LIÈGE université
Sciences Appliquées

Densys immersive week

Introduction to mixed-integer programming

Bertrand Cornélusse

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Overview

1. Integer and Mixed-Integer programming

2. Modeling techniques

3. Cutting planes

4. Branch and bound

5. Hands-on session C
Distribution network expansion planning

Lecture overview

- The goal of this lecture is to introduce the concept of mixed-integer programming, that is, when we have discrete decision variables in addition to variables that can take continuous values.
- We also quickly review solution methods for these problems.
- These tools can then be used for network design, approximation of non-linear functions, etc.

The lecture is based on slides from Pr. Quentin Louveaux at University of Liège.

Integer and Mixed-Integer programming

Modeling a discrete problem I

Consider the problem

$$
\begin{aligned}
\min\ & c(x) \\
\text{s.t. } & f(x) \leq b \\
& g(x) = 0 \\
& x \in X.
\end{aligned}
$$

When

- c, f, g are **linear**
- $X = \mathbb{Z}_+^n$

Modeling a discrete problem II

This is called **Integer (Linear) Programming** (IP).

Remarks:

- **Mixed Integer Programming** (MIP) when some variables have a continuous domain.
- If c is **nonlinear**, very little has been done (except the quadratic case)
- If f or g is nonlinear: even less

Why is that so complicated?

- After all, there is a **finite number of solutions**
- In particular, $n!$ possible permutations

- Imagine we can check 10^{12} possibilities per second
  That is already a pretty amazing machine!

10!	0 sec
20!	28 days
30!	8400 billion years
40!	5 quadrillions times the age of the Earth…

- Let us not dare to continue…

Example: Uncapacitated Lot Sizing (ULS)

You are producing bikes and you know in advance the demand d_t for T time steps ahead. Producing at time t induces a fixed cost f_t, and the variable cost per bike produced is c_t. There is no storage cost. Formulate the MIP that allows you to compute the production plan that minimizes the total production cost to satisfy the demand.

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Formulation as a MIP:

$$
\begin{aligned}
\min_{x,y} \; & \sum_{t=1}^{T} f_t x_t + \sum_{t=1}^{T} c_t y_t \\
s.t. \; & \sum_{u=1}^{t} y_u \geq \sum_{u=1}^{t} d_u, \; \forall t \in \mathcal{T} \\
& y_t \leq \left(\sum_{u=t}^{T} d_u \right) x_t, \; \forall t \in \mathcal{T} \\
& y_t \geq 0, \; \forall t \in \mathcal{T} \\
& x_t \in \{0, 1\}, \; \forall t \in \mathcal{T}
\end{aligned}
$$

Modeling techniques

Binary choice

A **choice** between 2 alternatives is modeled through a 0, 1-variable.

Example: the knapsack problem

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{n} c_i x_i \\ \text{subject to} \quad & \sum_{i=1}^{n} a_i x_i \leq b \\ & x_i \in \{0, 1\} \text{ for all } i = 1, \ldots, n. \end{aligned}$$

Forcing constraints I

If decision A is made **then** decision B must be made as well.

$x = 1$	if decision A is taken	$y = 1$	if decision B is taken
$x = 0$	otherwise	$y = 0$	otherwise

The constraint reads

$$x \leq y$$

Forcing constraints II

Example: Facility Location problem

- m clients ($i = 1, 2, \ldots, m$) to satisfy (demand $= 1$)
- n **potential** locations for facilities ($j = 1, 2, \ldots, n$)
- Can serve client i from facility j only if facility j is open:

$$x_{ij} \leq y_j$$

  - x_{ij} fraction of demand of client i served by facility j
  - $y_j \in \{0, 1\}$, 1 if facility is open.

Restricted range of values

Suppose we want to formulate $x \in \{a_1, a_2, \ldots, a_m\}$.

We introduce m **binary variables** y_j.

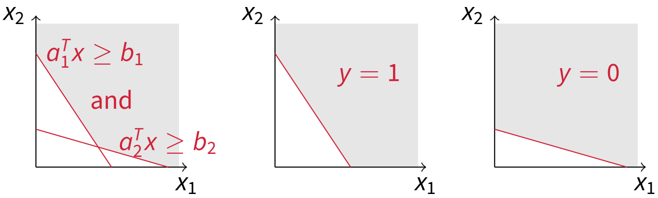
$$x = \sum_{j=1}^{m} a_j y_j, \quad \sum_{j=1}^{m} y_j = 1, \quad y_j \in \{0, 1\}$$

Disjunctive constraints

- Consider a variable $x \geq 0$,
- we want to model that **either** $a_1^T x \geq b_1$ **or** $a_2^T x \geq b_2$,
- and $a_1 \geq 0, a_2 \geq 0$.

