Problem 1. Commutators in the Heisenberg Picture

Let $\hat{t}$ be the coordinate operator for a free particle in one dimension in the Heisenberg Picture. Evaluate the following commutator:

$$[\hat{x}(t), \hat{x}(0)] \tag{1}$$

Solution

In the Heisenberg picture, the wave functions are stationary in time, and the operators evolve with time (whereas its the other way around in the S-Picture). Using the Heisenberg Equation of Motion,

$$i\hbar\frac{d\hat{A}_H}{dt} = [\hat{A}_H, \hat{H}_H] + i\hbar\left(\frac{\partial \hat{A}}{\partial t}\right)_H \tag{2}$$

with the Hamiltonian for the free particle

$$\hat{H} = \frac{\hat{p}^2}{2m} \tag{3}$$

we can find $x\hat{(t)}$ and then demonstrate the desired commutator relationship.

$$i\hbar\frac{d\hat{x}}{dt} = [\hat{x}, \hat{H}_H] + i\hbar\left(\frac{\partial \hat{x}}{\partial t}\right)_H \tag{4}$$

$$i\hbar\frac{d\hat{x}}{dt} = [\hat{x}, \frac{\hat{p}^2}{2m}] + i\hbar\left(\frac{\partial \hat{x}}{\partial t}\right)_H \tag{5}$$

$$\frac{d\hat{x}}{dt} = \frac{1}{i\hbar 2m}[\hat{x}, \hat{p}^2] + \left(\frac{\partial \hat{x}}{\partial t}\right)_H \tag{6}$$

$$\frac{d\hat{x}}{dt} = \frac{1}{i\hbar 2m}\left([\hat{x}, \hat{p}]\hat{p} + \hat{p}[\hat{x}, \hat{p}]\right) + \left(\frac{\partial \hat{x}}{\partial \hat{t}}\right)_H \tag{7}$$

$$\frac{d\hat{x}}{dt} = \frac{1}{i\hbar 2m}\left(2i\hbar\hat{p}\right) + \left(\frac{\partial \hat{x}}{\partial t}\right)_H \tag{8}$$

$$\frac{d\hat{x}}{dt} = \frac{\hat{p}}{m} + \left(\frac{\partial \hat{x}}{\partial t}\right)_H \tag{9}$$

Since $\left(\frac{\partial \hat{x}}{\partial t}\right)_H$ does not have a direct dependance on time, this term goes to zero, leaving a quick separation of variables to solve for $\hat{x}(t)$.

$$\frac{d\hat{x}}{dt} = \frac{\hat{p}}{m} \tag{10}$$

$$d\hat{x} = \frac{p}{m}dt \tag{11}$$

$$\int d\hat{x} = \int \frac{\hat{p}}{m}dt \tag{12}$$

$$\hat{x}(t) + C = \frac{\hat{p}}{m}t \tag{13}$$

$$\boxed{\hat{x}(t) = \frac{\hat{p}}{m}t + \hat{x}(0)} \tag{14}$$

We can recognize C as $\hat{x}(0)$, and now we have our time dependent operator, which will allow us to perform the commutation relationship.

$$[\hat{x}(t), \hat{x}(0)] = \left(\frac{\hat{p}}{m}t + \hat{x}(0)\right)\hat{x}(0) - \hat{x}(0)\left(\frac{\hat{p}}{m}t + \hat{x}(0)\right) \tag{15}$$

$$= \frac{\hat{p}}{m}t\hat{x}(0) + \hat{x}^2(0) - \hat{x}(0)\frac{\hat{p}}{m}t - \hat{x}^2(0) \tag{16}$$

$$= \frac{\hat{p}}{m}t\hat{x}(0) - \hat{x}(0)\frac{\hat{p}}{m}t \tag{17}$$

$$= \frac{t}{m}(\hat{p}\hat{x}(0) - \hat{x}(0)\hat{p}) \tag{18}$$

$$= \frac{t}{m}[\hat{p}, \hat{x}(0)] \tag{19}$$

$$\boxed{[\hat{x}(t), \hat{x}(0)] = i\hbar\frac{t}{m}} \tag{20}$$

Problem 2. Schrödinger vs Heisenberg Picture

Using the one-dimensional simple harmonic oscillator as an example, illustrate the differences between the Schrödinger (S) and the Heisenberg (H) picture by completing the following:

In the S-Picture:

Part (a)

Write down the fundamental equation of motion for the time evolution of a general state vector.

Solution

We simply pull a Griffiths page one and write down the TDSE:

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle \tag{21}$$

Part (b)

Find an expression for the time evolution operator $\hat{V}(t, t_0)$ and hence determine for the time evolution of a general state vector.

Solution

We start by introducing the time evolution operator such that

$$|\psi(t)\rangle = \hat{V}(t, t_0)\,|\psi(t0)\rangle \tag{22}$$

which, along with Equation (21), we get

$$i\hbar\frac{\partial}{\partial t}\hat{V}(t, t_0) = H\hat{V}(t, t_0) \tag{23}$$

where

$$\lim_{t\to t_0}\hat{V}(t, t_0) = 1 \tag{24}$$

At this point we can use separation of variables to solve for an expression for $\hat{V}(t, t_0)$.

$$i\hbar\frac{\partial}{\partial t}\hat{V}(t, t_0) = H\hat{V}(t, t_0) \tag{25}$$

$$\int i\hbar\frac{\partial(\hat{V}(t, t_0))}{\hat{V}(t, t_0)} = \int H\partial t \tag{26}$$

$$i\hbar\ln(\hat{V}(t, t_0)) = H(t + c) \qquad \text{where } c = t_0 \tag{27}$$

$$e^{\ln(\hat{V}(t,t_0))} = e^{H(t-t_0)\frac{1}{i\hbar}} \tag{28}$$

$$\boxed{\hat{V}(t, t_0) = e^{\frac{-iH(t-t_0)}{\hbar}}} \tag{29}$$

Part (c)

Comment on the time evolution of the dynamic variables $\hat{x}$ and $\hat{p}$.

