Measuring Harmonics using a Lock-In Amplifier

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Abstract

Using highly precise function generators and a lock in amplifier, it is possible to identify and measure the harmonics of periodic signals. These results collected reflect the analytical and numerical Fourier synthesis of the signal in questions. AM signals can also be analyzed, although later analysis is less straight forward.

I. INTRODUCTION

The purpose of this experiment was to measure the amplitudes of the sinusoidal harmonics of a square wave and triangle wave. These results were then analyzed in order to compare to the prediction of the analytical Fourier synthesis of the same signal. The equipment consisted of two HP digital function generators with frequency stability to 1 part in a million, a phase shifter, a lock in amplifier and a standard oscilloscope.

Working with simple wave signals like triangle and square waves ensures that the Fourier synthesis of the signal is fairly straight forward. This allows for the use of analytical solutions for verifying the measured results.

II. PROCEDURE

A single, phase shifted sine wave is used as the reference signal for the lock in amplifier. A second periodic function is then used as the analyzed signal in the lock in amplifier. After setting the signals to the same initial frequency, the reference frequency is the increased in search of harmonics. Harmonics occur when the lock in amplifier detect a voltage. This voltage will slowly oscillate from a positive amplitude to an equally negative aptitude. Often, it is possible to vary the reference by .0001 hrz to achieve a beat pattern that is observed at the different harmonics. The V_{pp} of this beat pattern should equal the total voltage difference between the max amplitudes of the harmonic when the signal is not offset and offers a quick and efficient way to take measurements of the harmonic amplitude. Chaining the phase of the reference signal while on a harmonic should change the phase of the observed voltage. Since the signal will oscillate between two values, shifting the phase can help speed up the process of measuring the maximums. It should have little affect when looking that the V_{pp} of the beat pattern of the harmonic. Any variance when shifting the phase while observing the beat pattern was set to maximize the V_{pp} but the variance that was observed was very minor. The harmonic frequencies will often be found at regular intervals.

III. SQUARE WAVE

Our square wave signal and initial sinusoidal reference signal was set to 1KHz. We increased the signal by 0.1KHz until the next harmonic was found. Harmonics were found to be at every nKhz for odd values of n. An exponential decrease in amplitude on the odd values of our frequency was observed. The even values were also recorded, but all were close to zero. Measurements of the positive and negative maximums of the harmonic were recorded, and later averaged. This takes into account the slight offset in aptitude between the positive and negative maximums at a harmonic, as well as minimized the slight offset from zero that our even valued measurements had.

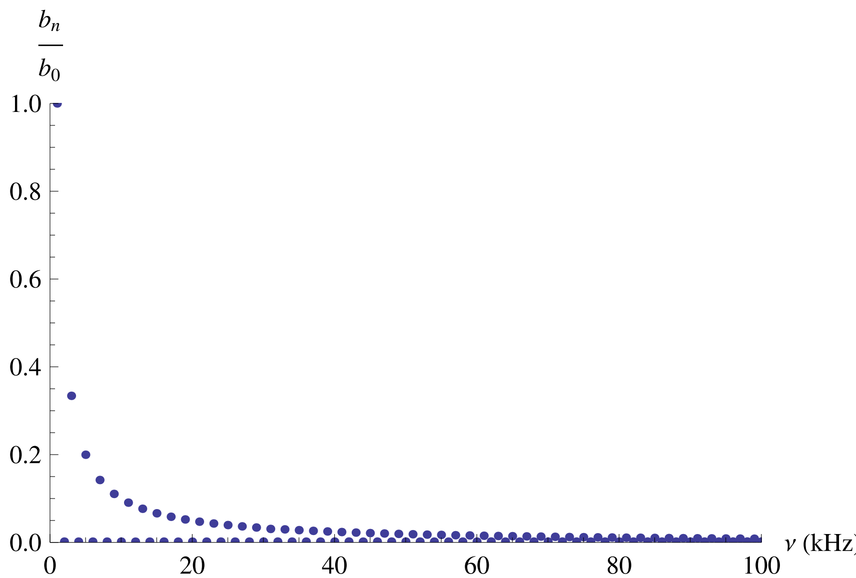

FIG. 1. Relative Measured Amplitudes of Square Wave Harmonics

Because the absolute magnitudes of our harmonic frequencies were arbitrarily set in order to meed the needs of our phase shifter, it is useful to look at the relative magnitudes of the harmonics. These can be calculated by dividing the data set by the peak amplitude of the first harmonic. The relative aptitudes of our data matched well with our theoretical prediction.

