PH 618 Quantum Mechanics
Problem Set 2: Scattering based on stationary states
(Due: February 26, 2013)

1. (30 points) Show that by direct differentiation, the Green function given by Eq. (11.8) in the notes:

$$G(\vec{r}, \vec{r}') = -\frac{e^{ik|\vec{r} - \vec{r}'|}}{4\pi|\vec{r} - \vec{r}'|}$$

does satisfy the Helmholtz equation with a point source as follows:

$$(\nabla^2 + k^2)\, G(\vec{r}, \vec{r}') = \delta(\vec{r} - \vec{r}').$$

[HINT: $\nabla^2 \frac{1}{R} = -4\pi\, \delta(\vec{R})$.]

2. (60 points) <u>The Yukawa / Screened Coulomb potential scattering</u>

For the Yukawa-type potential: $V(r) = \frac{V_0 e^{-\lambda r}}{r}$, prove that according to the first Born approximation, the so-called plane wave Born approximation (PWBA):

(a) the scattering amplitude is given by:

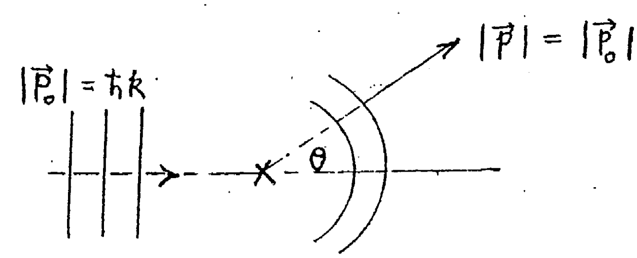
$$f(\theta, \varphi) = -\frac{2mV_0}{4p^2 \sin^2\frac{\theta}{2} + \hbar^2\lambda^2}.$$

(b) the differential and total cross-section are given respectively by:

$$\frac{d\sigma}{d\Omega} = \frac{4m^2V_0^2}{(4p^2\sin^2\frac{\theta}{2} + \hbar^2\lambda^2)^2},$$

$$\sigma = \frac{16\pi m^2 V_0^2}{(4p^2 + \hbar^2\lambda^2)\hbar^2\lambda^2}.$$

(c) Discuss the above results in (a) and (b) in the case of Coulomb scattering according to PWBA.



3. (50 points) The "Soft-Sphere" Scattering [Ref. Griffiths: p. 414 - 416]

Consider the "soft-sphere" potential (i.e. finite spherical potential barrier):

$$V(\vec{r}) = \begin{cases} V_0 \,, & \text{if } r \le a \\ 0 \,, & \text{if } r > a \end{cases}$$

(a) Using the first Born approximation, i.e. the PWBA, show that the scattering amplitude can be obtained in the following form:

$$f(\theta) = -\frac{2mV_0}{\hbar^2 K^3}\left[\sin Ka - Ka\cos Ka\right],$$

where $K = 2k\sin\dfrac{\theta}{2}$.

(b) Show that in the low energy limit, the total cross section can be obtained approximately as: $\sigma \cong Ca^6$, where the constant C is given by: $C = \dfrac{16\pi m^2 V_0^2}{9\hbar^4}$. Note that the exact total cross section can also be integrated, though pretty tedious.

(Remark: the simpler problem of a "hard sphere", $V_0 \to \infty$, cannot be treated in the Born approximation. This is clear from the limit of validity of this approximation. Later we shall solve this problem by solving it exactly.)

4. (50 points) The Optical Theorem

In class, we have proven that from partial wave analysis for central force potentials V(r), the scattering amplitude can be expanded in terms of the "phase shifts" δ_ℓ and the Legendre polynomials in the following form:

$$\mathrm{f}(\theta) = \sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{\mathrm{k}} \sin\delta_\ell\, e^{i\delta_\ell}\, \mathrm{P}_\ell(\cos\theta)\,.$$

Using the orthogonal property of P_ℓ in the following form:

$$\int \mathrm{P}_\ell(\cos\theta)\,\mathrm{P}_{\ell'}(\cos\theta)\, d\Omega = \frac{4\pi}{2\ell+1}\delta_{\ell\ell'}\,,$$

derive the following optical theorem:

$$\sigma = \frac{4\pi}{\mathrm{k}^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell = \frac{4\pi}{\mathrm{k}}\,\mathrm{Im}\,\mathrm{f}(0)\,,$$

where σ is the total cross-section and Im f(0) is the imaginary part of the forward scattering amplitude (i.e. $\theta = 0$).