1. Classification of PDEs
   (a) 2nd-order linear equation in 2-dimensions
   $$a(x,y)\partial_{xx}u+2b(x,y)\partial_{xy}u+c(x,y)\partial_{yy}y+\cdots=\cdots$$
      i. $b^2-ac<0$: elliptic region
      ii. $b^2-ac>0$: hyperbolic region
      iii. $b^2-ac=0$: parabolic region
   (b) 2nd-order linear equation in higher dimension
   $$\sum a_{ij}\partial_{ij}u+\cdots=\cdots$$
      i. $\det\left(\underline{\underline{A}}\right)=0$: parabolic region
      ii. $\underline{\underline{A}}$ positive definite : elliptic region
      iii. otherwise : hyperbolic
   (c) Nonlinear equations
      i. Linearization
      ii. Quasi-linear : principal order term is linear
      iii. Semi-linear
         A. is quasi-linear
         B. principal order coefficient has no dependence on lower orders
2. Elliptic Equation
   (a) Laplace fundamental $\Delta\Phi=\delta$
   $$\Phi=\begin{cases}-\frac{1}{2\pi}\ln r & n=2\\ \frac{1}{n(n-1)\alpha(n)}r^{2-n} & n\geq 3\end{cases}$$
   (b) Poisson equation $\Delta u=f$
   $$u=\Phi*f$$
3. Mean Value Theorem : $\Delta u=0$ (u is a harmonic function)
   $$\frac{1}{A}\int_{\partial B(x,r)}u(y)\mathrm{d}y=\frac{1}{V}\int_{B(x,r)}u(y)\mathrm{d}y$$
   is a constant of r.
4. Maximum Principle : $\Delta u=0$ (follows from Mean Value Theorem)
   $$\max_{\Omega}u=\max_{\partial\Omega}u$$
   $$\min_{\Omega}u=\min_{\partial\Omega}u$$
5. Finite Differences
   (a) $\Delta_+u=u_{i+1}-u_i$
   (b) $\Delta_-u=u_i-u_{i-1}$
   (c) $\Delta_0u=u_{i+1}-u_{i-1}$
   (d) $\mathcal{E}u=u_{i+1}$: $\partial_x=\frac{\ln\mathcal{E}}{h}$ can be used to get approximations
6. Fourier Method
   (a) Suppose $\phi_m=\sin(m\pi x)$, $m=0,1,\ldots$, $x\in[0,1]$
   (b) Plug into numerical scheme to find eigenvalues, λ_m of numerical operator
   (c) $|\lambda_m|>\eta$ then $\|U\|_2^2\leq\frac{1}{\eta^2}\|f\|_2^2$
7. Fundamental Theorem : consistency + stability $\implies$ convergence
8. Finite Element Method
   (a) Variational Formultation
      i. (D) : $\mathcal{L}u=f$, $\mathcal{B}u=g$
      ii. (M) : $u=\arg\min_{v\in V}F[v]$, $F[v]=\frac{1}{2}b(v,v)-l(v)$
      iii. (V) : $b(u,v)=l(v)$, $\forall v\in V$
      iv. (D) $\iff$ (M) $\iff$ (V)
   (b) Assumptions
      i. V is a Hilber Space with $\|\dot{}\|_V$ norm
      ii. b is a bilinear operator
         A. continuous/bounded : $b(u,v)\leq M\|u\|_V\|v\|_V$
         B. coercive : $b(u,u)\geq\alpha\|u\|_V^2$
      iii. l is a linear operator
         A. continuous/bounded : $l(u)\leq\Lambda\|u\|_V$
   (c) Lax-Milgram (existence) : $\langle Au,v\rangle=\langle v,w\rangle$
      i. injective (b coercive) : $\alpha\|u\|\leq\|Au\|$
      ii. surjective (b coercive) : $\alpha\|z\|^2\leq\langle Az,z\rangle=0$
   (d) Conclusions
      i. (V) has a unique solution : uses b coercivity, l continuity, Lax-Milgram for existence
      ii. (M) $\iff$ (V)
         A. $g(\epsilon)=F[u+\epsilon v]$, then $\left.\frac{\mathrm{d}g}{\mathrm{d}\epsilon}\right|_{\epsilon=0}=0$
         B. (V) $\implies$ (M) : $F[u]\leq F[u+v]$
      iii. (M_h) $\iff$ (V_h)
      iv. $\underline{\underline{A}}$ is positive definite
      v. quasi-optimality : $\|u-u_h\|_V\leq\frac{M}{\alpha}\|u-v\|_V$