Solution

In the S-Picture, operators vary with time, where the wave function does not!

Part (d)

Now In the H-Picture

Obtain the equations of motion for the time evolution of the dynamic variables $\hat{x}$ and $\hat{p}$.

Solution

Similar to Problem 1, we can run our operators through the Heisenberg Equation of Motion seen in Equation (2). We must note that the Hamiltonian operator is a little different for the SHO compared to the free particle:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \tag{30}$$

Lets find $\hat{x}$ first. Since there is no explicate time dependance on time for $\hat{x}$, we can argue that $\frac{\partial \hat{x}}{\partial t}$ goes to 0 right out of the gate.

$$i\hbar\frac{d\hat{x}}{dt} = [\hat{x}, \hat{H}_H] + i\hbar\cancelto{0}{\left(\frac{\partial \hat{x}}{\partial t}\right)_H} \tag{31}$$

$$\frac{d\hat{x}}{dt} = \frac{1}{i\hbar}[\hat{x}, \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2] \tag{32}$$

$$= \frac{1}{i\hbar}\left(\hat{x}\left(\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\right) - \left(\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\right)\hat{x}\right) \tag{33}$$

$$= \frac{1}{i\hbar}\left(\frac{\hat{x}\hat{p}^2}{2m} + \cancelto{0}{\frac{1}{2}m\omega^2\hat{x}^3} - \frac{\hat{p}^2\hat{x}}{2m} - \cancelto{0}{\frac{1}{2}m\omega^2\hat{x}^3}\right) \tag{34}$$

$$= \frac{1}{i\hbar 2m}\left(\hat{x}\hat{p}^2 - \hat{p}^2\hat{x}\right) \tag{35}$$

$$= \frac{1}{i\hbar 2m}(\cancelto{i\hbar}{[\hat{x},\hat{p}]}\hat{p} + \hat{p}\cancelto{i\hbar}{[\hat{x},\hat{p}]}) \qquad \text{by distributivity :)} \tag{36}$$

$$\boxed{\frac{d\hat{x}}{dt} = \frac{\hat{p}}{m}} \tag{37}$$

We can do the same thing for $\hat{p}$, making the same argument about $\frac{\partial \hat{p}}{\partial t}$ going to 0.

$$i\hbar\frac{d\hat{p}}{dt} = [\hat{p}, \hat{H}_H] + i\hbar\cancelto{0}{\left(\frac{\partial \hat{p}}{\partial t}\right)_H} \tag{38}$$

$$\frac{d\hat{p}}{dt} = \frac{1}{i\hbar}[\hat{p}, \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2] \tag{39}$$

$$= \frac{1}{i\hbar}\hat{p}\left(\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\right) - \left(\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\right)\hat{p} \tag{40}$$

$$= \frac{1}{i\hbar}\left(\cancelto{0}{\frac{\hat{p}^3}{2m}} + \frac{1}{2}m\omega^2\hat{p}\hat{x}^2 - \cancelto{0}{\frac{\hat{p}^3}{2m}} - \frac{1}{2}m\omega^2\hat{x}^2\hat{p}\right) \tag{41}$$

$$= \frac{m\omega^2}{i\hbar 2}\left(\hat{p}\hat{x}^2 - \hat{x}^2\hat{p}\right) \tag{42}$$

$$= \frac{m\omega^2}{i\hbar 2}[\hat{p}, \hat{x}^2] \tag{43}$$

$$= \frac{m\omega^2}{i\hbar 2}(\cancelto{-i\hbar}{[\hat{p}, \hat{x}]}\hat{x} + \hat{x}\cancelto{-i\hbar}{[\hat{p}, \hat{x}]}) \tag{44}$$

$$\frac{d\hat{p}}{dt} = -m\omega^2\hat{x} \tag{45}$$

$$\boxed{\frac{d\hat{p}}{dt} = -\beta\hat{x}} \tag{46}$$

Part (e)

Solve the above equations in part (d)

Solution

We need to solve the following differential equation:

$$\dot{x} = \frac{p}{m} \tag{47}$$
$$\dot{p} = -\beta x \tag{48}$$

which happens to be solvable using the general solution to Hook's law:

$$x(t) = A\sin(\omega t) + B\cos(\omega t) \tag{49}$$

where

$$\omega^2 = \frac{k}{m} = \frac{\beta}{m} \tag{50}$$

We can solve for B right off the bat by setting $t = 0$ in $x(t)$:

$$x(0) = \cancelto{0}{A\sin(0)} + B\cancelto{1}{\cos(0)} \tag{51}$$
$$x(0) = B \tag{52}$$
$$x(t) = A\sin(\omega t) + x_0\cos(\omega t) \tag{53}$$

Next we relate the two equations by solving for $\dot{p}$ and integrating:

$$\dot{p} = -\beta x(t) = -\beta(A\sin(\omega t) + x_0\cos(\omega t)) \tag{54}$$
$$\int \dot{p}dt = \int -\beta A\sin(\omega t)dt + \int x_0\cos(\omega t))dt \tag{55}$$
$$p = \frac{\beta A}{\omega}\cos\omega t - \frac{\beta x_0}{\omega}\sin\omega t + C \tag{56}$$

Now lets find A by setting $t = 0$ in $p(t)$ and dissolving C into A:

$$p(0) = \frac{\beta A}{\omega}\cancelto{1}{\cos 0} - \cancelto{0}{\frac{\beta x_0}{\omega}\sin 0} + C \tag{57}$$
$$p(0) = \frac{\beta A}{\omega} \tag{58}$$
$$A = \frac{p_0\omega}{\beta} \tag{59}$$

