# 1 Preliminaries

- Consider a limb in a fixed posture and a set of $n$ muscles acting on it. We study a set of $d$ output variables, $d \leq n$.
  - **Example:** Fix a point on the limb and look at the 3-dimensional space of output forces forces or the 6-dimensional space of output wrenches.
  - **Example:** Fix a pair of points and look at their combined wrench space (12 dimensions).
- The *muscle activation space* is the unit $n$-cube,

$$[0,1]^n = \underbrace{[0,1] \times \cdots \times [0,1]}_{n \text{ times}}.$$

- For

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in [0,1]^n,$$

  $x_i \in [0,1]$ is the *activation* of the $i$-th muscle, $i \in \{1, \ldots, n\}$.
- $V = [\mathbf{v}_1, \ldots, \mathbf{v}_n] \in \mathbb{R}^{d \times n}$ is the matrix of *generators*; i.e. the mapping from muscle activation space to output space, where $\mathbf{v}_i$ is the *contribution* of muscle $i$, $i = 1, \ldots, n$.
- The output space is

$$\begin{aligned} Z(V) &:= V\,[0,1]^n \\ &= \{V\mathbf{x} : \mathbf{x} \in [0,1]^n\} \\ &= \left\{ \sum_{i=1}^{n} x_i \mathbf{v}_i : x_i \in [0,1], i \in \{1, \ldots, n\} \right\} \subseteq \mathbb{R}^d. \end{aligned}$$

- $Z(V)$ is a particular type of a *convex polytope* called a *zonotope*, and if $V$ is full rank (i.e. its rows are linearly independent), then $Z(V)$ is $d$-dimensional and called a $d$-zonotope.

## 2 Introduction

In [1], Sohn et al looked at the cat leg and computed the feasible activation ranges of individual muscles for a fixed output force direction and looked at how it varied with output magnitude. In other words, they computed the sets of the form

$$\{x_i : V\mathbf{x} = \alpha\mathbf{y}, \mathbf{x} \in [0,1]^n\} =: \left[x_i^{(\min)}, x_i^{(\max)}\right], \quad i = 1, \ldots, n,$$

where $\alpha \in [0,1]$ (the *magnitude*) and $\mathbf{y} \in \mathbb{R}^d$ (the *output direction* of maximal force), are fixed. They did this by solving linear programs of the form

$$\begin{aligned} \text{maximize} \quad & x_i \\ \text{subject to} \quad & V\mathbf{x} = \alpha\mathbf{y} \\ & \mathbf{x} \in [0,1]^n \end{aligned} \tag{LP$_{\max}$}$$

with optimum $x_i^{(\max)}$ and

$$\begin{aligned} \text{minimize} \quad & x_i \\ \text{subject to} \quad & V\mathbf{x} = \alpha\mathbf{y} \\ & \mathbf{x} \in [0,1]^n \end{aligned} \tag{LP$_{\min}$}$$

with optimum $x_i^{(\min)}$, $i = 1, \ldots, n$. Thus, they actually computed a bounding box

$$\left[x_1^{(\min)}, x_1^{(\max)}\right] \times \cdots \times \left[x_n^{(\min)}, x_n^{(\max)}\right] \subseteq [0,1]^n$$

of the set

$$z^{-1}(\alpha\mathbf{y}) = \{\mathbf{x} \in [0,1]^n : z(\mathbf{x}) = \alpha\mathbf{y}\} = \{\mathbf{x} \in [0,1]^n : V\mathbf{x} = \alpha\mathbf{y}\},$$

where

$$z : [0,1]^n \to \mathbb{R}^d, \quad \mathbf{x} \mapsto V\mathbf{x}.$$

They then solved these linear programs for various $\alpha \in [0,1]$ to see how the bounding box changes as the output magnitude varies.

## 3 Opportunities for improvement

The primary limitation of this approach is the fact that they only consider isolated muscles and thus completely miss the co-dependencies of activation patterns imposed by the physical constraints in output space. Unfortunately, the activation space is $n$ dimensional, and the preimage

$$z^{-1}(\alpha \mathbf{y}) = \{\mathbf{x} \in [0,1]^n : V\mathbf{x} = \alpha \mathbf{y}\}$$

could still be $(n-d)$ dimensional. In case of the cat leg, we have $n = 31$ and $d = 3$, so this space is very hard to visualize. Another problem is that the combinatorial complexity of the $n$-cube is $2^n$, so $z^{-1}(\alpha \mathbf{y})$ may have exponential complexity.

### 3.1 Visualize the co-activation patterns of two or three muscles

Instead of computing only

$$\{x_i : V\mathbf{x} = \alpha \mathbf{y}, \mathbf{x} \in [0,1]^n\}, \quad i \in \{1, \ldots, n\}$$

we can study the co-activation polytopes of two muscles,

$$\{(x_i, x_j) : V\mathbf{x} = \alpha \mathbf{y}, \mathbf{x} \in [0,1]^n\}, \quad 1 \leq i < j \leq n$$

and those of three muscles:

$$\{(x_i, x_j, x_k) : V\mathbf{x} = \alpha \mathbf{y}, \mathbf{x} \in [0,1]^n\}, \quad 1 \leq i < j < k \leq n.$$

These are "trivially" obtained via projection of the full pre-image $z^{-1}(\alpha \mathbf{y})$, but computing the full $z^{-1}(\alpha \mathbf{y})$ may be prohibitively expensive (though we won't know unless we try; state of the art algorithms for such computations have been implemented in Komei Fukuda's cddlib (`http://www.inf.ethz.ch/personal/fukudak/cdd_home/`), and Parma Polyhedra Library (`http://bugseng.com/products/ppl/`)). I do believe, however, that it should be possible to obtain the co-activation of two and three muscles more efficiently than via full enumeration (just like a single muscle can be obtained more efficiently via LP), though this is something I would have to look further into.

### 3.2 Work only in the output space $Z(V)$

This is the approach taken in by Jason Kutch and Francisco Valero-Cuevas in [2]. At the time, they ran into problems with computational complexity of their algorithms, which is the primary motivation behind my master's thesis and the accompanying software library I've been working on for the past two months `https://github.com/vindvaki/libzonotope`. The cat leg analysis in[1] is only done in activation space, so we could follow up with analysis similar to that from [2] for robustness in output space.

## References

[1] Sohn, M. H., McKay, J. L., & Ting, L. H. (2013). Defining feasible bounds on muscle activation in a redundant biomechanical task: practical implications of redundancy. Journal of biomechanics, 46(7), 1363-1368.

[2] Kutch, J. J., & Valero-Cuevas, F. J. (2011). Muscle redundancy does not imply robustness to muscle dysfunction. Journal of biomechanics, 44(7), 1264-1270.