LIÈGE université
Sciences Appliquées

Three-phase systems in sinusoidal steady state, per-unit analysis

ELEC0447 - Analysis of electric power and energy systems

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Overview I

1. Three-phase systems (reminder)

2. Neutral break exercise

3. Per-phase analysis

4. Power transfer between AC systems

5. Per-unit normalization

What will we learn today?

Mostly from Chapter 2 of Ned Mohan's book:

Mohan, Ned. Electric power systems: a first course. John Wiley & Sons, 2012.

- 3-phase systems
- Power transfer between AC systems
- Per unit normalization

You will be able to do exercises 2.1, 2.2, 2.4, 2.5, 2.9, 2.11, 2.12, 2.14, 2.16, 2.17, 2.18, 2.19 and 2.20 from the Ned Mohan's book.

Three-phase systems (reminder)

Three-phase system

Here is an example of a simplified three-phase system with one generator, a transmission line, and a load.

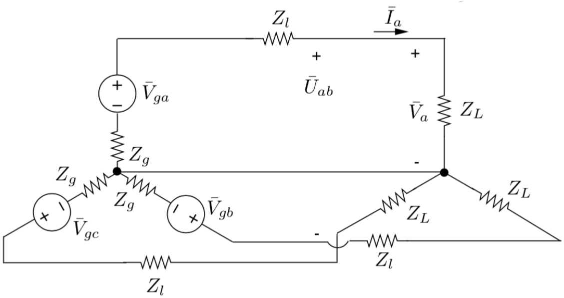


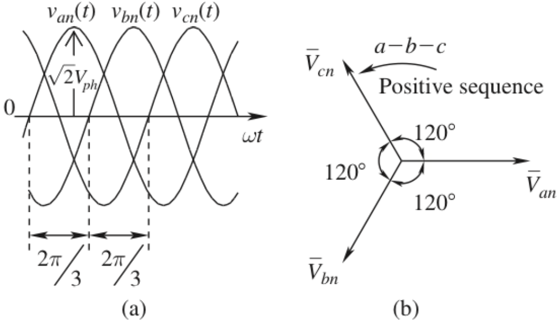
Figure 1: Generation -> transmission -> load

Here the load is connected as a *star*. A neutral point is present. The neutral conductor is not necessarily implemented, depending on the voltage level and the unbalanced nature of the network.

By design the voltage sources are shifted by 120° and of the same magnitude

$$\bar{V}_{ga} = V e^{j\phi_u}$$
$$\bar{V}_{gb} = V e^{j(\phi_u - 2\pi/3)} = \bar{V}_{ga} e^{-j2\pi/3}$$
$$\bar{V}_{gc} = V e^{j(\phi_u - 4\pi/3)} = \bar{V}_{ga} e^{-j4\pi/3}$$

In this course, we will always assume that the voltage sources generate a **positive sequence**, unless explicitly stated.



Three-phase voltages in time and phasor domain.

Phase voltage vs. line to line voltage

These voltages represent the **phase voltages** (between a phase and the ground). If we now look at the **line to line voltages**:

$$\bar{U}_{ab} = \bar{V}_{ga} - \bar{V}_{gb} = \sqrt{3}\bar{V}_{ga}e^{j\pi/6}$$

and similarly for $\bar{U}_{bc}$ and $\bar{U}_{ca}$.

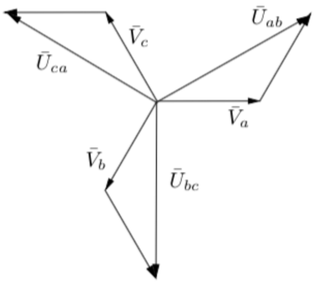


Figure 2: Line vs. phase voltages

Example: My house is fed by a 400V three-phase system. This means the line voltages are 400V (rms), and thus phase voltages are 230V (approximately). Typically, the phase voltages are distributed independently in the house, each with the neutral.

Total power

The total complex power transmitted to the load is

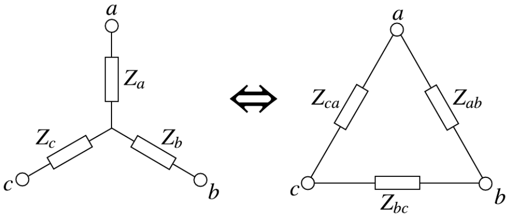
$$S = \sum_{k \in \{a,b,c\}} \bar{V}_{gk} \bar{I}_k^*$$

Hence in a balanced system the total active power to the load is $3VI\cos\phi$, with 3 or 4 wires (instead of 2 in a single-phase system).

