



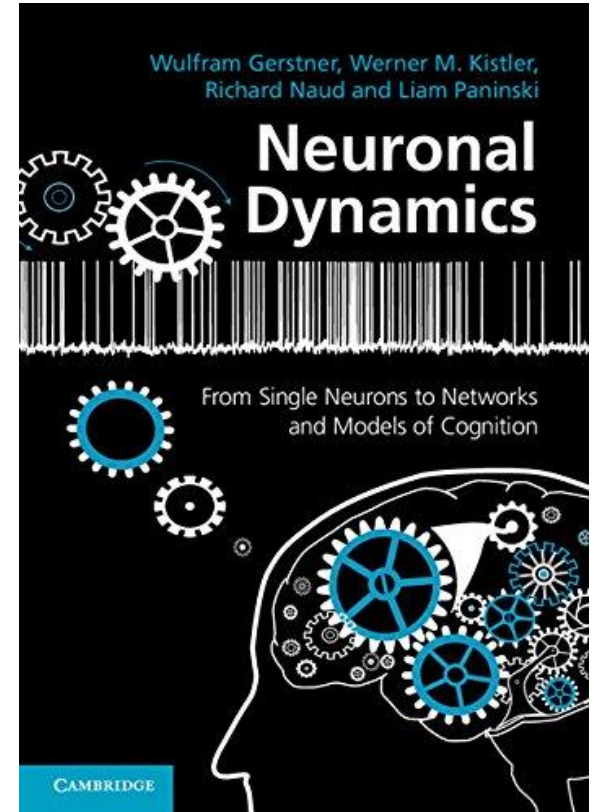
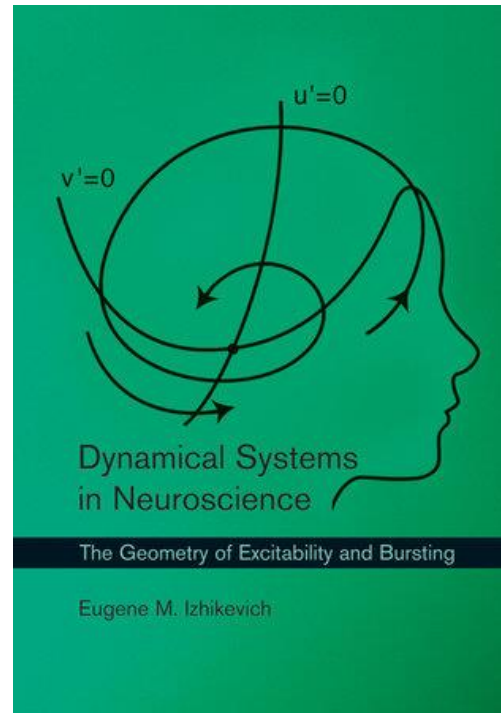
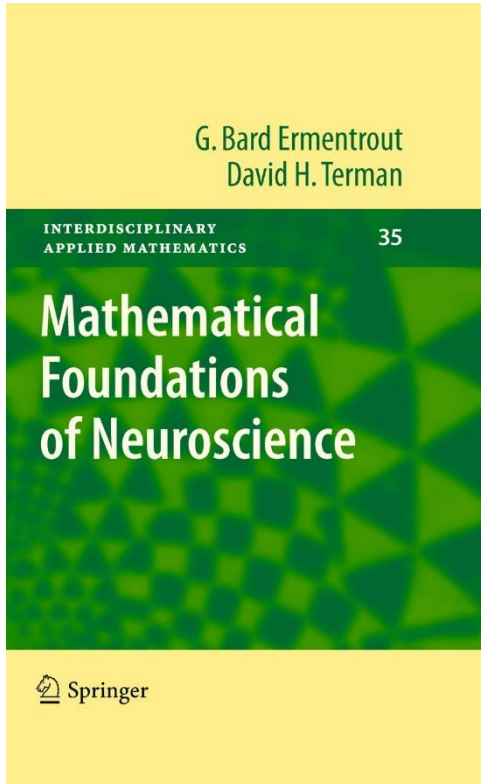
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Neural Dynamics

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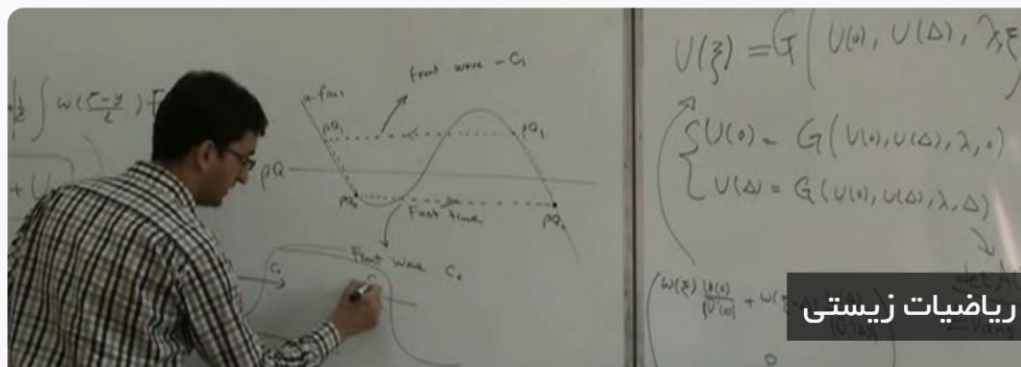
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صفحه اصلی درس ها درباره ما ارتباط با ما



ریاضیات زیستی

مشخصات درس

ترم ارائه: بهار ۱۳۹۱

مقطع: کارشناسی

استاد درس

مرتضی فتوحی



محتوای جلسات و ویدیوها

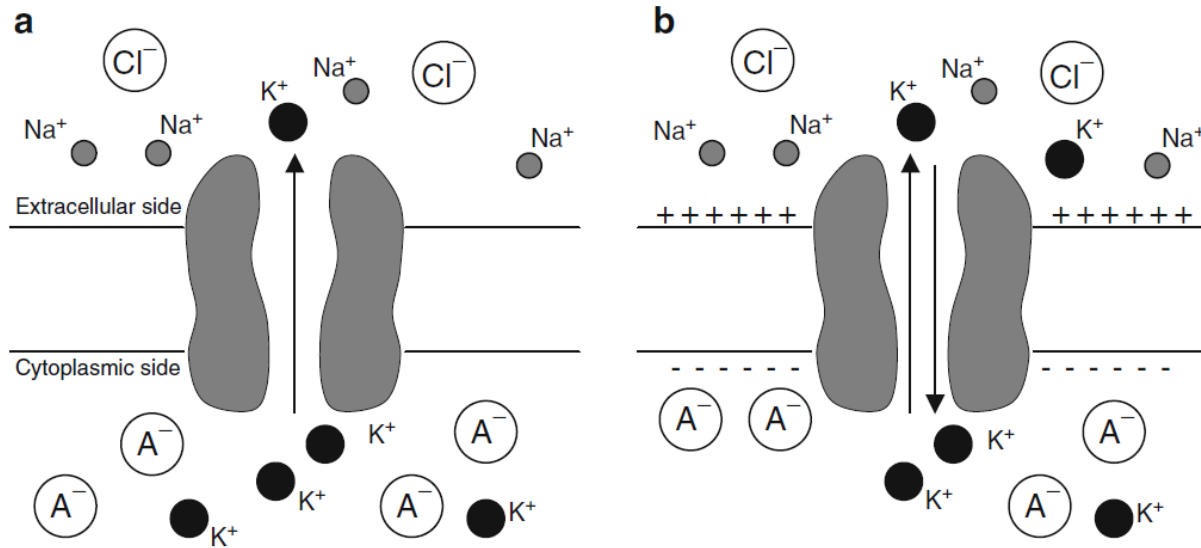
شماره	بخش	ویدیوی جلسه	جزوه ی درس
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جلسه ۳	---	دانلود (فیلم)	---
جلسه ۴	---	دانلود (فیلم)	---
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جلسه ۶	---	دانلود (فیلم)	---
جلسه ۷	---	دانلود (فیلم)	---

The Resting Potential



$$V_M = V_{in} - V_{out},$$

$$E_K = -\frac{RT}{zF} \ln \frac{[K^+]_{in}}{[K^+]_{out}}. \quad (1.1)$$



The Goldman–Hodgkin–Katz Equation



- Currents flow according to:
 - The permeabilities of ion channels
 - Concentration gradients across the cell membrane

$$V_M = \frac{RT}{F} \ln \frac{P_K[K^+]_{\text{out}} + P_{\text{Na}}[\text{Na}^+]_{\text{out}} + P_{\text{Cl}}[\text{Cl}^-]_{\text{in}}}{P_K[K^+]_{\text{in}} + P_{\text{Na}}[\text{Na}^+]_{\text{in}} + P_{\text{Cl}}[\text{Cl}^-]_{\text{out}}},$$

Different channels



- Non gated (leaky)
- Gated \rightarrow changes in the permeabilities \rightarrow action potentials

Equivalent circuit model

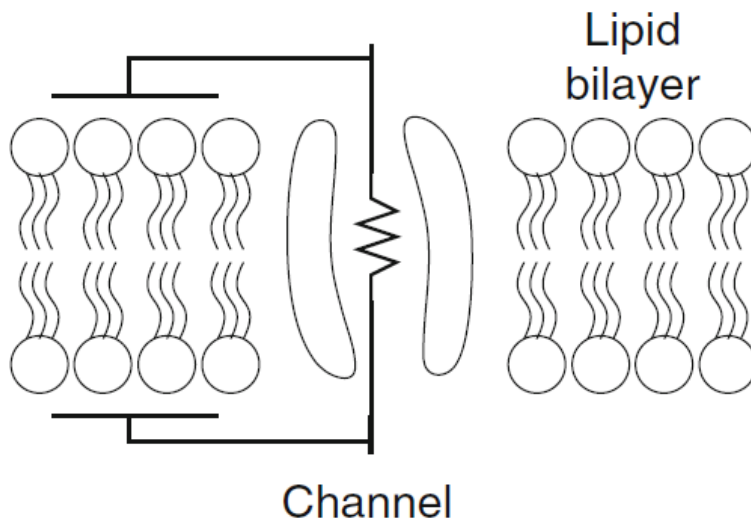


- (1) **Conductors** or resistors, representing the **ion channels**;
- (2) **Batteries**, representing the **concentration gradients** of the ions;
- (3) **Capacitors**, representing the ability of the membrane to **store charge**.
- **Lipid bilayer** has dielectric properties
 - Recall that capacitors store charge and then release it in the **form of currents**.

