

Variational Formulation

1. Suppose $u \in V$ is the solution to the following problem in its variational form

$$b(u, v) = l(v) \quad v \in V$$

where $b(u, v)$ is a bilinear form that maps $V \times V$ to \mathbb{R} . l is a linear operator that maps V to \mathbb{R} . Here V is a Hilbert space. To obtain a numerical solution, we use the Galerkin method, and search for a solution in a subspace $V_h = \text{span}\{\phi_1, \phi_2, \dots, \phi_N\} \subset V$

$$b(u_h, v) = l(v) \quad v \in V_h$$

In the linear algebra form, it is written as

$$\underline{\underline{A}} \cdot \underline{U} = \underline{F}$$

- (a) Find the specific form of $\underline{\underline{A}}$ and \underline{F} .

$$(\underline{\underline{A}})_{i,j} = b(\phi_j, \phi_i)$$

and

$$(\underline{F})_i = l(\phi_i)$$

- (b) If b is symmetric, show $\underline{\underline{A}}$ is symmetric.

Assume b is symmetric, i.e. $b(\phi_j, \phi_i) = b(\phi_i, \phi_j)$. Then $(\underline{\underline{A}})_{i,j} = (\underline{\underline{A}})_{j,i}$, i.e. $\underline{\underline{A}} = \underline{\underline{A}}^T$, so $\underline{\underline{A}}$ is symmetric.

- (c) Give the definition of the coercivity condition, and show if b is coercive, $\underline{\underline{A}}$ is positive definite.

Assume b is coercive, i.e. $\gamma \|u\|^2 \leq b(u, u)$. We can transform the basis, $\phi_1, \phi_2, \dots, \phi_N \rightarrow \tilde{\phi}_1, \tilde{\phi}_2, \dots, \tilde{\phi}_N$, such that $b(\tilde{\phi}_j, \tilde{\phi}_i) = 0$ for $i \neq j$, which is equivalent to diagonalizing $\underline{\underline{A}}$. (We know that $\underline{\underline{A}}$ is diagonalizable since our problem is well-posed, thus $\underline{\underline{A}}$ is invertible.) Now with this diagonalized basis set, the eigenvalues of $\underline{\underline{A}}$ are $\lambda_i = b(\tilde{\phi}_i, \tilde{\phi}_i) \geq \gamma \|\tilde{\phi}_i\|^2 > 0$. Therefore, $\underline{\underline{A}}$ is positive definite.

- (d) Still assume b is coercive and denote M and γ the bounded coefficient and coercive coefficient, respectively, show

$$\|u - u_h\| \leq \frac{M}{\gamma} \inf_{v \in V_h} \|u - v\|$$

Assume b is coercive and bounded, i.e. $\gamma \|u\|^2 \leq b(u, u)$ and $b(u, v) \leq M \|u\| \|v\|$. Using $b(u - u_h, v) = 0 \forall v \in V_h$, it can be shown that $b(u - u_h, u - v) = b(u - u_h, u - u_h)$. Then,

$$\gamma \|u - u_h\|^2 \leq b(u - u_h, u - u_h) = b(u - u_h, u - v) \leq M \|u - u_h\| \|u - v\| \quad \forall v \in V_h$$

$$\|u - u_h\| \leq \frac{M}{\gamma} \inf_{v \in V_h} \|u - v\|$$

- (e) For the following equation in $\Omega \in \mathbb{R}^2$ with zero Dirichlet boundary condition (Ω is compactly supported), write down its variational form, determine the space V , and show whether b satisfies the coercivity condition. Find M and γ , respectively.

$$-\vec{\nabla} \cdot (a \vec{\nabla} u) + cu = f$$

with $0 < \underline{a} \leq a < \bar{a}$ and $0 \leq \underline{c} \leq c < \bar{c}$.

Find some $u \in H_0^1$ such that

$$\langle a \vec{\nabla} u, \vec{\nabla} v \rangle + \langle cu, v \rangle = \langle f, v \rangle \quad \forall v \in H_0^1$$

i.e.

$$b(u, v) = \langle a \vec{\nabla} u, \vec{\nabla} v \rangle + \langle cu, v \rangle$$

To check coercivity,

$$b(u, u) = \langle a \vec{\nabla} u, \vec{\nabla} u \rangle + \langle cu, u \rangle \geq \underline{a} \langle \vec{\nabla} u, \vec{\nabla} u \rangle + \underline{c} \langle u, u \rangle = \underline{a} \|u\|_{H^1}^2 + (\underline{c} - \underline{a}) \|u\|_{L^2}^2 \geq (\underline{a} + \underline{c}) \|u\|_{H^1}^2$$

so $\gamma = \underline{a} + \underline{c}$. For the bounded condition,

$$b(u, v) = \langle a \vec{\nabla} u, \vec{\nabla} v \rangle + \langle cu, v \rangle \leq \bar{a} \langle \vec{\nabla} u, \vec{\nabla} v \rangle + \bar{c} \langle u, v \rangle$$

noting that $\langle \vec{\nabla} u, \vec{\nabla} v \rangle \leq \|u\|_{H^1} \|v\|_{H^1}$ and $\langle u, v \rangle \leq \|u\|_{H^1} \|v\|_{H^1}$,

$$b(u, v) \leq (\bar{a} + \bar{c}) \|u\|_{H^1} \|v\|_{H^1}$$

so $M = \bar{a} + \bar{c}$.

