# Variational Formulation

1. Suppose $u \in V$ is the solution to the following problem in its variational form

$$b(u,v) = l(v) \qquad v \in V$$

where $b(u,v)$ is a bilindear form that maps $V \times V$ to $\mathbb{R}$. $l$ is a linear operator that maps $V$ to $\mathbb{R}$. Here $V$ is a Hilbert space. To obtain a numerical solution, we use the Galerkin method, and search for a solution in a subspace $V_h = \text{span}\{\phi_1, \phi_2, \ldots, \phi_N\} \subset V$

$$b(u_h, v) = l(v) \qquad v \in V_h$$

In the linear algebra form, it is written as

$$\underline{\underline{A}} \cdot \underline{U} = \underline{F}$$

(a) Find the specific form of $\underline{\underline{A}}$ and $\underline{F}$.

$$\left(\underline{\underline{A}}\right)_{i,j} = b\left(\phi_j, \phi_i\right)$$

and

$$\left(\underline{F}\right)_i = l\left(\phi_i\right)$$

(b) If $b$ is symmetric, show $\underline{\underline{A}}$ is symmetric.

Assume $b$ is symmetric, i.e. $b\left(\phi_j, \phi_i\right) = b\left(\phi_i, \phi_j\right)$. Then $\left(\underline{\underline{A}}\right)_{i,j} = \left(\underline{\underline{A}}\right)_{j,i}$, i.e. $\underline{\underline{A}} = \underline{\underline{A}}^T$, so $\underline{\underline{A}}$ is symmetric.

(c) Give the definnition of the coercivity condition, and show if $b$ is coercive, $\underline{\underline{A}}$ is positive definite.

Assume $b$ is coercive, i.e. $\gamma\|u\|^2 \leq b(u,u)$. We can transform the basis, $\phi_1, \phi_2, \ldots, \phi_N \to \tilde{\phi}_1, \tilde{\phi}_2, \ldots, \tilde{\phi}_N$, such that $b\left(\tilde{\phi}_j, \tilde{\phi}_i\right) = 0$ for $i \neq j$, which is equivalent to diagonalizing $\underline{\underline{A}}$. (We know that $\underline{\underline{A}}$ is diagonalizable since our problem is well-posed, thus $\underline{\underline{A}}$ is invertible.) Now with this diagonalized basis set, the eigenvalues of $\underline{\underline{A}}$ are $\lambda_i = b\left(\tilde{\phi}_i, \tilde{\phi}_i\right) \geq \gamma \left\|\tilde{\phi}_i\right\|^2 > 0$. Therefore, $\underline{\underline{A}}$ is positive definite.

(d) Still assume $b$ is coercive and denote $M$ and $\gamma$ the bounded coefficient and coercive coefficient, respectively, show

$$\|u - u_h\| \leq \frac{M}{\gamma} \inf_{v \in V_h} \|u - v\|$$

Assume $b$ is coercive and bounded, i.e. $\gamma\|u\|^2 \leq b(u,u)$ and $b(u,v) \leq M\|u\|\|v\|$. Using $b\left(u - u_h, v\right) = 0$ $\forall v \in V_h$, it can be shown that $b\left(u - u_h, u - v\right) = b\left(u - u_h, u - u_h\right)$. Then,

$$\gamma \left\|u - u_h\right\|^2 \leq b\left(u - u_h, u - u_h\right) = b\left(u - u_h, u - v\right) \leq M \left\|u - u_h\right\| \left\|u - v\right\| \qquad \forall v \in V_h$$
$$\|u - u_h\| \leq \frac{M}{\gamma} \inf_{v \in V_h} \|u - v\|$$

(e) For the following equation in $\Omega \in \mathbb{R}^2$ with zero Dirichlet boundary condition ($\Omega$ is compactly supported), write down its variational form, determine the space $V$, and show whether $b$ satisfies the coercivity condition. Find $M$ and $\gamma$, respectively.

$$-\vec{\nabla} \cdot (a\vec{\nabla} u) + cu = f$$

with $0 < \underline{a} \leq a < \bar{a}$ and $0 \leq \underline{c} \leq c < \bar{c}$.

Find some $u \in H_0^1$ such that

$$\left\langle a\vec{\nabla} u, \vec{\nabla} v \right\rangle + \langle cu, v\rangle = \langle f, v\rangle \qquad \forall v \in H_0^1$$

i.e.

$$b(u,v) = \left\langle a\vec{\nabla} u, \vec{\nabla} v \right\rangle + \langle cu, v\rangle$$