Comments

- In every phase there is a current flowing. In a balanced system, currents are phase shifted by 120° and sum to zero. Thus the neutral can in theory be removed. This is done in some portions of the global system (typically at high voltage), where neutral points are grounded.
- Some loads can also be connected in "delta", hence the neutral is not accessible.
- Finally, in unbalanced systems, currents are dictated by the impedances seen in the different phases. There is no perfectly known relation, a priori.

Useful formulas: from star to delta connection (and back)



$$Z_a = \frac{Z_{ab}Z_{ca}}{Z_{ab} + Z_{bc} + Z_{ca}}$$

$$Z_b = \frac{Z_{bc}Z_{ab}}{Z_{ab} + Z_{bc} + Z_{ca}}$$

$$Z_c = \frac{Z_{ca}Z_{bc}}{Z_{ab} + Z_{bc} + Z_{ca}}$$

$$Z_{ab} = \frac{Z_aZ_b + Z_bZ_c + Z_cZ_a}{Z_c}$$

$$Z_{bc} = \frac{Z_aZ_b + Z_bZ_c + Z_cZ_a}{Z_a}$$

$$Z_{ca} = \frac{Z_aZ_b + Z_bZ_c + Z_cZ_a}{Z_b}$$

What happens in a balanced system?

Neutral break exercise

Neutral break: problem statement I

Consider a 3-phase Y-connected resistive circuit with unbalanced resistors in the phases. In normal operation there is a neutral wire. Then the neutral breaks (opens).

- Compute in both cases the voltages across the resistors, the currents and the consumed powers.
- What can you observe?

Assumptions:

- 3-phase balanced voltage supply: $U_{ab} = U_{bc} = U_{ca} = U_L$ (line-to-line voltage)

Neutral break: problem statement II

- Phase voltages (line-to-neutral) are given by $V_{an} = V_{bn} = V_{cn} = \frac{U_L}{\sqrt{3}}$, and the angle between them is $120°$ (direct sequence).
- Unbalanced resistances: R_a, R_b, R_c (different resistances in each phase).

Make a schematic of the two cases (neutral present or absent)

Case 1: Neutral Connected

Voltages: In normal operation, with the neutral connected, each resistor has its corresponding phase voltage across it:

$$\bar{V}_{R_a} = \bar{V}_{an}, \quad \bar{V}_{R_b} = \bar{V}_{bn}, \quad \bar{V}_{R_c} = \bar{V}_{cn}$$

Currents: The current in each phase is given by Ohm's law:

$$\bar{I}_a = \frac{\bar{V}_{an}}{R_a}, \quad \bar{I}_b = \frac{\bar{V}_{bn}}{R_b}, \quad \bar{I}_c = \frac{\bar{V}_{cn}}{R_c}$$

The neutral current, $\bar{I}_n$, is the sum of the phase currents:

$$\bar{I}_n = \bar{I}_a + \bar{I}_b + \bar{I}_c$$

Due to the unbalanced resistances, $\bar{I}_n$ will not be zero.

Powers

The power consumed in each phase is:

$$P_a = V_{an} I_a = \frac{V_{an}^2}{R_a}, \quad P_b = \frac{V_{bn}^2}{R_b}, \quad P_c = \frac{V_{cn}^2}{R_c}$$

Total power consumed:

$$P_{\text{total}} = P_a + P_b + P_c$$

Case 2: Neutral Broken

When the neutral breaks, the three resistors form a system without a direct connection to the neutral point. The current through each resistor still needs to sum to zero because the current has no return path through the neutral. This changes the voltage distribution across the resistors.

Voltages and currents I

Here, $\bar{V}_{N'}$ is an unknown voltage offset at the floating point N' (the shifted neutral).

Since the currents in each phase are given by:

$$\bar{I}_a = \frac{\bar{V}_{aN'}}{R_a}, \quad \bar{I}_b = \frac{\bar{V}_{bN'}}{R_b}, \quad \bar{I}_c = \frac{\bar{V}_{cN'}}{R_c}$$

and, if the neutral is broken, the sum of these currents must equal zero:

$$\bar{I}_a + \bar{I}_b + \bar{I}_c = 0,$$

then

$$\frac{\bar{V}_{aN'}}{R_a} + \frac{\bar{V}_{bN'}}{R_b} + \frac{\bar{V}_{cN'}}{R_c} = 0 \tag{1}$$

