GP 9505A Problem Set #4: Fourier Transforms

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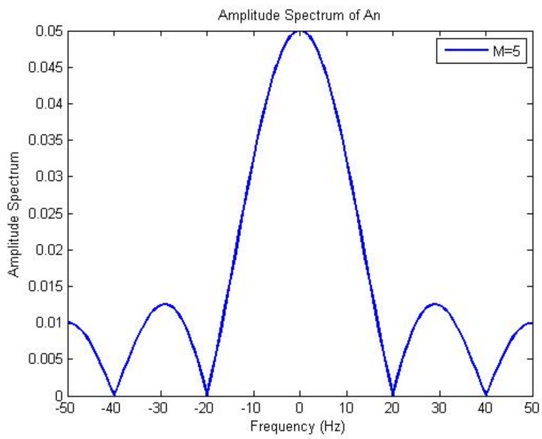
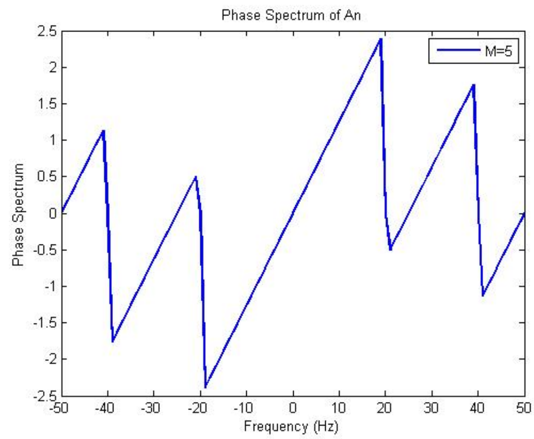
Submitted on 15 October 2015

1. Figures 1-3 answer the question. See attached MATLAB code in the Appendix for how plots were made.



(a)



(b)

Figure 1: (a) shows the amplitude spectrum of a boxcar function, where $N = 100$, $M = 5$. (b) shows the phase spectrum of the same function.

Let us consider Figure 1 when $M = 5$. We see that the zeros of the amplitude spectrum are at -40, -20, 20, and 40. The ratio of $N/M = 20$, which is the same interval we find for the frequency index, (n).

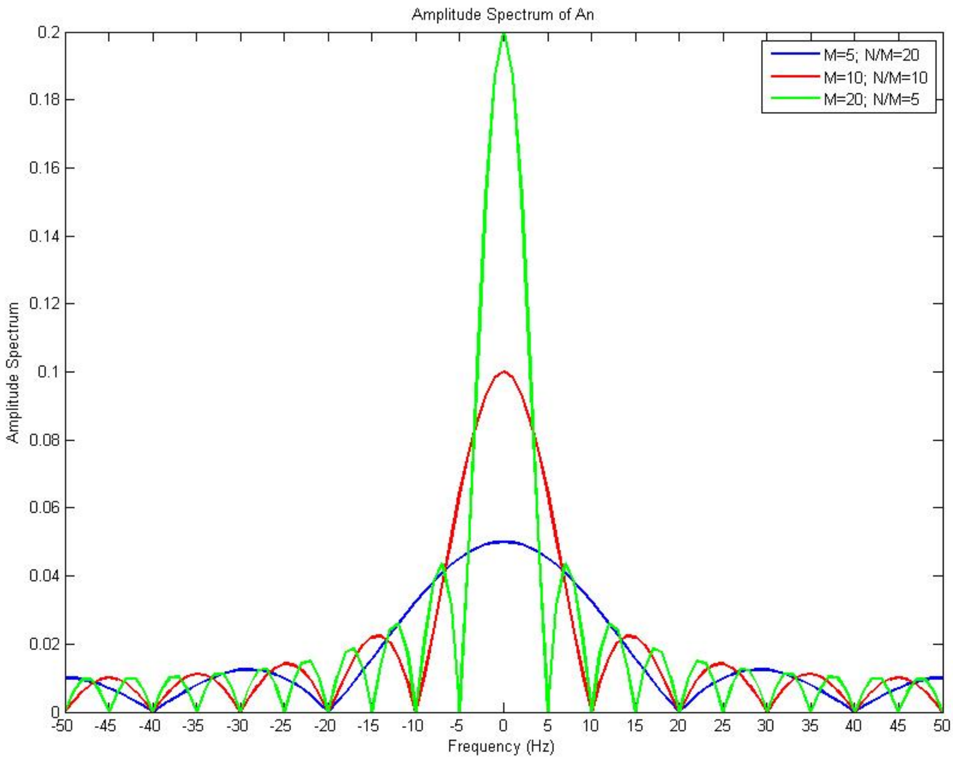
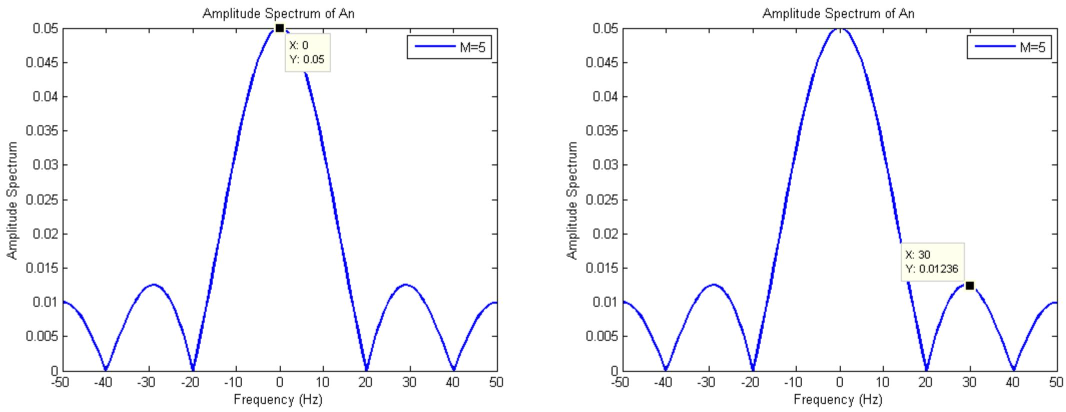


Figure 2: Amplitude spectrum of three boxcar functions, with $N = 100$; $M = 5$ (blue), $M = 10$ (red), $M = 20$ (green).

We notice again that the frequency index, (n), displays the same interval as the ratio N/M at the zeros, for each value of M. We also notice that as M increases, the ratio of N/M becomes smaller. The intervals where the frequency index equals 0 becomes more narrow.