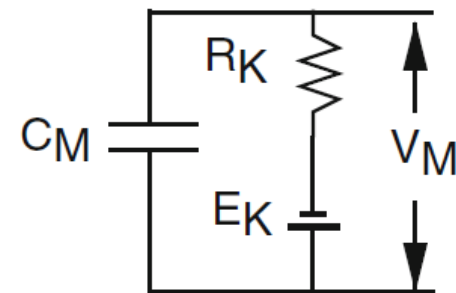
$$q = C_M V_M;$$

- c_M : *specific membrane capacitance*: **capacitance per square centimeter** ($\sim 1F/cm^2$);
 - The **bigger** cell, the **bigger** C_M

Simple model



$$i_{\text{cap}} = c_M \frac{dV_M}{dt}.$$



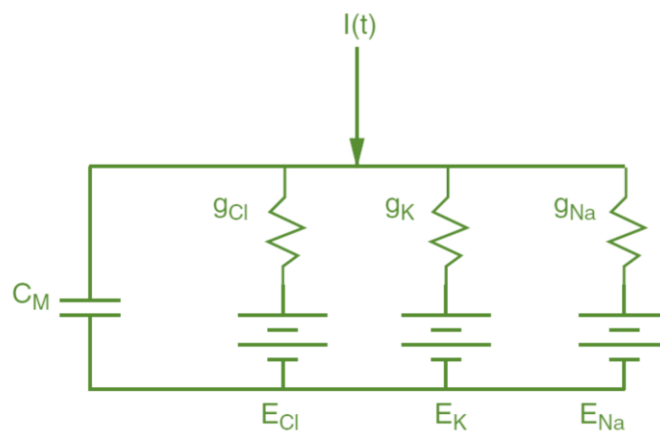
$$\hat{I}_K = \hat{g}_K (V_M - E_K).$$

$$I_K = g_K (V_M - E_K) = \frac{V_M - E_K}{r_K}.$$

g_K : conductance per unit area

$$0 = i_{\text{cap}} + I_K = c_M \frac{dV_M}{dt} + \frac{V_M - E_K}{r_K}$$

$$c_M \frac{dV_M}{dt} = -\frac{V_M - E_K}{r_K} = -g_K(V_M - E_K).$$




$$i_{\text{ion}} = -g_{Cl}(V_M - E_{Cl}) - g_K(V_M - E_K) - g_{Na}(V_M - E_{Na}).$$



current source is not typically expressed as current per unit area:

$$c_M \frac{dV_M}{dt} = -g_{Cl}(V_M - E_{Cl}) - g_K(V_M - E_K) - g_{Na}(V_M - E_{Na}) + I(t)/A.$$

total surface area of the neuron 

$$c_M \frac{dV_M}{dt} = -\frac{(V_M - E_R)}{r_M} + I(t)/A,$$

where

$$E_R = (g_{Cl}E_{Cl} + g_K E_K + g_{Na}E_{Na})r_M$$

cell's resting potential

Specific membrane resistance

$$r_M = \frac{1}{g_{Cl} + g_K + g_{Na}}$$



For a **passive membrane**:

$$V_{ss} = \frac{g_{Cl}E_{Cl} + g_K E_K + g_{Na}E_{Na} + I/A}{g_{Cl} + g_K + g_{Na}}.$$

steady state

Weighted sum of the **equilibrium potentials** of the three currents

Similar to the GHK contribution to the resting potential by each ion is **weighted** in **proportion to the permeability**

Conductance and permeability are related concepts:

- Permeability depends on the state of the membrane (**number of open channels**)
- Conductance depends on both the state of the membrane and the **concentration** of the ions



The Membrane Time Constant:

How a passive, isopotential cell responds to an **applied current**.

Passive:

If its electrical properties **do not change** during signaling. Such a cell cannot generate an action potential;

Isopotential

If the membrane potential is **uniform** at all points of the cell

We will consider a spherical cell with radius r

$$I_M(t) = \frac{I(t)}{4\pi\rho^2} = \begin{cases} \frac{I_0}{4\pi\rho^2} & \text{if } 0 < t < T \\ 0 & \text{otherwise.} \end{cases}$$



$$c_M \frac{dV_M}{dt} = -\frac{(V_M - E_R)}{r_M} + I(t)/A,$$

To simplify things, we take $E_R = 0$ so that V_M measures the deviation of the membrane potential from rest

$$E_R = 0$$



$$c_M \frac{dV_M}{dt} = -\frac{V_M}{r_M} + I_M(t).$$

If the cell starts at rest

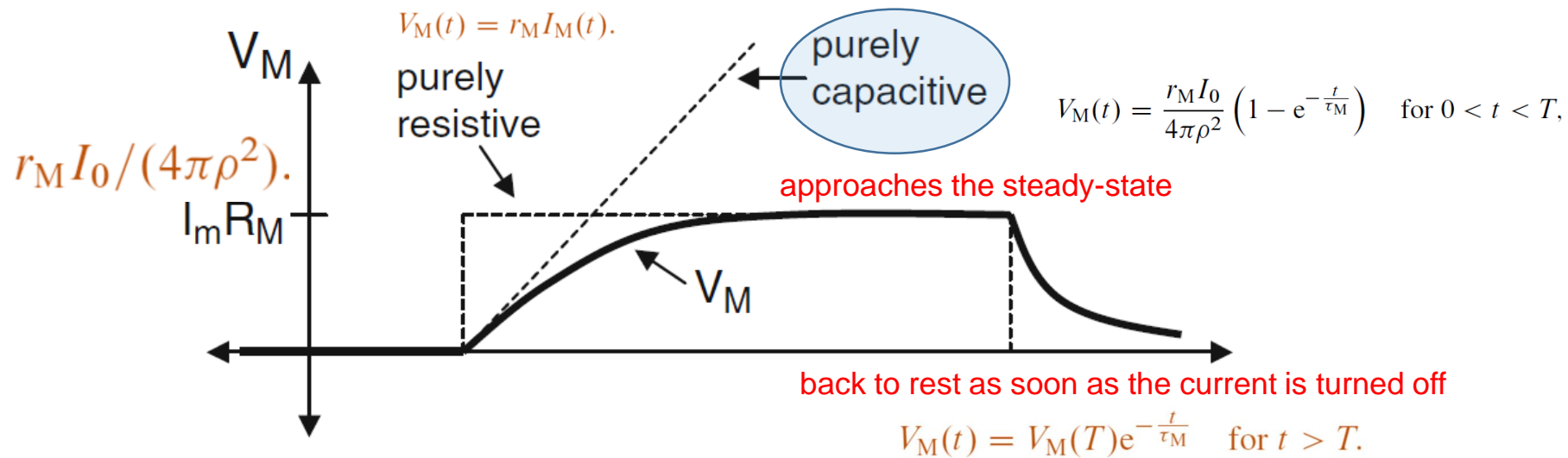
$$V_M(t) = \frac{r_M I_0}{4\pi\rho^2} \left(1 - e^{-\frac{t}{\tau_M}}\right) \quad \text{for } 0 < t < T,$$

$\tau_M \equiv c_M r_M$ Membrane time constant

$$c_M \frac{dV_M}{dt} = -\frac{V_M}{r_M} + \overset{0}{\cancel{I_M(t)}}.$$

$$V_M(t) = V_M(T)e^{-\frac{t}{\tau_M}} \quad \text{for } t > T.$$

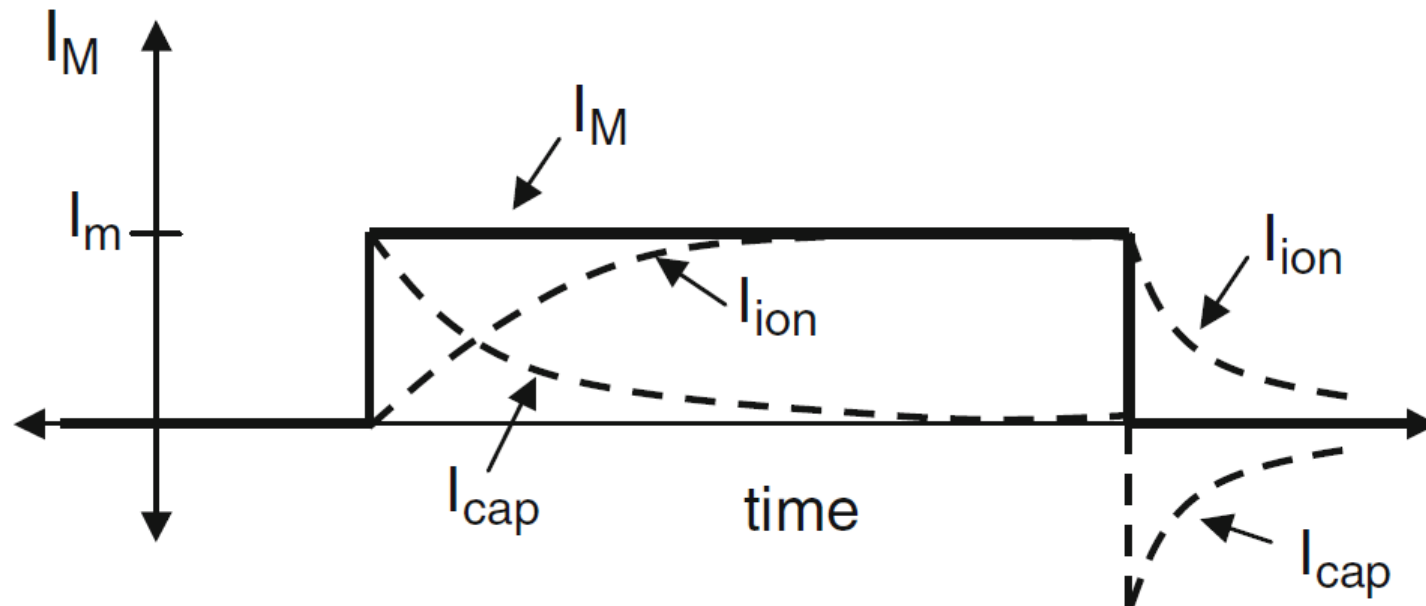
The change of membrane potential in response to a step of current.