Plugging A (59) back into Equation (53) and (56):

$$x(t) = \frac{p_0\omega}{\beta}\sin(\omega t) + x_0\cos(\omega t) \tag{60}$$
$$p(t) = p_0\cos\omega t - \frac{\beta x_0}{\omega}\sin\omega t \tag{61}$$

Plugging in Equation (50) we get the final solution:

$$\boxed{x(t) = \frac{p_0}{\sqrt{\beta m}} \sin(\sqrt{\frac{\beta}{m}} t) + x_0 \cos(\sqrt{\frac{\beta}{m}} t)} \tag{62}$$

$$\boxed{p(t) = p_0 \cos \sqrt{\frac{\beta}{m}} t - \sqrt{\beta m} x_0 \sin \sqrt{\frac{\beta}{m}} t} \tag{63}$$

Part (f)

Comment on both the classical correspondence and the time evolution of the quantum state in this picture.

Solution

The solutions we find here are the same solutions we find in classical mechanics for the SHO! The wave function does not evolve in time, but rather the operators do.

Problem 3. Density Matrix for spin $1/2$ particles

As shown in the Lecture Notes, the eigenvalue equation for spin-1/2 particles can indeed be given by the Pauli matrices as follows:

$$S\left|\chi\right\rangle = \frac{\hbar}{2}\sigma\left|\chi\right\rangle \tag{64}$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_x = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{65}$$

Part (a)

Show that the above spin eigenvectors along each of the x and z directions with eigenvalues $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ are given by:

$$\left|\chi_z+\right\rangle \equiv \left|+\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \left|\chi_z-\right\rangle \equiv \left|-\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{66}$$

$$\left|\chi_x+\right\rangle = \frac{1}{\sqrt{2}}(\left|+\right\rangle + \left|-\right\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \left|\chi_x-\right\rangle = \frac{1}{\sqrt{2}}(\left|+\right\rangle - \left|-\right\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{67}$$

Solution

Since we know the eigenvalues already, all we have to do is solve a really basic eigenvalue problem and then we will have shown how to get our spin eigenvectors.

In the Z direction, the eigenvalue problem can be reduced to the following form:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix} \tag{68}$$

Solving the characteristic equation

$$\left|\underline{\underline{M}} - \lambda \underline{\underline{I}}\right| = 0 \tag{69}$$

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0 \tag{70}$$

$$(1-\lambda)(-1-\lambda) = -1-\lambda^2 = 0 \tag{71}$$

$$\lambda = \pm 1 \tag{72}$$

The sign on λ represents our spin up and spin down state.

When $\lambda = +1$:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = +1 \begin{pmatrix} a \\ b \end{pmatrix} \tag{73}$$

$$a = +a \qquad -b = b \tag{74}$$

These results, under normalization, yield:

$$\begin{pmatrix} a \\ b \end{pmatrix}_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{75}$$

Similarly when $\lambda = -1$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -1 \begin{pmatrix} a \\ b \end{pmatrix} \tag{76}$$

$$a = -a \qquad\qquad -b = -b \tag{77}$$

Again, under normalization, these results yield our eigenvectors:

$$\begin{pmatrix} a \\ b \end{pmatrix}_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{78}$$

Putting these back into Equation (64), yields the following spin eigenvectors in the z direction.

$$\boxed{|\chi_z+\rangle \equiv |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}} \qquad \boxed{|\chi_z-\rangle \equiv |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}} \tag{79}$$

Similarly in the x-direction, we solve our characteristic equation:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix} \tag{80}$$

$$\left| \underline{\underline{M}} - \lambda \underline{\underline{I}} \right| = 0 \tag{81}$$

$$\begin{vmatrix} 0-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = 0 \tag{82}$$

$$(-\lambda)(-\lambda) - 1 = 0 \tag{83}$$

$$\lambda = \pm 1 \tag{84}$$

We again find our eigenvectors for the given eigenvalues.

For $\lambda = +1$:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = +1 \begin{pmatrix} a \\ b \end{pmatrix} \tag{85}$$

$$a = b \tag{86}$$

Again, we find the the eigenvector

$$\begin{pmatrix} a \\ b \end{pmatrix}_+ = \begin{pmatrix} a \\ a \end{pmatrix} \tag{87}$$

however under normalization

$$\sqrt{a^2 + a^2} = 1 \tag{88}$$

$$a = \frac{1}{\sqrt{2}} \tag{89}$$

resulting in our desired eigenvector seen in Equation (67)

$$\boxed{|\chi_x+\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}} \tag{90}$$

Now for $\lambda = -1$:

$$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = -1\begin{pmatrix}a\\b\end{pmatrix} \tag{91}$$

$$\begin{pmatrix}b\\a\end{pmatrix} = \begin{pmatrix}-a\\-b\end{pmatrix} \tag{92}$$

$$\begin{pmatrix}a\\b\end{pmatrix}_- = \begin{pmatrix}a\\-a\end{pmatrix} \tag{93}$$

Under normalization:

$$\sqrt{(a)^2 + (-a)^2} = 1 \tag{94}$$

$$a = \frac{1}{\sqrt{2}} \tag{95}$$

which brings us to the final eigenvector we seek:

$$\boxed{|\chi_x-\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}} \tag{96}$$

Part (b)

Show that for a pure ensemble of a completely polarized beam of 1/2 particles with spin up along the z-direction, i.e. $|\chi_z+\rangle = |+\rangle$, the density matrix can be expressed as:

$$\rho_z = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{97}$$

Solution

We need to use the definition of the density operator from equation (32.7) in the qm-619 notes,

$$\rho \equiv \sum_i f_i \rho^{(i)} \tag{98}$$

where

$$\rho^{(i)} \equiv |\Psi^{(i)}\rangle \langle \Psi^{(i)}| \tag{99}$$

and

$$f_i \equiv \frac{N_i}{N} \tag{100}$$

where N is the total number of identical particles making up the ensemble and N_i is the number of particles in state i within the ensemble.

