LIÈGE université
Sciences Appliquées

# Introduction to the power flow analysis

## ELEC0447 - Analysis of electric power and energy systems

Bertrand Cornélusse

September 5, 2025

# Overview

# Introduction

# What is a power flow analysis? I

Power flow (or load flow) analysis is about determining the *electrical state of an electrical power system*, when information about power generated or consumed is available at nodes of the network, and considering that the voltage level is regulated at some buses.

This type of analysis is commonly used by power companies for planning and operation purposes.

- If voltage magnitude and angles were measured at all buses,
  - then it would boil down to solving a set of simple linear equations.
- In a similar way, mesh or nodal analysis could be used if we had a full model of the system,

# What is a power flow analysis? II

  - even without all voltage measurements.
- But here the situation is different, because we mainly have access to *power* measurements.
  - The system is no more linear.

# Power flow problem statement I

Determine **the voltage at every bus**, assuming we have a power system composed of transmission lines connecting the following bus types:

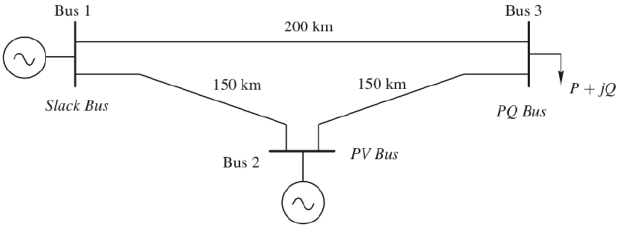
- *PQ buses* are typically loads where active and reactive power are measured
  - it can also be generation where voltage is not regulated (e.g. renewable generation)
- *PV buses* where the active power and the voltage are specified
  - these are typically generators
- one *slack bus* that sets the reference for the voltage magnitudes and angles (it is usually at 1 pu)
  - P and Q can take any value to reach the power balance in the system.

# Power flow problem statement II

Branch currents and losses can be determined from the voltages (magnitudes and phases).

Note: as we will see, PV buses must be swithed to PQ buses in case they reach a limit of their capability curve.

# A first tiny example I



# A first tiny example II

Buses:

- Bus 1 is the slack, with V = 1 pu
- Bus 2 is a PV bus, with V regulated at 1.05 pu and drawing P= 2 pu
- Bus 3 is a PQ bus, consumes P = 5 pu and Q = 1 pu.

Lines:

- X = 0.376 Ohm/km (at 60 Hz)
- R = 0.037 Ohm/km
- Shunt susceptances are ignored (4.5e-6 S/km)

Voltage base (3-phase): 345 kV, Power base (3-phase): 100 MVA

# Result of the tiny example using pandapower I

`https://colab.research.google.com/drive/`
`103VIzly2huoS-PjnYdbCaeBS1dunJS7H?usp=sharing`

| | vm_pu | va_degree | p_mw | q_mvar |
|---|---|---|---|---|
| 0 | 1.00 | 0.00 | -308.38 | 81.61 |
| 1 | 1.05 | -2.07 | -200.00 | -266.74 |
| 2 | 0.98 | -8.79 | 500.00 | 100.00 |

Are there losses?

# Result of the tiny example using pandapower II

**Results for the lines:**

| | p_from_mw | q_from_mvar | p_to_mw | q_to_mvar | pl_mw | ql_mvar | i_from_k |
|---|---|---|---|---|---|---|---|
| 0 | 68.99 | -110.87 | -68.20 | 118.95 | 0.80 | 8.08 | 0.2 |
| 1 | 268.20 | 147.79 | -264.23 | -107.49 | 3.97 | 40.30 | 0.4 |
| 2 | -235.77 | 7.49 | 239.38 | 29.26 | 3.62 | 36.75 | 0.4 |

# The power flow equations

# The power flow equations I

- Let $\mathcal{N}$ be the set of buses of the network
- Some buses are interconnected by transmission lines, given by their $\pi$ models
- Let $Y_{kG}$ be the sum of admittances connected between node $k$ and the ground:
  - the shunt admittances of the lines incident to $k$, and the admittances of the devices connected at node $k$ if any.
- For two nodes $k$ and $m$, let $Z_{km}$ be the series impedance of the line connecting them and $Y_{km} = Z_{km}^{-1}$ ($Y_{km} = 0$ if there is no line)

# The power flow equations II

The current injection at node $k$ is

$$\bar{I}_k = Y_{kG}\bar{V}_k + \sum_{m \in \mathcal{N} \setminus k} (\bar{V}_k - \bar{V}_m) Y_{km} \tag{1}$$

This last equation can be rewritten as

$$\bar{I}_k = \left( Y_{kG} + \sum_{m \in \mathcal{N} \setminus k} Y_{km} \right) \bar{V}_k - \sum_{m \in \mathcal{N} \setminus k} Y_{km} \bar{V}_m$$

# The power flow equations III

which highlights the possibility to write in matrix form

$$\bar{\mathbf{I}} = \mathbf{Y}\bar{\mathbf{V}} \tag{2}$$

with $\bar{\mathbf{I}}$ and $\bar{\mathbf{V}}$ the vectors of bus current injections and bus voltages, respectively.

