

Analysis of electric power and energy systems

Three-phase systems in sinusoidal steady state, per-unit analysis

Bertrand Cornélusse
bertrand.cornelusse@uliege.be

What will we learn today?

Mostly from Chapter 2 of Ned Mohan's book.

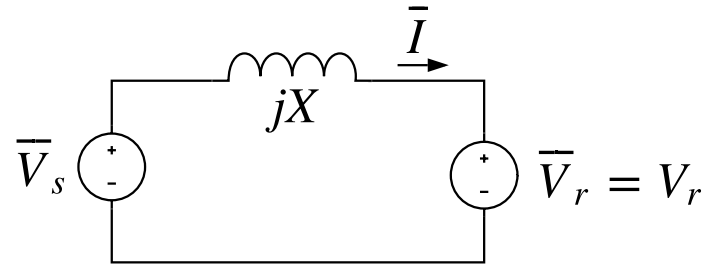
- Power transfer between AC systems
- 3-phase systems
- Per unit normalization

You will be able to do exercises 2.1, 2.2, 2.4, 2.5, 2.9, 2.11, 2.12, 2.14, 2.16, 2.17, 2.18, 2.19 and 2.20 from the Ned Mohan's book.

Power transfer between AC systems

Consider the following simple system

We have $\bar{I} = \frac{\bar{V}_s - \bar{V}_r}{jX}$



Let δ be the angle between \bar{V}_r and \bar{V}_s , then

$$\begin{aligned} S_r &= \bar{V}_r \bar{I}^* = V_r \left(\frac{V_s \angle -\delta - V_r}{-jX} \right) \\ &= \frac{V_s V_r \sin \delta}{X} + j \frac{V_s V_r \cos \delta - V_r^2}{X} \end{aligned}$$

Let's remember two things:

- The **active** power is highly sensitive to δ
- The **reactive** power acts on the **voltage magnitude** (look at what happens for $\delta = 0$)