Let us consider the examples where $M = 20$, then $N/M = 5$, and $M = 5$, then $N/M = 20$. We see that as we decrease our M in the time series, the frequency amplitude spectrum broadens (blue and green, respectively). "Reciprocal broadening" says that as we make something narrower in the time domain, the wider it is in the frequency (Fourier) domain; and the wider in the time domain, the narrower in the frequency domain. This is shown clearly in these amplitude spectr.



(a) (b)

Figure 3: (a) and (b) display the amplitude spectrum of the original $M = 5$, $N = 100$ boxcar function, with two points highlighted, where $n = 0$, and $n = 1.5(N/M) = 1.5(100/5) = 30$.

The ratio of the amplitude spectrum at $n = 30$ and $n = 0$ is:

$$
\begin{aligned}
R_{amp} &= \frac{0.01236}{0.05} \\
&= 0.2472
\end{aligned}
$$

Now, let us compare that ratio to

$$
\begin{aligned}
\left| \frac{1}{M} \frac{sin(1.5\pi)}{sin(1.5\pi/M)} \right| &= \left| \frac{1}{5} \frac{sin(1.5\pi)}{sin(1.5\pi/5)} \right| \\
&= 0.2472
\end{aligned}
$$

2. a) Gubbins (2004) offers a proof of the FT of Gaussian time series, $e^{-\alpha t^2}$, is $\sqrt{\frac{\pi}{\alpha}}e^{-\omega^2/4\alpha}$. The work shown here is that proof expanded:

$$
\begin{aligned}
G(\omega) &= \int_{-\infty}^{\infty} e^{-\alpha t^2} e^{iwt} dt \\
G(\omega) &= \int_{-\infty}^{\infty} e^{-\alpha t^2 + iwt} dt
\end{aligned}
$$

Let us complete the square of the exponent of e:

$$
\begin{aligned}
-\alpha t^2 + iwt & \\
-\alpha t^2 + iwt + \frac{\omega^2}{4\alpha} &= -\alpha\left(t^2 - \frac{i\omega t}{\alpha} - \frac{\omega^2}{4\alpha^2}\right) \\
-\alpha t^2 + iwt + \frac{\omega^2}{4\alpha} &= -\alpha\left(t - \frac{i\omega}{2\alpha}\right)^2
\end{aligned}
$$

Let

$$
\begin{aligned}
-x^2 &= -\alpha\left(t - \frac{i\omega}{2\alpha}\right)^2 \\
x &= \sqrt{\alpha}\left(t - \frac{i\omega}{2\alpha}\right)
\end{aligned}
$$

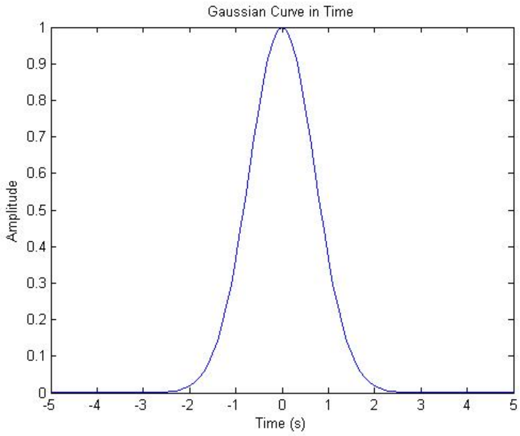
Now, let us find $\dfrac{dx}{dt}$ and dt:

$$
\begin{aligned}
\frac{dx}{dt} &= \sqrt{\alpha} \\
dt &= \frac{dx}{\sqrt{\alpha}}
\end{aligned}
$$

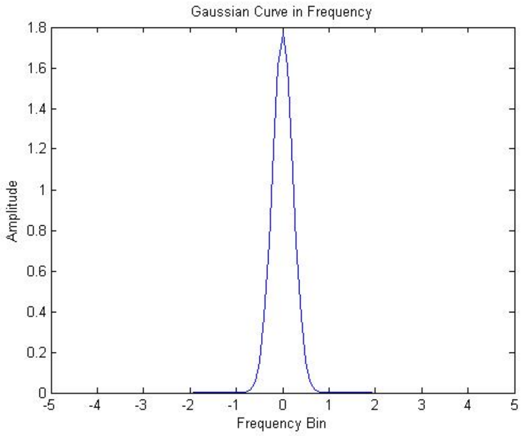
Now, we must take away the "completed square" component, $\frac{-\omega^2}{4\alpha}$. Our integral then becomes

$$
\begin{aligned}
G(\omega) &= e^{\frac{-\omega^2}{4\alpha}} \int_{-\infty}^{\infty} e^{-x^2} \frac{dx}{\sqrt{\alpha}} \\
G(\omega) &= \frac{1}{\sqrt{\alpha}} e^{\frac{-\omega^2}{4\alpha}} \sqrt{\pi} \\
G(\omega) &= \sqrt{\frac{\pi}{\alpha}} e^{-\omega^2/4\alpha}
\end{aligned}
$$

Standard deviation of the width of a Gaussian in time, $e^{-\alpha t^2}$, is $\sigma_t = \sqrt{\frac{1}{2\alpha}}$. The standard deviation of the width of a Gaussian in frequency, $\sqrt{\frac{\pi}{\alpha}}e^{-\omega^2/4\alpha}$, is $\sigma_f = \sqrt{2\alpha}$.



(a)



(b)

Figure 4: (a) displays the Gaussian function in time, $e^{-\alpha t^2}$, with a standard deviation width of $\sigma_t = \sqrt{\frac{1}{2\alpha}}$. (b) displays the associated Gaussian in frequency, $\sqrt{\frac{\pi}{\alpha}}e^{-\omega^2/4\alpha}$, with a standard deviation width of $\sigma_f = \sqrt{2\alpha}$.

