GP 9505A Problem Set #3: Fourier Transforms
Completed by Benjamin Consolvo
Submitted on 8 October 2015

1.

$$
\begin{aligned}
a_k &= (0, -1, 1, 0) \\
A_n &= \frac{1}{N}\sum_{k=0}^{N-1} a_k e^{i2\pi nk/N}
\end{aligned}
$$

$$
\begin{aligned}
A_0 &= \frac{1}{4}\left[0 - 1e^{i2\pi(0)(1)/4} + 1e^{i2\pi(0)(2)/4} + 0\right] \\
A_0 &= \frac{1}{4}\left[-1 + 1\right] \\
A_0 &= 0
\end{aligned}
$$

$$
\begin{aligned}
A_1 &= \frac{1}{4}\left[0 - 1e^{i2\pi(1)(1)/4} + 1e^{i2\pi(1)(2)/4} + 0\right] \\
A_1 &= \frac{1}{4}\left[-e^{i\pi/2} + e^{i\pi}\right] \\
A_1 &= \frac{1}{4}\left[-(cos(\tfrac{\pi}{2}) + isin(\tfrac{\pi}{2})) + (cos(\pi) + isin(\pi))\right] \\
A_1 &= \frac{1}{4}\left[-1 - i\right]
\end{aligned}
$$

$$
\begin{aligned}
A_2 &= \frac{1}{4}\left[0 - 1e^{i2\pi(2)(1)/4} + 1e^{i2\pi(2)(2)/4} + 0\right] \\
A_2 &= \frac{1}{4}\left[-e^{i\pi} + e^{i2\pi}\right] \\
A_2 &= \frac{1}{4}\left[2\right] \\
A_2 &= \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
A_3 &= \frac{1}{4}\left[0 - 1e^{i2\pi(3)(1)/4} + 1e^{i2\pi(3)(2)/4} + 0\right] \\
A_3 &= \frac{1}{4}\left[-e^{i3\pi/2} + e^{i3\pi}\right] \\
A_3 &= \frac{1}{4}\left[-1 + i\right] \\
A_n &= \left[0, \frac{1}{4}(-1-i), \frac{1}{2}, \frac{1}{4}(-1+i)\right]
\end{aligned}
$$

Using the Gubbins' sign convention, we would simply get the complex conjugate of each value in A_n:

$$
A_n^* = \left[0, \frac{1}{4}(-1+i), \frac{1}{2}, \frac{1}{4}(-1-i)\right]
$$

a)

$$\begin{aligned}
A_n &= \left[0, \frac{1}{4}(-1-i), \frac{1}{2}, \frac{1}{4}(-1+i)\right] \\
\text{Show } A_{N-n} &= A_n^* \text{ for our results} \\
N &= 4
\end{aligned}$$

$$\begin{aligned}
A_{4-3} &= A_3^* \\
A_1 &= A_3^* \\
\frac{1}{4}(-1-i) &= \left[\frac{1}{4}(-1+i)\right]^* \\
\frac{1}{4}(-1-i) &= \frac{1}{4}(-1-i)
\end{aligned}$$

$$\begin{aligned}
A_{4-2} &= A_2^* \\
A_2 &= A_2^* \\
\frac{1}{2} &= \left[\frac{1}{2}\right]^* \\
\frac{1}{2} &= \frac{1}{2}
\end{aligned}$$