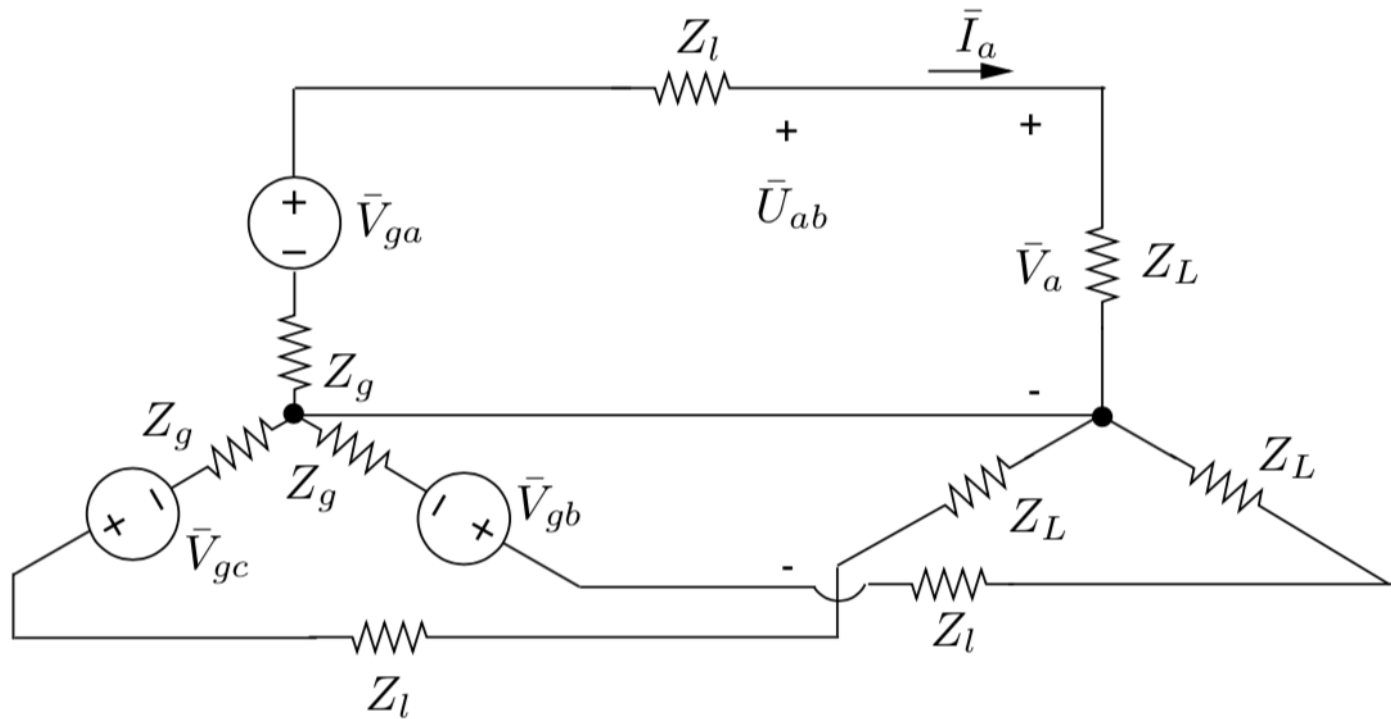
Demo

See

<https://colab.research.google.com/drive/1wrJYI082Y6qE6TyaT5a7eWO1IB0CCWiu?usp=sharing>

Three-phase systems

Three-phase system



Generation -> transmission -> load

Here the load is connected as a [star](#). A neutral point is present.

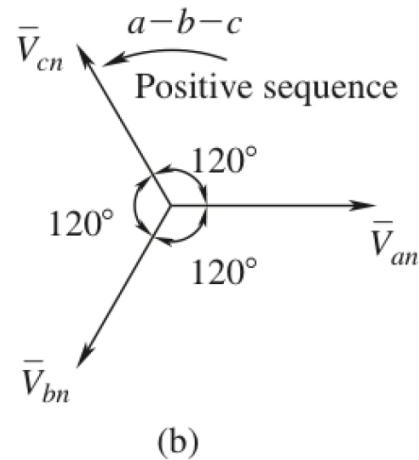
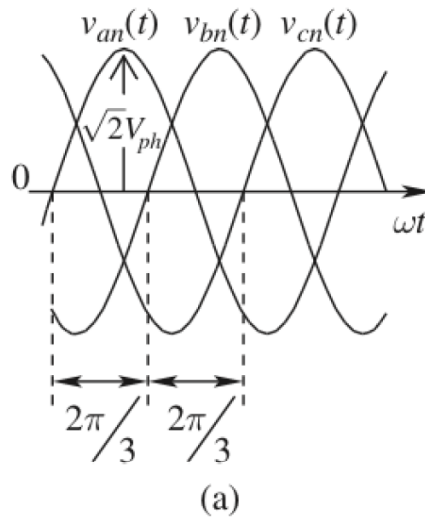
The neutral conductor is not necessarily implemented.

By design the voltage sources are shifted by 120° :

$$\bar{V}_{ga} = V e^{j\phi_u}$$

$$\bar{V}_{gb} = V e^{j(\phi_u - 2\pi/3)} = \bar{V}_{ga} e^{-j2\pi/3}$$

$$\bar{V}_{gc} = V e^{j(\phi_u - 4\pi/3)} = \bar{V}_{ga} e^{-j4\pi/3}$$



Three-phase voltages in time and phasor domain.

Phase voltage vs line to line voltage

These voltages represent the **phase voltages**. If we now look at the **line to line voltages**:

$$\bar{U}_{ab} = \bar{V}_{ga} - \bar{V}_{gb} = \sqrt{3}\bar{V}_{ga}e^{j\pi/6}$$

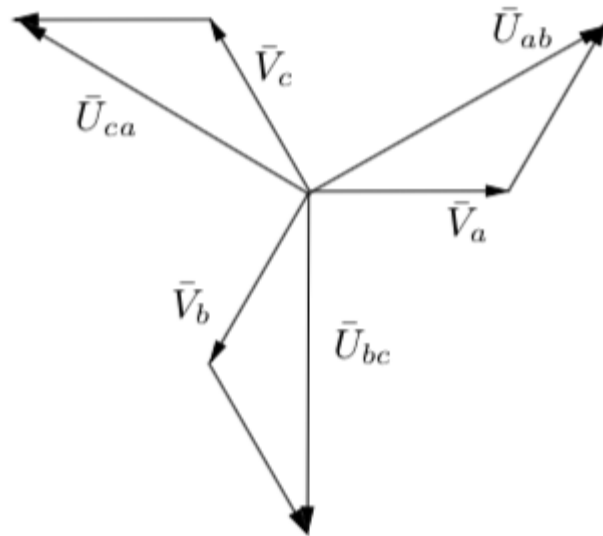
and similarly for \bar{U}_{bc} and \bar{U}_{ca} .

Total power

The total complex power transmitted to the load is

$$S = \sum_{k \in \{a,b,c\}} \bar{V}_{gk} \bar{I}_k^*$$

Hence in a balanced system the total active power to the load is $3VI \cos \phi$, with 3 or 4 wires (instead of 2 in a single-phase system).