The standard deviation for (a) can be found 61% up the curve ($s.d. = \dfrac{1}{\sqrt{e}} = 0.61$). We find the width of the curve for (a) to be equal to 2, and the width of the curve for (b) to be 0.5. If we multiply the width of both curves, we get

$$(2)(0.5) = 1$$

which is what we expect as we know that

$$\sigma_f \sigma_t = 1$$

3. a)

$$
\begin{aligned}
F(\omega) &= \frac{1}{2a}[H(\omega+a) - H(\omega-a)] \\
F(\omega) &= \frac{1}{2a}B(\omega)
\end{aligned}
$$

where $B(\omega)$ is a boxcar function. Then

$$
\begin{aligned}
FT^{-1}\{F(\omega)\} &= \frac{1}{2\pi}\frac{1}{2a}\int_{-\infty}^{\infty} B(\omega)e^{-iwt}dt \\
&= \frac{1}{4\pi a}\int_{-a}^{a} e^{-i\omega t}dt \\
&= \frac{1}{4\pi a}\left[\frac{e^{-i\omega t}}{-i\omega}\right]_{-a}^{a} \\
&= \frac{1}{4\pi a}\left[\frac{e^{-i\omega a} - e^{i\omega a}}{-i\omega}\right] \\
&= \frac{1}{4\pi a}\left[\frac{(cos(-\omega a) + isin(-\omega a)) - (cos(\omega a) + isin(\omega a))}{-i\omega}\right] \\
&= \frac{1}{4\pi a}\left[\frac{-2isin(\omega a)}{-i\omega}\right] \\
&= \frac{1}{2\pi a}\left[\frac{sin(\omega a)}{\omega}\right] \\
&= \frac{1}{2\pi}\frac{sin(\omega a)}{\omega a}
\end{aligned}
$$

b) Using the same argument (taking the Heaviside function as a boxcar function), but now taking the forward Fourier transform, we get

$$\begin{aligned}
FT\{F(t)\} &= \frac{1}{2\tau}\int_{-\infty}^{\infty} B(t)e^{i\omega t}dt \\
&= \frac{1}{2\tau}\int_{-\tau}^{\tau} e^{i\omega t}dt \\
&= \frac{1}{2\tau}\left[\frac{e^{i\omega t}}{i\omega}\right]_{-\tau}^{\tau} \\
&= \frac{1}{2\tau}\left[\frac{e^{i\omega\tau}-e^{-i\omega\tau}}{-i\tau}\right] \\
&= \frac{1}{2\tau}\left[\frac{2isin(\omega\tau)}{i\omega}\right] \\
&= \frac{1}{\tau}\left[\frac{sin(\omega\tau)}{\omega}\right] \\
&= \frac{sin(\omega\tau)}{\omega\tau}
\end{aligned}$$

c) Both Fourier transforms in (a) and (b) are real-valued because any $sinc(x)$ function is even, and real; and if a function is even and real, its Fourier transform is also even and real. This property is true going from the time domain to the frequency domain, and from the frequency to the time domain. We know that a boxcar function is even and real, and so we must end up with a $sinc(x)$ function.

d) The principle of duality states that if

$$FT\{f(t)\} = F(\omega)$$

then

$$FT^{-1}\{F(t)\} = \frac{1}{2\pi}f(\omega)$$

Our results from (a) and (b) directly demonstrate the principle, as

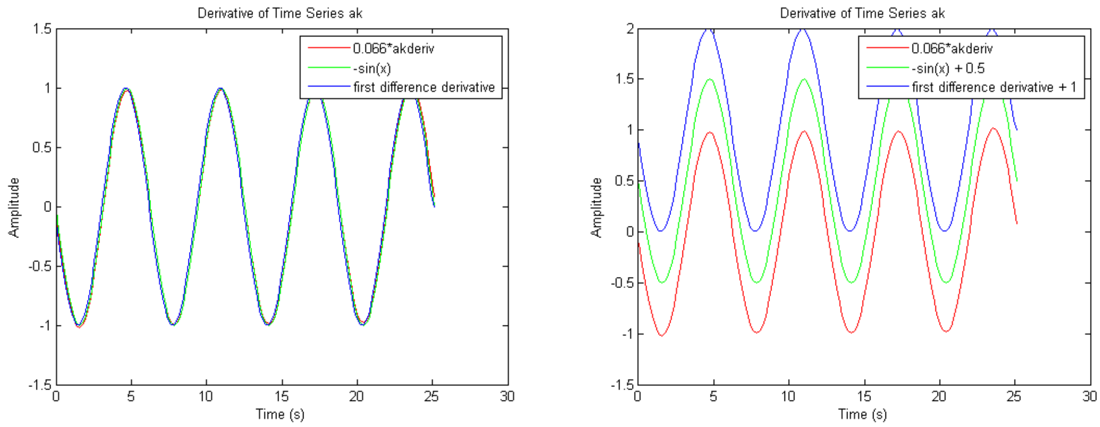
$$\begin{aligned}
FT^{-1}\{F(\omega)\} &= FT^{-1}\left\{\frac{1}{2a}[H(\omega+a)-H(\omega-a)]\right\} \\
&= \frac{1}{2\pi}\frac{sin(\omega a)}{\omega a} \\
&= \frac{1}{2\pi}f(\omega)
\end{aligned}$$

and

$$\begin{aligned}
FT\{F(t)\} &= FT\left\{\frac{1}{2\tau}[H(t+\tau)-H(t-\tau)]\right\} \\
&= \frac{sin(\omega\tau)}{\omega\tau} \\
&= F(\omega)
\end{aligned}$$

"Reciprocal broadening" says that as we make a narrower function the time domain, the wider it is in the frequency (Fourier) domain; and the wider we make a funcion in the time domain, the narrower in the frequency domain. Because both of the functions we are dealing with are boxcar functions, as in question 1, this property still holds.

4. a) Now we will use the MATLAB programs used in part a) to evaluate the horizontal derivative of g. We are not interested in the $cos\theta$ - only the shape of the gravity profile and its horizontal derivative. The horizontal derivative, by the Poisson relationship, gives us a pseudo-magnetic profile. See Appendix for MATLAB code. Plots are shown in Figure 5.



(a) (b)

Figure 5: (a) shows the results of the derivative formula for discrete Fourier transforms, as applied to a $cos(x)$ function; a $-sin(x)$ function; and a derivative of $cos(x)$ using the approximate formula "first difference." (b) shows the same results as (a) but shifted on the y-axis for viewing purposes.

b)

$$\begin{aligned} B &= -\nabla W \\ B &= \frac{\mu_0}{4\pi}\frac{1}{G\rho}\nabla(M \cdot g) \end{aligned}$$

Let

$$\frac{\mu_0}{4\pi}\frac{1}{G\rho} = 1$$

then

$$\begin{aligned} B &= \nabla(M \cdot g) \\ B &= \nabla\left(|M|\,|g|\,cos\theta\right) \\ B &= \frac{d\left(|M|\,|g|\,cos\theta\right)}{dx} + \frac{d\left(|M|\,|g|\,cos\theta\right)}{dy} + \frac{d\left(|M|\,|g|\,cos\theta\right)}{dz} \end{aligned}$$

We are only interested in the horizontal component:

$$B = cos\theta\,|M|\,\frac{d\,(|g|)}{dx}$$

Let $M = 1$:

$$B = cos\theta\frac{d\,(|g|)}{dx}$$

We are not interested in the $cos\theta$ - only the shape of the horizontal derivative of the gravity profile, $\frac{dg}{dx}$. Figure 6 is a plot of the original gravity profile sampled at 1.4 km, and its first derivative using the derivative formula for discrete Fourier transforms.