Voltages and currents II

that we can now decompose into

$$\frac{\bar{V}_{an} - \bar{V}_{nN'}}{R_a} + \frac{\bar{V}_{bn} - \bar{V}_{nN'}}{R_b} + \frac{\bar{V}_{cn} - \bar{V}_{nN'}}{R_c} = 0$$

which finaly yields

$$\bar{V}_{nN'} \left(\frac{1}{R_a} + \frac{1}{R_b} + \frac{1}{R_c} \right) = \frac{\bar{V}_{an}}{R_a} + \frac{\bar{V}_{bn}}{R_b} + \frac{\bar{V}_{cn}}{R_c}$$

Powers

- The power consumed in each phase is:

$$P_a = V_{aN'} I_a = \frac{(V_{aN'})^2}{R_a}, \quad P_b = \frac{(V_{bN'})^2}{R_b}, \quad P_c = \frac{(V_{cN'})^2}{R_c}$$

- Total power consumed:

$$P_{\text{total}} = P_a + P_b + P_c$$

Numerical example I

With $R_a = 5\Omega$, $R_b = 10\Omega$, $R_c = 20\Omega$ in a 400 V system (Link to the Python notebook):

Parameter	Case 1: Neutral Connected	Case 2: Neutral Broken
Total Power (W)	18515.00	15870.00
Phase A Power (W)	10580.00	4534.29
Phase B Power (W)	5290.00	6801.43
Phase C Power (W)	2645.00	4534.29
Phase A Current (A)	46.00	30.11
Phase B Current (A)	23.00	26.08
Phase C Current (A)	11.50	15.06
Phase A Voltage (V)	230.00	**150.57**
Phase B Voltage (V)	230.00	**260.80**
Phase C Voltage (V)	230.00	**301.14**

Observations

When the neutral is connected, the currents and powers are straightforwardly determined.

When the neutral is broken,

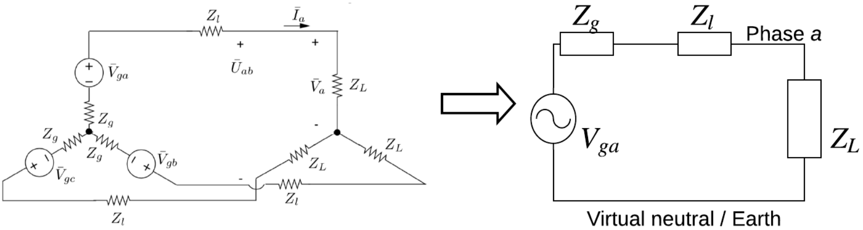
- the voltage distribution becomes more complex, and the voltages across the resistors are no longer the same as the phase-to-neutral voltages
- Some large overvoltages (equipment might not function as expected) and undervoltages (equipment might be damaged) can appear
- the total power consumed changes

Per-phase analysis

Per-phase analysis I

In a **balanced** system, analyses can be simplified by representing only one phase.

This is straightforward if there are no couplings between phases.



Per-phase analysis II

In case there is a coupling, and that for instance the voltage drop $\bar{V}_{aA}$ along a line presenting an impedance Z_{self} traversed by a current $\bar{I}_a$ is also function of the currents in the other phases:

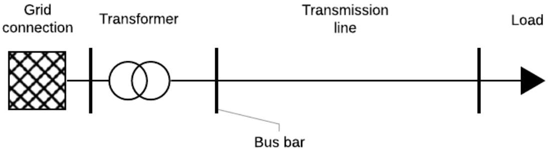
$$\bar{V}_{aA} = Z_{self}\bar{I}_a + Z_{mutual}\bar{I}_b + Z_{mutual}\bar{I}_c$$

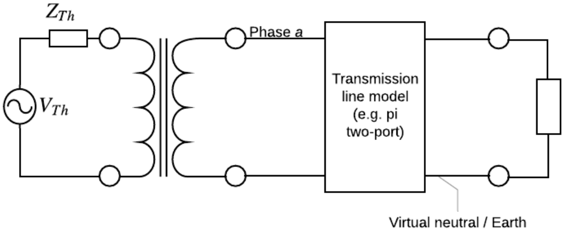
then the per-phase equivalent impedance (for phase a) is

$$Z_{aA} = Z_{self} - Z_{mutual}$$

since $\bar{I}_a + \bar{I}_b + \bar{I}_c = 0$

One-line diagram

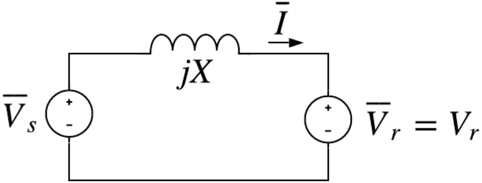




Power transfer between AC systems

Power transfer between AC systems I

Consider the following simple system. We have $\bar{I} = \frac{\bar{V}_s - \bar{V}_r}{jX}$