Line vs. phase voltages

Example:

My house is fed by a 400V three-phase system. This means the line voltages are 400V (rms), and thus phase voltages are 230V (approximately). Typically, the phase voltages are distributed independently in the house, each with the neutral.

Comments

In every phase there is a current flowing. In a balanced system, currents are phase shifted by 120° and sum to zero. Thus the neutral can in theory be removed. This is done in some portions of the global system (typically at high voltage), where neutral points are grounded.

Some loads can also be connected in "delta", hence the neutral is not accessible.

Finally, in unbalanced systems, currents are dictated by the impedances seen in the different phases. There is no perfectly known relation, a priori.

Exercise: Neutral break

Consider a 3-phase Y-connected resistive circuit with unbalanced resistors in the phases. In normal operation there is a neutral wire. Then the neutral breaks.

- Compute in both cases the voltages across the resistors, the currents and the consumed powers.
- What can you observe?

First, we'll analyze the circuit when the neutral is connected, and then we'll analyze what happens when the neutral wire breaks.

Assumptions:

- 3-phase balanced voltage supply: $V_{ab} = V_{bc} = V_{ca} = V_L$ (line-to-line voltage)
- Phase voltages (line-to-neutral) are given by $V_{an} = V_{bn} = V_{cn} = \frac{V_L}{\sqrt{3}}$, and the angle between them is 120° .
- Unbalanced resistances: R_a, R_b, R_c (different resistances in each phase).

Case 1: Neutral Connected

Voltages:

- In normal operation, with the neutral connected, each resistor has its corresponding phase voltage across it:

$$V_{R_a} = V_{an}, \quad V_{R_b} = V_{bn}, \quad V_{R_c} = V_{cn}$$

Currents:

- The current in each phase is given by Ohm's law:

$$I_a = \frac{V_{an}}{R_a}, \quad I_b = \frac{V_{bn}}{R_b}, \quad I_c = \frac{V_{cn}}{R_c}$$

- The neutral current, I_n , is the sum of the phase currents:

$$I_n = I_a + I_b + I_c$$

Due to the unbalanced resistances, I_n will not be zero.

Powers:

- The power consumed in each phase is:

$$P_a = V_{an} I_a = \frac{V_{an}^2}{R_a}, \quad P_b = \frac{V_{bn}^2}{R_b}, \quad P_c = \frac{V_{cn}^2}{R_c}$$

- Total power consumed:

$$P_{\text{total}} = P_a + P_b + P_c$$

Case 2: Neutral Broken

When the neutral breaks, the three resistors form a system without a direct connection to the neutral point. The current through each resistor still needs to sum to zero because the current has no return path through the neutral. This changes the voltage distribution across the resistors.

Voltages:

- With the neutral broken, the phase voltages no longer remain symmetrical across the resistors. The voltages across each resistor will now depend on the values of R_a , R_b , and R_c . The voltage at the common point (where the three resistors connect) will shift depending on the relative resistances.

To compute the new phase voltages, you can set up Kirchhoff's Current Law (KCL) at the common point (let's call it point N'):

$$V_{aN'} = V_{an} - V_{nN'}, \quad V_{bN'} = V_{bn} - V_{nN'}, \quad V_{cN'} = V_{cn} - V_{nN'}$$

Here, $V_{nN'}$ is an unknown voltage offset at the floating point N' (the shifted neutral). Using KCL, you solve for this voltage by ensuring the sum of the currents at N' is zero:

$$\frac{V_{aN'}}{R_a} + \frac{V_{bN'}}{R_b} + \frac{V_{cN'}}{R_c} = 0$$

Solving this system gives you the new voltages across the resistors.

Currents:

- The currents in each phase are then given by:

$$I_a = \frac{V_{aN'}}{R_a}, \quad I_b = \frac{V_{bN'}}{R_b}, \quad I_c = \frac{V_{cN'}}{R_c}$$

Since the neutral is broken, the sum of these currents must equal zero:

$$I_a + I_b + I_c = 0$$

Powers:

- The power consumed in each phase is:

$$P_a = V_{aN'} I_a = \frac{(V_{aN'})^2}{R_a}, \quad P_b = \frac{(V_{bN'})^2}{R_b}, \quad P_c = \frac{(V_{cN'})^2}{R_c}$$

- Total power consumed:

$$P_{\text{total}} = P_a + P_b + P_c$$

Observations:

- When the neutral is connected, the voltages across the resistors are straightforwardly determined by the phase-to-neutral voltages.
- When the neutral is broken, the voltage distribution becomes more complex, and the voltages across the resistors are no longer the same as the phase-to-neutral voltages. The total current in the system remains zero, and the power distribution can also change depending on the values of the resistors.