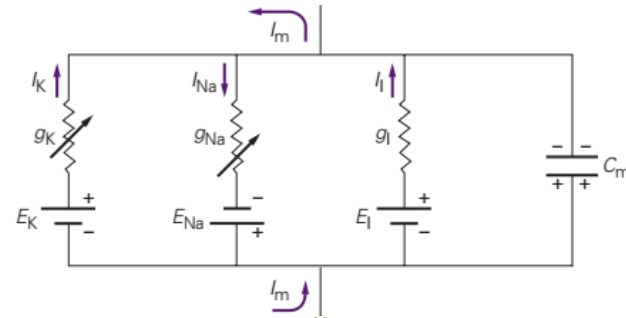
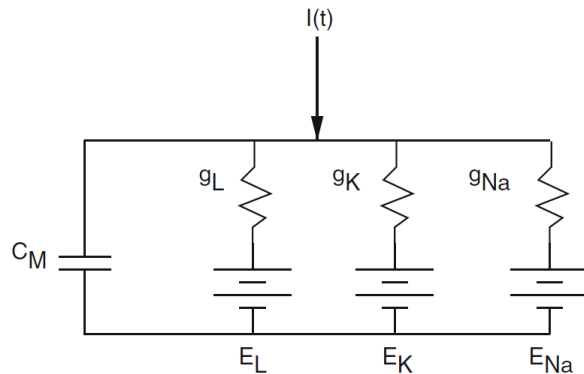
The steady-state membrane potential

$$I_0 \frac{r_M}{4\pi\rho^2} \equiv I_0 R_{\text{INP}}, \text{ input resistance}$$

The time course of the total membrane current, the ionic current, and the capacitive current



Equivalent circuit underlying the **Hodgkin–Huxley** equations

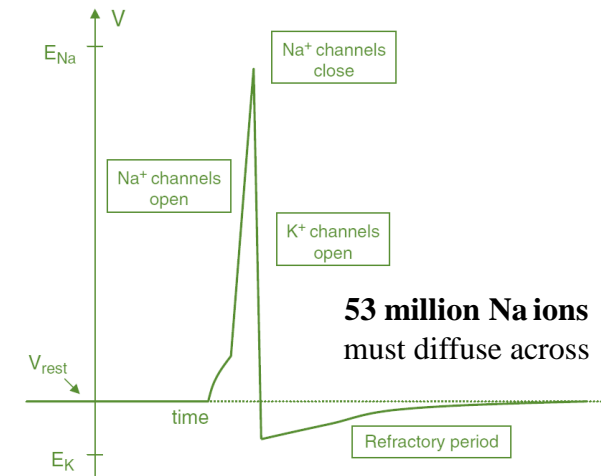


$$C_M \frac{dV}{dt} = -g_{Na}(V - E_{Na}) - g_K(V - E_K) - g_L(V - E_L).$$

$I_L \equiv g_L(V - E_L)$ is called the *leak current*.

Since most **nongated** channels are permeable to K ions, E_L is close to E_K .

AP: changes in the **relative conductances** of the dominant ionic species.

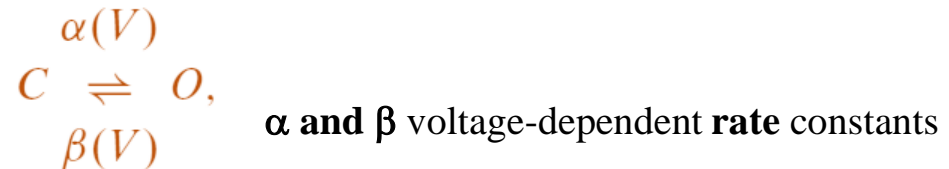




Voltage-Gated Channels

The **pores** have gates that can be either **open or closed**

The **probability** that a gate is open or closed depends on the membrane potential



If we let **m** be the fraction of open gates, then **1-m** is the fraction of closed gates

Law of mass action: the principle that the **rate of a chemical** reaction is proportional to the **concentrations of the reacting substances**.

$$\frac{dm}{dt} = \alpha(V)(1 - m) - \beta(V)m = (m_{\infty}(V) - m)/\tau(V),$$

where

$$m_{\infty}(V) = \frac{\alpha(V)}{\alpha(V) + \beta(V)} \quad \text{and} \quad \tau(V) = \frac{1}{\alpha(V) + \beta(V)}.$$



It is easy to solve this equation if V is constant. The solution starting at $m(0)$ is

$$m(t) = m_{\infty}(V) + (m(0) - m_{\infty}(V))e^{-t/\tau(V)}.$$

voltage-dependent rate constants α and β .

Based on **thermodynamics**: the probability of opening or closing a channel depends **exponentially on the potential**

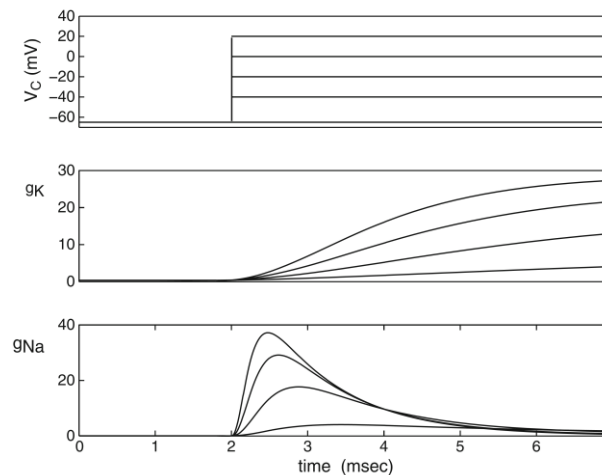
$$\alpha(V) = A_{\alpha} \exp(-B_{\alpha} V) \quad \text{and} \quad \beta(V) = A_{\beta} \exp(-B_{\beta} V).$$

$$m_{\infty}(V) = \frac{1}{1 + \exp(-(V - V_h)/V_s)},$$

Hodgkin and Huxley were able to isolate the K current by replacing Na ions in the external bathing solution with a larger, impermeant cation. This *eliminated the inward Na current*

Now we can use TTX

$$g_K(t) = \frac{I_K(t)}{(V_M - E_K)} \quad \text{and} \quad g_{Na}(t) = \frac{I_{Na}(t)}{(V_M - E_{Na})}.$$



inactivated



Using the voltage-clamp data, Hodgkin and Huxley derived expressions for the K^+ and Na^+ conductances. They proposed that

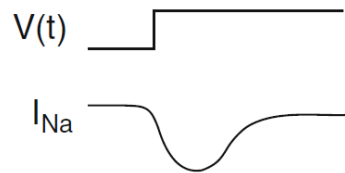
$$g_K = \bar{g}_K n^4 \quad \text{and} \quad g_{Na} = \bar{g}_{Na} m^3 h,$$

where \bar{g}_K and \bar{g}_{Na} are maximum conductances and n , m , and h are gating variables. Values between 0 and 1.

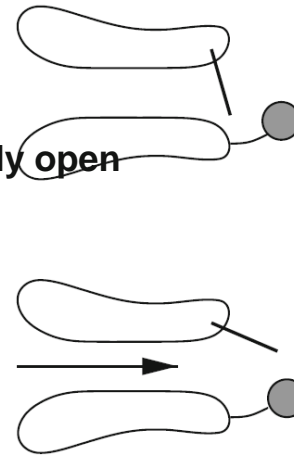
probability that the sodium activation gate is open is m^3
probability that the sodium inactivation gate is open is h
 n^4 represents the probability that a K^+

Na channel's activation gate (*line*) is closed but the inactivation gate (*ball*) is open.

a

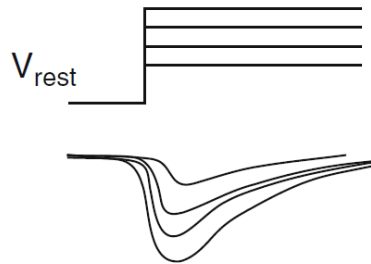


d



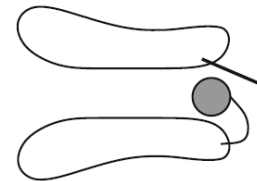
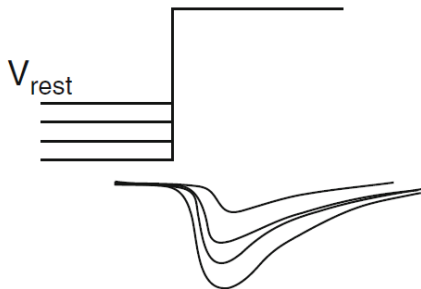
b

