# Variational Formulation

1. Suppose $u \in V$ is the solution to the following problem in its variational form

$$b(u,v) = l(v) \qquad v \in V$$

where $b(u,v)$ is a bilindear form that maps $V \times V$ to $\mathbb{R}$. $l$ is a linear operator that maps $V$ to $\mathbb{R}$. Here $V$ is a Hilbert space. To obtain a numerical solution, we use the Galerkin method, and search for a solution in a subspace $V_h = \text{span}\{\phi_1, \phi_2, \ldots, \phi_N\} \subset V$

$$b(u_h,v) = l(v) \qquad v \in V_h$$

In the linear algebra form, it is written as

$$\underline{\underline{A}} \cdot \underline{U} = \underline{F}$$

(a) Find the specific form of $\underline{\underline{A}}$ and $\underline{F}$.

$$\left(\underline{\underline{A}}\right)_{i,j} = b\left(\phi_j, \phi_i\right)$$

and

$$\left(\underline{F}\right)_i = l\left(\phi_i\right)$$

(b) If $b$ is symmetric, show $\underline{\underline{A}}$ is symmetric.

Assume $b$ is symmetric, i.e. $b\left(\phi_j,\phi_i\right) = b\left(\phi_i,\phi_j\right)$. Then $\left(\underline{\underline{A}}\right)_{i,j} = \left(\underline{\underline{A}}\right)_{j,i}$, i.e. $\underline{\underline{A}} = \underline{\underline{A}}^T$, so $\underline{\underline{A}}$ is symmetric.

(c) Give the definition of the coercivity condition, and show if $b$ is coercive, $\underline{\underline{A}}$ is positive definite.

Assume $b$ is coercive, i.e. $\gamma\|u\|^2 \le b(u,u)$. We can transform the basis, $\phi_1, \phi_2, \ldots, \phi_N \to \tilde{\phi}_1, \tilde{\phi}_2, \ldots, \tilde{\phi}_N$, such that $b\left(\tilde{\phi}_j, \tilde{\phi}_i\right) = 0$ for $i \neq j$, which is equivalent to diagonalizing $\underline{\underline{A}}$. (We know that $\underline{\underline{A}}$ is diagonalizable since our problem is well-posed, thus $\underline{\underline{A}}$ is invertible.) Now with this diagonalized basis set, the eigenvalues of $\underline{\underline{A}}$ are $\lambda_i = b\left(\tilde{\phi}_i, \tilde{\phi}_i\right) \ge \gamma \left\|\tilde{\phi}_i\right\|^2 > 0$. Therefore, $\underline{\underline{A}}$ is positive definite.

(d) Still assume $b$ is coercive and denote $M$ and $\gamma$ the bounded coefficient and coercive coefficient, respectively, show

$$\|u - u_h\| \le \frac{M}{\gamma} \inf_{v \in V_h} \|u - v\|$$

Assume $b$ is coercive and bounded, i.e. $\gamma\|u\|^2 \le b(u,u)$ and $b(u,v) \le M\|u\|\|v\|$. Using $b\left(u - u_h, v\right) = 0$ $\forall v \in V_h$, it can be shown that $b\left(u-u_h, u-v\right) = b\left(u-u_h, u-u_h\right)$. Then,

$$\gamma\left\|u-u_h\right\|^2 \le b\left(u-u_h,u-u_h\right) = b\left(u-u_h,u-v\right) \le M\left\|u-u_h\right\|\left\|u-v\right\| \qquad \forall v \in V_h$$
$$\left\|u-u_h\right\| \le \frac{M}{\gamma}\inf_{v \in V_h}\left\|u-v\right\|$$

(e) For the following equation in $\Omega \in \mathbb{R}^2$ with zero Dirichlet boundary condition ($\Omega$ is compactly supported), write down its variational form, determine the space $V$, and show whether $b$ satisfies the coercivity condition. Find $M$ and $\gamma$, respectively.

$$-\vec{\nabla} \cdot (a\vec{\nabla}u) + cu = f$$

with $0 < \underline{a} \le a < \bar{a}$ and $0 \le \underline{c} \le c < \bar{c}$.