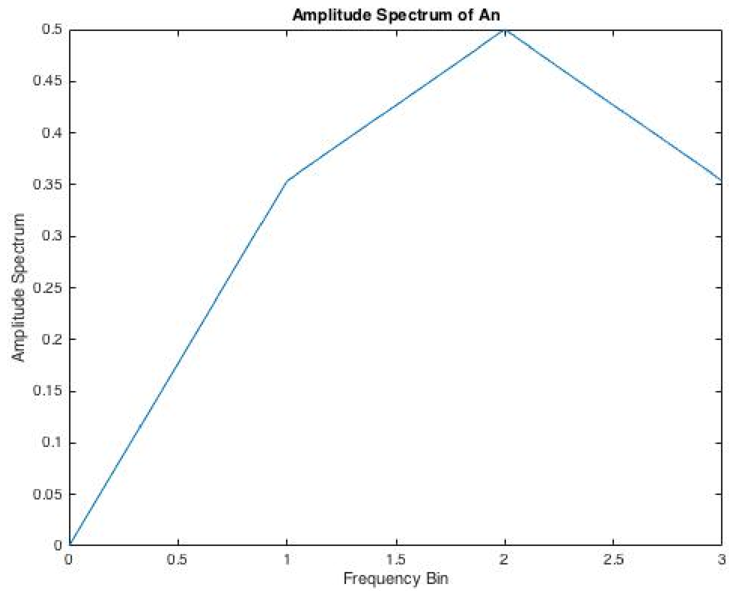
$$\begin{aligned}
A_{4-1} &= A_1^* \\
A_3 &= A_1^* \\
\frac{1}{4}(-1+i) &= \left[\frac{1}{4}(-1-i)\right]^* \\
\frac{1}{4}(-1+i) &= \frac{1}{4}(-1-i)
\end{aligned}$$

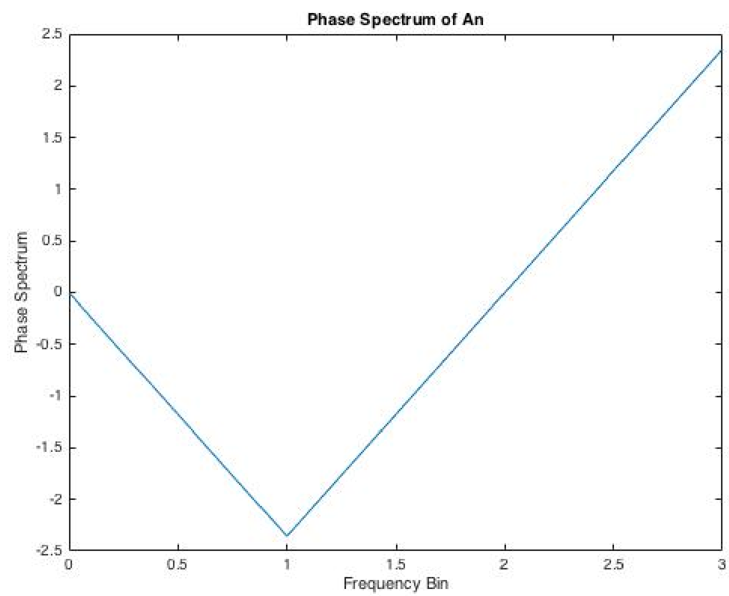
b)

$$\begin{aligned}
\omega_{Ny} &= \frac{\pi}{\Delta t} \\
\Delta w &= \frac{2\pi}{N\Delta t} \\
\frac{\omega_{Ny}}{\Delta w} &= \frac{\pi/\Delta t}{2\pi/(N\Delta t)} \\
\frac{\omega_{Ny}}{\Delta w} &= \frac{N}{2}
\end{aligned}$$

The $\frac{N}{2}$ sample in the Fourier transform corresponds to the "Nyquist" frequency. Both the Nyquest frequency component and the zero frequency component are real, and are their own complex conjugates.

c) The amplitude and phase spectra for A_n are shown below.





See accompanying MATLAB code in the Appendix for how plots created. By looking at the amplitude spectrum, it is clear that the time series, A_n, is symmetric about the Nyquist frequency, with complex conjugates on either side. Recall that

$$A_n \quad = \quad \left[0, \frac{1}{4}(-1-i), \frac{1}{2}, \frac{1}{4}(-1+i)\right]$$

In the case of the phase spectrum, A_n is symmetric about frequency = 1.

2. a) Consider an arbitary z-transform polynomial, $A(z)$, and let $z = exp(i\omega\Delta t)$:

$$\begin{aligned}
A(z) &= a_0\,(z)^0 + a_1\,(z)^1 + a_2\,(z)^2 + ... \\
A_\omega &= a_0\left(e^{(i\omega\Delta t)}\right)^0 + a_1\left(e^{(i\omega\Delta t)}\right)^1 + a_2\left(e^{(i\omega\Delta t)}\right)^2 + ... \\
A_\omega &= a_0 e^0 + a_1 e^{i\omega(1)\Delta t} + a_2 e^{i\omega(2)\Delta t} + ... \\
\\
A_\omega &= \frac{1}{N}\sum_{k=0}^{N-1} a_k e^{i\omega(k)\Delta t}
\end{aligned}$$

where $\frac{1}{N}$ is a normalizing factor. The property of the original time series that causes the periodicity in frequency is that our samples are taken at equally spaced time intervals. When we introduce Δt, we introduce periodicity.