We introduce a variable $y \in \{0, 1\}$ that represents whether **constraint 1** ($y = 1$) or **constraint 2** is satisfied, and replace both constraints by

$$a_1^T x \geq y b_1 \quad \text{and} \quad a_2^T x \geq (1 - y) b_2.$$



Disjunctive constraints (...) I

- Now consider $0 \leq x \leq U$,
- we want to express either $a_1^T x \leq b_1$ or $a_2^T x \leq b_2$,
- without restriction on a_1 nor a_2.

Again, introduce variable $y \in \{0, 1\}$ and parameter M defined as

$$M = \max_{m, 0 \leq x \leq U} \left\{ m : m \geq a_i^T x - b_i, \quad i = 1, 2 \right\},$$

then

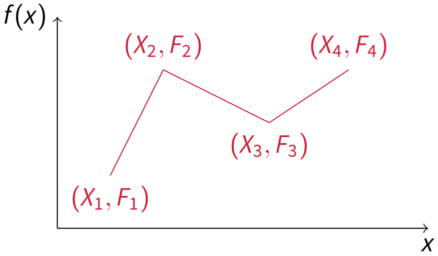
$$a_1^T x - b_1 \leq My \text{ and } a_2^T x - b_2 \leq M(1 - y).$$

Disjunctive constraints (...) II

Example: scheduling problem

- Two tasks, starting time $t_1, t_2 \geq 0$, duration $P_1, P_2 \geq 0$
- either task 1 is performed before task 2, or the opposite
- hence either $t_1 \geq t_2 + P_2$, or $t_2 \geq t_1 + P_1$

Arbitrary piecewise linear cost functions



Introduce variables $b_i \in \{0, 1\}$ such that

$$b_i = 1 \quad \text{if } x \in [X_i, X_{i+1}]$$
$$b_i = 0 \quad \text{if } x \notin [X_i, X_{i+1}]$$

Formulation 1:

- $\sum_i b_i = 1$
- $x_i \leq b_i$
- $f = \sum_i (F_i b_i + x_i(F_{i+1} - F_i))$

Formulation 2:

- $\sum_i b_i = 1$
- $\lambda_i \leq b_{i-1} + b_i$, $i = 2, \ldots, n-1$
- $\lambda_1 \leq b_1, \lambda_n \leq b_{n-1}$
- $\sum_i \lambda_i = 1$

Cutting planes

The linear (continuous) relaxation I

Definition

Given the Mixed Interger Program:

$$\begin{aligned} z_{MIP} = \min\ & c^T x + d^T y \\ \text{s.t.}\ & Ax + By = b \\ & x, y \geq 0 \\ & y \in \mathbb{R}^n \\ & x \in \mathbb{Z}^n, \end{aligned}$$

its linear relaxation is defined as

$$\begin{aligned} z_{LP} = \min\ & c^T x + d^T y \\ \text{s.t.}\ & Ax + By = b \\ & x, y \geq 0 \\ & y \in \mathbb{R}^n \\ & x \in \mathbb{R}^n. \end{aligned}$$

The linear (continuous) relaxation II

- The linear relaxation gives important information about the optimal value of an integer program:

$$z_{LP} \leq z_{MIP},$$

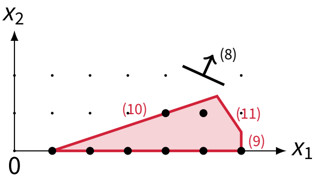
- hence, it is **easy** to obtain a lower bound (solving the relaxation is “easy"),
- but in general **hard** to obtain an integer solution from the solution of the relaxation without elaborated techniques.

Relaxation strength

- Alternative formulations of a problem may lead to different linear relaxations.
- If the formulation is ideal, that is, the LP relaxation defines the convex hull of the feasible set of the Integer Program, we need nothing else than Linear Programming algorithms. This often requires an exponential number of constraints.
- Here, we consider a different approach: **automatically** derive **valid inequalities** from the original constraints of the model in order to approximate the convex hull of the feasible points of the IP.

