# Theoretical Background

1. $a$ is an unknown parameter. On $\mathbb{R}^2$, find out the region where the following equation is elliptic, hyperbolic, or parabolic, and study their dependence on $a$.

$$(a+x)\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} - y^2\frac{\partial^2 u}{\partial y^2} = 0$$

To classify a second order PDE of the form

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + 2b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2}$$

we must consider $b^2 - ac$. For the given equation this yeilds $x^2y^2 + (a+x)y^2$. Thus the regions depend on the parameter $a$. When $a > \frac{1}{4}$ the equation is hyperbolic everywhere. When $a = \frac{1}{4}$ the equations is parabolic at $x = -\frac{1}{2}$ and hyperbolic elsewhere. When $a < \frac{1}{4}$ then

$$\begin{cases} x = \dfrac{-1 \pm \sqrt{1-4a}}{2} & \text{parabolic} \\ \dfrac{-1-\sqrt{1-4a}}{2} < x < \dfrac{-1+\sqrt{1-4a}}{2} & \text{elliptic} \\ otherwise & \text{hyperbolic} \end{cases}$$

2. We prove the maximum principle for the solution to the Poisson equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Prove that there exists a constant $C$ depending only on $\Omega$ such that

$$\max_{\bar{\Omega}} |u| \leq C\left(\max_{\partial\Omega}|g| + \max_{\bar{\Omega}}|f|\right)$$

Here $\Omega$ is a bounded open subset of $\mathbb{R}^n$ with $\Gamma$ being its boundary. $\bar{\Omega} = \Omega \cup \partial\Omega$ is its closure. Several hints:

(a) You could use the following property: assume $v$ is subharmonic, i.e. $-\Delta v \leq 0$ in $\Omega$, then

$$v(x) \leq \frac{1}{V}\int_{B(x,r)} v(y)\mathrm{d}y$$

Here $B(x,y)$ is a ball centered at $x$ with radius r, and $V$ is the volume of the ball. Given this, you could prove the maximum of $v$ is achieved at the boundary $\partial\Omega$.

(b) Try to show the function $u + \frac{|x|^2}{2n}\lambda$ is subharmonic. Here $\lambda = \max_{\bar{\Omega}}|f|$.

# Finite Differencing

3. (a) Prove $\Delta_- + \Delta_+ = \left(\mathcal{E}^{-\frac{1}{2}} + \mathcal{E}^{\frac{1}{2}}\right)\Delta_0$ Here $\mathcal{E}$ is the shifting operator, $(\mathcal{E}u)_j = u_{j+1}$, and the definitions for $\Delta_+$ and $\Delta_-$ are consistent with what we has in class. $(\Delta_0 u)_j = u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}$

(b) Determine the constants $c$ and $d$ so that

$$\partial_x^2 u(x) - \frac{1}{h^2}\left(\Delta_+^2 - \Delta_+^3\right)u(x) = ch^2\partial_x^4 u(x) + \mathcal{O}(h^3)$$
$$\partial_x^2 u(x) - \frac{1}{h^2}\Delta_0^2 u(x) = dh^2\partial_x^4 u(x) + \mathcal{O}(h^4)$$

Here we assume $u(x)$ is smooth enough.

(c) The two identities above tell you how to approximate $\partial_x^2$ using forward differencing and central differencing. How many grid points do you need to get the second order approximation respectively using these two methods?

4. Write a computer code (using your favorite language) to determine, to highest possible order, a finite difference approximation to $u''(x)$ based on the 5-point stencil $\{x-h, x-\frac{1}{2}h, x, x+h, x+2h\}$

$$u''(x) \approx c_0 u(x-h) + c_1 u\left(x - \frac{h}{2}\right) + c_2 u(x) + c_3 u(x+h) + c_4 u(x+2h)$$

   (a) Compute $c_j$.

   (b) Check the order.

Finite difference method for an elliptic equation

5. We used Fourier method for stability analysis in 1D in class. Carry out the same analysis for 2D.

6. Derive the expldit formulae for Green's functions in 1D and prove they are piecewise linear function.

7. In 2D, to compute the Poisson equation, $u'' = f$ with zero boundary condition on a rectangular domain, we discretize the domain by even grid points with mesh size $h$. Denote $A$ the associated discretization matrix with central differencing (5-stencil). Show $||A^{-1}||_\infty$ is bounded independent of $h$. Explain why it suggests that the order of accuracy of the numerical method is second order. (Hint: (a) $\left\|A^{-1}\right\|_\infty = \sup \dfrac{\left\|A^{-1}v\right\|_\infty}{\|v\|_\infty}$. (b) Fundeamental theorem for numerical convergence.)

8. Prove the discrete version of the Poincaré inequality,

$$\sum_{m,n} |U_{m,n}|^2 \leq \sum_{m,n} |\partial_x U_{m,n}|^2$$

   Here $U$ is a matrix on 2D with zero boundary condition, and $\partial_x U_{m,n}$ is the forward Euler representation of differentiation, definde by, $\partial_x U_{m,n} = \frac{1}{h}\left(U_{m+1,n} - U_{m,n}\right)$.