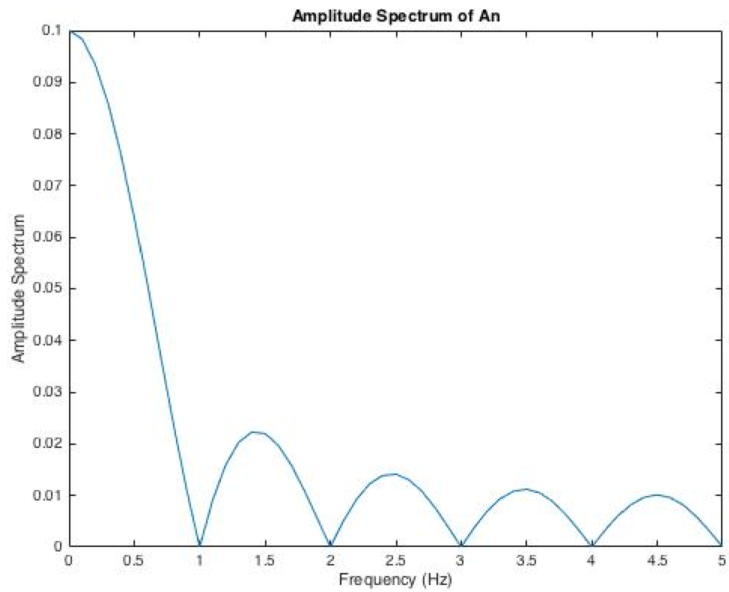
b) The principle of duality states that the discrete Fourier transform (DFT) of a sinusoid is a spike, and the DFT of a spike is a sinusoid.

$$\begin{aligned}
\omega_n &= n\Delta\omega \\
\omega_n &= n\frac{2\pi}{T} \\
\omega_n &= \frac{n2\pi}{N\Delta t}
\end{aligned}$$

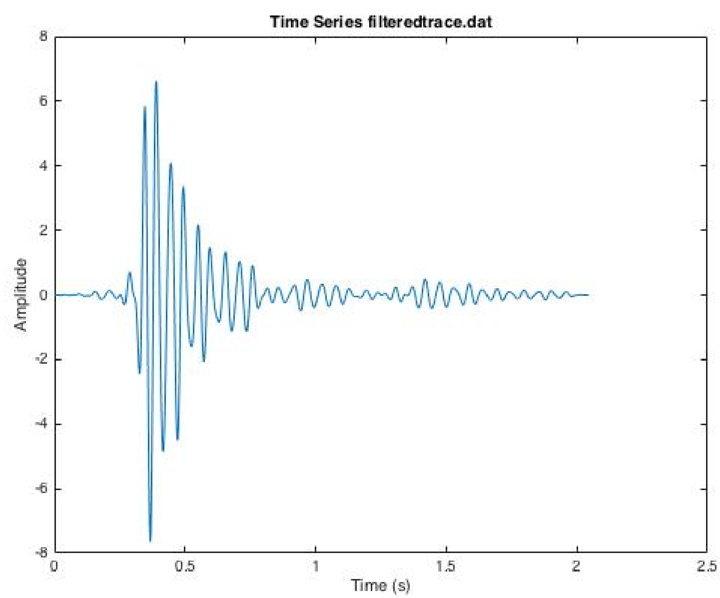
We are assuming that the original time series is either a sinusoid or a spike.

3. a) See MATLAB code in Appendix for all work behind the graphs that follow.