To check coercivity,

$$b(u,u) = \left\langle a\vec{\nabla}u, \vec{\nabla}u \right\rangle + \langle cu, u\rangle \geq \underline{a} \left\langle \vec{\nabla}u, \vec{\nabla}u \right\rangle + \underline{c}\,\langle u,u\rangle = \underline{a}\|u\|_{H^1}^2 + (\underline{c}-\underline{a})\,\|u\|_{L^2}^2 \geq (\underline{a}+\underline{c})\,\|u\|_{H^1}^2$$

so $\gamma = \underline{a} + \underline{c}$. For the bounded condition,

$$b(u,v) = \left\langle a\vec{\nabla}u, \vec{\nabla}v \right\rangle + \langle cu, v\rangle \leq \bar{a} \left\langle \vec{\nabla}u, \vec{\nabla}v \right\rangle + \bar{c}\,\langle u,v\rangle$$

noting that $\left\langle \vec{\nabla}u, \vec{\nabla}v \right\rangle \leq \|u\|_{H^1}\|v\|_{H^1}$ and $\langle u,v\rangle \leq \|u\|_{H^1}\|v\|_{H^1}$,

$$b(u,v) \leq (\bar{a}+\bar{c})\,\|u\|_{H^1}\|v\|_{H^1}$$

so $M = \bar{a} + \bar{c}$.

(f) For the problem above, if we choose $V_h$ to be a piecewise linear function space, show

$$\|u - u_h\|_{H^1} = \mathcal{O}(h)$$

and that if $a$ is a constant and $c = 0$

$$\|u - u_h\|_{L_2} = \mathcal{O}(h^2)$$

Using $b$ is coercive and bounded along with previous analysis, it can be shown that

$$\|u - u_h\|_{H^1} \leq \frac{M}{\gamma}\|u - v\|_{H^1} \qquad \forall v \in V_h$$

Simply pick $v = Iu$, where $Iu$ is the interpolation of $u$ using piecewise linear functions ($Iu \in V - h$). Then

$$\|u - u_h\|_{H^1} \leq \frac{M}{\gamma}\|u - Iu\|_{H^1} = \mathcal{O}(h)$$

(g) For the same equation in $\Omega \in \mathbb{R}^2$ with Neumann boundary contition

$$\partial_n u|_{\partial\Omega} = g$$

Write down its variational form, determine $V$, and show whether $b$ satisfies the coercivity condition.

## Euler-Bernoulli equation

2. Consider the Euler-Bernoulli equation

$$\frac{\partial^4 u}{\partial x^4} = f(x) \qquad 0 < x < 1$$

It is used to describe the deflection of $u$ of a clamped beam subject to a transversal force with intensity $f$.

(a) Show the equivalent varational form would be to find $u$ such that

$$\langle u'', v''\rangle = \langle f, v\rangle \qquad \forall v \in V$$

where $V = \{v : v \in C_1[0,1], v(0) = v'(0) = v(1) = v'(1) = 0, v \text{ piecewise continuous and bounded}\}$

(b) For an interval, $I = [a,b]$, define $P_3(I) = \left\{v : v(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3, x \in I\right\}$. Show that $v \in P_3(I)$ is uniquely determined by the values $v(a)$, $v'(a)$, $v(b)$, and $v'(b)$. Find the corresponding local basis functions. (Hint: count the nuber of degrees of freedom and use the values to fix the coefficients.)

(c) Construct a finite-dimensional subspace $V_h$ consisting of piecewise cubic polynomials on the mesh $0 = x_0 < x_1 < \cdots < x_{N+1} = 1$.

(d) Derive the error estimate

$$\left\|(u-u_h)''\right\|_2 \leq \left\|(u-v)''\right\|_2 \qquad \forall v \in V_h$$

You are given the estimate that cubic Hermite interpolation of $u$, denoted as $I_h u \in V_h$, satisfies the following

$$\left\|u''(x) - (I_h u)''(x)\right\| \lesssim h^2 \max_{0\leq\xi\leq 1} \left|u^{(4)}(\xi)\right|$$

show that

$$\left\|(u-u_h)''\right\| \leq Ch^2 \max_{0\leq\xi\leq 1} \left|u^{(4)}(\xi)\right|$$

(e) Write a computer program to solve

$$\begin{cases} \dfrac{\mathrm{d}^4 u}{\mathrm{d}x^4} = g(x) \\ u(0) = u'(0) = u'(1) = u'(1) = 0 \end{cases}$$

If we use

$$g(x) = \frac{\mathrm{d}^4}{\mathrm{d}x^4}\left(e^x x^2 (1-x)^2\right) = e^x\left(x^4 + 14x^3 + 49x^2 + 32x - 12\right)$$

the exact solution is $u(x) = e^x x^2 (1-x)^2$.

i. Give a brief description of your algorithm, in particular, the method you use to evaluate the load vector $\underline{b}$ (choose your favorite numerical integral method, but make sure the error here is not too big, and the error from $\underline{\underline{A}}$ still dominates)

ii. Tabulate the max-norm errors $e_N = \max|u_h(x_j) - u(x_j)|$ and show the numerical convergence order by performing linear regression of $\log e_N$ vs $\log N$

iii. Plot your finite element solution $u_h$ along with the real solution.

iv. Attach your code.