1. Matrix Splitting Methods
   (a) $A \to M - N$
   (b) $u^{n+1} = M^{-1}Nu^n + M^{-1}f$
   (c) Convergence based on $\left|\rho\left(M^{-1}N\right)\right|$
   (d) Jacobi: M is diagonal part of A
   (e) Gauss-Seidel: M is lower triangular part of A

2. Steepest Descent Method
   (a) $\min \phi$ where $\phi(u) = \frac{1}{2}u^T Au - b^T u \overset{A \text{ p.d.}}{\iff} Au = b$
      i. $\min \phi \iff \vec{\nabla}_u \phi = 0 = Au - b$
   (b) Algorithm:

$$\begin{aligned} r_k &= Au_k - b \\ u_{k+1} &= u_k + \alpha_k r_k \end{aligned}$$

   (c) $\alpha_k = -\frac{r_k^T r_k}{r_k^T A p_k}$
   (d) convergence $\sim \kappa \equiv \mathrm{cond}(A)$

3. Krylov Space
   (a) $\mathrm{span}\left\{p_0, \ldots, p_{k-1}\right\} = \mathrm{span}\left\{r_0, \ldots, A^{k-1}r_0\right\} = \mathrm{span}\left\{Ae_0, \ldots, A^k e_0\right\}$
      i. $\mathrm{span}\left\{p_0, p_1, \ldots, p_{k-1}\right\} = \mathrm{span}\left\{r_0, r_1, \ldots, r_{k-1}\right\} = \mathrm{span}\left\{r_0, Ap_0, \ldots, Ap_{k-2}\right\}$
   (b) $r_i^T r_j = 0$ for $i \neq j$
      i. assume $r_{k-1}^T r_j = 0$ for $j = 0, \ldots, k-2$
      ii. leads to alternative form of α_k
      iii. used to get better $c_j^{(k)}$

4. Conjugate Gradient Method
   (a) Algorithm:

$$\begin{aligned} p_k &= r_k + \sum_{j=0}^{k-1} c_j^{(k)} p_j \\ u_{k+1} &= u_k + \alpha_k p_k \\ r_{k+1} &= r_k + \alpha_k A p_k \end{aligned}$$

   (b) $\alpha_k = -\frac{p_k^T r_k}{p_k^T A p_k} = -\frac{r_k^T r_k}{r_k^T A p_k}$
   (c) $c_j^{(k)} = -\frac{r_k^T A p_j}{p_j^T A p_j} = \begin{cases} 0 & j < k-1 \\ \frac{r_k^T r_k}{r_{k-1}^T r_{k-1}} & j = k-1 \end{cases}$
   (d) convergence $\sim \frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}$