The *admittance matrix* $\mathbf{Y}$ can be determined by inspection:

- Element $y_{kk}$ is the sum of the admittances incident to bus $k$

# The power flow equations IV

- Element $y_{km}, m \neq k$, is the opposite of the sum of the admittances connecting bus $k$ to bus $m$

But remember that we have power measurements only (and voltage magnitudes at a few buses). So we can derive

$$\mathbf{P} + j\mathbf{Q} = \bar{\mathbf{V}} \circ \bar{\mathbf{I}}^{\star} = \bar{\mathbf{V}} \circ \mathbf{Y}^{\star}\bar{\mathbf{V}}^{\star} \tag{3}$$

where $\mathbf{P}$ and $\mathbf{Q}$ are the vectors of active and reactive power injections, respectively, and $\circ$ denotes the elementwise product.

# The power flow equations V

If we develop this relation for a node $k$, we have:

$$P_k = G_{kk} V_k^2 + V_k \sum_{m \in \mathcal{N} \setminus k} V_m (G_{km} \cos\theta_{km} + B_{km} \sin\theta_{km}) = p_k(\bar{\mathbf{V}})$$

$$Q_k = -B_{kk} V_k^2 + V_k \sum_{m \in \mathcal{N} \setminus k} V_m (G_{km} \sin\theta_{km} - B_{km} \cos\theta_{km}) = q_k(\bar{\mathbf{V}})$$

with

- $Y_{km} = G_{km} + jB_{km}$
- $Y_{kk} = G_{kk} + jB_{kk}$ is the sum of the admittances from bus $k$ to ground
- $\theta_{km} = \theta_k - \theta_m$ the phase difference between voltages at nodes $k$ and $m$

# Number of equations and unknowns I

If there are $n$ buses in total, among which $n_{PQ}$ PQ buses, $n_{PV}$ PV buses and one slack bus, hence

$$n = n_{PQ} + n_{PV} + 1,$$

then

- $\mathbf{P}$ is known for $n_{PQ} + n_{PV}$ buses (all but the slack)
- Elements of $\mathbf{Q}$ are known for the $n_{PQ}$ PQ buses
- Voltage magnitude is known at PV buses and at the slack bus
- Voltage angle is known at the slack bus.

# Number of equations and unknowns II

In total, there are $2n$ equations for $2n$ unknowns: $n-1$ voltage angles, $n_{PQ}$ voltage magnitudes, $n_{PV}+1$ reactive powers, and 1 active power.

# Power flow solution method

# Power flow solution method

Let

- $\mathbf{P}^0$ be the active powers specified at the $\mathcal{N}_{PQ} \cup \mathcal{N}_{PV}$ buses
- $\mathbf{Q}^0$ be the reactive powers specified at the $\mathcal{N}_{PQ}$ buses.

To find $\bar{\mathbf{V}}$, we must solve

$$
\begin{aligned}
P_k^0 - p_k(\bar{\mathbf{V}}) = 0,\ \forall k \in \mathcal{N}_{PQ} \cup \mathcal{N}_{PV} \\
Q_k^0 - q_k(\bar{\mathbf{V}}) = 0,\ \forall k \in \mathcal{N}_{PQ}
\end{aligned}
$$

which is a set of $2n_{PQ} + n_{PV}$ non-linear equations.

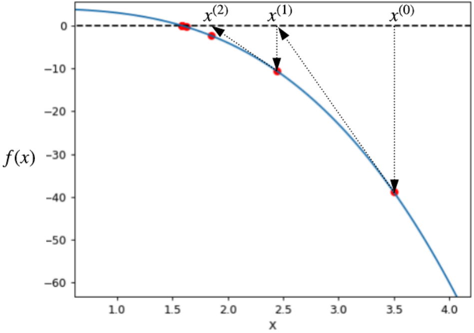
The most widespread method to solve this system is the *Newton-Raphson method*.