A. Fourier Analysis of a Square Wave

The sinusoidal harmonics of the square wave can be described using Fourier analysis. Any piecewise continuous periodic waveform with period $2L$ can be represented as a sum of sins and cosines. The general form of a Fourier series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \frac{n\pi x}{L} \right) \quad (1)$$

Where

$$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\frac{n\pi x}{L}dx \tag{2}$$

$$b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin\frac{n\pi x}{L}dx \tag{3}$$

In the case of a square wave, which is an odd function, we immediately see that we will only have b_n terms as all of the a_n terms will be zero. Since the relative harmonics of a square wave are independent of the frequency, we can choose an arbitrary square wave with a period of 2π and an amplitude of -1 to 1. We have to integrate twice in order to find be since the function is discontinuous.

$$b_{n=}\frac{1}{\pi}\int_0^{\pi} 1\sin(nx)dx = \frac{1}{n\pi}(1-\cos n\pi) \text{ from}(0<x<\pi)$$

and

$$b_{n=}\frac{1}{\pi}\int_{-\pi}^{0} -1\sin(nx)dx = \frac{1}{n\pi}(1-\cos n\pi) \text{ from}(-\pi<x<0)$$

Our total b_n is then the sum of these two.

$$b_n = \frac{2}{n\pi}(1-\pi n\cos) \tag{4}$$

Plugging this back into Eqn 1 we get an infinite sum of terms that construct a square wave as seen in FIG. 2. In effect, we are summing the different harmonics of the square wave in order to construct the square wave.

We can compare the relative magnitudes of the Fourier series of a square wave to the relative magnitudes of our harmonics data and see that the harmonics making up the square wave we measured indeed contained the same sinusoidal waves of the same relative magnitudes and relative frequencies. The two are overlaid in FIG 3 and the two can hardly be differentiated. The average percent error of all the points was %4.6.

IV. TRIANGLE WAVE

Procedurally, the triangle wave was performed in the same manner as the square wave. The initial frequencies for the signal and reference was 1KHz. Harmonics were found at every nKHz for odd values of n. The relative amplitudes of the harmonics are shown in FIG. 4