Let δ be the angle between $\bar{V}_r$ and $\bar{V}_s$, then

$$S_r = \bar{V}_r \bar{I}^* = V_r \left(\frac{V_s \angle - \delta - V_r}{-jX} \right)$$

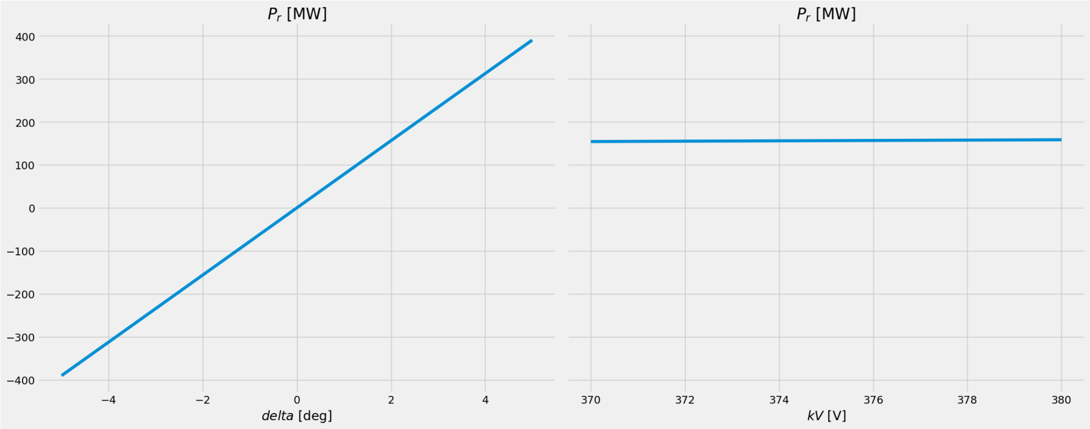
$$= \frac{V_s V_r \sin\delta}{X} + j \frac{V_s V_r \cos\delta - V_r^2}{X}$$

Power transfer between AC systems II

Let's remember two things:

- The **active** power is highly sensitive to δ
- The **reactive** power acts on the **voltage magnitude** (look at what happens for $\delta = 0$)

Illustration [(link to Python notebook)]



Per-unit normalization

Per-unit values

Per-unit values are the ratio between the actual value and the base value.

$$\text{Value}_{pu} = \frac{\text{Actual value}}{\text{Base value}}$$

It is useful in electrical power systems for two reasons:

- The parameters of rotating machines and transformers provided by the manufacturers are often given in pu.
- In a transformer, the impedance (in ohm) changes according to the square of the voltage ratio. If we express the impedance in pu, the value is invariant from one side of the transformer to the other.

In an electric network, a single base power is sufficient for the whole system, but for a system with transformers, one base voltage per voltage level is preferable.

Example I

It is **known** that the internal reactance of a synchronous machine lies typically in the range $[1.5, 2.5]$ pu (on the machine base)!

A machine with the characteristics $(20$ kV, 300 MVA$)$ has a reactance of $2.667\,\Omega$.
Is this a normal value?

- (Here we do not need a base value for time)
- The base impedance is $Z_B = 20^2/300 = 1.333\,\Omega$
- Hence the value of the reactance in per unit is $2.667/1.333 = 2\,pu$
- This is a quite normal value!

Example II

Same question for a machine with the characteristics $(15 \text{ kV}, 30 \text{ MVA})$

- The base impedance is now $Z_B = 15^2/30 = 7.5\,\Omega$
- The value of the reactance in per unit is $2.667/7.5 = 0.356\,pu$
- Hence an abnormal small value!

Per unit in three-phase systems

Let the base power S_B be the three-phase power, and $U_b = \sqrt{3}V_B$ be the line to line voltage base.

The (single-phase) base current is

$$I_B = \frac{S_B}{3V_B} = \frac{S_B}{\sqrt{3}U_B}$$

The base impedance is

$$Z_B = \frac{V_B}{I_B} = \frac{3V_B^2}{S_B} = \frac{U_B^2}{S_B}$$

In a single phase equivalent representation, the power values in per unit can be multiplied by S_b to get the total three-phase power.