both gates are partially open



c

the activation gate is open and the inactivation gate is closed



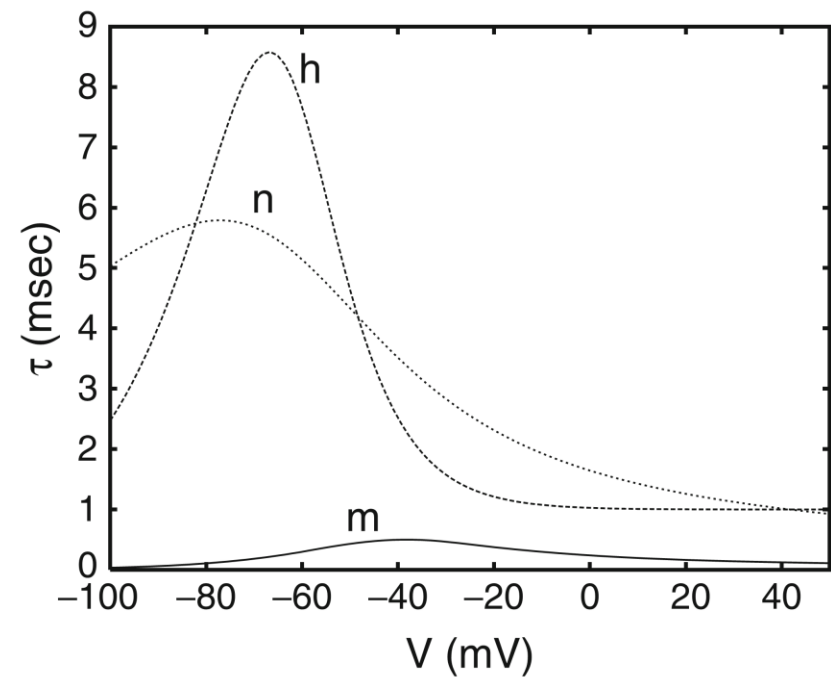
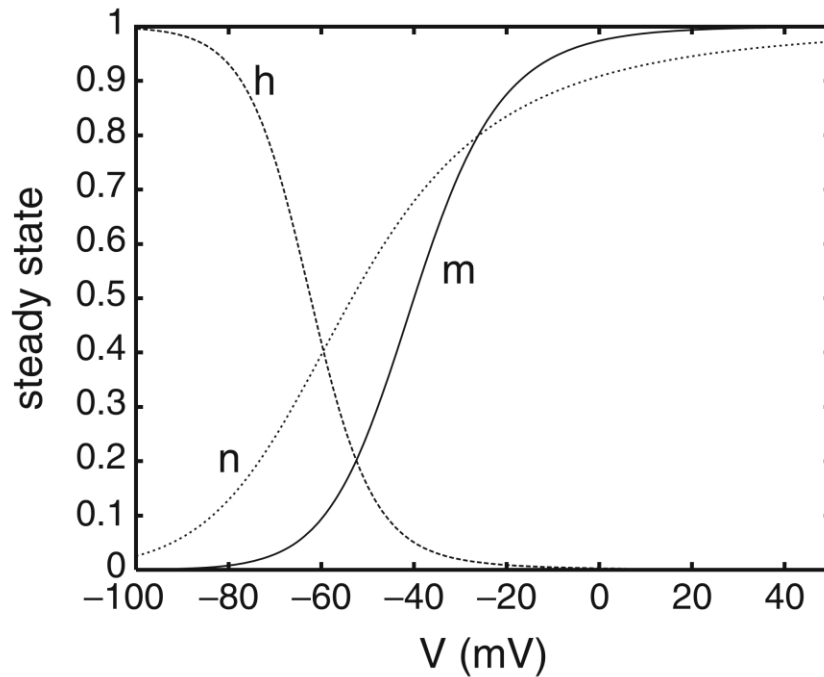
$$\begin{aligned}\frac{dn}{dt} &= \alpha_n(V)(1-n) - \beta_n(V)n = (n_\infty(V) - n)/\tau_n(V), \\ \frac{dm}{dt} &= \alpha_m(V)(1-m) - \beta_m(V)m = (m_\infty(V) - m)/\tau_m(V), \\ \frac{dh}{dt} &= \alpha_h(V)(1-h) - \beta_h(V)h = (h_\infty(V) - h)/\tau_h(V).\end{aligned}$$

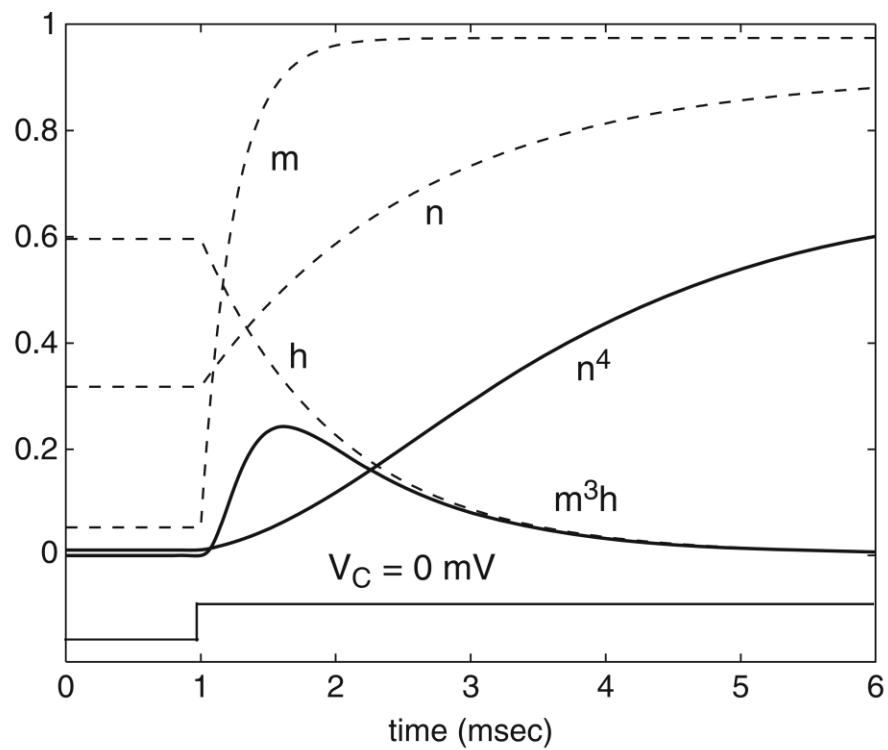
If $X = n, m$, or h , then

$$X_\infty(V) = \frac{\alpha_X(V)}{\alpha_X(V) + \beta_X(V)} \quad \text{and} \quad \tau_X(V) = \frac{1}{\alpha_X(V) + \beta_X(V)}.$$

Hodgkin–Huxley functions:

Left: the steady-state opening of the gates and
Right: the time constants





Response of the activation and inactivation variables m , h , and n to a step in voltage

Hodgkin–Huxley model is a system of four differential equations;
there is **one equation for the membrane potential** and **three equations for channel gating variables**.

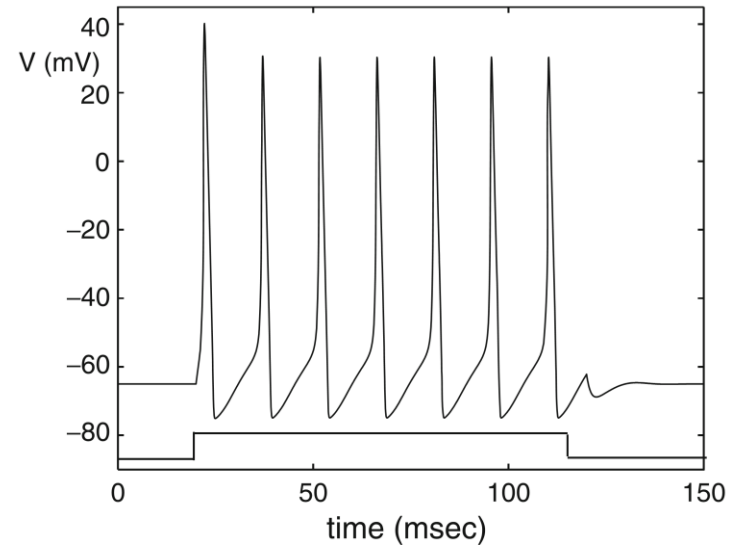
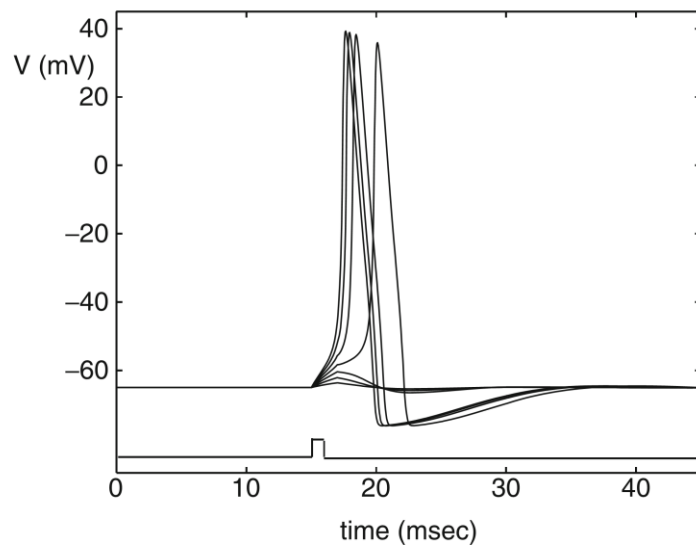
$$\begin{aligned}c_M \frac{dV}{dt} &= -\bar{g}_{\text{Na}} m^3 h (V - E_{\text{Na}}) - \bar{g}_{\text{K}} n^4 (V - E_{\text{K}}) - \bar{g}_{\text{L}} (V - E_{\text{L}}), \\ \frac{dn}{dt} &= \phi [\alpha_n(V)(1 - n) - \beta_n(V)n], \\ \frac{dm}{dt} &= \phi [\alpha_m(V)(1 - m) - \beta_m(V)m], \\ \frac{dh}{dt} &= \phi [\alpha_h(V)(1 - h) - \beta_h(V)h].\end{aligned}$$

Temperature factor:

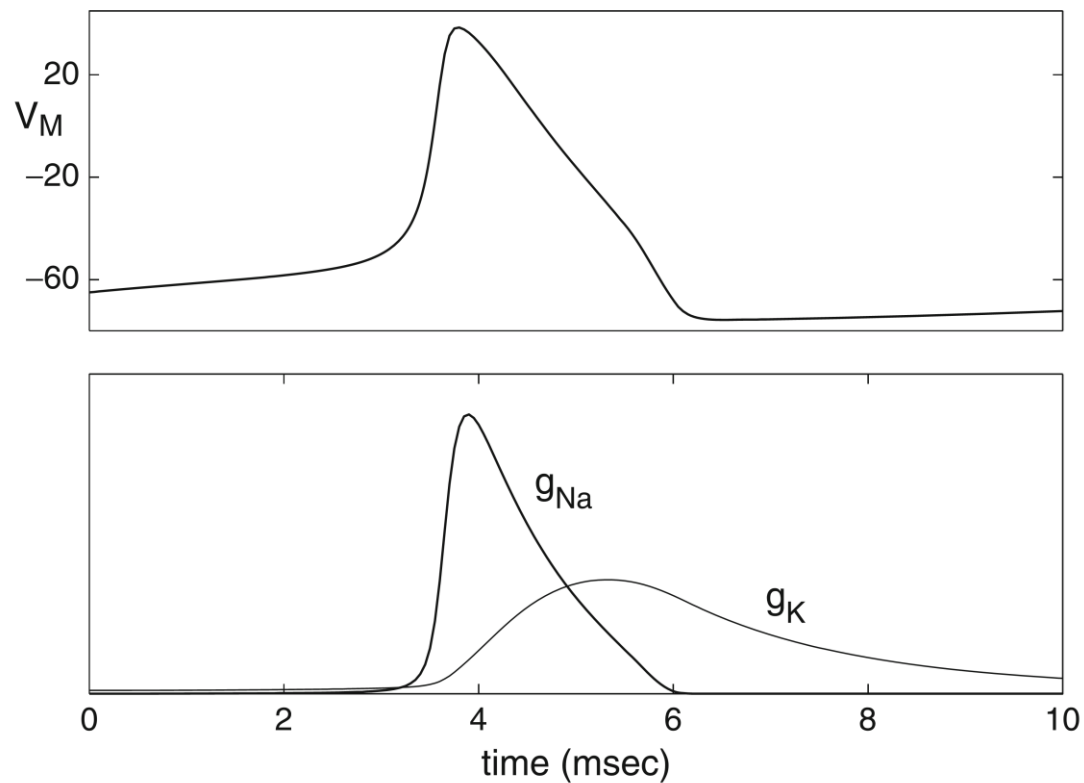
$$\phi = Q_{10}^{(T - T_{\text{base}})/10}.$$

Q_{10} is the ratio of the rates for an increase in temperature of 10°C. For the squid giant axon, $T_{\text{base}} = 6.3^\circ\text{C}$ and $Q_{10} = 3$.

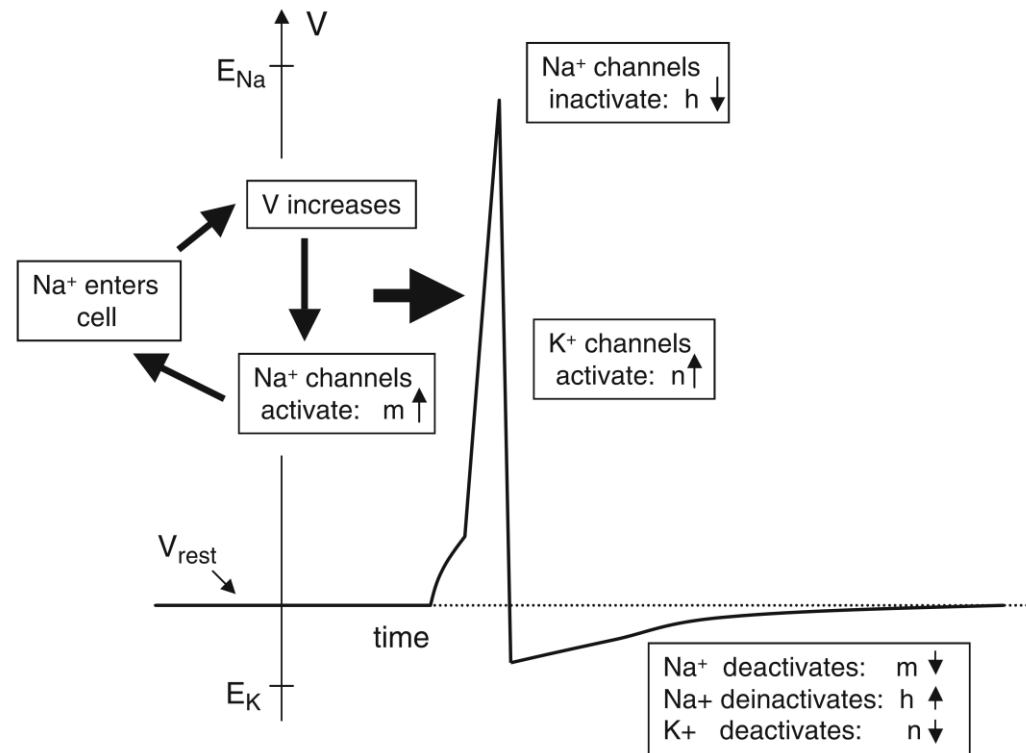
Responses of the Hodgkin–Huxley model to applied **currents**. *Left* transient responses showing “all-or-none” behavior and *right* sustained periodic response



Solution of the Hodgkin–Huxley equations showing an action potential. Also shown are the Na^+ and K^+ conductances



Mechanisms underlying the action potential



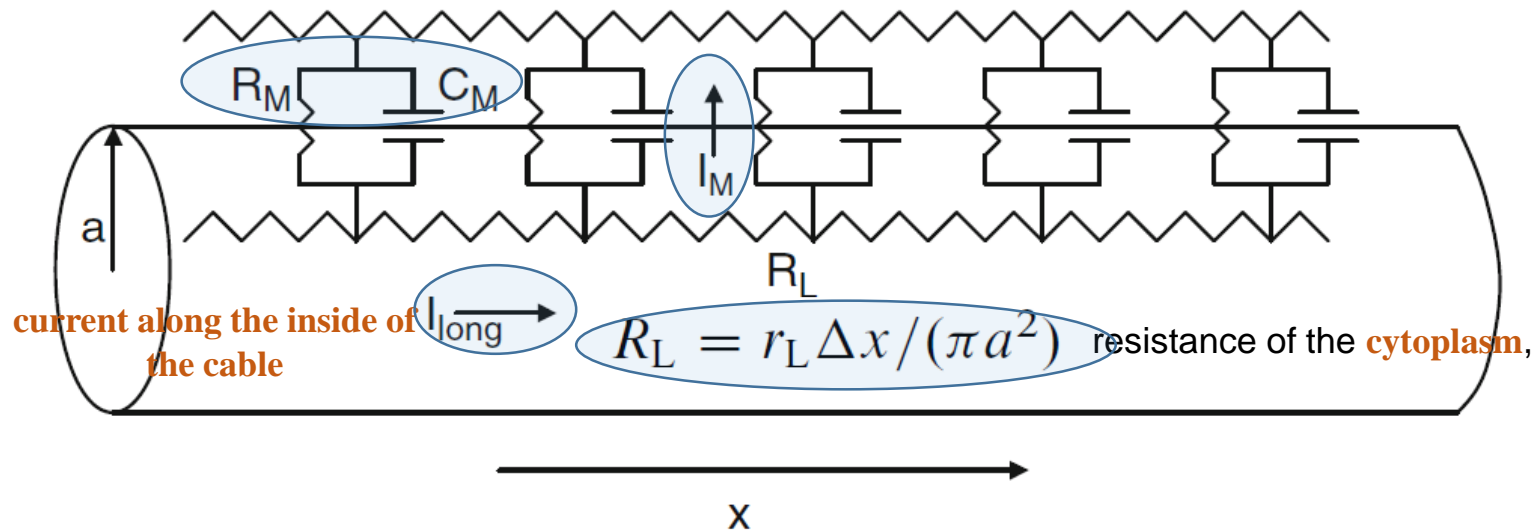
The Cable Equation

It is important to understand how the **geometry of the cell** affects the spread of the signal.

Axon approximated by **cylinders**

I_M is the current across the membrane.

$R_e = 0$, extracellular space is **isopotential**



R_M is the membrane resistance, and C_M is the membrane capacitance

$$V_M(x + \Delta x, t) - V_M(x, t) = -I_{\text{long}}(x, t) R_L = -I_{\text{long}}(x, t) \frac{\Delta x}{\pi a^2} r_L.$$



$$V_M(x + \Delta x, t) - V_M(x, t) = -I_{\text{long}}(x, t)R_L = -I_{\text{long}}(x, t)\frac{\Delta x}{\pi a^2}r_L.$$

In the limit $\Delta x \rightarrow 0$

$$I_{\text{long}}(x, t) = -\frac{\pi a^2}{r_L} \frac{\partial V_M}{\partial x}(x, t).$$

i_{ion} be the current per unit area

$$I_{\text{ion}} = (2\pi a \Delta x)i_{\text{ion}}.$$

$$C_M = (2\pi a \Delta x)c_M$$

$$I_{\text{cap}}(x, t) = (2\pi a \Delta x)c_M \frac{\partial V_M}{\partial t}.$$



$$I_{\text{cap}}(x, t) + I_{\text{ion}}(x, t) = -I_{\text{long}}(x + \Delta x, t) + I_{\text{long}}(x, t),$$

$$(2\pi a \Delta x) c_M \frac{\partial V_M}{\partial t} + (2\pi a \Delta x) i_{\text{ion}} = \frac{\pi a^2}{r_L} \frac{\partial V_M}{\partial x}(x + \Delta x, t) - \frac{\pi a^2}{r_L} \frac{\partial V_M}{\partial x}(x, t).$$

We divide both sides of this equation by $2\pi a \Delta x$ and let $\Delta x \rightarrow 0$

$$c_M \frac{\partial V_M}{\partial t} = \frac{a}{2r_L} \frac{\partial^2 V_M}{\partial x^2} - i_{\text{ion}}.$$

$$i_{\text{ion}} = V_M(x, t) / r_M, \text{ specific membrane resistance}$$

$$c_M \frac{\partial V_M}{\partial t} = \frac{a}{2r_L} \frac{\partial^2 V_M}{\partial x^2} - \frac{V_M}{r_M}.$$



$$c_M \frac{\partial V_M}{\partial t} = \frac{a}{2r_L} \frac{\partial^2 V_M}{\partial x^2} - \frac{V_M}{r_M}.$$