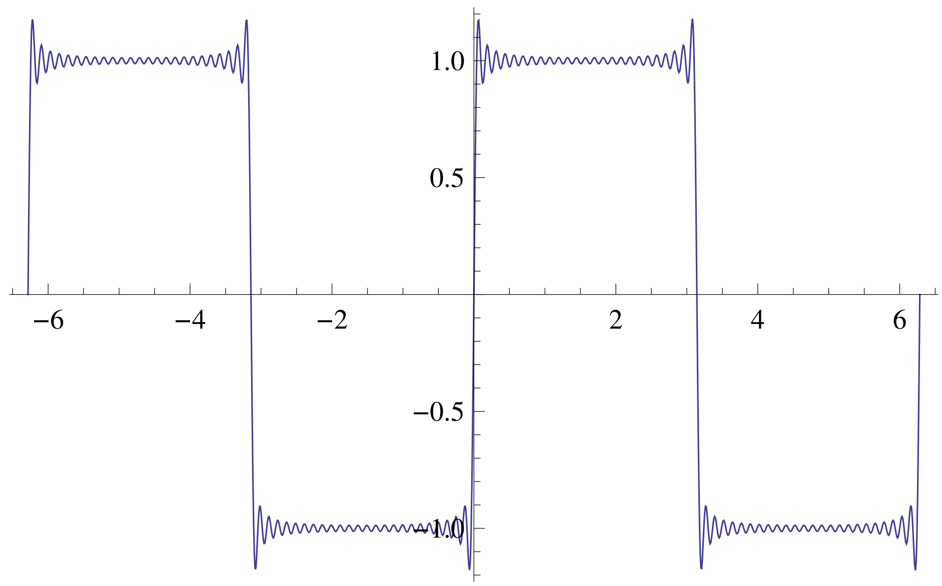

FIG. 2. First 50 terms of Square Wave Fourier Series

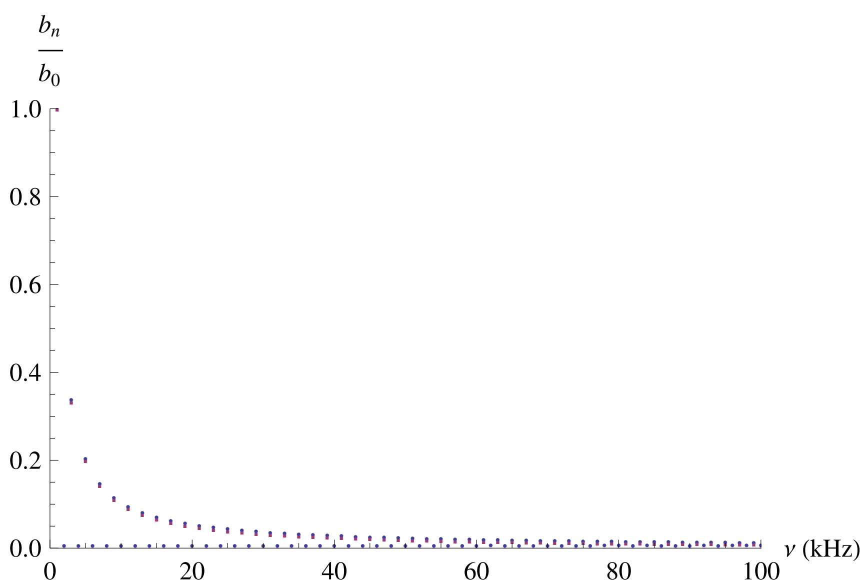

FIG. 3. Theoretical and Measured Relative Amplitudes of a Square Wave

A. Fourier Analysis of a Triangle Wave

The Fourier analysis is similar to the Square wave only the triangle wave is an even functions such that all values of b_n are zero and we will only have values for a_n. In comparison to the harmonics we measured, all we are interested in are these coefficients that determine what sinusoidal functions are used in our Fourier representation.

FIG. 4. Relative Measured Amplitudes of Triangle Wave Harmonics

Solving for a_n is much the same as solving for b_n.

$$a_{n=}\frac{1}{\pi}\int_0^{\pi} x\cos(nx)dx = \frac{1}{n^2\pi}(-1+\cos(n\pi)+n\pi\sin(n\pi)) \text{ from}(0<x<\pi) \tag{5}$$

and

$$a_{n=}\frac{1}{\pi}\int_0^{\pi} x\cos(nx)dx = \frac{1}{n^2\pi}(-1+\cos(n\pi)+n\pi\sin(n\pi)) \text{ from}(0<x<\pi) \tag{6}$$

Plugging these terms into EQN. 1, we can construct a triangle wave.

Comparing our relative a_n coefficients to our relative measured harmonic amplitudes of the triangle wave we again see that the follow exactly.

V. FURTHER ANALYSIS

A more complex AM signal was analyzed during the procedure by using an AM signal and a sinusoidal reference. Harmonics were observed, although more complex signals were observed between the main harmonics. Analysis of this is still being performed. We did collect a data run while varying the reference signal.