# Newton-Raphson example in 1D I

- Let's assume we want to solve $c - f(x) = 0$ with $f$ a non-linear function.
- We start with a first guess for $x$, $x^{(0)}$, at iteration $i = 0$
- Then, while $|c - f(x^{(i)})| > \epsilon$:
  - $x^{(i+1)} = x^{(i)} + \frac{c - f(x^{(i)})}{f'(x^{(i)})}$
  - $i \leftarrow i + 1$

For $c = 4$ and $f(x) = x^3$ (`https://colab.research.google.com/drive/12tCcO6kPxkoScAGnUYxgumS3CVSBSmtJ?usp=sharing`):

# Newton-Raphson example in 1D II



# Newton-Raphson example in 1D III

The *convergence is quadratic* if we start with x(0) "close” to the solution.

**TODO add NR example from transient analysis lecture**

# Newton-Raphson for the power flow problem I

We apply exactly the same idea to our problem, except that we are in dimension $2n_{PQ} + n_{PV}$.

Hence we must compute partial derivatives to compute the update steps:

$$\bar{\mathbf{V}}_x^{(i+1)} = \bar{\mathbf{V}}_x^{(i)} + \underbrace{\left[\mathbf{J}(\bar{\mathbf{V}}^{(i)})\right]^{-1} (\mathbf{F}^0 - \mathbf{f}(\bar{\mathbf{V}}^{(i)}))}_{\Delta \bar{\mathbf{V}}_x}$$

where

- $\mathbf{F}^0$ gathers the measured active powers at buses in $\mathcal{N}_{PQ} \cup \mathcal{N}_{PV}$ and reactive powers at buses $\mathcal{N}_{PQ}$

# Newton-Raphson for the power flow problem II

- $\mathbf{f}(\bar{\mathbf{V}})$ gathers the active and power flow equations at the corresponding buses
- $\bar{\mathbf{V}}_x$ is the subvector of $\bar{\mathbf{V}}$ that gathers the unknwon voltage magnitudes and angles at the the corresponding buses
- $\mathbf{J}(\bar{\mathbf{V}})$ is the jacobian of $\mathbf{f}$, of size $(2n_{PQ} + n_{PV}) \times (2n_{PQ} + n_{PV})$

**add NR code example applied to a tiny case (former homework)**

# Remarks

- In practice, instead of computing the inverse of the Jacobian, we solve the system

$$\mathbf{J}(\bar{\mathbf{V}}^{(i)})\Delta\bar{\mathbf{V}}_x = \mathbf{F}^0 - \mathbf{f}(\bar{\mathbf{V}}^{(i)})$$

  to get the update step
- The Jacobian is often sparse, since a bus is connected to a few neighbors; it is very important to take into account the sparsity properties in practical implementations
- The Jacobian is not necessarily updated at every iteration, especially close to convergence

# Fast decoupled power flow I

Remember that

- active power flow is mostly a function of voltage angles
- reactive power flow is mostly a function of voltage magnitudes

If we apply these ideas stricly, we can subdivide the problem in two much simpler subproblems:

- one problem for angles, based on the active power measurements and the sub-Jacobian containing the partial derivatives of the active power flow equations w.r.t. angles

# Fast decoupled power flow II

- one problem for magnitudes, based on the reactive power measurements and the sub-Jacobian containing the partial derivatives of the reactive power flow equations w.r.t. magnitudes

This procedure, through the sub-Jacobian that are computed, also provide information useful for *sensitivity analysis*.

# DC power flow I

"Direct Current" power flow is a further simplification:

- it is assumed that the impact of the reactance of lines is much bigger than the impact of their resistance, and shunt conductances are neglected
- voltage magnitudes are assumed equal to $1pu$
- angle differences are small
- active power losses are neglected, reactive power flows as well

# DC power flow II

$$P_k = \sum_{m \in \mathcal{N} \setminus k} B_{km} \theta_{km}$$

for every bus but the slack bus, which sets the angle difference, and collects the algebraic sum of all other injected powers.

In matrix form, with $\mathbf{Y}$ the admittance matrix defined before:

$$\mathbf{P} = \Im(\mathbf{Y})\theta$$

This is usefull for fast simulations, or when including a power flow model in an optimization problem, e.g. day-ahead market coupling.

# DC power flow III

**TODO add example, apply DC power flow to previous case with three buses**

# References

# References


- Mohan, Ned. Electric power systems: a first course. John Wiley & Sons, 2012.
- Course notes of ELEC0014 by Pr. Thierry Van Cutsem.
- L. Thurner, A. Scheidler, F. Schäfer et al, pandapower - an Open Source Python Tool for Convenient Modeling, Analysis and Optimization of Electric Power Systems, in IEEE Transactions on Power Systems, vol. 33, no. 6, pp. 6510-6521, Nov. 2018.