We can rewrite this equation as

$$\tau_M \frac{\partial V_M}{\partial t} = \lambda^2 \frac{\partial^2 V_M}{\partial x^2} - V_M,$$

where

$$\lambda = \sqrt{\frac{a r_M}{2 r_L}} \quad \text{and} \quad \tau_M = c_M r_M$$

space or length constant and the *membrane time constant*

steady-state solutions

$$\cancel{\tau_M}^0 \frac{\partial V_M}{\partial t} = \lambda^2 \frac{\partial^2 V_M}{\partial x^2} - V_M,$$

we inject a step of current, I_0 , at $x = 0$. As $t \rightarrow \infty$,

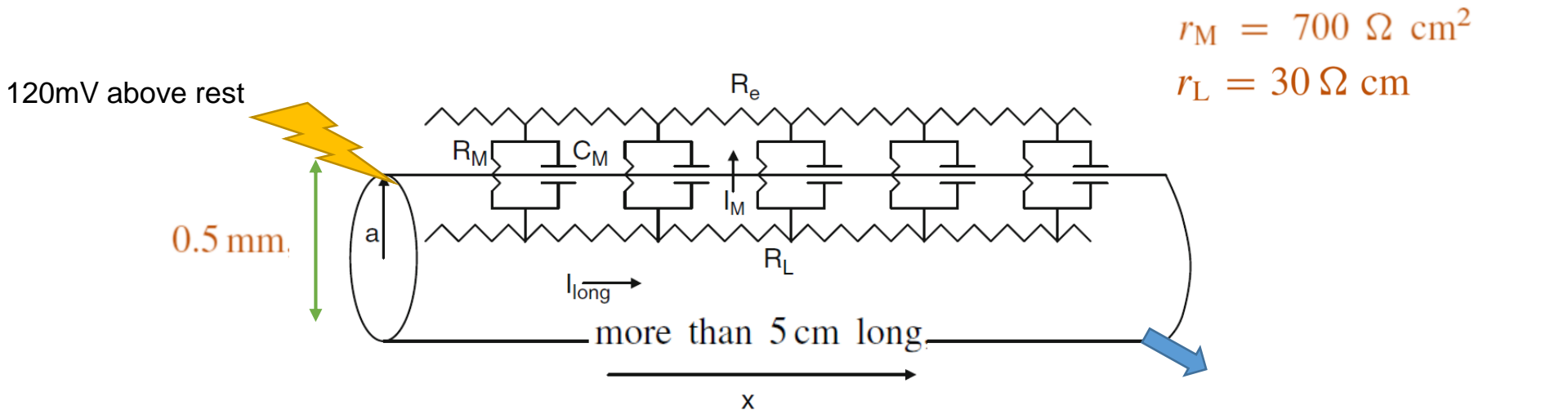
$$\lambda^2 \frac{d^2 V_{ss}}{dx^2} - V_{ss} = 0.$$

we need boundary conditions

$$I_{\text{long}}(x, t) = -\frac{\pi a^2}{r_L} \frac{\partial V_M}{\partial x}(x, t). \quad \longrightarrow \quad I_0 = -\frac{\pi a^2}{r_L} \frac{\partial V_M}{\partial x} \quad \longrightarrow \quad \frac{dV_{ss}}{dx}(0) = -\frac{r_L}{\pi a^2} I_0.$$

$$\longrightarrow \quad V_{ss}(x) = \frac{\lambda r_L}{\pi a^2} I_0 e^{-x/\lambda}.$$

The Squid Action Potential



$$r_M = 700 \, \Omega \, \text{cm}^2$$

$$r_L = 30 \, \Omega \, \text{cm}$$

10 μV above the rest, a 10,000-fold decrement

$$\lambda = \sqrt{\frac{ar_M}{2r_L}} \longrightarrow \lambda = 5.4 \, \text{mm}.$$

order of magnitude **smaller than the length**

$$c_M \frac{\partial V_M}{\partial t} = \frac{a}{2r_L} \frac{\partial^2 V_M}{\partial x^2} - \frac{V_M}{r_M}, \quad i_{\text{cap}} = c_M \frac{dV_M}{dt}.$$

$$I_L = I_{\text{cap}} + I_{\text{ion}}$$

$$\frac{a}{2r_L} \frac{\partial^2 V_M}{\partial x^2} = c_M \frac{\partial V_M}{\partial t} + I_K + I_{\text{Na}} + I_L.$$

$$c_M \frac{\partial V_M}{\partial t} = \frac{a}{2r_L} \frac{\partial^2 V_M}{\partial x^2} - g_K(V_M - E_K) - g_{\text{Na}}(V_M - E_{\text{Na}}) - g_L(V_M - E_L).$$

Voltage clamp: capacitive could be zero in constant voltage $I_{\text{cap}} = C_M dV_M/dt = 0.$

Space-clamped: inserting a highly conductive axial wire inside the fiber $\frac{\partial^2 V_M}{\partial x^2} = 0.$

The leaky integrate-and-fire (LIF)

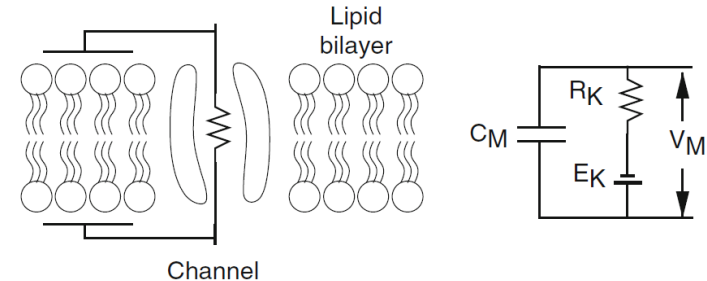
$$C_M \frac{dV_M}{dt} = -\frac{(V_M - E_R)}{r_M} + I(t)/A,$$

$$\tau_m \frac{dv}{dt} = -v(t) + RI(t)$$

τ_m is the membrane time constant

R is the membrane resistance.

simple resistor-capacitor (RC) circuit



Leakage term is due to the **resistor** and the **integration** of $I(t)$ is due to the **capacitor**

Spiking events are not explicitly modeled in the LIF model.



When the membrane potential $v(t)$ reaches a certain threshold V_{th} (**spiking threshold**), it is instantaneously reset to a lower value V_r (**reset potential**)

absolute refractory period Δ_{abs} immediately after $v(t)$ hits v_{th} .



Stimulation by a constant input current

$$I(t) = I.$$

assume $v_r = 0$.

The solution of Equation 1 is then given by:

$$v(t) = RI[1 - \exp(-\frac{t}{\tau_m})]$$

Asymptotic value of the membrane potential is RI .

If this value is less than the spiking threshold, v_{th} ,



Time of the first spike

Assuming $v(0) = v_r = 0$,

$t^{(1)}$, can be found by solving:

$$v_{th} = RI[1 - \exp(-\frac{t^{(1)}}{\tau_m})]$$

This yields:

$$t^{(1)} = \tau_m \ln \frac{RI}{RI - v_{th}}$$

Firing rate of the neuron



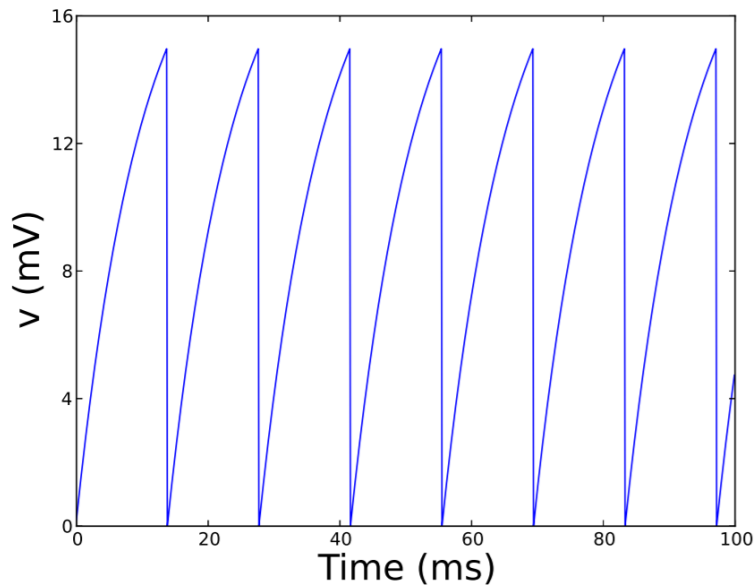
$$T = \Delta_{abs} + \tau_m \ln \frac{RI}{RI - v_{th}}$$

$$f = [\Delta_{abs} + \tau_m \ln \frac{RI}{RI - v_{th}}]^{-1}$$

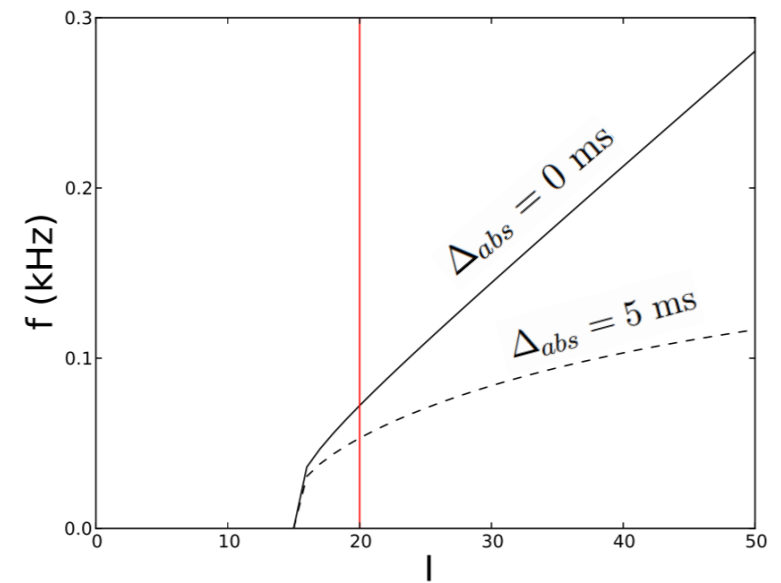
Stimulation of a LIF neuron by a constant input current