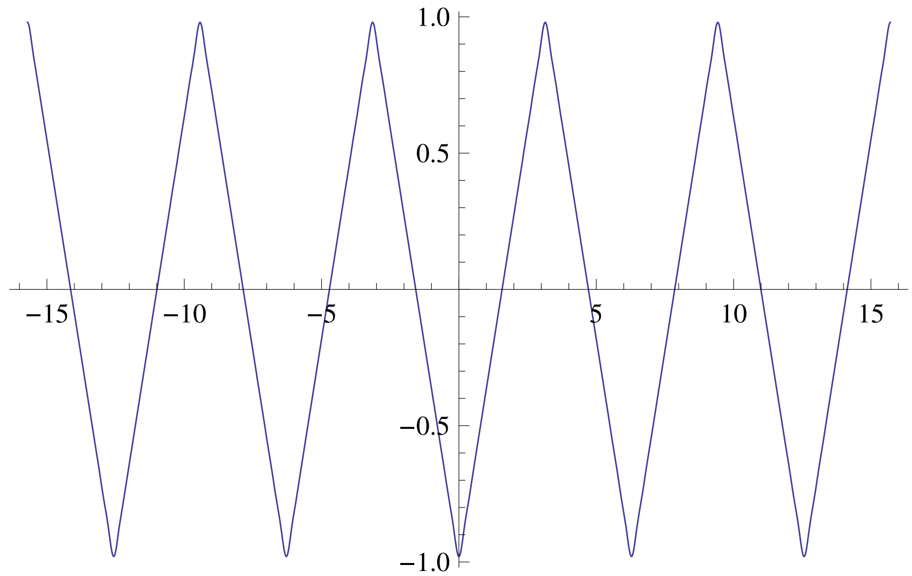

FIG. 5. First 20 terms of Square Wave Fourier Series

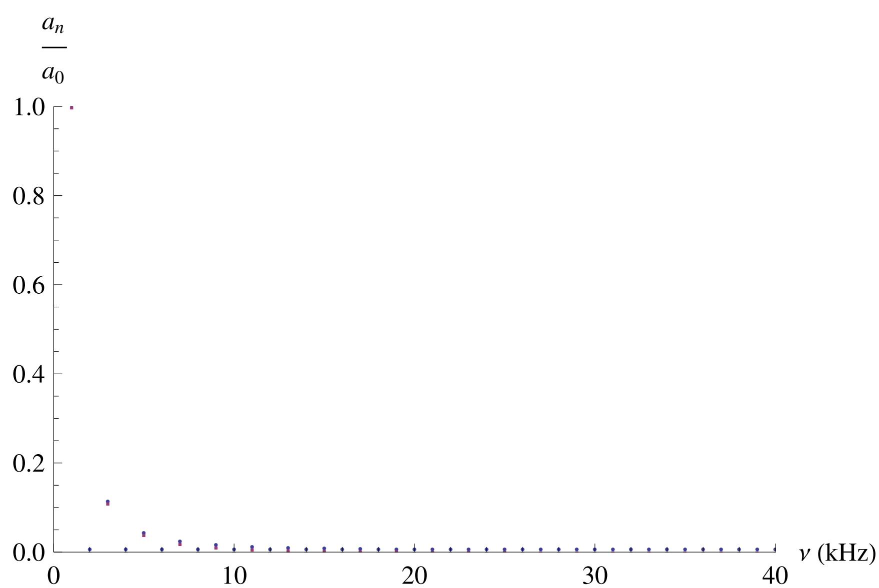

FIG. 6. Theoretical and Measured Relative Amplitudes of a Triangle Wave

VI. CONCLUSION

The harmonics that make up a square and and triangle wave were successfully observed and verified analytically. This procedure appears to work with any signals that exist predominantly in one frequency. AM signals, which are a super position of two signals super positioned on each other, appears to require a more in depth procedure which is currently

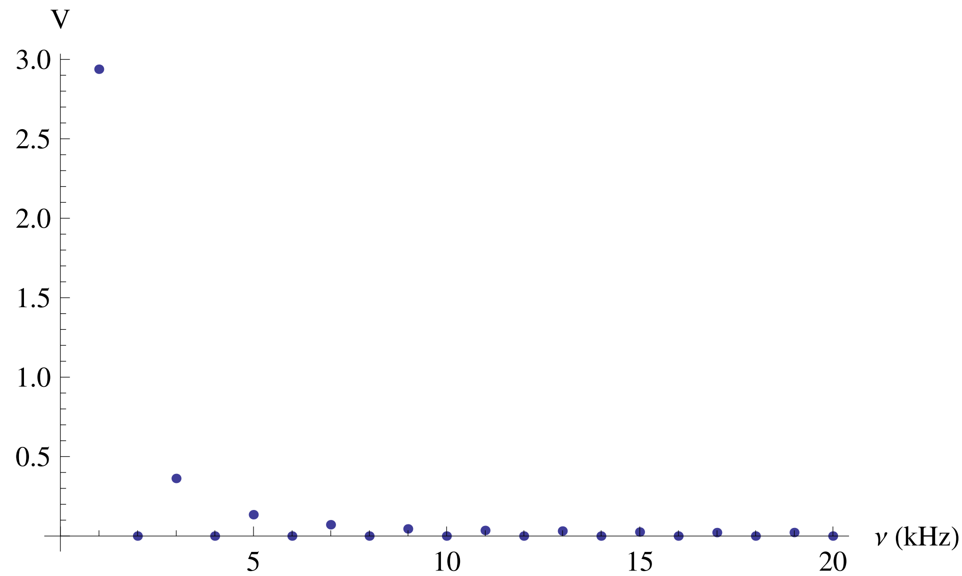

FIG. 7. AM Harmonics

being investigated and will be the subject of a separate report. (if possible?)

[1] Richard Haberman. *Applied Partial Differential Equations*. Pearson Prentice Hall, fourth edition edition, 2004.

[2] John Liu Murray R. Speigel, Seymour Lipschutz. *Mathematical Handbook of Formulas and Tables*. Schaum's Outlines. McGraw-Hill, 2009.