x}$.

$$\int 2x \cos(x^2) \, dx = \int 2x \cos u \frac{du}{2x} = \sin u + C = \sin(x^2) + C.$$

5. (8 points) Let $f(x) = x(6 - x)$.

(a) Find the area under the curve $y = x(6 - x)$ from $x = 0$ to $x = 3$.

Solution:

$$\int_0^3 x(6 - x) dx = \int_0^3 6x - x^2 dx = \left[3x^2 - \frac{x^3}{3} \right]_0^3 = 27 - 9 = 18.$$

(b) Find average value of $f(x) = x(6 - x)$ on the interval $[0, 3]$.

Solution:

$$\frac{1}{3 - 0} \int_0^3 x(6 - x) dx = \frac{1}{3}(18) = 6.$$

6. (6 points) Find the value of c that makes the following function continuous.

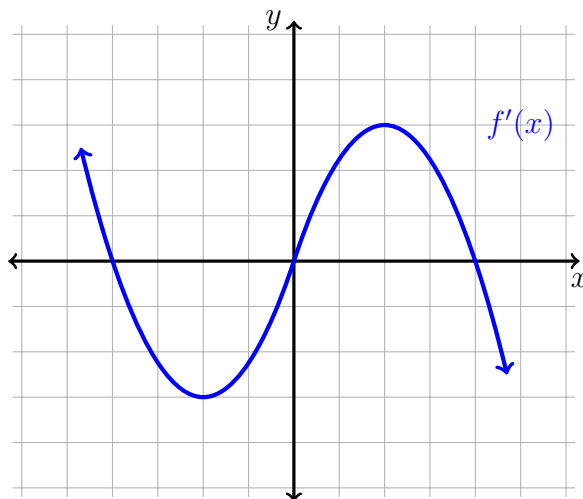
$$f(x) = \begin{cases} cx - 2 & \text{if } x > 3 \\ 1 - x^2 & \text{if } x \leq 3 \end{cases}$$

Solution: We just need the two pieces to touch when $x = 3$. At $x = 3$, the left piece is $1 - x^2 = -8$. The right piece is $cx - 2 = 3c - 2$. So we solve

$$3c - 2 = -8$$

to find $c = \frac{-6}{3} = -2$.

7. (8 points) The graph below shows the **derivative** $f'(x)$ of another function $f(x)$.



- (a) Find the intervals where the original function $f(x)$ (which is not shown) is increasing.

Solution: The original function is increasing when $f'(x)$ is positive. That looks like the intervals $(0, 4)$ and $(-\infty, -4)$. It is decreasing when $f'(x)$ is negative, i.e., $(-4, 0)$ and $(4, \infty)$.

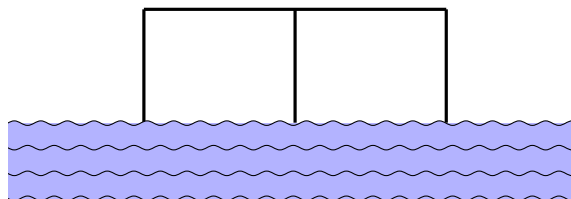
- (b) Find the x -values of all local maxes and local mins for $f(x)$.

Solution: Local extrema happen at critical points where the derivative is zero. Those are $x = -4$, $x = 0$, and $x = 4$. $x = -4$ and $x = 4$ are both local maxes and $x = 0$ is a local minimum.

- (c) Use the graph of $f'(x)$ to find the intervals of concavity for $f(x)$.

Solution: To find concavity, we need the derivative of $f'(x)$. Really, we just need to find 2nd order critical points, which are when the derivative of $f'(x)$ is zero. Those appear to be $x = -2$ and $x = +2$ from the graph above. In the interval $(-2, 2)$, $f'(x)$ is increasing so $f''(x) > 0$ and $f(x)$ is concave up. In both intervals $(-\infty, -2)$ and $(2, \infty)$, $f'(x)$ is decreasing so $f''(x) < 0$ and $f(x)$ is concave down.

8. (8 points) A farmer has 2400 ft. of fence and wants to fence off a rectangular field that borders a straight river. He also wants to use fence to divide the field into two halves as shown below. No fence is needed along the side of the river.