Illustration

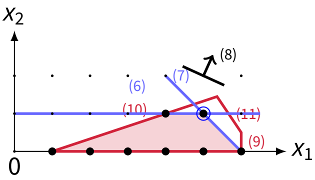
$$
\begin{aligned}
\max \quad & 5x_1 + 11x_2 && (1)\\
\text{s.t.} \quad & x_1 \leq 6 && (2)\\
& x_1 - 3x_2 \geq 1 && (3)\\
& 3x_1 + 2x_2 \leq 19 && (4)\\
& x_1, x_2 \in \mathbb{Z}_+ && (5)
\end{aligned}
$$



Illustration

$$\frac{(9)-(10)}{3} \text{ and } (12) \Rightarrow x_2 \leq 1 \tag{6}$$

$$\frac{(6)+(11)}{3} \text{ and } (12) \Rightarrow x_1+x_2 \leq 6 \tag{7}$$



Valid inequalities

Definition

Let $P \subseteq \mathbb{R}^n$. An inequality $\sum_{j=1}^{n} a_j x_j \leq b$ is **valid** if it is satisfied by all points $x \in P$.

Typically,

- we want to derive valid inequalities for the set of **integral solutions**
- and at the same time **cut off** some part of the **linear programming relaxation**.

Branch and bound

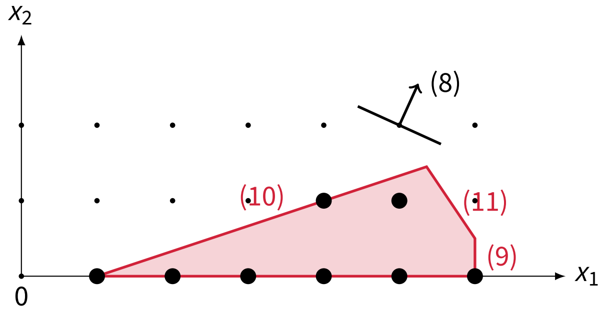
Introduction

- Here, we see an algorithm for solving to optimality an Integer Program when its formulation is not ideal: **branch-and-bound**.
- Other algorithms such as cutting-planes, are almost always used in conjunction with branch-and-bound (leading to the well known branch-and-cut algorithm).

Consider the following problem

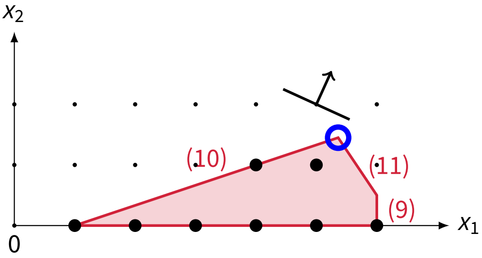
$$
\begin{aligned}
\max \quad & 5x_1 + 11x_2 && (8)\\
\text{s.t.} \quad & x_1 \leq 6 && (9)\\
& x_1 - 3x_2 \geq 1 && (10)\\
& 3x_1 + 2x_2 \leq 19 && (11)\\
& x_1, x_2 \in \mathbb{Z}_+ && (12)
\end{aligned}
$$

Geometrical view



What information does the LP relaxation yield?

- Objective: $z^{\star,0} \approx 42.82$
- Solution: $x^{\star,0} \approx (5.36, 1.45)$



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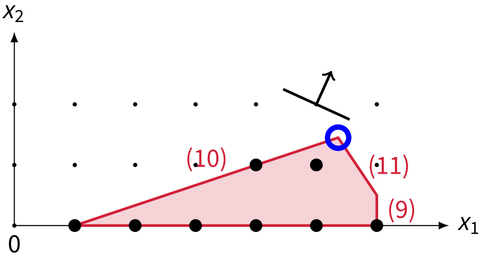
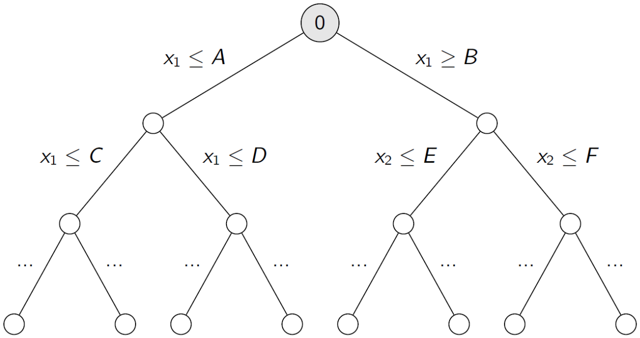


- Idea: enumerate, i.e. iteratively restrict the domain of x, but **using the information** of the linear relaxation.
- The enumeration yields a search tree.