the time-course of the membrane potential, $v(t)$



f-I curve for a LIF neuron



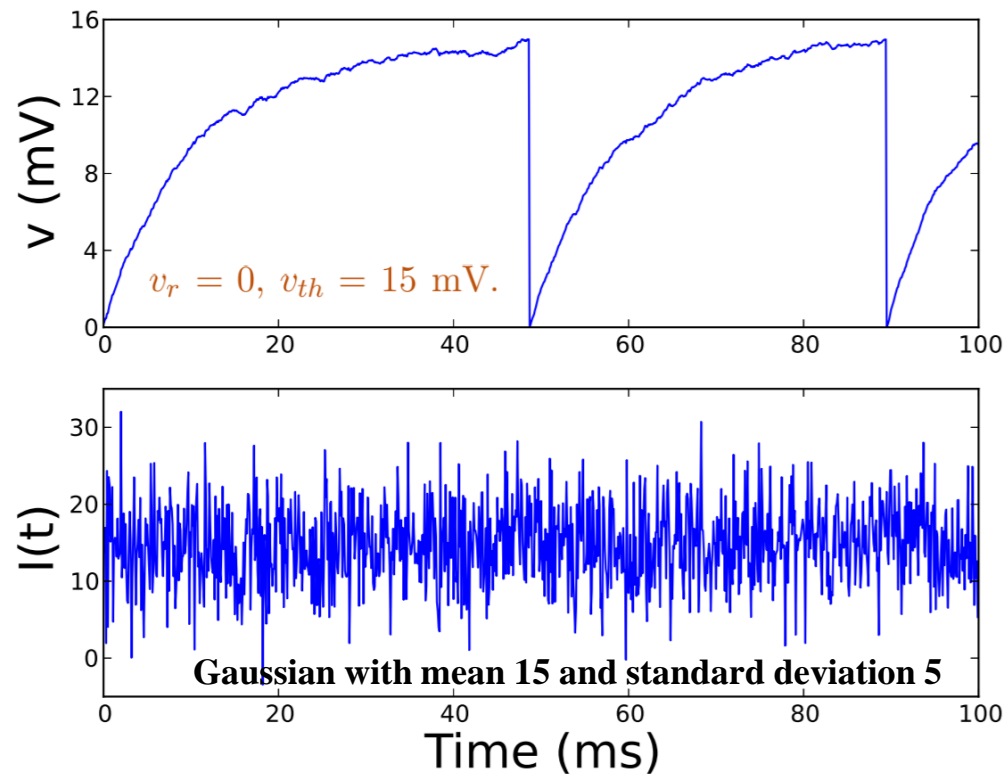


Stimulation by a time-varying input current

For a general time-varying input current $I(t)$, the solution LIF neuron Equation, with the initial condition $v(t_0) = v_r$, is given by:

$$v(t) = v_r \exp\left(-\frac{t - t_0}{\tau_m}\right) + \frac{R}{\tau_m} \int_0^{t-t_0} \exp\left(-\frac{s}{\tau_m}\right) I(t - s) ds$$

Stimulation of a LIF neuron by a time-varying input current





Stimulation by synaptic currents

Each **pre-synaptic spike** makes a stereotyped contribution, described by a function $\alpha(t)$, to the post-synaptic current and contributions of different pre-synaptic spikes are linearly summed to obtain the total post-synaptic current.

Total post-synaptic current to the i -th neuron:

$$I_i(t) = \sum_j w_{ij} \sum_f \alpha(t - t_j^{(f)})$$

where $t_j^{(f)}$ represents the time of the f -th spike of the j -th pre-synaptic neuron;

w_{ij} is the **strength of synaptic efficacy**

Common choice for a

Dirac δ -pulse:

$$\alpha(t) = q\delta(t)$$

the alpha synapse

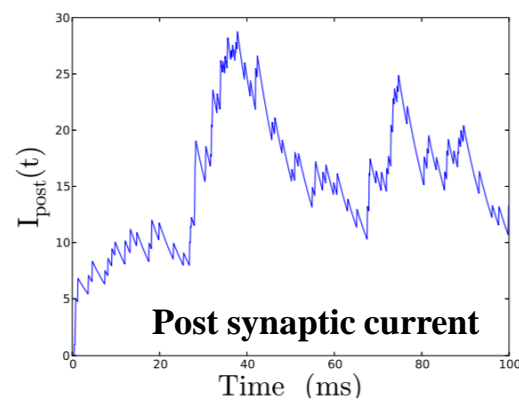
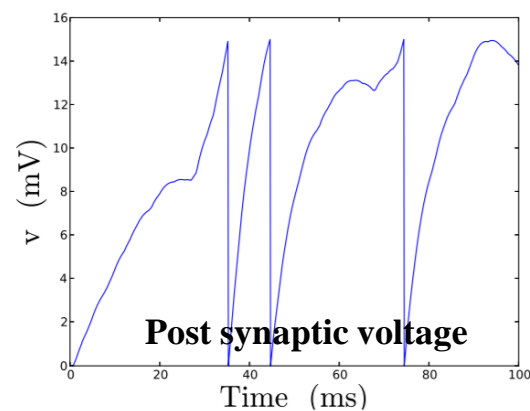
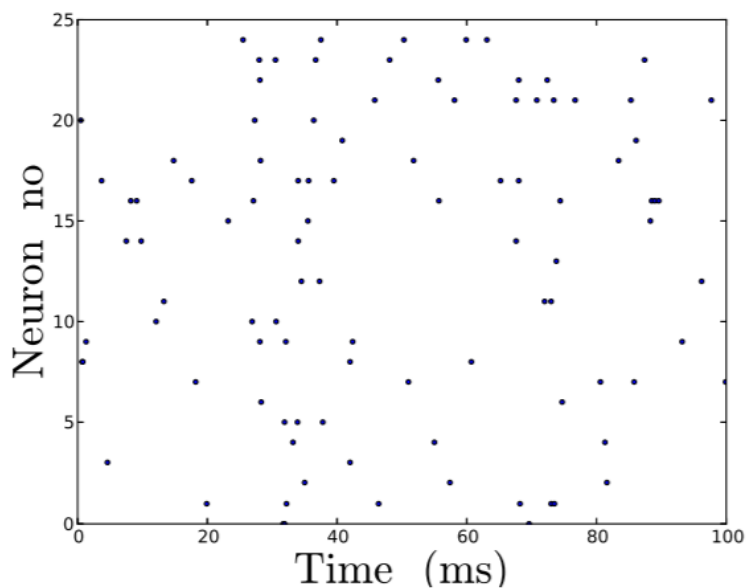
$$\alpha(t) = \alpha \frac{t}{\tau} \exp(1 - \frac{t}{\tau})$$

the bi-exponential synapse:

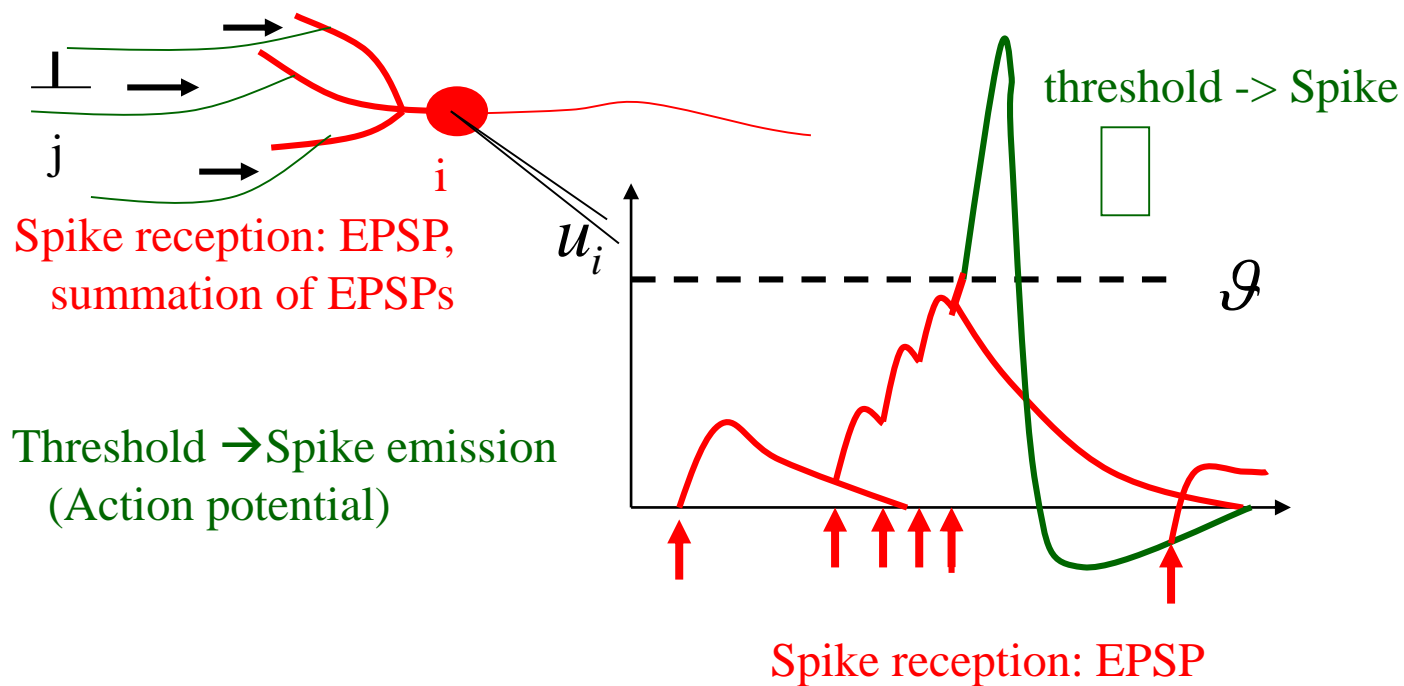
$$\alpha(t) = \beta \frac{\tau_2}{\tau_2 - \tau_1} [\exp(-\frac{t}{\tau_1}) - \exp(-\frac{t}{\tau_2})]$$

where α and β are normalizing constants and τ , τ_1 and τ_2 are the time constants of the synapses