For an ensemble of 1/2 particles all with spin up, Equation (98) reduces to:

$$\rho = \rho^{(z+)} = |+\rangle \langle +| \tag{101}$$

$$\rho^{(z)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{102}$$

$$\boxed{\rho^{(z)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} \tag{103}$$

Part (c)

Same problem as in (b), but now for a pure state $|\chi_x+\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$, show that

$$\rho_x = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{104}$$

Solution

This is very similar to the previous part.

$$\rho = \rho^{(x)} = |\chi_x+\rangle \langle \chi_x+| \tag{105}$$

$$\rho^{(x)} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} \tag{106}$$

$$\boxed{\rho^{(x)} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}} \tag{107}$$

Part (d)

Comment on the values for $\mathrm{Tr}(\rho)$ and $\mathrm{Tr}(\rho^2)$ in (b) and (c), respectively.

Solution

$$\boxed{\mathrm{Tr}(\rho_{(z)}) = \mathrm{Tr}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} = 1 + 0 = 1} \tag{108}$$

$$\boxed{\mathrm{Tr}(\rho^2_{(z)}) = \mathrm{Tr}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}^2 = \mathrm{Tr}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} = 1 + 0 = 1} \tag{109}$$

$$\boxed{\mathrm{Tr}(\rho_{(x)}) = \mathrm{Tr}\,\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} = \frac{1}{2} + \frac{1}{2} = 1} \tag{110}$$

$$\boxed{\mathrm{Tr}(\rho^2_{(x)}) = \mathrm{Tr}\left(\left(\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\right)^2\right) = \mathrm{Tr}\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix} = \mathrm{Tr}\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix} = \frac{1}{2} + \frac{1}{2} = 1} \tag{111}$$

They are all 1!

Part (e)

For a completely unpolarized beam of spin 1/2 particles, show that

$$\rho_x = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \rho_x = \frac{1}{4}\left[\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} + \begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\right] \qquad \rho_z = \rho_x \tag{112}$$

Solution

For an unpolarized beam, we have an equal chance of finding particles in the spin up or spin down state, so $\frac{N_i}{N} = \frac{1}{2}$ for $i = \pm$. Looking at $\rho^{(z\pm)}$ first:

$$\rho = \frac{1}{2}\rho^{(z+)} + \frac{1}{2}\rho^{(z-)} = \frac{1}{2}((|+\rangle\langle+|) + (|-\rangle\langle-|)) \tag{113}$$

$$= \frac{1}{2}\left(\begin{pmatrix}1\\ 0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}\right) = \frac{1}{2}\left(\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right) \tag{114}$$

$$\boxed{\rho^{(z\pm)} = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}} \tag{115}$$

Similarly looking $\rho^{x\pm}$:

$$\rho = \frac{1}{2}\rho^{(x+)} + \frac{1}{2}\rho^{(x-)} = \frac{1}{2}((|\chi_x+\rangle\langle\chi_x+|) + (|\chi_x-\rangle\langle\chi_x-|)) \tag{116}$$

$$= \frac{1}{2}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\ 1\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix} + \begin{pmatrix}1\\ -1\end{pmatrix}\begin{pmatrix}1 & -1\end{pmatrix}\right) \tag{117}$$

$$\boxed{\rho^{x\pm} = \frac{1}{4}\left[\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} + \begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\right]} \tag{118}$$

Which allows us to conclude

$$\boxed{\rho^{z\pm} = \rho^{x\pm}} \tag{119}$$

Part (f)

Show also that in the case of (e): the quantum ensemble average of each of the spin components of the beam is zero, i.e.:

$$\langle \bar{S}_x \rangle = \langle \bar{S}_y \rangle = \langle \bar{S}_z \rangle = 0 \tag{120}$$

Solution

Calculating the expectation value of S is simple. All we need to perform is $\langle \chi | \hat{S} | \chi \rangle$.

$$\langle \chi | \hat{S}_z | \chi \rangle = \langle + | \hat{S} | + \rangle + \langle - | \hat{S} | - \rangle \tag{121}$$

For example in the z direction

$$\begin{aligned}
\langle \bar{S}_z \rangle &= \sum_i f_i \operatorname{Tr}(\hat{S}_z \rho^{(i)}) && (122)\\
&= \frac{\hbar}{2}\left(f_+ \operatorname{Tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \right) + f_- \operatorname{Tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \right) \right) && (123)\\
&= \frac{\hbar}{2}\left(f_+ \operatorname{Tr}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + f_- \operatorname{Tr}\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \right) && (124)\\
&= \frac{\hbar}{2}(f_+ - f_-) && (125)\\
&\boxed{\langle \bar{S}_z \rangle = 0} && (126)
\end{aligned}$$

since the beam is unpolarized so $f_- = f_+$.