[A numerical example here.](#)



Why 3 Phase Power? Why not 6 or 12?



Watch Later

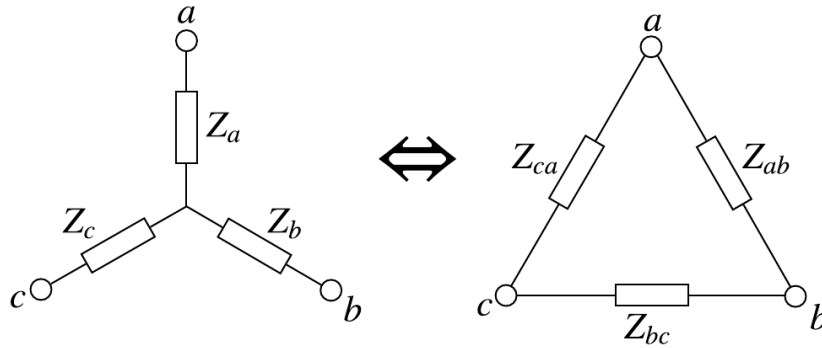


Share



Why not 6 or 12?

Useful formulas: from star to delta connection (and back)



$$Z_a = \frac{Z_{ab} Z_{ca}}{Z_{ab} + Z_{bc} + Z_{ca}}$$

$$Z_b = \frac{Z_{bc} Z_{ab}}{Z_{ab} + Z_{bc} + Z_{ca}}$$

$$Z_c = \frac{Z_{ca} Z_{bc}}{Z_{ab} + Z_{bc} + Z_{ca}}$$

$$Z_{ab} = \frac{Z_a Z_b + Z_b Z_c + Z_c Z_a}{Z_c}$$

$$Z_{bc} = \frac{Z_a Z_b + Z_b Z_c + Z_c Z_a}{Z_a}$$

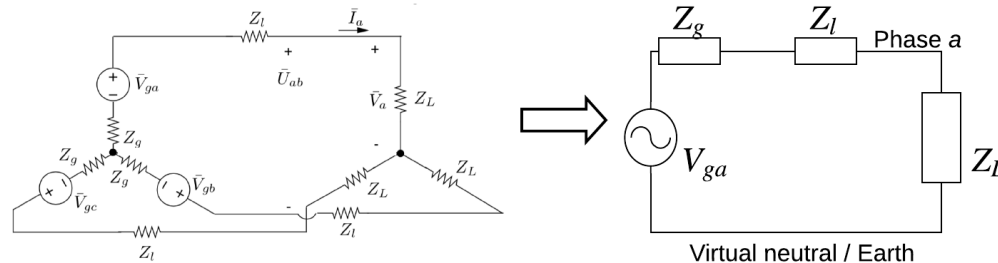
$$Z_{ca} = \frac{Z_a Z_b + Z_b Z_c + Z_c Z_a}{Z_b}$$

What happens in a balanced system?

Per-phase analysis

In a **balanced** system, analyses can be simplified by representing only one phase

- this is straightforward if there are no couplings between phases



- in case there is a coupling, and that for instance the voltage drop \bar{V}_{aA} along a line presenting an impedance Z_{self} traversed by a current \bar{I}_a is also function of the currents in the other phases:

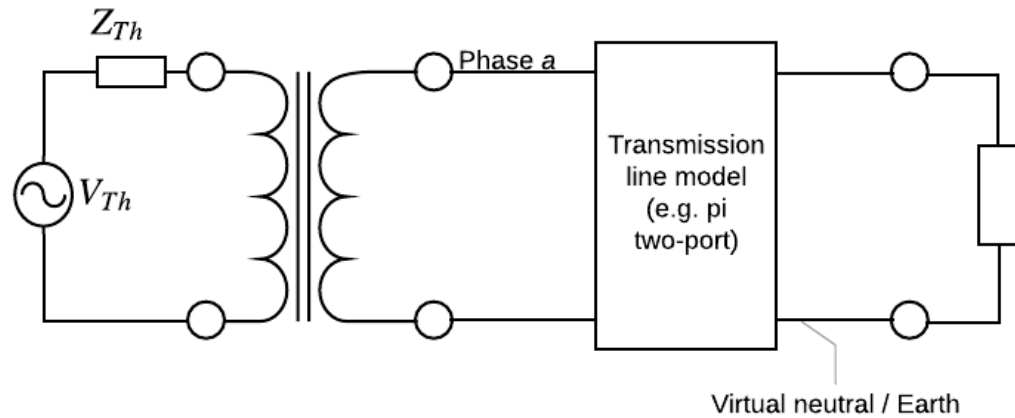
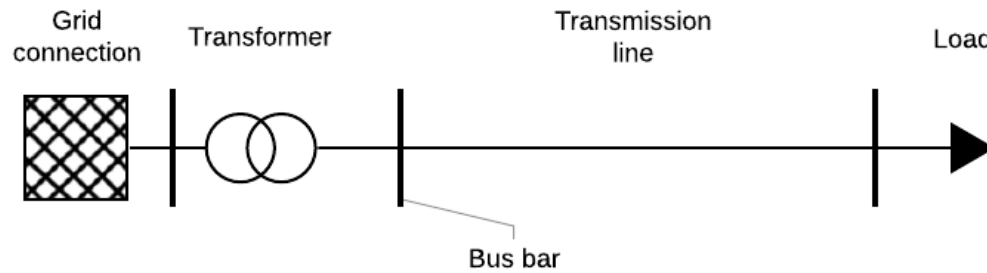
$$\bar{V}_{aA} = Z_{self} \bar{I}_a + Z_{mutual} \bar{I}_b + Z_{mutual} \bar{I}_c$$