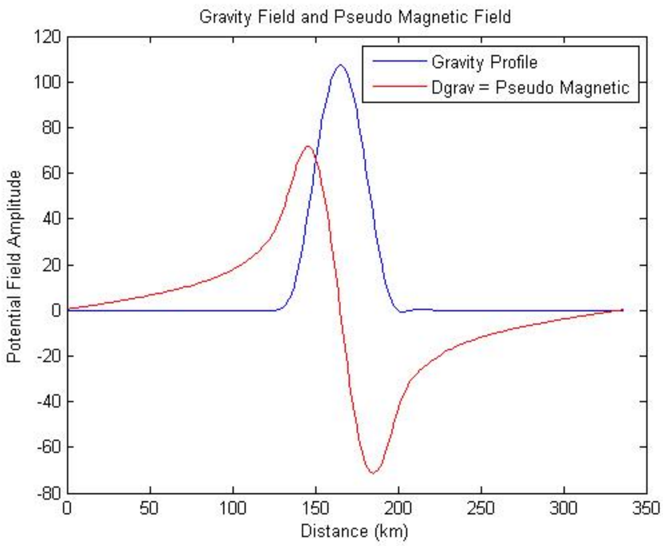


Figure 6: Gravity profile from 'gprofile.dat', and its first derivative, which by the Poisson formula is a pseudo-magnetic profile.

APPENDIX

The appendix displays an alternate (failed) solution to question (3), and the MATLAB code associated to the problems.

The following is an alternate attempt at question (3), but I failed to get the right answer, and made a poor assumption.

3. a) Let us go backwards; and let $x = \omega - a$ and $y = \omega + a$.

$$
\begin{aligned}
F(\omega) &= \frac{1}{2a}[H(y) - H(x)] \\
FT^{-1}\{F(\omega)\} &= \frac{1}{2\pi}\frac{1}{2a}\left[\int H(y)e^{-i(y-a)t}dy - \int H(x)e^{-i(a+x)t}dx\right] \\
f(t) &= \frac{1}{4\pi a}\left[\int H(y)e^{-iyt}e^{iat}dy - \int H(x)e^{-iat}e^{-ixt}dx\right] \\
f(t) &= \frac{1}{4\pi a}\left[\left(e^{iat}\int H(y)e^{-iyt}dy\right) - \left(e^{-iat}\int H(x)e^{-ixt}dx\right)\right] \\
f(t) &= \frac{1}{4\pi a}\left[\left(e^{iat}\left[\frac{1}{2}\delta(t) - \frac{i}{2\pi t}\right]\right) - \left(e^{-iat}\left[\frac{1}{2}\delta(t) - \frac{i}{2\pi t}\right]\right)\right] \\
f(t) &= \frac{1}{4\pi a}\left[\left[\frac{1}{2}\delta(t) - \frac{i}{2\pi t}\right]\left(e^{iat} - e^{-iat}\right)\right] \\
f(t) &= \frac{1}{4\pi a}\left[\left[\frac{1}{2}\delta(t) - \frac{i}{2\pi t}\right][[cos(at) + isin(at)] - [cost(-at) + isin(-at)]]\right]
\end{aligned}
$$

Using the even and odd properties of cos and sin functions, we get

$$
f(t) = \frac{1}{4\pi a}\left[\left[\frac{1}{2}\delta(t) - \frac{i}{2\pi t}\right]2isin(at)\right]
$$

Let $\delta(t) = 0$:

$$
\begin{aligned}
f(t) &= \frac{1}{4\pi a}\left[\left[\frac{-i}{2\pi t}\right]2isin(at)\right] \\
f(t) &= \frac{1}{4\pi a}\left[\frac{sin(at)}{\pi t}\right] \\
f(t) &= \frac{1}{4\pi}\frac{sin(at)}{at} \\
f(t) &= \frac{1}{4\pi}sinc(at) \\
&?
\end{aligned}
$$

$\therefore$ If we trace the proof backwards, we have proved that if $f(t) = \frac{1}{2\pi}\frac{sin(at)}{at}$, then $F(\omega) = \frac{1}{2a}[H(\omega + a) - H(\omega - a)]$.

b) Verify that if $f(t) = \frac{1}{2\tau}[H(\omega + \tau) - H(\omega - \tau)]$, then $F(\omega) = \frac{sin(\omega\tau)}{\omega\tau}$. First, let $x = \omega - \tau$ and $y = \omega + \tau$.

$$
\begin{aligned}
f(t) &= \frac{1}{2\tau}[H(y) - H(x)] \\
FT\{f(t)\} &= \frac{1}{2\tau}\left[\int H(y)e^{i(y-\tau)\omega}dy - \int H(x)e^{i(\tau+x)\omega}dx\right] \\
F(\omega) &= \frac{1}{2\tau}\left[\left(e^{-i\tau\omega}\int H(y)e^{iy\omega}dy\right) - \left(e^{i\tau\omega}\int H(x)e^{ix\omega}dx\right)\right] \\
F(\omega) &= \frac{1}{2\tau}\left[\left(e^{-i\tau\omega}\left[\frac{1}{2}\delta(\omega) + \frac{i}{2\omega}\right]\right) - \left(e^{i\tau\omega}\left[\frac{1}{2}\delta(\omega) + \frac{i}{2\omega}\right]\right)\right] \\
F(\omega) &= \frac{1}{2\tau}\left[\frac{1}{2}\delta(\omega) + \frac{i}{2\omega}\right]\left[e^{-i\tau\omega} - e^{i\tau\omega}\right] \\
F(\omega) &= \frac{1}{2\tau}\left[\frac{1}{2}\delta(\omega) + \frac{i}{2\omega}\right][[cos(-\tau\omega) + isin(-\tau\omega)] - [cos(\tau\omega) + isin(\tau\omega)]] \\
F(\omega) &= \frac{1}{2\tau}\left[\frac{1}{2}\delta(\omega) + \frac{i}{2\omega}\right](-2isin(\tau\omega))
\end{aligned}
$$

Let $\delta(\omega) = 0$:

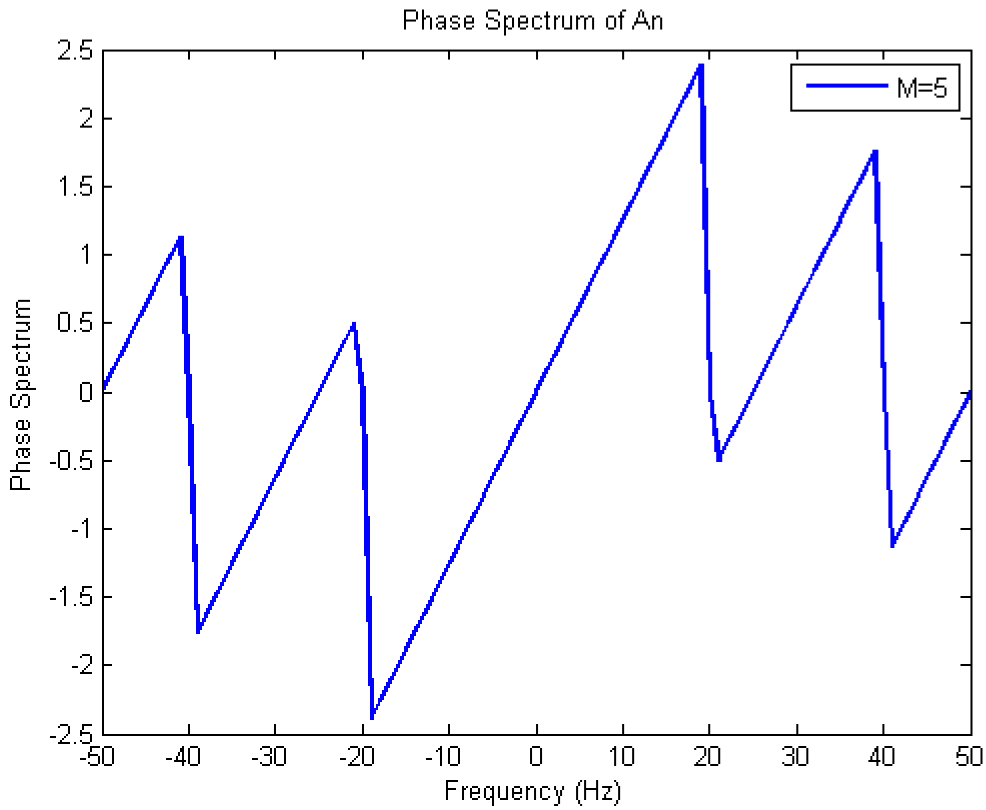
$$
\begin{aligned}
F(\omega) &= \frac{1}{2\tau}\left[\frac{i}{2\omega}\right](-2isin(\tau\omega)) \\
F(\omega) &= \frac{1}{2\tau}\frac{sin(\tau\omega)}{\omega\tau} \\
&?
\end{aligned}
$$

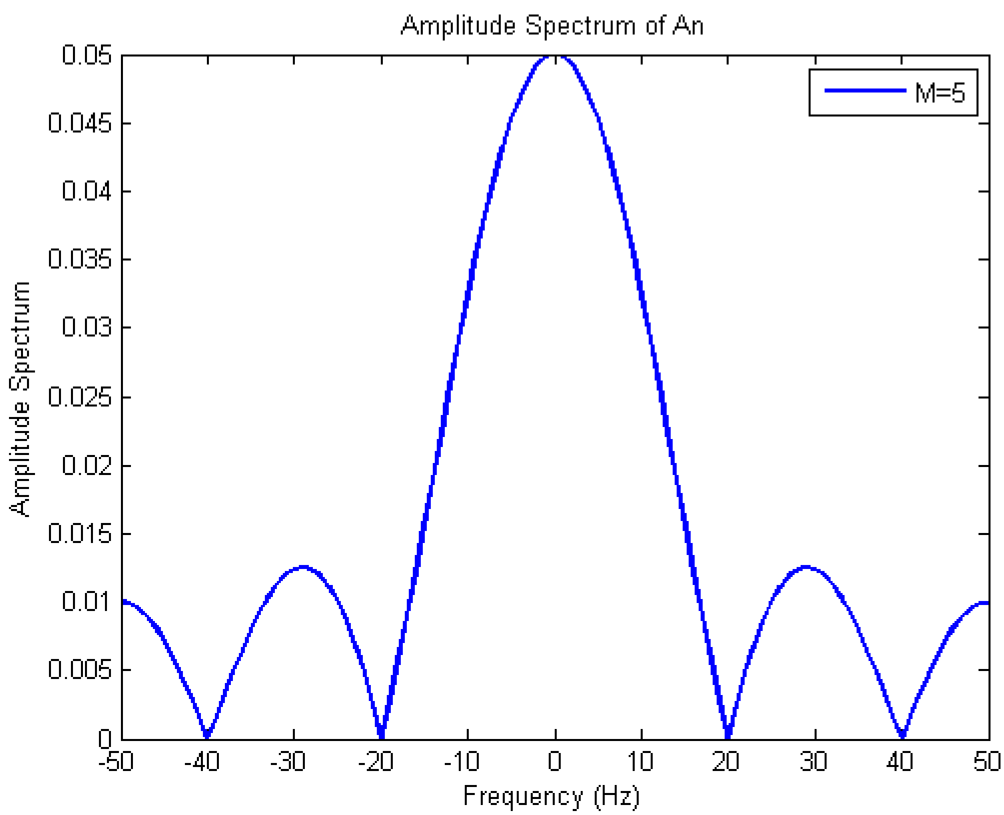
Inverse Problem Set 4

Table of Contents

1. % M = 5

```
%Taking a Fourier transform of the Boxcar function
% Writing code to carry out the DFT

M = 5;
box = zeros(100,1);
box(1:M)=1;
N = length(box);              % Find the length of time series
X = zeros(N,1);             % Set up an N-by-1 matrix of zeros

X=ifft(box);

int=0:N;
freq=-max(int/2):1:max(int/2);
freq=freq';

g1=zeros(N+1,1);

g1(1:50)=X(51:100);
g1(51:100)=X(1:50);
g1(101)=0.0099;

ampspect=abs(g1);

ps=angle(g1);
figure;
plot(freq,ps,'linewidth',2);
title('Phase Spectrum of An');
xlabel('Frequency (Hz)');
ylabel('Phase Spectrum');
legend('M=5');

figure;
plot(freq,ampspect,'linewidth',2);
title('Amplitude Spectrum of An');
xlabel('Frequency (Hz)');
ylabel('Amplitude Spectrum');
legend('M=5');
```

