Problem set 1: Basics, and Finite Difference Method

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Theoretical background

1. a is an unknown parameter. On $\mathbb{R}^2$, find out the region where the following equation is elliptic, hyperbolic or parabolic, and study their dependence on a.

$$(a+x)\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} - y^2\frac{\partial^2 u}{\partial y^2} = 0. \tag{1}$$

2. We prove the maximum principle for solution to Poisson equation:

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega \\ u = g & \text{on} \quad \partial\Omega \end{cases} . \tag{2}$$

Prove that there exists a constant C depending only on Ω such that:

$$\max_{\bar{\Omega}} |u| \leq C(\max_{\partial\Omega} |g| + \max_{\bar{\Omega}} |f|)$$

Here Ω be a bounded open subset of $\mathbb{R}^n$ with Γ being its boundary. $\bar{\Omega} = \Omega \cup \partial\Omega$ is its closure. Several hints:

   1. You could use the following property: assume v is subharmonic, i.e. $-\Delta v \leq 0$ in Ω, then $v(x) \leq \frac{1}{V}\int_{B(x,r)} v(y)\mathrm{d}y$. Here $B(x,r)$ is a ball centered at x with radius r, and V is the volume of the ball. Given this, you could prove the maximum of v is achieved at the boundary $\partial\Omega$.
   2. Try to show function $u + \frac{|x|^2}{2n}\lambda$ is subharmonic. Here $\lambda = \max_{\bar{\Omega}} |f|$.

Finite differencing

3.
   1. Prove: $\Delta_- + \Delta_+ = \left(\mathcal{E}^{-1/2} + \mathcal{E}^{1/2}\right)\Delta_0$ and $\Delta_-\Delta_+ = \Delta_0^2$. Here $\mathcal{E}$ is the shifting operator: $(\mathcal{E}u)_j = u_{j+1}$, and the definitions for Δ_+ and Δ_- are consistent with what we had in class. $(\Delta_0 u)_j = u_{j+1/2} - u_{j-1/2}$.
   2. Determine the constants c and d so that:

$$\partial_x^2 u(x) - \frac{1}{h^2}\left(\Delta_+^2 - \Delta_+^3\right)u(x) = ch^2\partial_x^4 u(x) + \mathcal{O}(h^3), \quad h \to 0$$
$$\partial_x^2 u(x) - \frac{1}{h^2}\Delta_0^2 u(x) = dh^2\partial_x^4 u(x) + \mathcal{O}(h^4), \quad h \to 0$$

   Here we assume $u(x)$ is smooth enough.

3. The two identities above tell you how to approximate ∂_x^2 using forward differencing and central differencing. How many grid points you need to get the second order approximation respectively using these two methods?

4. Write a computer code (using your favorite language) to determine, to highest possible order, a finite difference approximation to $u''(x)$ based on the 5-point stencil $\{x-h, x-\frac{1}{2}h, x, x+h, x+2h\}$:

$$u''(x) \approx c_0 u(x-h) + c_1 u(x-\frac{h}{2}) + c_2 u(x) + c_3 u(x+h) + c_4 u(x+2h). \tag{3}$$

   1. Computed c_j;
   2. Check the order: what is the highest order of error between the left/right hand side of (3);

Finite difference method for elliptic equation

5. We used Fourier method for stability analysis in $1D$ in class. Carry out the same analysis for $2D$.

6. Derive the explicit formulae for Green's functions in $1D$ and prove they are piecewise linear functions.

7. In 2D, to compute the Poisson equation $u'' = f$ with zero boundary condition on a rectangular domain, we discretize the domain by even grid points with mesh size h. Denote A the associated discretization matrix with central differencing (five-stencils). Show $\|A^{-1}\|_\infty$ is bounded independence of h. Explain why it suggests that the order of accuracy of the numerical method is second order. (Hint: 1. $\|A^{-1}\|_\infty = \sup \frac{\|A^{-1}v\|_\infty}{\|v\|_\infty}$; 2. Fundamental theorem for numerical convergence.)

8. Prove the discrete version of Poincaré inequality: $\sum_{mn} |U_{m,n}|^2 \leq \sum_{mn} |\partial_x U_{m,n}|^2$. Here U is a matrix on 2D with zero boundary condition, and $\partial_x U_{m,n}$ is the forward Euler presentation of differentiation, defined by: $\partial_x U_{mn} = \frac{1}{h}[U_{m+1,n} - U_{m,n}]$.