Theoretical Background

1. a is an unknown parameter. On $\mathbb{R}^2$, find out the region where the following equation is elliptic, hyperbolic, or parabolic, and study their dependence on a.

$$(a+x)\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} - y^2\frac{\partial^2 u}{\partial y^2} = 0$$

To classify a second order PDE of the form

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + 2b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2}$$

we must consider $b^2 - ac$. For the given equation this yeilds $x^2y^2 + (a+x)y^2$. Thus the regions depend on the parameter a. When $y = 0$ the equation is always parabolic. When $a > \frac{1}{4}$ the equation is hyperbolic everywhere. When $a = \frac{1}{4}$ the equations is parabolic at $x = -\frac{1}{2}$ and hyperbolic elsewhere. When $a < \frac{1}{4}$ then

$$\begin{cases} x = \dfrac{-1 \pm \sqrt{1-4a}}{2} & \text{parabolic} \\ \dfrac{-1-\sqrt{1-4a}}{2} < x < \dfrac{-1+\sqrt{1-4a}}{2} & \text{elliptic} \\ otherwise & \text{hyperbolic} \end{cases}$$

2. We prove the maximum principle for the solution to the Poisson equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

   Prove that there exists a constant C depending only on Ω such that

$$\max_{\bar{\Omega}} |u| \leq C\left(\max_{\partial\Omega}|g| + \max_{\bar{\Omega}}|f|\right)$$

   Here Ω is a bounded open subset of $\mathbb{R}^n$ with Γ being its boundary. $\bar{\Omega} = \Omega \cup \partial\Omega$ is its closure. Several hints:

   (a) You could use the following property: assume v is subharmonic, i.e. $-\Delta v \leq 0$ in Ω, then

$$v(x) \leq \frac{1}{V}\int_{B(x,r)} v(y)\mathrm{d}y$$

   Here $B(x,r)$ is a ball centered at x with radius r, and V is the volume of the ball. Given this, you could prove the maximum of v is achieved at the boundary $\partial\Omega$.

   (b) Try to show the function $u + \frac{|x|^2}{2n}\lambda$ is subharmonic. Here $\lambda = \max_{\bar{\Omega}}|f|$.

Assume v is subharmonic, then

$$v(x) \leq \frac{1}{V}\int_{B(x,r)} v(y)\mathrm{d}y$$

This implies that the maximum of v is achieved at the boundary $\partial\Omega$.

Finite Differencing

3. (a) Prove $\Delta_- + \Delta_+ = \left(\mathcal{E}^{-\frac{1}{2}} + \mathcal{E}^{\frac{1}{2}}\right)\Delta_0$ and $\Delta_-\Delta_+ = \Delta_0^2$. Here $\mathcal{E}$ is the shifting operator, $(\mathcal{E}u)_j = u_{j+1}$, and the definitions for Δ_+ and Δ_- are consistent with what we has in class. $(\Delta_0 u)_j = u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}$

First consider the left-hand side of the first relation.

$$(\Delta_- + \Delta_+)\, u_i = u_i - u_{i-1} + u_{i+1} - u_i = u_{i+1} - u_{i-1}$$

Then, the right-hand side of the first relation.

$$\left(\mathcal{E}^{-\frac{1}{2}} + \mathcal{E}^{\frac{1}{2}}\right)\Delta_0 u_i = \left(\mathcal{E}^{-\frac{1}{2}} + \mathcal{E}^{\frac{1}{2}}\right)\left(u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}\right) = u_i + u_{i+1} - u_{i-1} - u_i = u_{i+1} - u_{i-1}$$

Thus, the two operators are equal. Next, consider the left-hand side of the second relation.

$$\Delta_-\Delta_+ u_i = \Delta_-\left(u_{i+1} - u_i\right) = u_{i+1} - u_i - u_i + u_{i-1} = u_{i+1} - 2u_i + u_{i-1}$$

Then, the right-had side of the second relation.

$$\Delta_0^2 u_i = \Delta_0\left(u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}\right) = u_{i+1} - u_i - u_i + u_{i-1} = u_{i+1} - 2u_i + u_{i-1}$$

Thus, the two operators are equal.

(b) Determine the constants c and d so that

$$\partial_x^2 u(x) - \frac{1}{h^2}\left(\Delta_+^2 - \Delta_+^3\right)u(x) = ch^2\partial_x^4 u(x) + \mathcal{O}(h^3)$$
$$\partial_x^2 u(x) - \frac{1}{h^2}\Delta_0^2 u(x) = dh^2\partial_x^4 u(x) + \mathcal{O}(h^4)$$

Here we assume $u(x)$ is smooth enough.

