GP 9505A Problem Set 8
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1. a) See MATLAB code below for all calculations.

```
%% Question 1.a)
t1=[0; 3.0]; % defining time vector
d1=[2;46]; % defining distance vector

% least squares solution to solve for velocity
v1=(inv(t1'*t1))*t1'*d1

% Let us now calculate the covariance of the model
Cd1=[5^2 0; 0 5^2] % covariance of data
Cm1=inv(t1'*inv(Cd1)*t1) %covariance of model

% Let us now calculate the root mean squared of the prediction errors
p1=d1-t1*v1 % prediction error vector
rms1=sqrt((2^2+0^2)/2) % rms of prediction errors
```

q1a.m

Variables are named here similar to the MATLAB code for easy correlation. By using the method of least squares using the formula

$$d = v_0 t,$$

I calculate

$$\hat{v_1} = [v_0] = 15.3\text{m/s},$$

which is the same as the result calculated by the anti-Newtonians. I calculate the covariance of the model to be

$$C_{m1} = 2.78.$$

I then calculate the prediction error vector to be

$$p_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

The RMS of the prediction error is

$$RMS_1 = 1.41.$$

They did not get a zero prediction error, as we see in p_1. The root mean square (rms) of the prediction error is also then not 0. The anti-Newtonians have unfortunately not falsified Newtonian mechanics. They only used two data points, which are not sufficient. Their velocity estimate seems so certain because they only used two data points! One can make a lot of formulae work to fit two data points. b) See MATLAB code below for all calculations made.

```
%% Question 1.b)
clear all;
clc; close all;

% i) Calculating for model where d=vt
t2=[0; 1.5; 3.0]; % defining time vector
d2=[2; 12; 46]; % defining distance (data) vector
v2=(inv(t2'*t2))*t2'*d2
Cd2=[5^2 0 0; 0 5^2 0; 0 0 5^2] % covariance of data
Cm2=inv(t2'*inv(Cd2)*t2) %covariance of model
sigma2=sqrt(Cm2) % standard deviation of model
p2=d2-t2*v2 % prediction error vector
rms2=sqrt((2^2+(-8.8)^2+(4.4)^2)/3) % rms of prediction errors



% ii) Calculations for model where d=vt+.5at^2

ta=[0 .5*(t2(1))^2; 1.5 .5*(t2(2))^2; 3.0 .5*(t2(3))^2]  % defining time and acceleration matrix
v22=(inv(ta'*ta))*ta'*d2 % least squares solution to solve for velocity

% Let us now calculate the covariance of the model
Cd22=[5^2 0 0; 0 5^2 0; 0 0 5^2] % covariance of data
Cm22=inv(ta'*inv(Cd22)*ta) %covariance of model

% Standard deviation of velocity
sigma22=[sqrt(Cm22(1)); sqrt(Cm22(4))]

% standard deviation calculation that did not work :(
% sd2=5/(sqrt(0^2+1.5^2+3^2))

p22=d2-ta*v22 % prediction error vector
rms22=sqrt((2^2+0^2+0^2)/3) % rms of prediction errors
```

q1b.m

Variables are named here similar to the MATLAB code for easy correlation.
i) I first calculate v_0 in the least squared sense using the formula, as in (a).

$$d = v_0 t,$$

I calculate

$$\hat{v_2} = [v_0] = 13.9\text{m/s}$$

for these new data. I calculate the covariance of the model to be

$$C_{m2} = 2.22.$$

The standard deviation for our estimate of velocity (v_0) is then $\sqrt{2.22} = 1.49$. The prediction error vector is calculated as

$$p_2 = \begin{bmatrix} 2.0 \\ -8.8 \\ 4.4 \end{bmatrix}$$

and the associated RMS of the prediction error is

$$RMS_2 = 5.80.$$

The least squares model thus does not fit all the data to within the estimated uncertaintity. The second value of the predicted error vector is -8.8, while it is only ± 5 in their uncertainties.

ii) I then calculate v_0 and a in the least squares sense using the formula

$$d = v_0 t + \frac{1}{2} a t^2$$

resulting in

$$\hat{v_{22}} = \begin{bmatrix} v_0 \\ a \end{bmatrix},$$
$$\hat{v_{22}} = \begin{bmatrix} 0.67\text{m/s} \\ 9.8\text{m/s}^2 \end{bmatrix}.$$

The covariance model matrix is calculated to be

$$C_{m22} = \begin{bmatrix} 47.2 & -33.3 \\ -33.3 & 24.7 \end{bmatrix}.$$

The standard deviation for our estimate of velocity (v_0) is then $\sqrt{47.2} = 6.87$ and the standard deviation for our estimate of the acceleration (a) is then $\sqrt{24.7} = 4.97$. The prediction error vector is calculated as

$$p_{22} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

and the associated RMS of the prediction error is

$$RMS_{22} = 1.15.$$

I can be more certain that my analysis (ii) is more sound than the original analysis (i) because both my prediction error vector and RMS values are lower (cf. p_{22} to p_2; and RMS_{22} to RMS_2, indicating that my model which includes acceleration is a better fit.

2. a) i. "Additional equations are added to the original linear problem that set each of the model parameters to zero" is true of method C. We add another row after G, εI, and we add another row after our data vector, 0. If we were to multiply this out, we would get $\varepsilon \mathrm{I}(\mathrm{m}) = 0$, effectively setting the model parameters to zero. If we know certain

ii. "An additional term is added to the misfit function to penalize large model norms" is true of method A.

iii. "A small number is added to the diagonal elements of $\mathrm{G^T G}$ to stabilize its inverse" is true of method B. By adding a small white noise factor multiplied by the identity matrix we avoid having $det(\mathrm{G^T G}) = 0$. We would find that $det(\mathrm{G^T G} + \varepsilon^2 \mathrm{I}) \neq 0$, which is what we want in order to have an inverse that exists.

b) If we write the damped least squares as

$$\mathrm{m} \;=\; \left[\mathrm{F^T F}\right]^{-1} \mathrm{F^T h},$$

where

$$\mathrm{F} \;=\; \begin{bmatrix} \mathrm{G} \\ \varepsilon \mathrm{I} \end{bmatrix}$$

where I is the $M \times M$ identity matrix, and

$$\mathrm{h} \;=\; \begin{bmatrix} \mathrm{d} \\ 0 \end{bmatrix},$$

where the 0 matrix is $M \times M$, then

$$\mathrm{Fm} \;=\; \mathrm{h}$$

has $N+M$ rows (data), and M columns (or model parameters). The new problem is thus overdetermined, as we have more data than model parameters. The extra rows contain information about (εIm), the model norm. As we adjust ε, we adjust the weighting we put on these extra rows. **[All of the equations in the new problem will be satisfied by the solution.]?>????**

c) 10% bonus mark: see attached calculations.