- (a) Write down a formula for the area of the field. If you use variables, clearly indicate what each letter represents.

Solution: Let y be the length of the three vertical fences in the picture, and let x be the length of the long horizontal size. Then area is xy .

- (b) Find the dimensions of the field with maximum possible area given the amount of fence available.

Solution: The constraint is the amount of fence: $3y + x = 2400$, so $x = 2400 - 3y$. Therefore the area is

$$A(y) = (2400 - 3y)y = 2400y - 3y^2.$$

To find the max area, set the derivative to zero:

$$A'(y) = 2400 - 6y = 0 \Rightarrow y = 400$$

Then solve for $x = 2400 - 3(400) = 600$. Finally, we just need to double check that our answer really is the maximum. The easiest way is to use the 2nd derivative test.

$$A''(y) = -6$$

which is always concave down, so our critical point must be a max.

9. (6 points) Use linear approximation or differentials to approximate $\sqrt[3]{8.24}$.

Solution: Use the linear approximation formula

$$f(x) \approx f(a) + f'(a)(x - a).$$

Here $f(x) = x^{1/3}$ and $a = 8$ because that is an easy number to plug into the formula that is close to the actual input value of $x = 8.24$. The derivative is $f'(x) = \frac{1}{3}x^{-2/3}$ and so $f'(a) = \frac{1}{3}(8^{-2/3}) = \frac{1}{6}$. Also $f(a) = \sqrt[3]{8} = 2$. Putting it all together,

$$\sqrt[3]{8.24} \approx 2 + \frac{1}{6}(0.24) = 2.04.$$

10. (6 points) Use the *definition of derivative* to find the derivative of $f(x) = x^2 + 4$.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^2 + 4 - x^2 - 4}{h} = \\ &= \lim_{h \rightarrow 0} \frac{x^2 - 2xh + h^2 + 4 - x^2 - 4}{h} = \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} = \lim_{h \rightarrow 0} 2x + h = 2x. \end{aligned}$$

11. (6 points) A particle moves along the x -axis. The velocity of a particle is given by the formula $v(t) = 16t - 10$.
- (a) What is the acceleration of the particle as a function of time?

Solution: Acceleration is the derivative of velocity.

$$a(t) = v'(t) = 16$$

- (b) What is the position of the particle as a function of time, if the particle is at $x = 5$ when $t = 1$?

Solution: Position is the antiderivative of velocity:

$$x(t) = \int v(t) dt = 8t^2 - 10t + C.$$

To find C , use the initial condition $x(1) = 5$:

$$x(1) = 8 - 10 + C = C - 2 = 5$$

so $C = 7$ and $x(t) = 8t^2 - 10t + 7$.

12. (4 points) If $h'(t)$ represents the rate of growth of a tree in feet per year, then what does

$$\int_3^5 h'(t) dt$$

represent about the tree?

Solution: $\int_3^5 h'(t) dt$ is the total growth of the tree from year 3 until year 5.

13. (8 points) Consider the function $f(x) = \frac{2x^2 + 6}{x^2 + 2x - 3}$.

(a) Does $f(x)$ have any vertical asymptotes? If so, where?

Solution: You get a vertical asymptote when you have a nonzero number divided by 0. That happens in the expression above when the bottom $x^2 + 2x - 3 = (x + 3)(x - 1) = 0$. So when $x = 1$ and $x = -3$.

(b) Does $f(x)$ have any horizontal asymptotes? Where?

Solution: To find the horizontal asymptotes, calculate the limit of the function as x goes to $+\infty$ and $-\infty$. The easiest way to do that is to factor out the highest power of x :