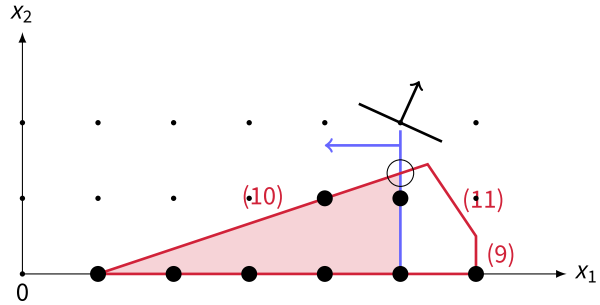
Use information of the relaxation to ...

- decide on which variables to branch
- set the thresholds
- prune parts of the search tree
- ...



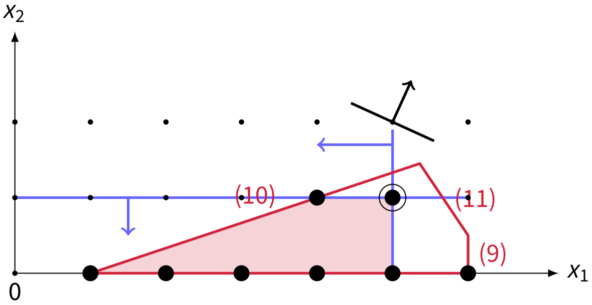
Back to our example, branch on x_1: $x_1 \leq 5$ (Node 1)

- $z^{\star,1} \approx 39.67$
- $x^{\star,1} = (5, 4/3)$



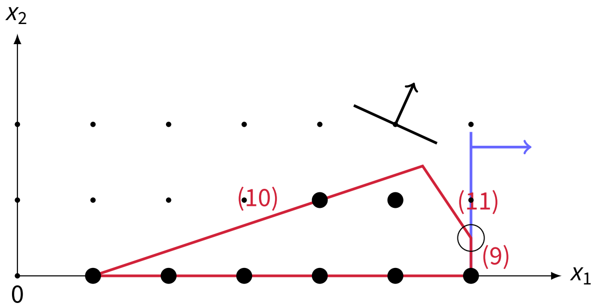
Node 2: from node 1, branch on x_2: $x_2 \leq 1$

- $z^{\star,2} = 36$
- $x^{\star,2} = (5, 1)$

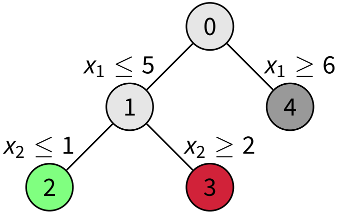


Node 4: the second alternative from the root node: $x_1 \geq 6$

- $z^{\star,4} = 35.5$
- $x^{\star,4} = (6, 1/2)$



Branch and bound tree



Remark: node index $=$ order of exploration ! $=$ order of creation.

Node	Nodes left	Objective	Nb integer infeasible variables	Best Integer	Best Bound	Gap	Decision
0	2	42,82	2	/	42,82	Infinite	Branch
1	2	39,67	1	/	42,82	Infinite	Branch
2	2	36	0	36	42,82	15,93%	prune by optimality
3	1	"-infinity"	/	36	42,82	15,93%	prune by infeasibility
4	0	35,5	1	36	35,5	0,00%	prune by bound

Remarks

- Opportunities to **prune** the search:
  - by bound,
  - by optimality,
  - by infeasibility
- Need of a good **primal bound** in the beginning
- Different strategies for the **node selection**:
  - depth-first-search (good to find quickly primal solutions)
  - breadth-first-search (good to increase the **dual bound**)
- Different strategies for **variable selection**:
  - Most fractional variable or least fractional variable
  - Take advantage of the **history of branching**
  - Look ahead for best improvement in the bound: **strong branching**

Hands-on session C

Distribution network expansion planning

DNEP