Midterm 3 Review - Math 243

1. Consider the system

$$\frac{dx}{dt} = x^2 - y$$
$$\frac{dy}{dt} = x^2 - 1$$

   (a) Find all equilibria.

   (b) Use linearization (i.e., the Jacobian matrix) to classify each equilibrium point.

2. Consider the following one-parameter family of systems.

$$\frac{dx}{dt} = x^2 - y$$

$$\frac{dy}{dt} = x^2 - a$$

(a) Find all equilibrium points.

(b) Identify all values of a at which a bifurcation occurs.

(c) Briefly describe how the equilibrium points change at each of the bifurcation values.

3. Consider the system

$$\frac{dx}{dt} = x\cos(xy)$$
$$\frac{dy}{dt} = -y\cos(xy)$$

(a) Show that the quantity $H(x, y) = \sin(xy)$ is a Hamiltonian function for the system.

(b) Sketch the level sets for this system. Hint: The level sets occur when xy is a constant. What does the graph of $xy = c$ look like?

(c) Sketch the solution curve that passes through the point $(1, \pi)$. Be sure to indicate the direction of the solution.

4. Consider the following forced harmonic oscillator.

$$y'' + 3y' + 2y = \cos 2t$$

(a) What is the complexification of this differential equation?

(b) Find a particular solution for this differential equation.

(c) Find the general solution for this differential equation.

(d) If this system represents a harmonic oscillator, is it overdamped or underdamped? How can you tell?

5. Consider the non-homogeneous linear system

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y} + \mathbf{b}, \text{ where } A = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} -6 \\ 2 \end{bmatrix}.$$

(a) Find the equilibrium point for the system.

(b) Find the general solution to the system above.

(c) If you replace the constant vector $\mathbf{b}$ above with the vector-valued function $\mathbf{b}(t) = \begin{bmatrix} -6e^{-t} \\ 2e^{-t} \end{bmatrix}$. What would make a good guess for a particular solution $\mathbf{y}_p$ for this new system?

6. Find general solutions to the following non-homogeneous equations.

(a) $y'' - y' - 6 = e^{4t}$

(b) $y'' - y' - 6 = e^{2t}$

(c) $y'' + 4y = \cos(2t)$

(d) $y'' + 2y' + 6y = 3t + 2$