We could continue this way, but finding $\langle \bar{S}_x \rangle$ and $\langle \bar{S}_y \rangle$ is a bit more involved. Instead lets notice that the density matrices $\rho^{x\pm}$ and $\rho^{z\pm}$ both equal $\frac{1}{2}\mathbf{I}$, and when running that through the double average of S, due to the known eigenvalues $\pm\frac{\hbar}{2}$, that summing this will always result in 0. We already demonstrated this in Equation (115) and (115) for $\langle \bar{S}_x \rangle$ and $\langle \bar{S}_z \rangle$. This leaves us to confirm $\rho^{(y\pm)} = \frac{1}{2}\mathbf{I}$. Looking up the spin-1/2 eigenstates in the y-direction[1]:

$$|\chi_y +\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ +i \end{pmatrix} \qquad |\chi_y -\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \tag{127}$$

we can quickly make said confirmation assuming:

$$\rho^{(y\pm)} = \frac{1}{2}\rho^{(y+)} + \frac{1}{2}\rho^{(y-)} = \frac{1}{2}((|\chi_y+\rangle\langle\chi_y+|) + (|\chi_y-\rangle\langle\chi_y-|)) \tag{128}$$

$$= \frac{1}{2}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\ i\end{pmatrix}\begin{pmatrix}1 & -i\end{pmatrix} + \begin{pmatrix}1\\ -i\end{pmatrix}\begin{pmatrix}1 & i\end{pmatrix}\right) \tag{129}$$

$$\rho^{y\pm} = \frac{1}{4}\left[\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix} + \begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}\right] \tag{130}$$

$$\boxed{\rho^{y\pm} = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}} \tag{131}$$

So thats it! $\rho^{(y)} = \frac{1}{2}\mathbf{I}$, and thus $\langle\bar{S}_y\rangle = 0$, but we can go ahead and do it anyway because I already typed it in before correcting the above statement.

$$\langle\bar{S}_y\rangle = \sum_i f_i \operatorname{Tr}(\hat{S}_y\rho^{(i)}) \tag{132}$$

$$= \frac{\hbar}{2}\left(f_+ \operatorname{Tr}\left(\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}\begin{pmatrix}1 & -i\end{pmatrix}\right) + f_- \operatorname{Tr}\left(\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}\begin{pmatrix}1 & i\end{pmatrix}\right)\right) \tag{133}$$

$$= \frac{\hbar}{2}\frac{1}{2}\left(f_+ \operatorname{Tr}\left(\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}\right) + f_- \operatorname{Tr}\left(\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}\right)\right) \tag{134}$$

$$= \frac{\hbar}{4}\left(\cancelto{0}{f_+ \operatorname{Tr}\left(\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}\right) + f_- \operatorname{Tr}\left(\begin{pmatrix}-1 & -i\\ i & -1\end{pmatrix}\right)}\right) \tag{135}$$

$$\boxed{\langle\bar{S}_y\rangle = 0} \tag{136}$$

So $\langle\bar{S}_y\rangle$ goes to 0 this way too. And we can also argue that once we know z and x we don't care about y since its going to be the same or something.

Part (g)

For a beam of spin 1/2 particles in a mixed state with 60% "spin up" along the z axis (i.e. in the $|\chi_z+\rangle$ state) and 40% in the $|\chi_z-\rangle$ state, find the density matrix and show that $\operatorname{Tr}(\rho^2) < 1$ in this case.

Solution

Using Equation (98)

$$\rho = (0.6)\left|\chi_z+\right\rangle\left\langle\chi_z+\right| + (0.4)\left|\chi_z-\right\rangle\left\langle\chi_z-\right| \tag{137}$$

$$= \left((0.6)\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + (0.4)\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}\right) \tag{138}$$

$$= \left((0.6)\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + (0.4)\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\right) \tag{139}$$

$$\rho = \begin{pmatrix}0.6 & 0\\0 & 0.4\end{pmatrix} \tag{140}$$

$$\rho^2 = \begin{pmatrix}0.6 & 0\\0 & 0.4\end{pmatrix}\cdot\begin{pmatrix}0.6 & 0\\0 & 0.4\end{pmatrix} = \begin{pmatrix}0.36 & 0\\0 & 0.16\end{pmatrix} \tag{141}$$

$$\boxed{\mathrm{Tr}(\rho^2) = 0.52} \tag{142}$$

Problem 4. Berry Phase

For the example discussed in class with a spin 1/2 particle in a magnetic field ($\vec{B}$) of constant magnitude precessing slowly about the z-axis:

Part (a)

Show that the Berry phase associated with the spin state aligning opposite to $\vec{B}$ with the eigenstate given in eq. (34.21) is by:

$$\gamma_-(T) = \frac{1}{2}\Omega - 2\pi \tag{143}$$

Solution

We will be following along with the procedure done in the notes on page [H-16], except calculating $\gamma_-(T)$ instead. Looking at Equation (34.21) from the notes:

$$\gamma_- = \begin{pmatrix} \sin\frac{\alpha}{2} \\ -\cos\frac{\alpha}{2}e^{i\omega t} \end{pmatrix} \tag{144}$$

where $\omega t = \psi$. Now we find $\vec{A}_+$:

$$\vec{A}_+ = i \langle \psi_- | \vec{\nabla}_r \psi_- \rangle \tag{145}$$

In spherical coordinates:

$$\vec{\nabla}_r = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\alpha \frac{1}{r}\frac{\partial}{\partial \alpha} + \hat{e}_\psi \frac{1}{r \sin\alpha}\frac{\partial}{\partial \psi} \tag{146}$$

so that Equation (145) turns into:

$$\vec{A}_+ = i \left(\sin\tfrac{\alpha}{2} \quad -\cos\tfrac{\alpha}{2}e^{-i\psi} \right) \frac{1}{r} \begin{pmatrix} \frac{1}{2}\cos\frac{\alpha}{2} \\ \frac{1}{2}e^{i\psi}\sin\frac{\alpha}{2} \end{pmatrix} \hat{e}_\alpha + \frac{i}{r\sin\alpha} \left(\sin\tfrac{\alpha}{2} \quad -\cos\tfrac{\alpha}{2}e^{-\psi} \right) \begin{pmatrix} 0 \\ -ie^{i\psi}\cos\frac{\alpha}{2} \end{pmatrix} \hat{e}_\psi \tag{147}$$