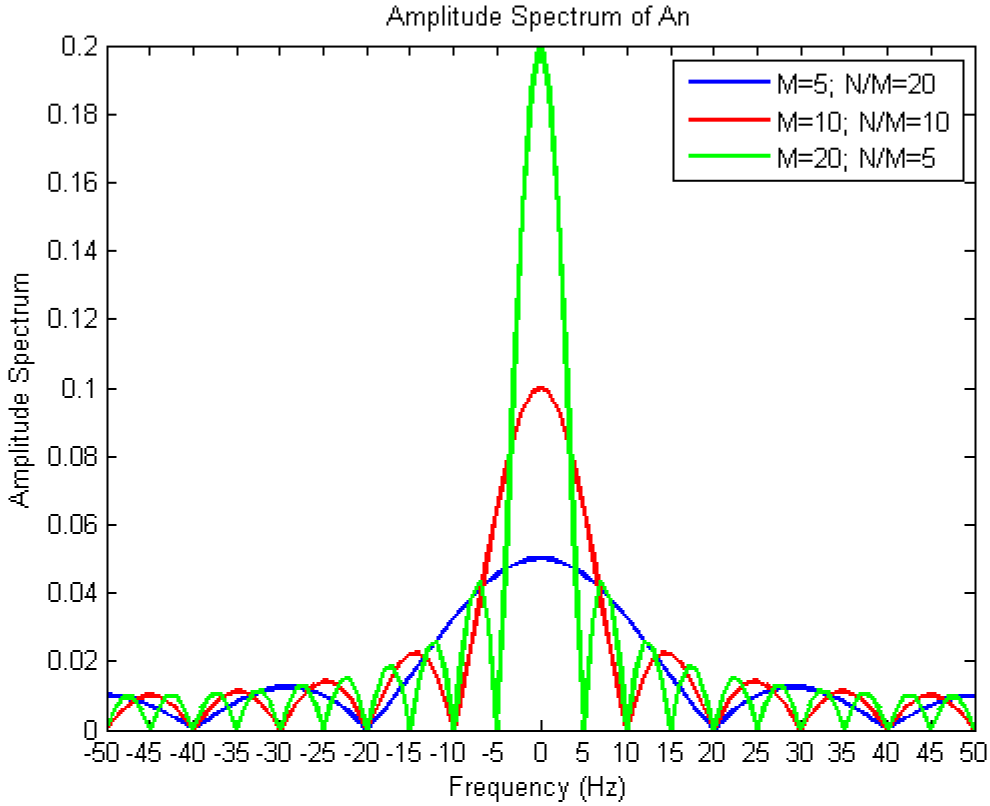

Phase Spectrum of An
M=5
Phase Spectrum
Frequency (Hz)
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-50
-40
-30
-20
-10
0
10
20
30
40
50


Amplitude Spectrum of An
M=5
Amplitude Spectrum
Frequency (Hz)
0.05
0.045
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
-50
-40
-30
-20
-10
0
10
20
30
40
50

1. ii) Testing different values of M

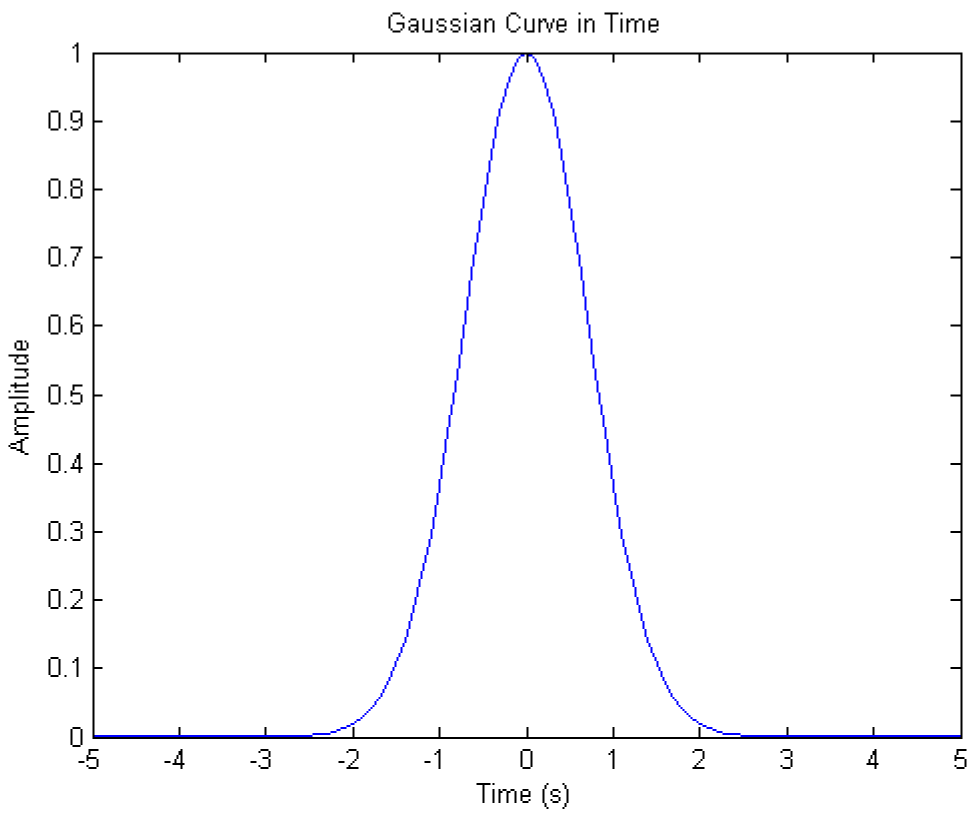
```
M = 5;
box = zeros(100,1);
box(1:M)=1;
N = length(box);              % Find the length of time series
X = zeros(N,1);             % Set up an N-by-1 matrix of zeros

X=ifft(box);

int=0:N;
freq=-max(int/2):1:max(int/2);
freq=freq';

g1=zeros(N+1,1);

g1(1:50)=X(51:100);
g1(51:100)=X(1:50);
g1(101)=0.0099;

ampspect=abs(g1);

figure;
plot(freq,ampspect,'linewidth',2);
title('Amplitude Spectrum of An');
xlabel('Frequency (Hz)');
ylabel('Amplitude Spectrum');
legend('M=5');
hold on;

M = 10;
box = zeros(100,1);
box(1:M)=1;
N = length(box);              % Find the length of time series
X = zeros(N,1);             % Set up an N-by-1 matrix of zeros

X=ifft(box);

int=0:N;
freq=-max(int/2):1:max(int/2);
freq=freq';

g1=zeros(N+1,1);

g1(1:50)=X(51:100);
g1(51:100)=X(1:50);

ampspect=abs(g1);

plot(freq,ampspect,'r','linewidth',2);
title('Amplitude Spectrum of An');
xlabel('Frequency (Hz)');
ylabel('Amplitude Spectrum');
```