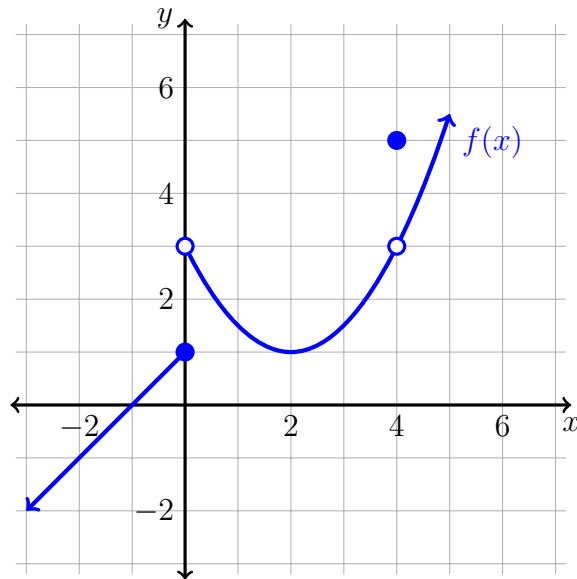
$$\lim_{x \rightarrow \infty} \frac{2x^2 + 6}{x^2 + 2x - 3} = \lim_{x \rightarrow \infty} \frac{x^2(2 + 6/x^2)}{x^2(1 + 2/x - 3/x^2)} = 2.$$

The limit as $x \rightarrow -\infty$ is the same, so there is only one horizontal asymptote at $y = 2$.

(c) Is $f(x)$ a continuous function everywhere? Why or why not?

Solution: No, the function is undefined at the vertical asymptotes $x = 1$ and $x = -3$. So it is not continuous at those two points.

14. (6 points) Use the graph below to find the indicated limits.



(a) $\lim_{x \rightarrow 0} f(x)$.

Solution: The limit from the left $\lim_{x \rightarrow 0^-} f(x) = 1$ and the limit from the right $\lim_{x \rightarrow 0^+} f(x) = 3$. Since the two sides don't agree, the limit does not exist.

(b) $\lim_{x \rightarrow 4} f(x)$.

Solution: The limit from the left $\lim_{x \rightarrow 4^-} f(x) = 3$ and the limit from the right $\lim_{x \rightarrow 4^+} f(x) = 3$. Since the two sides do agree, the limit is 3. It does not matter what the actual value is at $x = 4$.

Formula Sheet

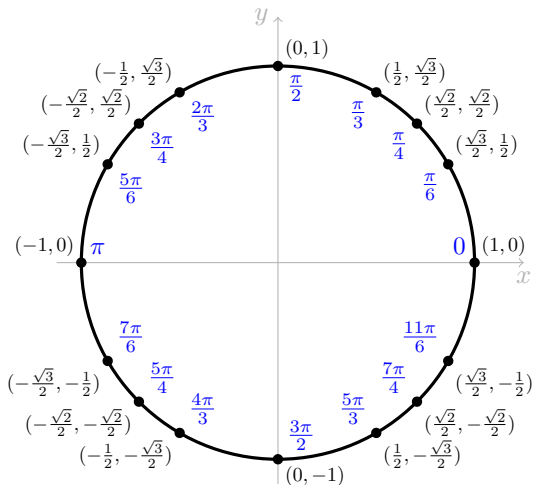
Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Point-Slope Form

$$y - y_1 = m(x - x_1)$$

Common Angles



Trigonometry Ratios

- $\tan x = \frac{\sin x}{\cos x} \quad \cot x = \frac{\cos x}{\sin x}$
- $\sec x = \frac{1}{\cos x} \quad \csc x = \frac{1}{\sin x}$

Angle Addition Identities

- $\cos(a + b) = \cos a \cos b - \sin a \sin b$
- $\sin(a + b) = \sin a \cos b + \sin b \cos a$

Trigonometry Limits

- $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$
- $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$

Definition of Derivative

- $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}, \text{ or}$
- $f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$

Selected Derivatives

- $\frac{d}{dx} \tan x = \sec^2 x$
- $\frac{d}{dx} \sec x = \sec x \tan x$
- $\frac{d}{dx} \cot x = -\csc^2 x$
- $\frac{d}{dx} \csc x = -\csc x \cot x$

Linear Approximation

- $f(x) \approx f(a) + f'(a)(x - a)$

Error and Relative Error

- $dy \approx$ the error in y
- $\frac{dy}{y} \approx$ the relative (percent) error in y

Newton's Method

- $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Summation Formulas

- $\sum_{i=1}^n i = \frac{n(n+1)}{2}$
- $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$

Riemann Sum

- $A \approx \sum_{n=1}^N f(x_n) \Delta x$ where
- $\Delta x = \frac{b-a}{N}$ and
- $x_n = a + n\Delta x$

Average Value of a Function

- $\frac{1}{b-a} \int_a^b f(x) dx$