$$= \left(\sin\frac{\alpha}{2}\frac{1}{r}\frac{1}{2}\cos\frac{\alpha}{2} - \cos\frac{\alpha}{2}\cancelto{1}{e^{-i\psi}}\frac{1}{r}\frac{1}{2}\cancelto{1}{e^{i\psi}}\sin\frac{\alpha}{2} \right)\hat{e}_\alpha + \frac{i}{\sin\alpha}\left(\cos\frac{\alpha}{2}\cancelto{1}{e^{-i\psi}} i \cancelto{1}{e^{i\psi}}\cos\frac{\alpha}{2} \right)\hat{e}_\psi \tag{148}$$

$$= \cancelto{0}{\left(\sin\frac{\alpha}{2}\frac{1}{r}\frac{1}{2}\cos\frac{\alpha}{2} - \cos\frac{\alpha}{2}\frac{1}{r}\frac{1}{2}\sin\frac{\alpha}{2} \right)}\hat{e}_\alpha + \frac{i}{\sin\alpha}(i\cos^2\frac{\alpha}{2})\hat{e}_\psi \tag{149}$$

$$\vec{A}_+ = \frac{-\cos^2\frac{\alpha}{2}}{\sin\alpha}\hat{e}_\psi \tag{150}$$

Remembering that

$$\sin\alpha = 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} \tag{151}$$

so that we finally get

$$\boxed{\vec{A}_+ = \frac{-\cos\frac{\alpha}{2}}{2r\sin\frac{\alpha}{2}}\hat{e}_\psi} \tag{152}$$

At this point we are ready to find $\gamma_-(T)$. Following Equation [34.11]

$$\gamma_-(T) = \oint \vec{A}_- \cdot d\vec{R} = \oint \vec{A}_- \cdot d\vec{B} \tag{153}$$

$$= -\frac{1}{2B_0}\frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}\int_0^{2\pi} B_0 \sin\alpha d\psi \tag{154}$$

$$= -\frac{1}{2}\frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}\sin\alpha 2\pi \tag{155}$$

$$= -\frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}} 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}(2\pi) \tag{156}$$

$$= -\cos^2\frac{\alpha}{2}(2\pi) \tag{157}$$

$$= -\frac{1}{2} - \frac{1}{2}\cos\alpha(2\pi) \tag{158}$$

$$\boxed{\gamma_-(T) = -(1+\cos\alpha)\pi} \tag{159}$$

To get this to its final form, lets solve for $\gamma_-(T)$ in terms of Ω

$$\Omega = (1-\cos\alpha)2\pi \tag{160}$$

$$\Omega/2 - 2\pi = (1-\cos\alpha)\pi - 2\pi \tag{161}$$

$$\Omega/2 - 2\pi = -(1+\cos\alpha)\pi \tag{162}$$

$$\boxed{\gamma_-(T) = \Omega/2 - 2\pi} \tag{163}$$

Part (b)

Now consider the more general situation when $\vec{B}$ does not precess with a constant azimuthal angle. Let us generalize the "motion of the tip" of $\vec{B}$ to "sweep" out an arbitrary closed loop on the surface of a sphere as shown. The eigenstate ψ_+ is given by:

$$\psi_+ = \begin{pmatrix} \cos\frac{\alpha}{2} \\ \sin\frac{\alpha}{2}e^{i\varphi} \end{pmatrix} \tag{164}$$

with $\alpha(t)$ and $\varphi(t)$ assuming instantaneous values. Using eq. (34.13) from the lecture notes, show that the Berry phase calculated for this case over one complete cycle is still given by (34.23), i.e.

$$\gamma_+(T) = -\frac{1}{2}\Omega \tag{165}$$

Where Ω is the solid angle subtended by the surface on the sphere swept out by $\vec{B}$.

Solution

Lets do this!

$$\psi_+ = \begin{pmatrix} \cos\frac{\alpha}{2} \\ \sin\frac{\alpha}{2}e^{i\psi} \end{pmatrix} \tag{166}$$

$$\vec{A}_+ = -\frac{1}{2B_0}\tan\frac{\alpha}{2}\hat{e}_\psi \tag{167}$$

Equation [34.13] states:

$$\gamma_n = \int_s (\vec{\nabla}_r \times \vec{A}_n) \cdot d\vec{S} \tag{168}$$

Looking up $\vec{\nabla}_r \times \vec{A}_n$ in Griffiths, dropping terms that do not have $\hat{e}_\psi$, and swapping r for B_0 we get:

$$\begin{aligned}
\vec{\nabla}_r \times \vec{A}_n &= \frac{1}{r\sin\alpha}\left[\frac{\partial}{\partial\alpha}(\sin(\alpha)A_\psi)\right]\hat{e}_r + \frac{1}{r}\left[-\frac{\partial}{\partial r}(rA_\psi)\right]\hat{e}_\alpha && (169)\\
&= \frac{1}{B_0\sin\alpha}\left[\frac{\partial}{\partial\alpha}(\sin(\alpha)\left(\frac{-1}{2B_0}\tan\frac{\alpha}{2}\right))\right]\hat{e}_{B_0} + \frac{1}{B_0}\left[-\frac{\partial}{\partial B_0}(B_0\left(\frac{-1}{2B_0}\tan\frac{\alpha}{2}\right))\right]\hat{e}_\alpha && (170)\\
&= \frac{1}{B_0\sin\alpha}\frac{-1}{2B_0}\frac{\partial}{\partial\alpha}(\sin\alpha\tan\frac{\alpha}{2})\hat{e}_{B_0} + \cancelto{0}{\frac{1}{B_0}\tan\frac{\alpha}{2}\frac{\partial}{\partial B_0}\frac{1}{2}}\hat{e}_\alpha && (171)\\
&= \frac{-1}{2B_0\sin\alpha}\frac{\partial}{\partial\alpha}(\sin\alpha\tan\frac{\alpha}{2})\hat{e}_{B_0} && (172)\\
&= \frac{-1}{2B_0\sin\alpha}\left(\frac{1}{2}\sec^2(\frac{\alpha}{2})\sin\alpha + \cos\alpha\tan\frac{\alpha}{2}\right)\hat{e}_{B_0} && (173)\\
&= \frac{-1}{2B_0}\left(\frac{1}{2}\sec^2(\frac{\alpha}{2}) + \cot\alpha\tan\frac{\alpha}{2}\right)\hat{e}_{B_0} && (174)
\end{aligned}$$