then the per-phase equivalent impedance (for phase *a*) is

$$Z_{aA} = Z_{self} - Z_{mutual}$$

since $\bar{I}_a + \bar{I}_b + \bar{I}_c = 0$

One-line diagram



The "per unit" representation

The per unit representation is a way to normalize data

It is used extensively in power systems. It has several advantages:

- Values (e.g. device parameters) stay in a narrow range (whatever their "size")
- Voltages tend to have values close to 1, which eases comparison
- With appropriate "normalization ratios", (ideal) transformers disappear
- Normalization facilitates numerical computation (no huge numbers, no tiny numbers)

Working in per unit in 5 steps

1. Chose 3* "fundamental" base values:
 - V_B for the voltage [kV]
 - S_B for the power [MVA]
 - t_B for the time [s], or ω_B for the angular frequency [rad/s]
2. Derive other needed base values from physical laws, e.g.
 - the base current: $I_B = S_B / V_B$ [A]
 - the base impedance: $Z_B = V_B^2 / S_B$ [Ω]
3. Normalize input data: divide parameters by their base value
4. Make your computations
5. (If necessary) Apply the inverse transformation

Working in per unit in 5 steps

1. Chose 3* "fundamental" base values:
 - V_B for the voltage [kV]
 - S_B for the power [MVA]
 - t_B for the time [s], or ω_B for the angular frequency [rad/s]
2. Derive other needed base values from physical laws, e.g.
 - the base current: $I_B = S_B / V_B$ [A]
 - the base impedance: $Z_B = V_B^2 / S_B$ [Ω]
3. Normalize input data: divide parameters by their base value
4. Make your computations
5. (If necessary) Apply the inverse transformation

Data in per unit is by definition dimensionless; pay attention to base value units w.r.t. to physical value units!

(*) When there are transformers, one base voltage per voltage level is preferable

Example

It is known that the internal reactance of a synchronous machine lies typically in the range $[1.5, 2.5] \text{ pu}$ (on the machine base)

- A machine with the characteristics $(20 \text{ kV}, 300 \text{ MVA})$ has a reactance of 2.667Ω . Is this a normal value?
 - (Here we do not need a base value for time)
 - the base impedance is $Z_B = 20^2/300 = 1.333 \Omega$
 - hence the value of the reactance in per unit is $2.667/1.333 = 2 \text{ pu}$
 - this is a quite normal value!
- Same question for a machine with the characteristics $(15 \text{ kV}, 30 \text{ MVA})$
 - The base impedance is now $Z_B = 15^2/30 = 7.5 \Omega$
 - the value of the reactance in per unit is $2.667/7.5 = 0.356 \text{ pu}$
 - hence an abnormal small value!

Per unit in three-phase systems

Let the base power S_B be the three-phase power, and $U_b = \sqrt{3}V_B$ be the line to line voltage base.

The (single-phase) base current is

$$I_B = \frac{S_B}{3V_B} = \frac{S_B}{\sqrt{3}U_B}$$

The base impedance is

$$Z_B = \frac{V_B}{I_B} = \frac{3V_B^2}{S_B} = \frac{U_B^2}{S_B}$$

In a single phase equivalent representation, the power values in per unit can be multiplied by S_b to get the total three-phase power.

References

- Mohan, Ned. Electric power systems: a first course. John Wiley & Sons, 2012.
- Course notes of ELEC0014 by Pr. Thierry Van Cutsem.