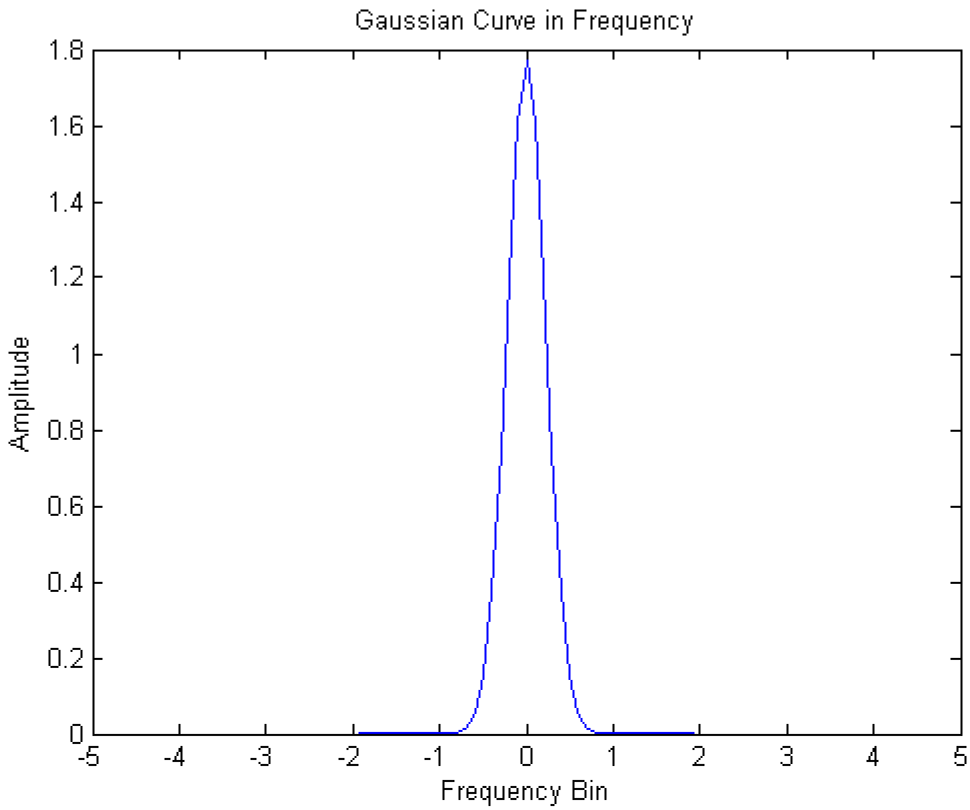
```
hold on;

M = 20;
box = zeros(100,1);
box(1:M)=1;
N = length(box);              % Find the length of time series
X = zeros(N,1);             % Set up an N-by-1 matrix of zeros

X=ifft(box);

int=0:N;
freq=-max(int/2):1:max(int/2);
freq=freq';

g1=zeros(N+1,1);
g1(1:50)=X(51:100);
g1(51:100)=X(1:50);

ampspect=abs(g1);

plot(freq,ampspect,'g','linewidth',2);
title('Amplitude Spectrum of An');
xlabel('Frequency (Hz)');
ylabel('Amplitude Spectrum');
legend('M=5; N/M=20','M=10; N/M=10','M=20; N/M=5');
set(gca,'xtick',[-100:5:100]);
```



2.

```
t=[-5:.1:5];
alpha=1;
gt=exp(-alpha*(t.^2));
figure;
plot(t,gt);
title('Gaussian Curve in Time');
xlabel('Time (s)');
ylabel('Amplitude');
%eb=0.707*ones(size(gt));
%errorbar(gt,eb);

fs=1/0.1;
N=length(gt);
n=[-50:1:50];
omega=n*2*pi/(N*0.1);
%gf=fft(gt);
f=(n*fs/N);
gf=(sqrt(pi/alpha))*exp(-(omega.^2)/(4*alpha));
figure;
plot(f,gf);
title('Gaussian Curve in Frequency');
xlabel('Frequency Bin');
ylabel('Amplitude');
```



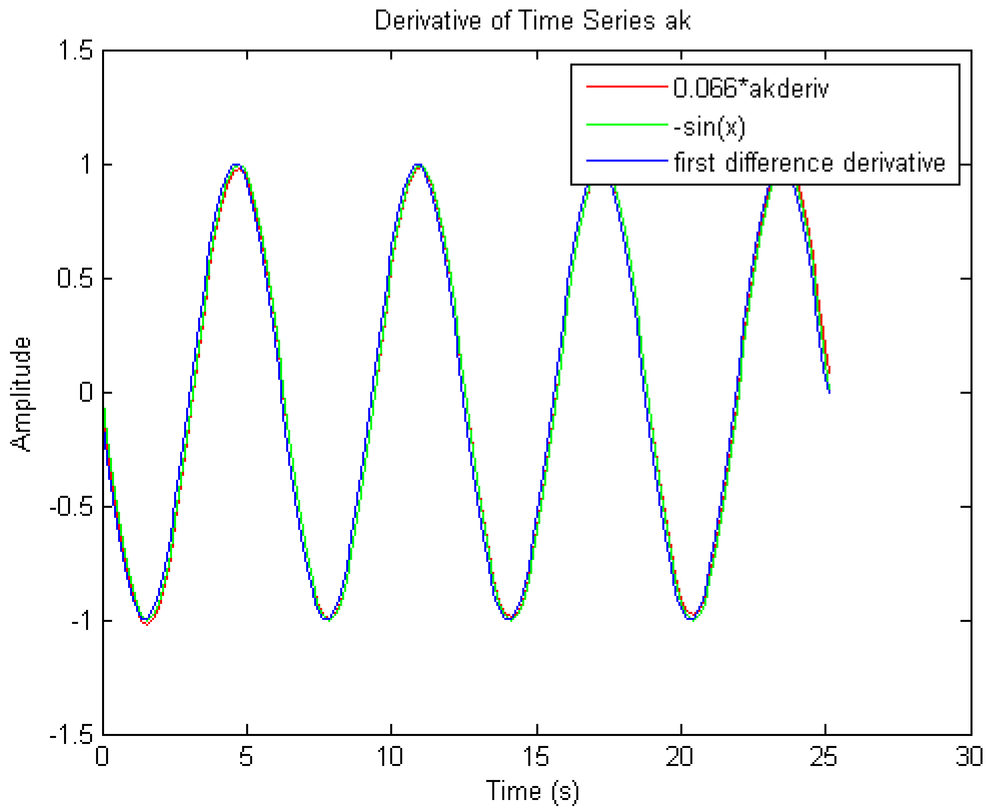


4.a)