Remembering our half angle theorems

$$\frac{1}{\cos^2\frac{a}{2}} = \frac{2}{1+\cos\alpha} \tag{175}$$

$$\tan a2 = \frac{\sin a}{1+\cos a} \tag{176}$$

we can continue reducing Equation (174)

$$\vec{\nabla}_r \times \vec{A}_n = \frac{-1}{2B_0}\left(\frac{1}{2}\frac{1}{\cos^2\frac{\alpha}{2}} + \frac{\cos\alpha}{\sin\alpha}\tan\frac{\alpha}{2}\right)\hat{e}_{B_0} \tag{177}$$

$$= \frac{-1}{2B_0}\left(\frac{1}{(1+\cos\alpha)} + \cos\alpha\frac{1}{1+\cos\alpha}\right)\hat{e}_{B_0} \tag{178}$$

$$\boxed{\vec{\nabla}_r \times \vec{A}_n = \frac{-1}{2B_0^2}\hat{e}_{B_0}} \tag{179}$$

Putting this result into Equation (168) we arrive at our final answer, where $d\vec{S} = B_0^2 d\Omega \hat{e}_{B_0}$

$$\gamma_n = \int_s (\vec{\nabla}_r \times \vec{A}_n)\cdot d\vec{S} \tag{180}$$

$$= \int_s \left(\frac{-1}{2B_0^2}\hat{e}_{B_0}\right)\cdot B_0^2 d\Omega \hat{e}_{B_0} \tag{181}$$

$$= \frac{-1}{2}\int d\Omega \tag{182}$$

$$= \frac{-1}{2}\int_0^{\alpha}\sin\alpha d\alpha \int_0^{2\pi} d\psi \tag{183}$$

$$= (1-\cos\alpha)\pi \tag{184}$$

$$\boxed{\gamma_n = -\frac{1}{2}\Omega} \tag{185}$$

Problem 5. Fermion or Boson

Explain how according to the algebra of quantum mechanical angular momentum, particles must exist in the form of either a fermion (with half-integral spin) or a boson (with integral spin), and cannot have any other values (rational or irrational) for their intrinsic spin such as $s = 2/3$, $7/5$, π, ..., etc.

Solution

If particles were allowed to have rational or irrational values for their intrinsic spin, the ladder operators can't lower m all the way down to m_{min} in integer steps without going beyond $m_m in$ or never quite reaching it.

Looking at our ladder operators derived in the notes[I-3]:

$$J_+ |jm\rangle = c^+_{jm} |jm+1\rangle \tag{186}$$
$$J_- |jm\rangle = c^-_{jm} |jm-1\rangle \tag{187}$$

and solving the eigenvalue problem to find the bounds on m we find

$$m_{\text{min}} = -m_{\text{max}} \tag{188}$$
$$m_{\text{min}} = -j \tag{189}$$

we are left to work on the limits of j, where the question lies. Why does j have to take a 1/2 integer value? The answer lies in the requirements for the minimum and maximum values of having to equal $|j|$, which depends on m_{min}. If we run our lowering operator all the way to the bottom, we quickly find that for all of these conditions to be met, j has to take on these positive 1/2 integer values.

$$J_- |jm_{\text{max}}\rangle = J_- |jj\rangle = c^- |j(j-1)\rangle \tag{190}$$
$$J_- |j(j-1)\rangle = c^- |j(j-2)\rangle \tag{191}$$
$$\vdots \tag{192}$$
$$J_- |j(j_{\text{min}}+1)\rangle = c^- |j(j_{\text{min}})\rangle \tag{193}$$
$$J_- |j(j_{\text{min}})\rangle = 0 \tag{194}$$

So if we run this process with some value with j, subtracting 1 from m_{max} each time, we find that we will only get to (j_{min}) if $j = \frac{1}{2}n$ where n is a positive integer. For example, if we try with $j = 1.1$, the lowest we can get is only $m_{\text{min}} = -0.9$ such that $m_{\text{max}} \neq -m_{\text{min}}$. Now that is some dense notation.

Problem 6. The W-Boson

The W-Boson is one of the fundamental mediator of quanta in the unified electroweak field theory. The W was discovered in CERN in 1983 and is known to have spin equal to 1. Derive explicit matrix representations for the spin operator of the W particle: i.e., let $\bar{S} = \hbar\bar{\Sigma}$, find all the component matrices of $\bar{\Sigma}$, which include: Σ_+,Σ_-,Σ_x,Σ_y,Σ_z,Σ^2.