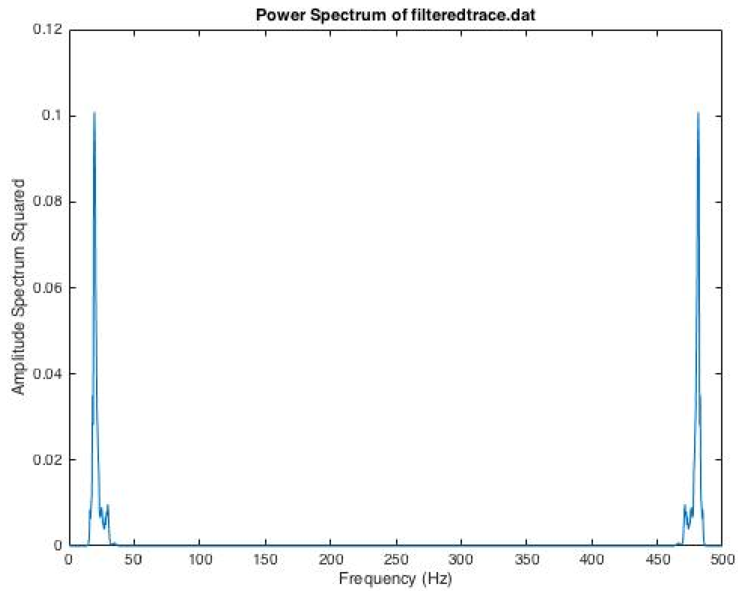
   b) The plot below depicts the amplitude spectrum of the Boxcar function. I have only plotted half of the amplitude spectrum here, as the second half is a mirror image of the plot.



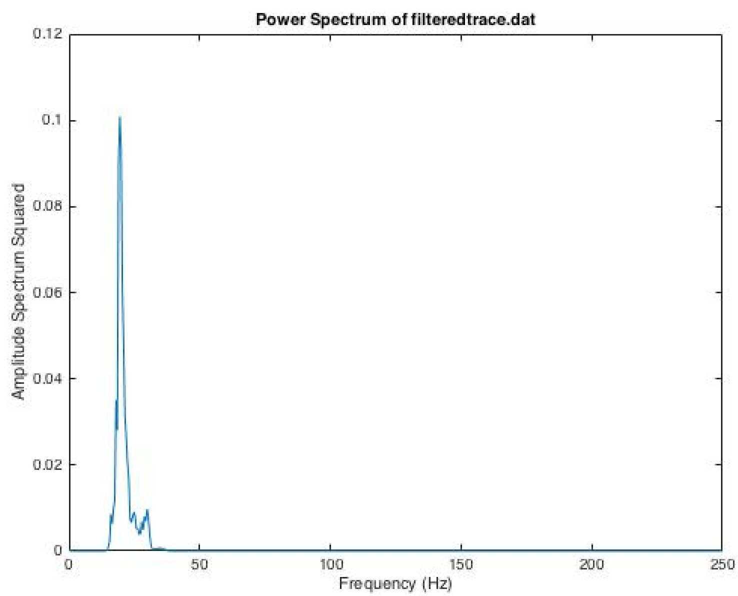
   c) The plot below depicts the time series "filteredtrace.dat":



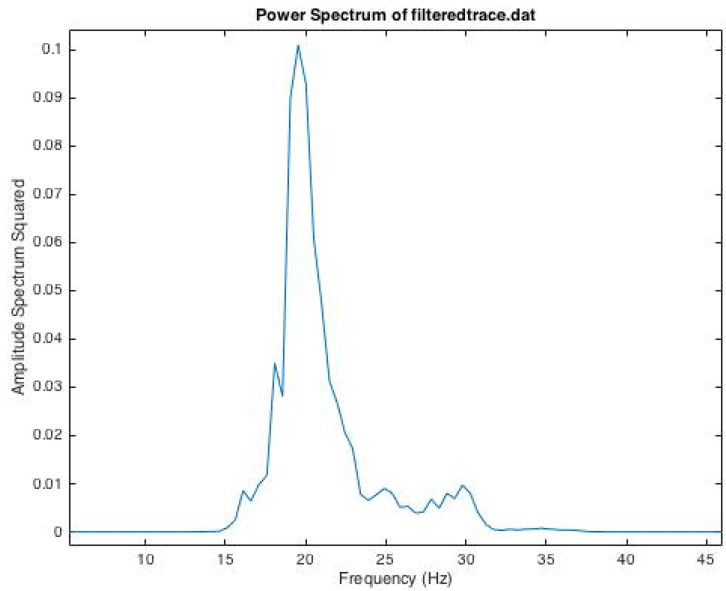
The plot below depicts the power spectrum of the DFT of "filteredtrace.dat":



The power spectrum is clearly a mirror image about 250 Hz.