First, we will determine c.

$$\begin{aligned}
&\partial_x^2 u(x) - \frac{1}{h^2}\left(\Delta_+^2 - \Delta_+^3\right)u(x)\\
&= \partial_x^2 u(x) - \frac{1}{h^2}\left[u(x+2h) - 2u(x+h) + u(x) - u(x+3h) + 3u(x+2h) - 3u(x+h) + u(x)\right]\\
&= \partial_x^2 u(x) - \frac{1}{h^2}\left[2u(x) - 5u(x+h) + 4u(x+2h) - u(x+3h)\right]\\
&= \partial_x^2 u(x) - \frac{1}{h^2}\Bigg[2u(x) - 5\left[u(x) + h\partial_x u(x) + \frac{h^2}{2}\partial_x^2 u(x) + \frac{h^3}{6}\partial_x^3 u(x) + \frac{h^4}{24}\partial_x^4 u(x) + \mathcal{O}(h^5)\right]\\
&\qquad +4\left[u(x) + 2h\partial_x u(x) + \frac{(2h)^2}{2}\partial_x^2 u(x) + \frac{(2h)^3}{6}\partial_x^3 u(x) + \frac{(2h)^4}{24}\partial_x^4 u(x) + \mathcal{O}(h^5)\right]\\
&\qquad -\left[u(x) + 3h\partial_x u(x) + \frac{(3h)^2}{2}\partial_x^2 u(x) + \frac{(3h)^3}{6}\partial_x^3 u(x) + \frac{(3h)^4}{24}\partial_x^4 u(x) + \mathcal{O}(h^5)\right]\Bigg]\\
&= \partial_x^2 u(x) - \frac{1}{h^2}\left[h^2\partial_x^2 u(x) - \frac{22}{24}h^4\partial_x^4 u(x) + \mathcal{O}(h^5)\right] = \frac{11}{12}h^2\partial_x^4 u(x) + \mathcal{O}(h^3)
\end{aligned}$$

Thus, $c = \frac{11}{12}$. Next, determine d.

$$\begin{aligned}
\partial_x^2 u(x) - \frac{1}{h^2}\Delta_0^2 u(x) &= \partial_x^2 u(x) - \frac{1}{h^2}\left[u(x+h) - 2u(x) + u(x-h)\right]\\
&= \partial_x^2 u(x) - \frac{1}{h^2}\Bigg[u(x) + h\partial_x u(x) + \frac{h^2}{2}\partial_x^2 u(x) + \frac{h^3}{6}\partial_x^3 u(x) + \frac{h^4}{24}\partial_x^4 u(x) + \frac{h^5}{120}\partial_x^5 u(x) + \mathcal{O}(h^6)\\
&\quad -2u(x) + u(x) - h\partial_x u(x) + \frac{h^2}{2}\partial_x^2 u(x) - \frac{h^3}{6}\partial_x^3 u(x) + \frac{h^4}{24}\partial_x^4 u(x) - \frac{h^5}{120}\partial_x^5 u(x) + \mathcal{O}(h^6)\Bigg]\\
&= \partial_x^2 u(x) - \frac{1}{h^2}\left[h^2\partial_x^2 u(x) + \frac{2}{24}h^4\partial_x^4 u(x) + \mathcal{O}(h^6)\right] = -\frac{1}{12}h^2\partial_x^4 u(x) + \mathcal{O}(h^4)
\end{aligned}$$

Thus, $d = -\frac{1}{12}$.

(c) The two identities above tell you how to approximate ∂_x^2 using forward differencing and central differencing. How many grid points do you need to get the second order approximation respectively using these two methods?

For forward differencing, you need 4 points to get a second-order approximation of ∂_x^2. Whereas, for central differencing, you only need 3 points to get a second-order approximation of ∂_x^2.

4. Write a computer code (using your favorite language) to determine, to highest possible order, a finite difference approximation to $u''(x)$ based on the 5-point stencil $\{x-h, x-\frac{1}{2}h, x, x+h, x+2h\}$

$$u''(x) \approx c_0 u(x-h) + c_1 u\left(x - \frac{h}{2}\right) + c_2 u(x) + c_3 u(x+h) + c_4 u(x+2h)$$

(a) Compute c_j.

(b) Check the order.

The linear system that must be solved is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & -\frac{1}{2} & 0 & 1 & 2 \\ 1 & \frac{1}{4} & 0 & 1 & 4 \\ -1 & -\frac{1}{8} & 0 & 1 & 8 \\ 1 & \frac{1}{16} & 0 & 1 & 16 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{2}{h^2} \\ 0 \\ 0 \end{bmatrix}$$

The solution to this linear system is

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \frac{1}{h^2} \begin{bmatrix} \frac{1}{3} \\ \frac{32}{15} \\ -4 \\ \frac{5}{3} \\ -\frac{2}{15} \end{bmatrix}$$

This method is $\mathcal{O}(h^3)$.

Finite difference method for an elliptic equation

5. We used Fourier method for stability analysis in 1D in class. Carry out the same analysis for 2D.

6. Derive the expldit formulae for Green's functions in 1D and prove they are piecewise linear function.

7. In 2D, to compute the Poisson equation, $u'' = f$ with zero boundary condition on a rectangular domain, we discretize the domain by even grid points with mesh size h. Denote A the associated discretization matrix with central differencing (5-stencil). Show $||A^{-1}||_\infty$ is bounded independent of h. Explain why it suggests that the order of accuracy of the numerical method is second order. (Hint: (a) $\left\|A^{-1}\right\|_\infty = \sup \dfrac{\left\|A^{-1}v\right\|_\infty}{\|v\|_\infty}$. (b) Fundeamental theorem for numerical convergence.)

8. Prove the discrete version of the Poincaré inequality,

$$\sum_{m,n} |U_{m,n}|^2 \leq \sum_{m,n} |\partial_x U_{m,n}|^2$$

Here U is a matrix on 2D with zero boundary condition, and $\partial_x U_{m,n}$ is the forward Euler representation of differentiation, definde by, $\partial_x U_{m,n} = \frac{1}{h}\left(U_{m+1,n} - U_{m,n}\right)$.