Introduction & problem statement

- $Z(V) = \sum_{i=1}^{n} [\mathbf{0}, \mathbf{v}_i]$
- $Z_i = Z\left(V \setminus \{\mathbf{v}_i\}\right)$ is the set of points reachable without the i-th muscle.
- $\mathbf{v}_i + Z_i$ is the set of points reachable with the i-th muscle fully activated.
- Is there a set where the i-th muscle *must* be fully activated? I.e. points $\mathbf{y} \in Z(V)$ s.t. if $\mathbf{y} = V\mathbf{x}$, $\mathbf{x} \in [0,1]^n$, then $x_i = 1$.

There is at least one such point

- Consider

$$\begin{aligned} \max \quad & \mathbf{v}_i^T \mathbf{y} \\ \text{s.t.} \quad & \mathbf{y} = V\mathbf{x}, \\ & \mathbf{x} \in [0,1]^n . \end{aligned}$$

- Clearly, the optimum of this problem always has $x_i = 1$:

$$\mathbf{v}_i^T \mathbf{y} = \sum_{j=1}^{n} x_j \mathbf{v}_i^T \mathbf{v}_j,$$

- More precisely, the optimal solutions $\mathbf{x}$ all satisfy

$$x_j = \begin{cases} 1, & \text{if } \mathbf{v}_i^T \mathbf{v}_j > 0, \\ 0, & \text{if } \mathbf{v}_i^T \mathbf{v}_j < 0, \\ \text{any value in } [0,1], & \text{if } \mathbf{v}_i^T \mathbf{v}_j = 0, \end{cases} \quad j = 1, \ldots, n.$$

- This specifies a face of $Z(V)$ of dimension $\leq \left|\left\{j \in [n] : \mathbf{v}_i^T \mathbf{v}_j = 0\right\}\right|$.
- In other words: For every muscle i, there is at least one maximal output for which that muscle is absolutely necessary.
- In fact, *every vertex with $\mathbf{v}_i$ as a component requires full activation of the i-th muscle.*

The volume of the set of those points is 0

- Note that

$$Z = \bigcup_{\lambda \in [0,1]} \left(\lambda \mathbf{v}_i + Z_i\right)$$

- The set of points where the i-th generator needs to be fully activated is the set

$$Z \setminus \bigcup_{\lambda \in [0,1)} \left(\lambda \mathbf{v}_i + Z_i\right).$$

- But

$$\bigcup_{\lambda \in [0,1)} \left(\lambda \mathbf{v}_i + Z_i\right)$$

  is an open dense set in Z with the same volume as Z, so its complement in Z must have volume 0.

What about other levels of activation?

- Similarly to above,
$$Z \setminus \bigcup_{\lambda \in [0, \lambda_{\min}]} (\lambda \mathbf{v}_i + Z_i)$$
is the set of points where the i-th generator needs to be activated above $\lambda_{\min}$.

- The set
$$\bigcup_{\lambda \in [0, \lambda_{\min}]} (\lambda \mathbf{v}_i + Z_i) = [\mathbf{0}, \lambda_{\min} \mathbf{v}_i] + Z_i$$
is a zonotope combinatorially equivalent to Z.

- We can associate with each i the sum
$$s_i := \sum_{I \in \binom{[n]}{d-1}} \left| \det \left(V_{I \cup \{i\}} \right) \right|$$
of determinants of linearly independent combinations containing $\mathbf{v}_i$.

- Then
$$\begin{aligned} \text{volume}\left([\mathbf{0}, \lambda \mathbf{v}_i] + Z_i\right) &= \text{volume}\left(Z_i\right) + \lambda s_i \\ &= \left(\text{volume}\left(Z\right) - s_i\right) + \lambda s_i \\ &= \text{volume}\left(Z\right) - \left(1 - \lambda\right) s_i \end{aligned}$$
and the volume of its complement is
$$\left(1 - \lambda\right) s_i.$$
In particular, s_i is the volume of the set of outputs where the i-th muscle is required to *some* extent.

- In other words, the volume of the set of outputs where the i-th muscle needs to have activation at least λ is $(1 - \lambda)\, s_i$.

- This adds a multiplicative overhead of $O(d)$ to the volume computation, and requires us to store the volume of Z and $s_1, \ldots, s_n$, but the volume query time is constant for any i and λ.

Operating on real data is important

- If the v_{ij} are i.i.d. $N(0,1)$ normals, then we can assume $|\det(V_I)| \approx c$ with high probability (as $d \to \infty$) for all $I \in \binom{[n]}{d}$ for some constant c (around $d^{o(1)}\sqrt{(d-1)!}$, see [1]), and thus
$$\frac{s_i}{\text{volume}\left(Z\right)} \approx \frac{\binom{n}{d-1}}{\binom{n}{d}} = \frac{d}{n - (d-1)}.$$

- The above does not actually hold in our desired use case, where $d \leq 6$ is constant.

- It does, however, suggest (but does not prove) that the above ratio might be $O(1/n)$; i.e. that only a small fraction of all feasible movements requires a fixed muscle i to be active at all, given enough muscles.

- I wouldn't be too surprised to find both, i.e.
  - An example driven by a very large number of muscles with similar properties (from that particular POV)
  - An example where only a few muscles account for almost all the output range
- These two examples may very well be two sides of the same example: Imagine a limb with one very strong muscle complemented by many smaller muscles for mostly orthogonal actions to those of the strong muscle.

References

[1] Kevin P. Costello and Van Vu. Concentration of random determinants and permanent estimators. `http://arxiv.org/abs/0905.1909`