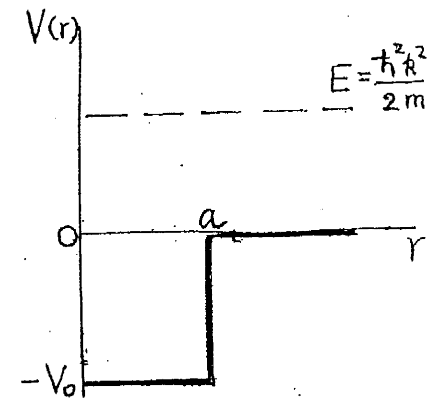
PH 618 Problem Set #3

Time independent scattering theory: Phase-shift Analysis
(Due: March 7, 2013)

1. Consider scattering by the following 3D attractive spherical potential well:

$$V(r) = \begin{cases} -V_0 , & 0 < r < a \\ 0, & a < r \end{cases}$$



The radial TISE can then be written as:

<u>For r < a</u>:

$$\frac{d^2R_\ell}{dr^2}+\frac{2}{r}\frac{dR_\ell}{dr}+[k^2+\frac{2m}{\hbar^2}V_0-\frac{\ell(\ell+1)}{r^2}]R_\ell=0$$

$$\Rightarrow$$

$$\frac{d^2R_\ell}{dr^2}+\frac{2}{r}\frac{dR_\ell}{dr}+[\alpha^2-\frac{\ell(\ell+1)}{r^2}]R_\ell=0, \text{ with } \alpha^2 \equiv k^2+\frac{2m}{\hbar^2}V_0 \quad (1)$$

<u>For r > a</u>:

$$\frac{d^2R_\ell}{dr^2}+\frac{2}{r}\frac{dR_\ell}{dr}+[k^2-\frac{\ell(\ell+1)}{r^2}]R_\ell=0. \quad (2)$$

Thus the general non-singular solutions for (1) and (2) are given by the following in terms of the spherical Bessel functions:

<u>For r < a</u>: $\quad R_\ell = A_\ell j_\ell(\alpha r), \quad (3)$

<u>For r > a</u>: $\quad R_\ell = B_\ell j_\ell(kr)+C_\ell n_\ell(kr) . \quad (4)$

(a) (10 points) Show that the logarithmic derivative which must be matched at r = a can be expressed as:

$$\frac{1}{R_\ell}\frac{dR_\ell}{dr}\bigg|_{r=a_-} \equiv \frac{1}{a\gamma_\ell(k)} \quad (5)$$

where $\gamma_\ell(k)=\dfrac{j_\ell(\alpha a)}{\alpha a j_\ell'(\alpha a)}$.

(b) (10 points) Hence find an <u>exact</u> expression for the phase shifts δ_ℓ using the result for tan δ_ℓ given in Eq. (15.2) in the lecture notes.

(c) (20 points) Show that the s-wave phase shift is given by:

$$\delta_0 = -ka+\tan^{-1}\left[\frac{k}{\alpha}\tan(\alpha a)\right].$$

(HINT: From table, one obtains: $j_0(x)=\dfrac{\sin x}{x}$, $n_0(x)=-\dfrac{\cos x}{x}$.)

(d) (10 points) Show that the <u>scattering length</u> defined for s-wave scattering is given by:

$$L = -\lim_{k \to 0} \frac{\tan \delta_0}{k} = a - \frac{\tan(\lambda a)}{\lambda}, \qquad (6)$$

where $\lambda = \sqrt{\frac{2m}{\hbar^2} V_0}$.

(e) (15 points) Discuss the dependence of L on the potential strength (λ) and potential range a.

(f) (15 points) For extremely low incident energy ($k \approx 0$), show that the total scattering cross-section is dominated by the s-wave and is given by (HINT: refer to Eq. (15.14), lecture notes):

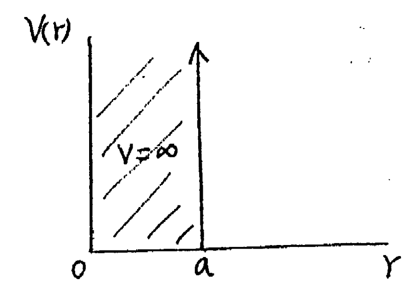
$$\sigma_{total} \approx \sigma_0 \approx 4\pi L^2. \qquad (7)$$

Show that the result in (7) is identical to the previous result obtained in #3(b) in Prob. Set 2 for the low energy limit of the same cross section obtained from the first Born approximation.

(g) (10 points) Comment briefly on the results in (6) and (7) for the special cases:

(*i*) $\lambda a = \tan \lambda a$, and

(*ii*) $\lambda a = n\pi / 2,\ n = 1, 3, 5, 7, \ldots etc.$



2. For the hard sphere potential defined as follows:

$$V(r) = \begin{cases} \infty & 0 \le r \le a \\ 0 & a < r \end{cases}$$

(a) (30 points) Recall that in this case, the phase shifts can again be obtained in exact form. From the definitions of the scattering length and effective range as given in Eq. (15.9) (lecture notes), calculate these quantities (L and γ_{eff}), as well as the total cross section (σ_0) for s-wave scattering for the above potential.

(b) (10 points) By letting $V_0 \to -\infty$ in no. 1, try to check the results obtained there against those obtained in part (a) in this problem.

(c) (10 points) Hence from (a), show that the total cross section in the low energy limit for a hard sphere potential is given by: $\sigma_{total} \approx \sigma_0 \approx 4\pi L^2 = 4\pi a^2$, which is 4 times the value expected from classical physics and turns out to be the "surface area" of the sphere (!).

***Bonus (+20 points)*:** It turns out that even in the high energy case, the QM scattering from a hard sphere is still different from the classical result πa^2! In this case, one has to sum over ALL the partial waves up to angular momentum $\approx ka$. For the simple hard sphere case, this summation can be done and the total cross section can be obtained as:

$$\sigma = \frac{4\pi}{k^2} \sum_{\ell=0}^{\ell \approx ka} (2\ell + 1) \sin^2 \delta_\ell \approx 2\pi a^2,$$

where the factor 2 can be understood as the sum of two contributions: πa^2 from reflection, and πa^2 from "shadow scattering" which is similar to Fraunhofer diffraction in optics (see Sakurai, pp 406-410). Try to give a derivation of the above result.