i) Derivative theorem test

```
close all;
clear all;
clc;
x=[0:(pi/16):8*pi];
an=cos(x); % Defining time series

% An=ifft(an); % Takes the forward DFT in Gerhard's notation
% First, let us take the FT of an.
N = length(an);              % Find the length of time series
An = zeros(N,1);             % Set up an N-by-1 matrix of zeros

for n = 1:N
    for k = 1:N
        compl=i*2*pi*(n-1)*(k-1)/N; % Calculating the complex exponent
        An(n)=An(n)+(1/N)*an(k)*exp(compl); %Formula for DFT
    end
end

% figure;
% plot(An);
% title('FT of ak');
% xlabel('Time (s)');
```

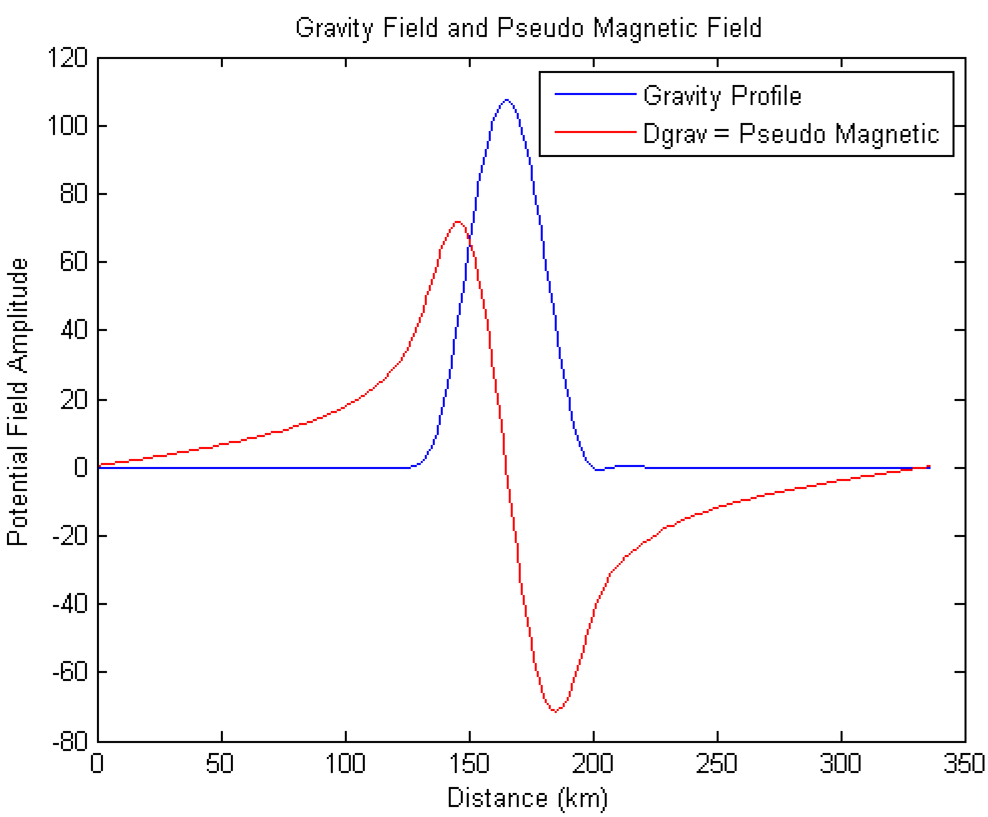
```
% ylabel('Amplitude');

% Code to carry out the derivative of an

N=length(An);
anderiv = zeros(N,1);
for n = 1:N
    for k = 1:N
        omegan=(2*pi*(n-1))/(N*(pi/16));
        compl=-i*2*pi*(n-1)*(k-1)/N; % Calculating the complex exponent
        anderiv(k)=anderiv(k)+0.066*i*omegan*An(n)*exp(compl); %Formula for invers
        % I had to multiply by 0.066 to get my derivative to match.
    end
end


figure;
plot(x,real(anderiv),'r',x,-sin(x),'g');
title('Derivative of Time Series ak');
xlabel('Time (s)');
ylabel('Amplitude');
hold on;

% ii) approximate derivative of ak - "first difference"

t=pi/16;
akderiv=zeros(length(an),1);
akderiv=akderiv';

for i=1:length(an)
    if i+1>length(an)
        an(i+1)=1;
    end
    akderiv(i)=akderiv(i)+((an(i+1)-an(i))/t);

end
x2=x(1:129);
akderiv=akderiv(1:129);

plot(x2,akderiv,'b')
title('Derivative of Time Series ak');
xlabel('Time (s)');
ylabel('Amplitude');
legend('0.066*akderiv','-sin(x)','first difference derivative')
```



4. b) Poisson relationship: horizontal position

```
close all;
clear all;
clc;
% Assume a value of unity for constants
load gprofile.dat;
grav=gprofile';
x=[0:1.4:336];
an=grav; % Defining time series

% An=ifft(an); % Takes the forward DFT in Gerhard's notation
% First, let us take the FT of an.
N = length(an);             % Find the length of time series
An = zeros(N,1);            % Set up an N-by-1 matrix of zeros

for n = 1:N
    for k = 1:N
        compl=i*2*pi*(n-1)*(k-1)/N; % Calculating the complex exponent
        An(n)=An(n)+(1/N)*an(k)*exp(compl); %Formula for DFT
    end
end

% figure;
% plot(An);
```

```
% title('FT of ak');
% xlabel('Time (s)');
% ylabel('Amplitude');

% Code to carry out the derivative of an

N=length(An);
anderiv = zeros(N,1);
for n = 1:N
    for k = 1:N
        omegan=(2*pi*(n-1))/(N*(pi/16));
        compl=-i*2*pi*(n-1)*(k-1)/N; % Calculating the complex exponent
        anderiv(k)=anderiv(k)+0.066*i*omegan*An(n)*exp(compl); %Formula for invers
        % I had to multiply by 0.066 to get my derivative to match.
    end
end


figure;
plot(x,grav,'b',x,real(anderiv),'r');
title('Gravity Field and Pseudo Magnetic Field');
xlabel('Distance (km)');
ylabel('Potential Field Amplitude');
legend('Gravity Profile','Dgrav = Pseudo Magnetic')
hold on;
```



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