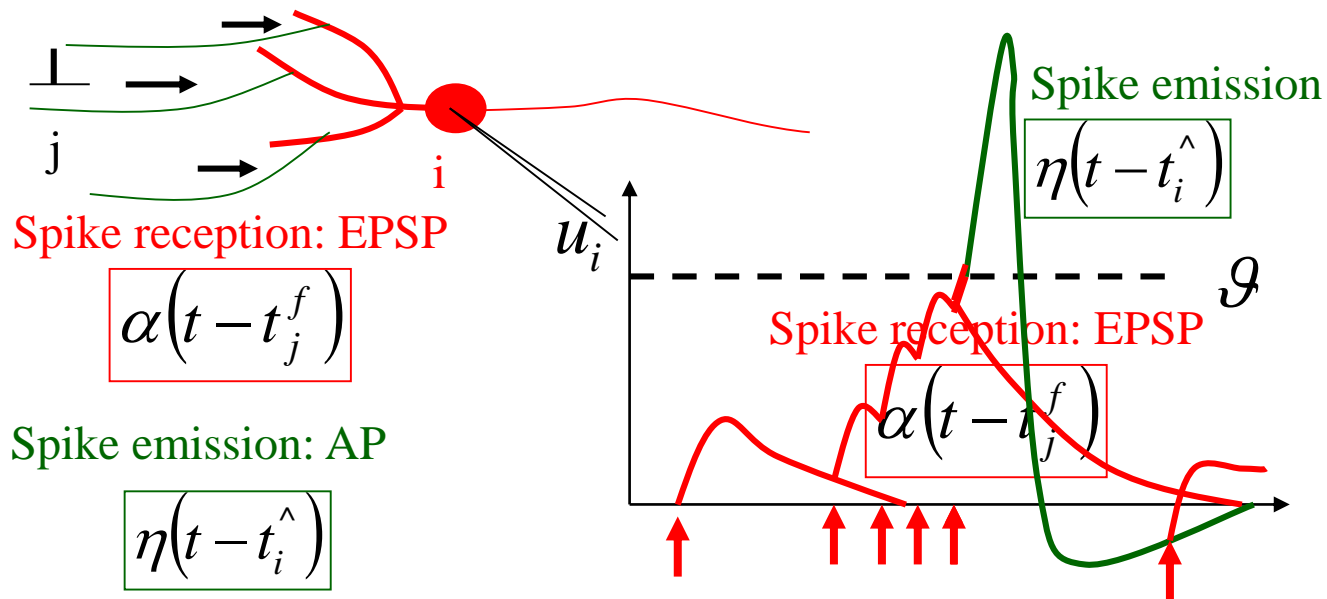
Stimulation of a LIF neuron by synaptic current



Phenomenology of spike generation



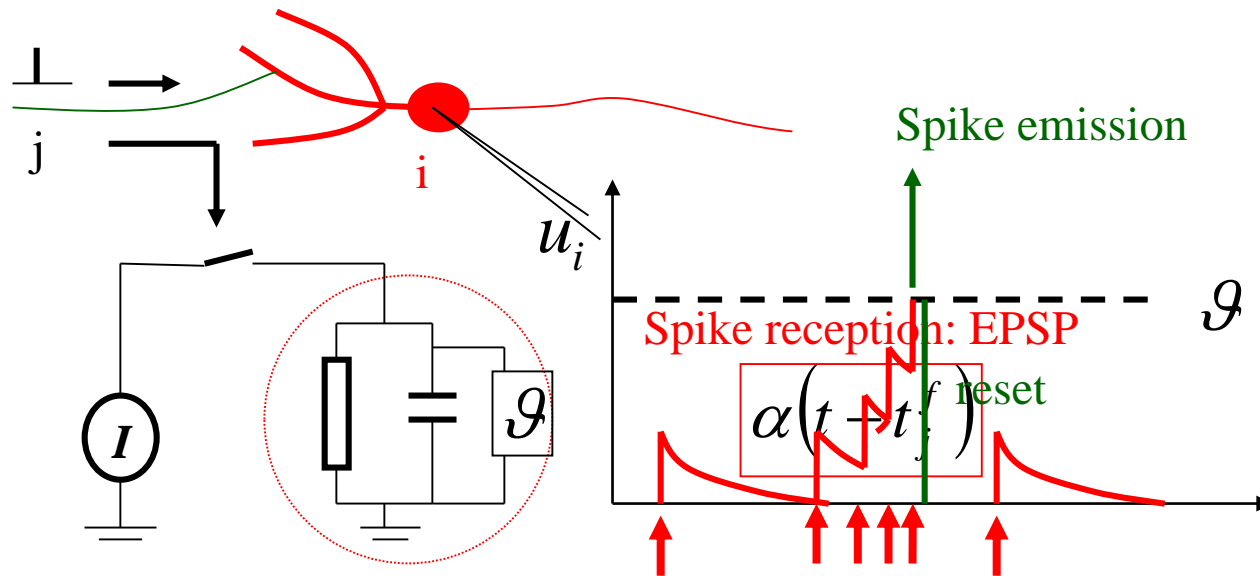
Spike Response Model



$$u_i(t) = \underbrace{\eta(t - t_i^{\wedge})}_{\text{Last spike of i}} + \sum_j \sum_f w_{ij} \underbrace{\alpha(t - t_j^f)}_{\text{All spikes, all neurons}} \quad \text{linear}$$

$$u_i(t) = \mathcal{G} \Rightarrow \text{Firing: } t_i^{\wedge} = t \quad \text{threshold}$$

Integrate-and-fire Model



$$\tau \cdot \frac{d}{dt} u_i = -u_i + RI(t) \quad \text{linear}$$

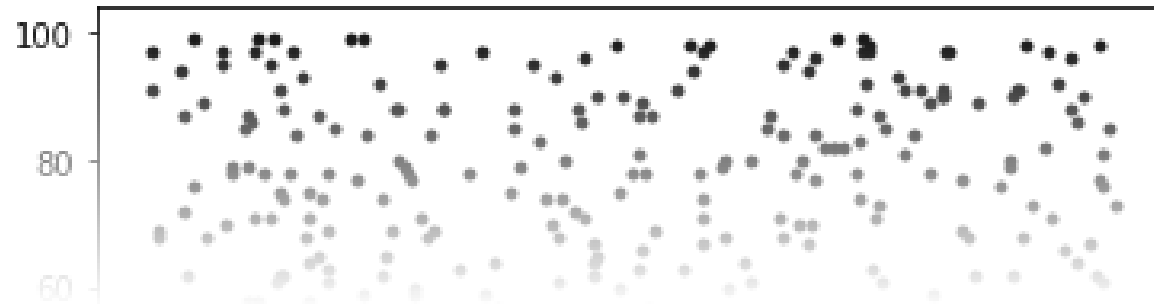
$$u_i(t) = \mathcal{G} \Rightarrow \text{Fire+reset} \quad \text{threshold}$$

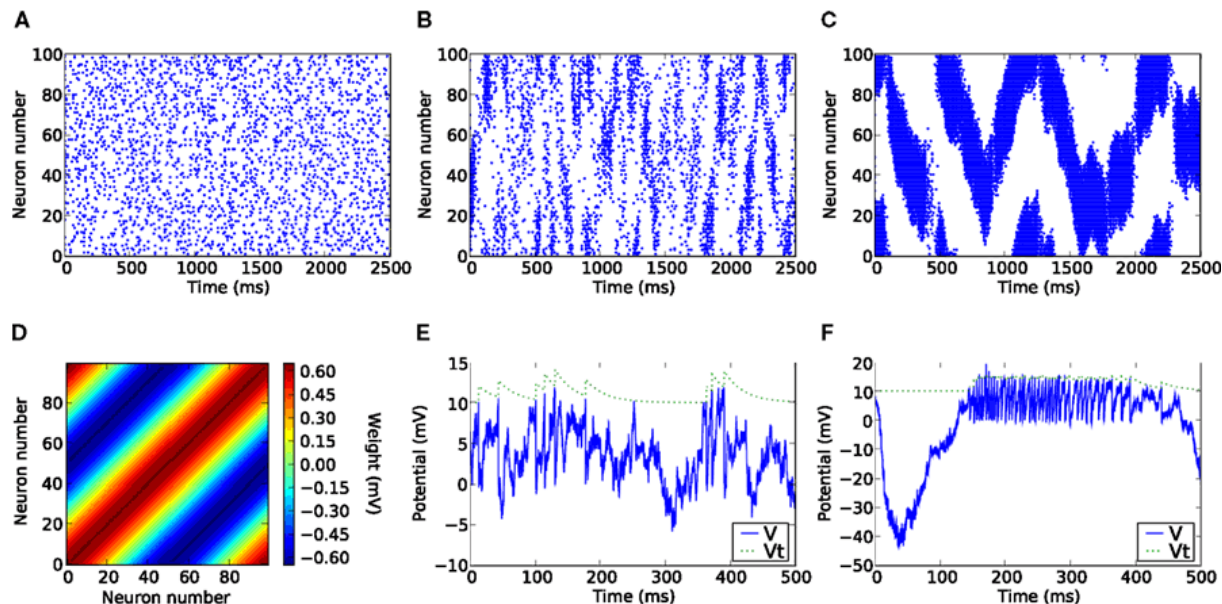


Lets try real program with Brian . .

■

```
In [6]: from brian2 import *  
G = PoissonGroup(100, rates=50*Hz)  
M = SpikeMonitor(G)  
run(100*ms)  
plot(M.t/ms, M.i, '.k');
```





```

from brian import *
w = .5*mV
def adaptive_threshold_reset(P, spikes):
    P.V[spikes] = 0*mV
    P.Vt[spikes] = clip(P.Vt[spikes]+2*mV, 10*mV, 15*mV)
eqs = ''' dV/dt = (5*mV-V)/(10*ms) + 4*mV*xi/(10*ms)**.5 : volt
          dVt/dt = (10*mV-Vt)/(30*ms) : volt '''
group=NeuronGroup(100, model=eqs,
                  threshold=lambda V,Vt:V>=Vt,
                  reset=adaptive_threshold_reset)
C = Connection(group, group, 'V', delay=2*ms)
S = SpikeMonitor(group)
C.connect_full(group, group, weight=lambda i,j:w*cos(2.*pi*(i-j)*1./100))
group.V = rand(100)*5*mV+5*mV
group.Vt = 10*mV
run(2.5*second)
raster_plot(S)
show()

```