(f) For the problem above, if we choose V_h to be a piecewise linear function space, show

$$\|u - u_h\|_{H^1} = \mathcal{O}(h)$$

and that if a is a constant and $c = 0$

$$\|u - u_h\|_{L_2} = \mathcal{O}(h^2)$$

Using b is coercive and bounded along with previous analysis, it can be shown that

$$\|u - u_h\|_{H^1} \leq \frac{M}{\gamma} \|u - v\|_{H^1} \quad \forall v \in V_h$$

Simply pick $v = Iu$, where Iu is the interpolation of u using piecewise linear functions ($Iu \in V - h$). Then

$$\|u - u_h\|_{H^1} \leq \frac{M}{\gamma} \|u - Iu\|_{H^1} = \mathcal{O}(h)$$

(g) For the same equation in $\Omega \in \mathbb{R}^2$ with Neumann boundary condition

$$\partial_n u|_{\partial\Omega} = g$$

Write down its variational form, determine V , and show whether b satisfies the coercivity condition.

The variational form is find some $u \in H^1$

$$\langle a \vec{\nabla} u, \vec{\nabla} v \rangle + \langle cu, v \rangle = \langle f, v \rangle + \int_{\partial\Omega} g v dS \quad \forall v \in H^1$$

The bilinear form b is the same as in part (e) so the coercivity proven there still holds.

Euler-Bernoulli equation

2. Consider the Euler-Bernoulli equation

$$\frac{\partial^4 u}{\partial x^4} = f(x) \quad 0 < x < 1$$

It is used to describe the deflection of u of a clamped beam subject to a transversal force with intensity f .

- (a) Show the equivalent variational form would be to find u such that

$$\langle u'', v'' \rangle = \langle f, v \rangle \quad \forall v \in V$$

where $V = \{v : v \in C_1[0, 1], v(0) = v'(0) = v(1) = v'(1) = 0, v \text{ piecewise continuous and bounded}\}$

- (b) For an interval, $I = [a, b]$, define $P_3(I) = \{v : v(x) = c_0 + c_1x + c_2x^2 + c_3x^3, x \in I\}$. Show that $v \in P_3(I)$ is uniquely determined by the values $v(a)$, $v'(a)$, $v(b)$, and $v'(b)$. Find the corresponding local basis functions. (Hint: count the number of degrees of freedom and use the values to fix the coefficients.)
- (c) Construct a finite-dimensional subspace V_h consisting of piecewise cubic polynomials on the mesh $0 = x_0 < x_1 < \dots < x_{N+1} = 1$.
- (d) Derive the error estimate

$$\|(u - u_h)''\|_2 \leq \|(u - v)''\|_2 \quad \forall v \in V_h$$

You are given the estimate that cubic Hermite interpolation of u , denoted as $I_h u \in V_h$, satisfies the following

$$\|u''(x) - (I_h u)''(x)\| \lesssim h^2 \max_{0 \leq \xi \leq 1} |u^{(4)}(\xi)|$$

show that

$$\|(u - u_h)''\| \leq Ch^2 \max_{0 \leq \xi \leq 1} |u^{(4)}(\xi)|$$

- (e) Write a computer program to solve

$$\begin{cases} \frac{d^4 u}{dx^4} = g(x) \\ u(0) = u'(0) = u'(1) = u(1) = 0 \end{cases}$$

If we use

$$g(x) = \frac{d^4}{dx^4} (e^x x^2 (1-x)^2) = e^x (x^4 + 14x^3 + 49x^2 + 32x - 12)$$

the exact solution is $u(x) = e^x x^2 (1-x)^2$.

- Give a brief description of your algorithm, in particular, the method you use to evaluate the load vector \underline{b} (choose your favorite numerical integral method, but make sure the error here is not too big, and the error from \underline{A} still dominates)
- Tabulate the max-norm errors $e_N = \max |u_h(x_j) - u(x_j)|$ and show the numerical convergence order by performing linear regression of $\log e_N$ vs $\log N$
- Plot your finite element solution u_h along with the real solution.
- Attach your code.