Problem set 2: Finite Element Method basics

good luck!

no due

Variational Formulation

Suppose $u \in V$ is the solution to the following problem in its variational form:

$$b(u,v) = l(v)\,, \quad v \in V\,,$$

where $b(u,v)$ is a bilinear form that maps $V \times V$ to $\mathbb{R}$. l is a linear operator that maps V to $\mathbb{R}$. Here V is a Hilbert space. To obtain a numerical solution, we use the Galerkin method, and seek for solution in a subspace $V_h = \text{Span}\{\phi_1\,, \phi_2\,, \cdots\,, \phi_N\} \subset V$:

$$b(u_h, v) = l(v)\,, \quad v \in V_h\,.$$

In the linear algebra form, it writes as:

$$A \cdot U = F\,.$$

1. Find the specific form of A and F;
2. If b is symmetric, show A is symmetric;
3. Give the definition of the coercivity condition, and show if b is coercive, A is positive definite;
4. Still assume b is coercive and denote M and γ the bounded coefficient and coercive coefficient respectively, show
$$\|u - u_h\| \leq \frac{M}{\gamma} \inf_{v \in V} \|u - v\|\,;$$
5. For the following equation in $\Omega \in \mathbb{R}^2$ with zero Dirichlet boundary condition (Ω is compactly supported), write down its variational form, determine the space V, and show whether b satisfies the coercivity condition. Find M and γ respectively.
$$-\nabla_x \cdot (a \nabla_x u) + cu = f\,, \quad \text{with} \quad 0 < \underline{a} \leq a < \bar{a}\,,\ 0 \leq \underline{c} \leq c < \bar{c}\,.$$
6. For the problem above, if we choose V_h to be piecewise linear function space, show:
$$\|u - u_h\|_{H^1} = \mathcal{O}(h)\,,$$
and that if a is a constant and $c = 0$:
$$\|u - u_h\|_{L_2} = \mathcal{O}(h^2)\,.$$

7. For the same equation in $\Omega \in \mathbb{R}^2$ with Neumann boundary condition

$$\partial_n u = g\,,$$

write down its variational form, determine V, and show whether b satisfies the coercivity condition.

Euler-Bernoulli equation

Consider the Euler-Bernoulli equation

$$\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} = f(x), \quad 0 < x < 1. \tag{1}$$

It is used to describe the deflection u of a clamped beam subject to a transversal force with intensity f.

(a) Show the equivalent variational form would be to find u such that:

$$(u'', v'') = (f, v), \forall v \in V, \tag{2}$$

where

$$V = \left\{v : v \in C_1[0,1], v(0) = v'(0) = v(1) = v'(1) = 0, v''\text{piecewise continuous and bounded}\right\}.$$

(b) For $I = [a, b]$ an interval, define $P_3(I) = \left\{v : v(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3, \quad x \in I\right\}$. Show that $v \in P_3(I)$ is uniquely determined by the values $v(a)$, $v'(a)$, $v(b)$ and $v'(b)$. Find the corresponding local basis functions. (Hint: count the number of degree of freedom and write use the values to fix the coefficients.)

(c) Construct a finite-dimensional subspace V_h consisting piecewise cubic polynomials on the mesh $0 = x_0 < x_1 < \cdots < x_{n+1} = 1$. Find suitable representations for the functions in V_h.

(d) Formulate a finite element method for on space V_h. Find out explicit expression for each entry of the stiff matrix and load vector.

(e) Derive the error estimate

$$\|(u - u_h)''\|_2 \le \|(u - v)''\|_2, \forall v \in V_h. \tag{3}$$

You are given the estimate that cubic Hermite interpolant of u, denoted as $I_h u \in V_h$, satisfies the following:

$$\|u''(x) - (I_h u)''(x)\| \lesssim h^2 \max_{0 \le \xi \le 1} |u^{(4)}(\xi)|, \tag{4}$$

show that

$$\|(u - u_h)''\| \le C h^2 \max_{0 \le \xi \le 1} |u^{(4)}(\xi)| \tag{5}$$

(f) (if you'd like to try some implementing) Write a computer program to solve

$$\begin{cases} \frac{\mathrm{d}^4 u}{\mathrm{d}x^4} = g(x), \\ u(0) = u'(0) = u(1) = u'(1) = 0 \end{cases} \tag{6}$$

If we use $g(x) = \frac{\mathrm{d}^4}{\mathrm{d}x^4} e^x x^2 (1-x)^2 = e^x (x^4 + 14x^3 + 49x^2 + 32x - 12)$, the exact solution is $u(x) = e^x x^2 (1-x)^2$.

- give a brief description of your algorithm, in particular, the method you use to evaluate the load vector b (choose your favorite numerical integral method, but make sure the error here is not too big, and the error from A still dominates).
- tabulate the max-norm errors $e_n = \max |u_h(x_j) - u(x_j)|$ and show the numerical convergence order by performing linear regression of $\log(e_n)$ v.s $\log(n)$.
- plot your finite element solution u_h along with the real solution.
- attach your code.