Find some $u \in H_0^1(\Omega)$ such that

$$\left\langle a\vec{\nabla}u, \vec{\nabla}v\right\rangle + \langle cu, v\rangle = \langle f, v\rangle \qquad \forall v \in H_0^1(\Omega)$$

i.e.

$$b(u,v) = \left\langle a\vec{\nabla}u, \vec{\nabla}v\right\rangle + \langle cu, v\rangle$$

To check coercivity,

$$b(u,u) = \left\langle a\vec{\nabla}u, \vec{\nabla}u \right\rangle + \langle cu, u\rangle \geq \underline{a} \left\langle \vec{\nabla}u, \vec{\nabla}u \right\rangle + \underline{c} \langle u, u\rangle \geq \min\{\underline{a}, \underline{c}\} \|u\|_{H^1}^2$$

so $\gamma = \min\{\underline{a}, \underline{c}\}$. For the bounded condition,

$$b(u,v) = \left\langle a\vec{\nabla}u, \vec{\nabla}v \right\rangle + \langle cu, v\rangle \leq \bar{a} \left\langle \vec{\nabla}u, \vec{\nabla}v \right\rangle + \bar{c} \langle u, v\rangle$$

noting that $\left\langle \vec{\nabla}u, \vec{\nabla}v \right\rangle \leq \|u\|_{H^1}\|v\|_{H^1}$ and $\langle u, v\rangle \leq \|u\|_{H^1}\|v\|_{H^1}$,

$$b(u,v) \leq (\bar{a} + \bar{c}) \|u\|_{H^1}\|v\|_{H^1}$$

so $M = \bar{a} + \bar{c}$.

(f) For the problem above, if we choose $V_h$ to be a piecewise linear function space, show

$$\|u - u_h\|_{H^1} = \mathcal{O}(h)$$

and that if $a$ is a constant and $c = 0$

$$\|u - u_h\|_{L_2} = \mathcal{O}(h^2)$$

Using $b$ is coercive and bounded along with previous analysis, it can be shown that

$$\|u - u_h\|_{H^1} \leq \frac{M}{\gamma}\|u - v\|_{H^1} \qquad \forall v \in V_h$$

Simply pick $v = Iu$, where $Iu$ is the interpolation of $u$ using piecewise linear functions ($Iu \in V - h$). Then

$$\|u - u_h\|_{H^1} \leq \frac{M}{\gamma}\|u - Iu\|_{H^1} = \mathcal{O}(h)$$

(g) For the same equation in $\Omega \in \mathbb{R}^2$ with Neumann boundary contition

$$\partial_n u|_{\partial\Omega} = g$$

Write down its variational form, determine $V$, and show whether $b$ satisfies the coercivity condition.

The variational form is find some $u \in H^1(\Omega)$

$$\left\langle a\vec{\nabla}u, \vec{\nabla}v \right\rangle + \langle cu, v\rangle = \langle f, v\rangle + \int_{\partial\Omega} gv\mathrm{d}S \qquad \forall v \in H^1(\Omega)$$

The bilinear form $b$ is the same as in part (e) so the coercivity proven there still holds.

## Euler-Bernoulli equation

2. Consider the Euler-Bernoulli equation

$$\frac{\partial^4 u}{\partial x^4} = f(x) \qquad 0 < x < 1$$

It is used to describe the deflection of $u$ of a clamped beam subject to a transversal force with intensity $f$.

(a) Show the equivalent varational form would be to find $u$ such that

$$\langle u'', v''\rangle = \langle f, v\rangle \qquad \forall v \in V$$

where $V = \{v : v \in C_1[0,1], v(0) = v'(0) = v(1) = v'(1) = 0, v \text{ piecewise continuous and bounded}\}$

Starting with the Euler-Bernoulli equation and $u(0) = u'(0) = u(1) = u'(1) = 0$

$$\left\langle u^{(4)}, v\right\rangle = \langle f, v\rangle$$
$$\left\langle u^{(3)}, v'\right\rangle = \langle f, v\rangle$$
$$\langle u'', v''\rangle = \langle f, v\rangle$$

where $v \in V = H_0^2[0,1]$

(b) For an interval, $I = [a,b]$, define $P_3(I) = \left\{v : v(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3, x \in I\right\}$. Show that $v \in P_3(I)$ is uniquely determined by the values $v(a)$, $v'(a)$, $v(b)$, and $v'(b)$. Find the corresponding local basis functions. (Hint: count the nuber of degrees of freedom and use the values to fix the coefficients.)