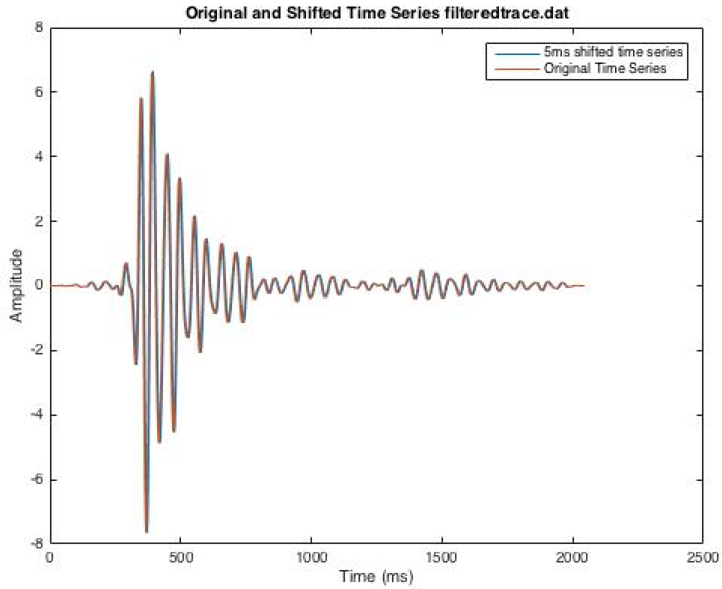
Here is a zoom-in on the spectrum for better detail:

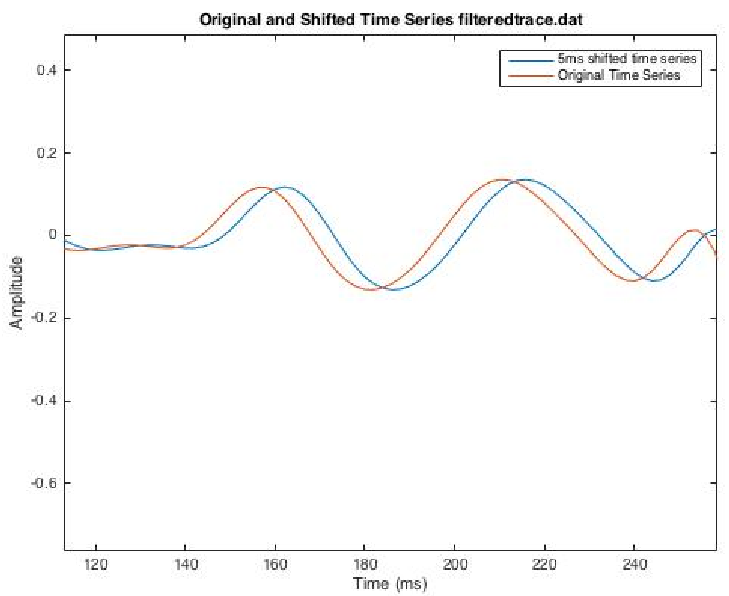


d) By looking at the power spectrum, we can see that the range of frequencies used in the filter is 14 Hz - 37 Hz, and 463 Hz - 486 Hz.

e) BONUS: Apply a phase shift in the frequency domain to advance the seismic trace 5 ms in time, then carry out the inverse DFT. Produce plots to verify the time shift.

In order to advance the seismic trace by 5 ms, our n in the shift must be 2.5 in the frequency domain. This causes a problem, as when we carry out the inverse DFT, we get complex values. To fix this, we need to adjust our index to [1 2 3 ... 1024] to [1 2 3 ... 511 512 -512 -511 ... -3 -2 -1] to create a Hermite-symmetric complex exponential. The resulting plot is shown, as well as a zoomed in portion:


Original and Shifted Time Series filteredtrace.dat
5ms shifted time series
Original Time Series
Amplitude
Time (ms)
8
6
4
2
0
-2
-4
-6
-8
0
500
1000
1500
2000
2500


Original and Shifted Time Series filteredtrace.dat
5ms shifted time series
Original Time Series
Amplitude
Time (ms)
0.4
0.2
0
-0.2
-0.4
-0.6
120
140
160
180
200
220
240