Solution

Lets assume here that we have a free particle such that the orbital angular momentum is zero: $\hat{L} = \cancelto{0}{\hat{J}} + \hat{S}$. This leaves us to find $\hat{S}$ for a spin 1 particle, which means we can follow the notes on page [I-8]. First, lets get our tools out, including our raising and lowering operators.

$$S_+ |jm\rangle = c^{+}_{jm} |j(m+1)\rangle \qquad c^{+}_{jm} = \hbar\sqrt{(j-m)(j+m+1)} \tag{195}$$

$$S_+ |jm\rangle = c^{-}_{jm} |j(m-1)\rangle \qquad c^{-}_{jm} = \hbar\sqrt{(j+m)(j-m+1)} \tag{196}$$

$$S_x |jm\rangle = \frac{1}{2}(S_+ + S_-) |jm\rangle \tag{197}$$

$$S_y |jm\rangle = \frac{i}{2}(S_- - S_+) |jm\rangle \tag{198}$$

$$S_z |jm\rangle = \hbar m |jm\rangle \tag{199}$$

$$S^2 |jm\rangle = \hbar^2 j(j+1) |jm\rangle \tag{200}$$

Lets go ahead and calculate our $c^{\pm}_{jm}$ values. For spin 1 particles, $j = 1$, $m_{\text{min}} = -j$ and $m_{\text{max}} = j$, with integer steps in between.

$$c^{+}_{11} = 0 \qquad c^{+}_{10} = \sqrt{2}\hbar \qquad c^{+}_{1(-1)} = \sqrt{2}\hbar \tag{201}$$

$$c^{-}_{11} = \sqrt{2}\hbar \qquad c^{-}_{10} = \sqrt{2}\hbar \qquad c^{+}_{1(-1)} = 0 \tag{202}$$

Our matrix elements take the form of:

$$\langle j'm'| S_{\pm} |jm\rangle = c^{\pm}_{jm}\delta_{jj'}\delta_{m'(m\pm 1)} \tag{203}$$

$j = j'$ for all elements however, so we can skip writing it in our matrix elements.

$$S_+ |jm\rangle = \begin{pmatrix} c^{+}_{11}\delta_{12} & c^{+}_{10}\delta_{11} & c^{+}_{1(-1)}\delta_{10} \\ c^{+}_{11}\delta_{02} & c^{+}_{10}\delta_{01} & c^{+}_{1(-1)}\delta_{00} \\ c^{+}_{11}\delta_{(-1)2} & c^{+}_{10}\delta_{(-1)1} & c^{+}_{1(-1)}\delta_{(-1)0} \end{pmatrix} |jm\rangle \tag{204}$$

$$= \begin{pmatrix} 0 & \sqrt{2}\hbar & 0 \\ 0 & 0 & \sqrt{2}\hbar \\ 0 & 0 & 0 \end{pmatrix} |jm\rangle \tag{205}$$

$$S_+ = \sqrt{2}\hbar \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{206}$$

$$\boxed{\Sigma_+ = \sqrt{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}} \tag{207}$$

Same procedure for S_+:

$$S_- |jm\rangle = \begin{pmatrix} c^-_{11}\delta_{10} & c^-_{10}\delta_{1(-1)} & c^-_{1(-1)}\delta_{1(-2)} \\ c^-_{11}\delta_{00} & c^-_{10}\delta_{0(-1)} & c^-_{1(-1)}\delta_{0(-2)} \\ c^-_{11}\delta_{(-1)0} & c^-_{10}\delta_{(-1)(-1)} & c^-_{1(-1)}\delta_{(-1)(-2)} \end{pmatrix} |jm\rangle \tag{208}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2}\hbar & 0 & 0 \\ 0 & \sqrt{2}\hbar & 0 \end{pmatrix} |jm\rangle \tag{209}$$

$$S_- = \sqrt{2}\hbar \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{210}$$

$$\boxed{\Sigma_- = \sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}} \tag{211}$$

Solving for S_x using Equation (197):

$$S_x |jm\rangle = \frac{1}{2}(S_+ + S_-) |jm\rangle \tag{212}$$

$$= \frac{1}{2} \left(\begin{pmatrix} 0 & \sqrt{2}\hbar & 0 \\ 0 & 0 & \sqrt{2}\hbar \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2}\hbar & 0 & 0 \\ 0 & \sqrt{2}\hbar & 0 \end{pmatrix} \right) |jm\rangle \tag{213}$$

$$= \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} |jm\rangle \tag{214}$$

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{215}$$

$$\boxed{\Sigma_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}} \tag{216}$$

Solving for S_y using Equation (198):

$$S_y |jm\rangle = \frac{i}{2}(S_- - S_+)|jm\rangle \tag{217}$$

$$= \frac{i}{2}\left(\begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2}\hbar & 0 & 0 \\ 0 & \sqrt{2}\hbar & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sqrt{2}\hbar & 0 \\ 0 & 0 & \sqrt{2}\hbar \\ 0 & 0 & 0 \end{pmatrix}\right)|jm\rangle \tag{218}$$

$$= \frac{i\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}|jm\rangle \tag{219}$$

$$S_y = \frac{i\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \tag{220}$$

$$\boxed{\Sigma_y = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}} \tag{221}$$

Solving for S_z using Equation (199):

$$S_z |jm\rangle = \hbar m |jm\rangle \tag{222}$$

$$= \begin{pmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{pmatrix}|jm\rangle \tag{223}$$

$$S_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{224}$$

$$\boxed{\Sigma_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}} \tag{225}$$

Solving for S^2 using Equation (200):

$$S^2 |jm\rangle = \hbar^2 j(j+1)|jm\rangle \tag{226}$$

$$= \begin{pmatrix} 2\hbar^2 & 0 & 0 \\ 0 & 2\hbar^2 & 0 \\ 0 & 0 & 2\hbar^2 \end{pmatrix}|jm\rangle \tag{227}$$

$$S_z = 2\hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{228}$$

$$\boxed{\Sigma^2 = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2\mathbf{I}} \tag{229}$$

Bibliography

[1] David H. McIntyre. *Quantum Mechanics*. Pearson Addison-Wesley, 1st edition, 2012.