There are 4 degrees of freedom and 4 constraints, so $v \in P_3(I)$ can be uniquely determined. They can be determined by solving the linear system

$$\begin{bmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & 2a & 3a^2 \\ 1 & b & b^2 & b^3 \\ 0 & 1 & 2b & 3b^2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} v(a) \\ v'(a) \\ v(b) \\ v'(b) \end{bmatrix}$$

To produce local basis functions we can produce a single set of basis functions for a reference element. This reference element will have $I = [0,1]$ and the basis functions will have a value of 1 at one of $v(0)$, $v'(0)$, $v(1)$, and $v'(1)$ with 0 at the others and will be denoted $h_{00}$, $h_{10}$, $h_{01}$, and $h_{11}$.

Table 1: Basis functions of the reference element.

| **Basis Function** | **Definition** |
|---|---|
| $h_{00}(x)$ | $2x^3 - 3x^2 + 1$ |
| $h_{10}(x)$ | $x^3 - 2x^2 + x$ |
| $h_{01}(x)$ | $-2x^3 + 3x^2$ |
| $h_{11}(x)$ | $x^3 - x^2$ |

(c) Construct a finite-dimensional subspace $V_h$ consisting of piecewise cubic polynomials on the mesh $0 = x_0 < x_1 < \cdots < x_{N+1} = 1$.

A coordinate transformation, $\mathcal{F} : \hat{x} \to x$, will be used to produce local basis functions from the reference set defined in part (b). This transformation can be explicitly written out in the affine mapping of $\hat{I} = [0,1] \xrightarrow{\mathcal{F}} I = [a,b]$ as $\mathcal{F}(\hat{x}) = (b-a)x + a$. Using this change of variables also allows integration over the reference element. The degrees of freedom for this problem will consist of the values of $u_h$ and $u_h'$ at the points $x_1, x_2, \ldots, x_N$. Thus the local basis functions are actually the sum of the corresponding basis functions from each of the neighboring elements. For example, denote $\phi_i$ as the basis function corresponding to the value of $u_h$ at $x_i$ (and $\tilde{\phi}_i$ as the basis function corresponding to the value of $u_h'$ at $x_i$), then $\phi_i = h_{01}|_{e_{i-1}} + h_{00}|_{e_i}$ (and $\tilde{\phi}_i = h_{11}|_{e_{i-1}} + h_{10}|_{e_i}$), where $e_i$ is the element living between $x_i$ and $x_{i+1}$.

(d) Derive the error estimate

$$\left\|(u - u_h)''\right\|_2 \leq \left\|(u - v)''\right\|_2 \qquad \forall v \in V_h$$

You are given the estimate that cubic Hermite interpolation of $u$, denoted as $I_h u \in V_h$, satisfies the following

$$\left\|u''(x) - (I_h u)''(x)\right\| \lesssim h^2 \max_{0 \leq \xi \leq 1} \left|u^{(4)}(\xi)\right|$$

show that

$$\left\|(u-u_h)''\right\| \le Ch^2 \max_{0\le\xi\le1}\left|u^{(4)}(\xi)\right|$$

(e) Write a computer program to solve

$$\begin{cases} \dfrac{\mathrm{d}^4u}{\mathrm{d}x^4} = g(x) \\ u(0) = u'(0) = u'(1) = u'(1) = 0 \end{cases}$$

If we use

$$g(x) = \frac{\mathrm{d}^4}{\mathrm{d}x^4}\left(e^x x^2(1-x)^2\right) = e^x\left(x^4 + 14x^3 + 49x^2 + 32x - 12\right)$$

the exact solution is $u(x) = e^x x^2(1-x)^2$.

i. Give a brief description of your algorithm, in particular, the method you use to evaluate the load vector $\underline{b}$ (choose your favorite numerical integral method, but make sure the error here is not too big, and the error from $\underline{\underline{A}}$ still dominates)
ii. Tabulate the max-norm errors $e_N = \max|u_h(x_j) - u(x_j)|$ and show the numerical convergence order by performing linear regression of $\log e_N$ vs $\log N$
iii. Plot your finite element solution $u_h$ along with the real solution.
iv. Attach your code.